POPULATION EMPIRICAL LIKELIHOOD FOR NONPARAMETRIC INFERENCE IN SURVEY SAMPLING

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Supplementary Material

In this supplement material, we provided proofs for Theorems in the paper.

S1 Proof of Theorem 1

We first prove the consistency of $(\hat{\theta}, \hat{\lambda})$. Let

$$\hat{Q}_1(\theta,\lambda) = \frac{1}{n_B}\sum_{i=1}^{N}\frac{I_i\pi_i^{-1}f_N U_i(\theta)}{1+\lambda'\Psi_i}, \quad \hat{Q}_2(\theta,\lambda) = \frac{1}{n_B}\sum_{i=1}^{N}\frac{\Psi_i}{1+\lambda'\Psi_i}, \tag{S1.1}$$

where $f_N = n_B/N$, $\Psi_i = f_N\big((I_i\pi_i^{-1}-1), I_i\pi_i^{-1}(h_i' - \bar{h}_N')\big)'$. Let $\hat{\lambda} = \rho\delta$, where $||\delta|| = 1$, so according to (S1.1), we have

$$\begin{aligned}
0 &= ||n_B^{-1}\sum_{i=1}^{N}\frac{\Psi_i}{1+\hat{\lambda}'\Psi_i}|| \geq \big|n_B^{-1}\sum_{i=1}^{N}\frac{\delta'\Psi_i}{1+\rho\delta'\Psi_i}\big| \\
&= \big|n_B^{-1}\sum_{i=1}^{N}\frac{\delta'\Psi_i(1+\rho\delta'\Psi_i-\rho\delta'\Psi_i)}{1+\rho\delta'\Psi_i}\big| \\
&= \big|n_B^{-1}\sum_{i=1}^{N}\delta'\Psi_i - n_B^{-1}\sum_{i=1}^{N}\frac{\rho\delta'\Psi_i\Psi_i'\delta}{1+\rho\delta'\Psi_i}\big| \\
&\geq \big||n_B^{-1}\sum_{i=1}^{N}\delta'\Psi_i| - |n_B^{-1}\sum_{i=1}^{N}\frac{\rho\delta'\Psi_i\Psi_i^T\delta}{1+\rho\delta'\Psi_i}|\big|.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\frac{1}{n_B}\sum_{i=1}^{N}\delta'\Psi_i| &= |\frac{1}{n_B}\sum_{i=1}^{N}\frac{\rho\delta'\Psi_i\Psi_i'\delta}{1+\rho\delta'\Psi_i}| \\
&\geq \big|\delta'\frac{1}{n_B}\sum_{i=1}^{N}\Psi_i\Psi_i'\delta\big|\frac{|\rho|}{1+|\rho|u^*},
\end{aligned} \tag{S1.2}$$

where $u^* = \max_{i\in A} ||\Psi_i||$.

Under assumption (C3), we have $n_B^{-1}\sum_{i=1}^N \Psi_i\Psi_i' = \Sigma_\Psi + o_p(1)$, and Σ_Ψ is a positive definite matrix. Let λ_p be the smallest eigenvalue of Σ_Ψ, then $\lambda_p > 0$. So, the following holds

$$\left|\delta' n_B^{-1}\sum_{i=1}^N \Psi_i\Psi_i'\delta\right| \geq \lambda_p + o_p(1). \tag{S1.3}$$

In addition, according to Assumption (C3),

$$\frac{1}{n_B}\sum_{i=1}^N \delta'\Psi_i = O_p(n_B^{-1/2}). \tag{S1.4}$$

By (S1.2), (S1.3), (S1.4) and assumptions (C5), (C6),

$$\lambda_p|\rho| = O_p(n_B^{-1/2}) + o_p(|\rho|).$$

Thus, we have $|\rho| = O_p(n_B^{-1/2})$, which means $||\hat\lambda|| = O_p(n_B^{-1/2})$.

Because $\max_{i\in A}|\hat\lambda'\Psi_i| = O_p(n_B^{-1/2})o_p(n_B) = o_p(1)$ and assumption (C4), we can apply Taylor expansion and get

$$\begin{aligned} 0 &= \frac{1}{n_B}\sum_{i=1}^N \frac{f_N I_i \pi_i^{-1} U_i(\hat\theta)}{1+\hat\lambda'\Psi_i} \\ &= \frac{1}{n_B}\sum_{i=1}^N f_N I_i\pi_i^{-1}U_i(\hat\theta) - \left\{\frac{1}{n_B}\sum_{i=1}^N f_N I_i\pi_i^{-1}U_i(\hat\theta)\Psi_i'\right\}\hat\lambda \\ &+ O_p(n_B^{-1}). \end{aligned} \tag{S1.5}$$

By assumption (C4), it can be shown that

$$n_B^{-1}\sum_{i=1}^N f_N I_i\pi_i^{-1}U_i(\hat\theta)\Psi_i' = O_p(1), \tag{S1.6}$$

and

$$\sup_{\theta\in\Theta}||\frac{1}{n_B}\sum_{i=1}^N f_N I_i\pi_i^{-1}U_i(\theta) - \frac{1}{N}\sum_{i=1}^N U_i(\theta)|| \to^p 0, \tag{S1.7}$$

so according to (S1.5), (S1.6) and (S1.7),

$$\begin{aligned} 0 &= p\lim\big|\frac{1}{n_B}\sum_{i=1}^{N} f_N I_i \pi_i^{-1} U_i(\hat\theta)\big| \\ &= p\lim\big|\frac{1}{n_B}\sum_{i=1}^{N} f_N I_i \pi_i^{-1} U_i(\hat\theta) - \frac{1}{N}\sum_{i=1}^{N} U_i(\hat\theta) + \frac{1}{N}\sum_{i=1}^{N} U_i(\hat\theta)\big| \\ &\geq p\lim\big||\frac{1}{n_B}\sum_{i=1}^{N} f_N I_i \pi_i^{-1} U_i(\hat\theta) - \frac{1}{N}\sum_{i=1}^{N} U_i(\hat\theta)| - |\frac{1}{N}\sum_{i=1}^{N} U_i(\hat\theta)|\big| \\ &= \big|\frac{1}{N}\sum_{i=1}^{N} U_i(\hat\theta)\big|. \end{aligned} \quad \text{(S1.8)}$$

By (S1.8), assumptions (C1) and (C2), we have $\hat\theta \to^p \theta_0$. Hence,

$$(\hat\theta_{POEL}, \hat\lambda) \to^p (\theta_0, 0). \quad \text{(S1.9)}$$

According to (S1.9), assumptions (C2) and (C4), we can apply the standard arguments using Taylor expansion to get

$$0 = \hat Q_1(\hat\theta, \hat\lambda) = \hat Q_1(\theta_0, 0) + \frac{\partial \hat Q_1(\theta_0, 0)}{\partial \theta'}(\hat\theta - \theta_0) + \frac{\partial \hat Q_1(\theta_0, 0)}{\partial \lambda'}(\hat\lambda - 0) + o_p(\delta_n),$$

and

$$0 = \hat Q_2(\hat\theta, \hat\lambda) = \hat Q_2(\theta_0, 0) + \frac{\partial \hat Q_2(\theta_0, 0)}{\partial \theta'}(\hat\theta - \theta_0) + \frac{\partial \hat Q_2(\theta_0, 0)}{\partial \lambda'}(\hat\lambda - 0) + o_p(\delta_n),$$

with $\delta_n = ||\hat\theta - \theta_0|| + ||\hat\lambda||$. Let

$$S_n = \begin{pmatrix} \partial \hat Q_1(\theta_0, 0)/\partial\lambda & \partial \hat Q_1(\theta_0, 0)/\partial\theta \\ \partial \hat Q_2(\theta_0, 0)/\partial\lambda & \partial \hat Q_2(\theta_0, 0)/\partial\theta \end{pmatrix}.$$

Under the existence of moments, we can obtain

$$S_n \to^p \begin{pmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & 0 \end{pmatrix},$$

and

$$||S_{11}^* - S_{11}|| = o_p(1), \quad ||S_{12}^* - S_{12}|| = o_p(1), \quad ||S_{21}^* - S_{21}|| = o_p(1), \quad \text{(S1.10)}$$

where

$$S_{11} = -\big(\frac{1}{N}\sum_{i=1}^{N} f_N(\frac{1}{\pi_i} - 1)U_i, \frac{1}{N}\sum_{i=1}^{N} f_N \frac{1}{\pi_i} U_i(h_i - \bar h_N)'\big), \quad S_{12} = N^{-1}\sum_{i=1}^{N} \frac{\partial U_i(\theta_0)}{\partial\theta}, \quad \text{(S1.11)}$$

and

$$S_{21} = -\begin{pmatrix} N^{-1}\sum_{i=1}^{N} f_N(\pi_i^{-1}-1) & N^{-1}\sum_{i=1}^{N} f_N(\pi_i^{-1}-1)(h_i-\bar{h}_N)' \\ N^{-1}\sum_{i=1}^{N} f_N(\pi_i^{-1}-1)(h_i-\bar{h}_N) & N^{-1}\sum_{i=1}^{N} f_N\pi_i^{-1}(h_i-\bar{h}_N)^{\otimes 2} \end{pmatrix}. \quad \text{(S1.12)}$$

According to assumption (C3),

$$\hat{Q}_1(\theta_0,0) = N^{-1}\sum_{i=1}^{N} I_i\pi_i^{-1}U_i(\theta_0) = O_p(n_B^{-1/2}), \quad \hat{Q}_2(\theta_0,0) = N^{-1}\sum_{i=1}^{N}\Psi_i(\theta_0,0) = O_p(n_B^{-1/2}),$$

so we have $\delta_n = O_p(n_B^{-1/2})$. Also, according to (S1.10), (S1.11) and (S1.12), after some algebra,

$$\hat{\lambda} = -S_{21}^{-1}\hat{Q}_2(\theta_0,0) + o_p(n_B^{-1/2}), \quad \text{(S1.13)}$$

and

$$\begin{aligned} \hat{\theta}-\theta_0 &= -S_{12}^{-1}\left\{\hat{Q}_1(\theta_0,0) - S_{11}S_{21}^{-1}\hat{Q}_2(\theta_0,0)\right\} + o_p(n_B^{-1/2}) \\ &= -\tau\left\{\hat{Q}_1(\theta_0,0) - B^*\hat{Q}_2(\theta_0,0)\right\} + o_p(n_B^{-1/2}), \quad \text{(S1.14)} \end{aligned}$$

where $\tau = S_{12}$, $B^* = \Omega_1\Omega_2^{-1}$, $\Omega_1 = -(N\alpha_N)^{-1}S_{11}$ and $\Omega_2 = -(N\alpha_N)^{-1}S_{21}$. Hence, (3.7) in Theorem 1 is proved. Result (3.10) can be obtained by (S1.14) and assumptions (C3), (C4).

S2 Proof of Theorem 2

Maximizing (3.1) subject to

$$\sum_{i=1}^{N}\omega_i = 1, \quad \sum_{i=1}^{N}\omega_i f_N(\frac{I_i}{\pi_i}-1) = 0, \quad \sum_{i=1}^{N}\omega_i\frac{I_i}{\pi_i}f_N(h_i-\bar{h}_N) = 0,$$

leads, after some algebra, to

$$l(\hat{\theta}) = \sum_{i=1}^{N}\log(\omega_i(\hat{\theta})) = -N\log(N) - \sum_{i=1}^{N}\log(1+\hat{\lambda}'\Psi_{i1}),$$

where $\Psi_{i1} = (f_N(I_i\pi_i^{-1}-1), I_i\pi_i^{-1}f_N(h_i-\bar{h}_N)')'$. Similarly, consider maximizing (3.1) subject to

$$\sum_{i=1}^{N}\omega_i = 1, \quad \sum_{i=1}^{N}\omega_i f_N(\frac{I_i}{\pi_i}-1) = 0, \quad \sum_{i=1}^{N}\omega_i\frac{I_i}{\pi_i}f_N(h_i-\bar{h}_N) = 0,$$

and

$$\sum_{i=1}^{N} \omega_i f_N r_i = 0,$$

with $r_i = I_i \pi_i^{-1} U_i(\theta_0) - B_1^*(I_i \pi_i^{-1} - 1) - B_2^* I_i \pi_i^{-1}(h_i - \bar{h}_N)$ and $B^* = (B_1^*, B_2^*) = S_{11} S_{21}^{-1}$, where S_{11}, S_{21} are defined in (S1.11) and (S1.12) of the proof of Theorem 1. After some algebra,

$$l(\theta_0) = \sum_{i=1}^{N} \log(\omega_i(\theta_0)) = -N \log(N) - \sum_{i=1}^{N} \log(1 + \lambda_0' \Psi_{i2}),$$

where $\Psi_{i2} = (f_N(I_i \pi_i^{-1} - 1), I_i \pi_i^{-1} f_N (h_i - \bar{h}_N)', f_N r_i')'$. So, we can write

$$R_n(\theta_0) = 2\big\{ \sum_{i=1}^{N} \log(1 + \lambda_0' \Psi_{i2}) - \sum_{i=1}^{N} \log(1 + \hat{\lambda}' \Psi_{i1}) \big\}, \tag{S2.1}$$

and λ_0 is the solution of $\hat{Q}_3(\theta_0, \lambda) = 0$ with

$$\hat{Q}_3(\theta_0, \lambda) = \frac{1}{n_B} \sum_{i=1}^{N} \frac{\Psi_{i2}(\theta_0)}{1 + \lambda' \Psi_{i2}(\theta_0)}.$$

By the same argument for (S1.9), we have $\lambda_0 \to^p 0$. We can apply a Taylor expansion to get

$$0 = \hat{Q}_3(\theta_0, \lambda_0) = \hat{Q}_3(\theta_0, 0) + \frac{\partial \hat{Q}_3(\theta_0, 0)}{\partial \lambda} \lambda_0 + o_p(||\lambda_0||).$$

According to assumption (C3), $\hat{Q}_3(\theta_0, 0) = n_B^{-1} \sum_{i=1}^{N} \Psi_{i2}(\theta_0, 0) = O_p(n_B^{-1/2})$, hence $||\lambda_0|| = O_p(n_B^{-1/2})$, so

$$\lambda_0 = -S^{-1} \hat{Q}_3(\theta_0, 0) + o_p(n_B^{-1/2}), \tag{S2.2}$$

with

$$S = \begin{pmatrix} S_{21} & 0 \\ 0 & S_r \end{pmatrix}, \tag{S2.3}$$

where

$$\begin{aligned} S_r &= f_N\{-N V_{poi}(\bar{r}_N) - N^{-1} \sum_{i=1}^{N} U_i^{\otimes 2} - B_2^* N^{-1} \sum_{i=1}^{N} (h_i - \bar{h}_N)^{\otimes 2} B_2^{*'} \\ &+ N^{-1} \sum_{i=1}^{N} U_i (h_i - \bar{h}_N)' B_2^{*'} + B_2^* N^{-1} \sum_{i=1}^{N} (h_i - \bar{h}_N) U_i' \}, \end{aligned} \tag{S2.4}$$

and $\bar{r}_N = N^{-1} \sum_{i=1}^{N} r_i$, S_{21} is defined in (S1.12) in the proof of Theorem 1 and V_{poi} is the variance under Poisson sampling.

According to assumption (C6), $n_B/N = o(1)$ and (S2.4), it can be shown that

$$||S_r + f_N N V_{poi}(\bar{r}_N)|| = o(1). \tag{S2.5}$$

Similarly, by a Taylor expansion with respect to $\lambda_0 = 0$,

$$2\sum_{i=1}^{N}\log(1+\lambda_0'\Psi_{i2}) = 2\sum_{i=1}^{N}\lambda_0'\Psi_{i2} - \sum_{i=1}^{N}\lambda_0'\Psi_{i2}\Psi_{i2}'\lambda_0 + o_p(1). \tag{S2.6}$$

According to (S2.2), we have

$$\partial\hat{Q}_3(\theta_0,0)/\partial\lambda = -n_B^{-1}\sum_{i=1}^{N}\Psi_{i2}(\theta_0)\Psi_{i2}'(\theta_0) \to^p S. \tag{S2.7}$$

By plugging (S2.2) into (S2.6) and according to (S2.7), we have

$$2\sum_{i=1}^{N}\log(1+\lambda_0'\Psi_{i2}(\theta_0)) = -n_B\hat{Q}_3'(\theta_0,0)S^{-1}\hat{Q}_3(\theta_0,0) + o_p(1). \tag{S2.8}$$

Similarly, according to (S1.13) and by using a Taylor expansion around $\hat{\lambda} = 0$,

$$2\sum_{i=1}^{N}\log(1+\hat{\lambda}'\Psi_{i1}(\hat{\theta})) \quad = \quad -n_B\hat{Q}_2'(\theta_0,0)S_{21}^{-1}\hat{Q}_2(\theta_0,0) + o_p(1). \tag{S2.9}$$

By assumption (C3) and (C4), we can apply the central limit theorem to get

$$V_{poi}^{-1/2}(\bar{r}_N)\bar{r}_N \to^d N(0,I). \tag{S2.10}$$

Therefore, plugging (S2.8) and (S2.9) into (S2.1) and by (S2.3), we have

$$\begin{aligned} R_n(\theta_0) &= -n_B\hat{Q}_3'(\theta_0,0)S^{-1}\hat{Q}_3(\theta_0,0) + n_B\hat{Q}_2'(\theta_0,0)S_{21}^{-1}\hat{Q}_2(\theta_0,0) + o_p(1) \\ &= \bar{r}_N(-n_B^{-1}S_r)^{-1}(\bar{r}_N)' + o_p(1). \end{aligned} \tag{S2.11}$$

According to (S2.5), (S2.11) and (S2.10),

$$R_n(\theta_0) = \bar{r}_N\left\{V_{poi}(\bar{r}_N)\right\}^{-1}(\bar{r}_N)' + o_p(1) \to^d \chi_p^2,$$

where $\bar{r}_N = N^{-1}\sum_{i=1}^{N} r_i$, and p is the dimension of θ_0.

S3 Proof of Theorem 3

Similar as the proof of Theorem 1, $\hat{\theta}$ can be obtained by solving $\hat{Q}_1(\theta,\lambda) = 0$ and $\hat{Q}_2(\theta,\lambda) = 0$, where $\hat{Q}_1(\theta,\lambda)$ and $\hat{Q}_2(\theta,\lambda)$ are defined in (S1.1) with π_i, Ψ_i replaced by p_i, Ψ_i^*, and $\Psi_i^* = \left(f_N(I_ip_i^{-1}-1)z_i^*, f_NI_ip_i^{-1}(h_i'-\bar{h}_N')\right)'$, with $z_i^* = (1, z_i')'$. Without loss of generality, we assume $\bar{z}_N = 0$ and $n_BN^{-2}\sum_{i=1}^{N}(1-p_i)p_i^{-1}z_i^2 = 1$. Hence, according to assumption (C9) and (C10) in Section 4,

$$\begin{aligned} \pi_i &= Pr(i\in s|\hat{Q}_{p,n}\le\gamma^2) = \frac{Pr(\hat{Q}_{p,n}\le\gamma^2|i\in s)Pr(i\in s)}{Pr(\hat{Q}_{p,n}\le\gamma^2)} \\ &= p_i\left\{1 + C_\gamma\eta_i + o_p(n_B^{-1})\right\}, \end{aligned} \tag{S3.1}$$

and

$$\begin{aligned}\pi_{ij} &= Pr(i,j\in s|\hat{Q}_{p,n}\leq\gamma^2)=\frac{Pr(\hat{Q}_{p,n}\leq\gamma^2|i,j\in s)Pr(i,j\in s)}{Pr(\hat{Q}_{p,n}\leq\gamma^2)}\\ &= p_{ij}\left\{1+C_\gamma(\eta_i+\eta_j)+o_p(n_B^{-1})\right\},\end{aligned}\tag{S3.2}$$

with $C_\gamma=g_{1N}(\gamma^2)G_N^{-1}(\gamma^2)$.

According to (S3.1), (S3.2) and by using a similar argument as the proof of Theorem 1, it can be shown that $(\hat{\theta},\hat{\lambda})\rightarrow^p(\theta_0,0)$. After some algebra,

$$\hat{\theta}-\theta_0=-S_{12}^{-1}\left\{\hat{Q}_1(\theta_0,0)-S_{11}S_{21}^{-1}\hat{Q}_2(\theta_0,0)\right\}+o_p(n_B^{-1/2}),\tag{S3.3}$$

where

$$S_{11}=-(N^{-1}\sum_{i=1}^N f_N(\pi_ip_i^{-2}-\pi_ip_i^{-1})U_iz_i^{*'},N^{-1}\sum_{i=1}^N f_N\pi_ip_i^{-2}U_i(h_i-\bar{h}_N)'),$$

$$S_{12}=\frac{1}{N}\sum_{i=1}^N\frac{\pi_i}{p_i}\frac{\partial U_i(\theta_0)}{\partial\theta}$$

and

$$S_{21}=-N^{-1}\begin{pmatrix}\sum_{i=1}^N f_N(\pi_ip_i^{-2}-2\pi_ip_i^{-1}+1)z_i^*z_i^{*'} & \sum_{i=1}^N f_N(\pi_ip_i^2-\pi_ip_i^{-1})z_i^*(h_i-\bar{h}_N)'\\ \sum_{i=1}^N f_N(\pi_ip_i^2-\pi_ip_i^{-1})(h_i-\bar{h}_N)z_i^{*'} & \sum_{i=1}^N f_N\pi_ip_i^{-2}(h_i-\bar{h}_N)^{\otimes 2}\end{pmatrix}.$$

By (S3.1) and (S3.2),

$$||\frac{1}{N}\sum_{i=1}^N\frac{I_i}{p_i}U_i(\theta_0)-\frac{1}{N}\sum_{i=1}^N\frac{I_i}{\pi_i}U_i(\theta_0)||=o_p(n_B^{-1/2}).$$

Hence,

$$\frac{1}{N}\sum_{i=1}^N\frac{I_i}{p_i}U_i(\theta_0)=\frac{1}{N}\sum_{i=1}^N\frac{I_i}{\pi_i}U_i(\theta_0)+o_p(n_B^{-1/2}).\tag{S3.4}$$

Similarly, it can be shown that

$$\frac{1}{N}\sum_{i=1}^N(\frac{I_i}{p_i}-1)z_i^*=\frac{1}{N}\sum_{i=1}^N(\frac{I_i}{\pi_i}-1)z_i^*+o_p(n_B^{-1/2}),\tag{S3.5}$$

$$\frac{1}{N}\sum_{i=1}^N\frac{I_i}{p_i}(h_i-\bar{h}_N)=\frac{1}{N}\sum_{i=1}^N\frac{I_i}{\pi_i}(h_i-\bar{h}_N)+o_p(n_B^{-1/2}),\tag{S3.6}$$

$$||S_{12}-S_{12}^*||=o_p(1),\quad ||S_{11}-S_{11}^*||=o_p(1),\quad ||S_{21}-S_{21}^*||=o_p(1),$$

where

$$S_{11}^* = -(N^{-1}\sum_{i=1}^N f_N(\pi_i^{-1}-1)U_i z_i^{*'}, N^{-1}\sum_{i=1}^N f_N \pi_i^{-1} U_i (h_i - \bar{h}_N)'),$$

$$S_{12}^* = \frac{1}{N}\sum_{i=1}^N \frac{\partial U_i(\theta_0)}{\partial \theta}$$

and

$$S_{21}^* = -\begin{pmatrix} N^{-1}\sum_{i=1}^N f_N(\pi_i^{-1}-1) z_i^* z_i^{*'} & N^{-1}\sum_{i=1}^N f_N(\pi_i^{-1}-1) z_i^* (h_i-\bar{h}_N)' \\ N^{-1}\sum_{i=1}^N f_N(\pi_i^{-1}-1)(h_i-\bar{h}_N) z_i^{*'} & N^{-1}\sum_{i=1}^N f_N \pi_i^{-1}(h_i-\bar{h}_N)^{\otimes 2} \end{pmatrix}.$$

Hence according to previous derivations and (S3.3),

$$\hat{\theta} - \theta_0 = -\tau \left\{ \frac{1}{N}\sum_{i=1}^N \frac{I_i}{\pi_i} U_i(\theta_0) - B(\frac{1}{N}\sum_{i=1}^N \frac{I_i}{\pi_i}\eta_i - \bar{\eta}_N) \right\} + o_p(n_B^{-1/2}) \qquad \text{(S3.7)}$$

with $\tau = S_{12}^*, \eta = (z_i^*, (h - \bar{h}_N)')'$, $B = \Omega_1 \Omega_2^{-1}$, $\Omega_1 = -(Nf_N)^{-1} S_{11}^*$, $\quad \Omega_2 = -(Nf_N)^{-1} S_{21}^*$. Thus, (4.9) in Theorem 3 is proved.

Let $e_i = U_i - B\eta_i$ and $\hat{\bar{e}}_p = N^{-1}\sum_{i=1}^N I_i p_i^{-1} e_i$. Next we want to prove

$$||V_{rej}(\hat{\bar{e}}_p) - V_{poi}(\hat{\bar{e}}_p)|| = o_p(n_B^{-1}), \qquad \text{(S3.8)}$$

where V_{rej} and V_{poi} denote the variances under rejective Poisson sampling and Poisson sampling, respectively. According to (S3.1) and (S3.2),

$$\begin{aligned} V_{rej}(\hat{\bar{e}}_p) &= \frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N \frac{\pi_{ij}-\pi_i\pi_j}{p_i p_j} e_i e_j' \\ &= \frac{1}{N^2}\sum_{i=1}^N \frac{\pi_i - \pi_i^2}{p_i^2} e_i^{\otimes 2} + \frac{1}{N^2}\sum_{i\neq j} \frac{\pi_{ij}-\pi_i\pi_j}{p_i p_j} e_i e_j' \\ &= \frac{1}{N^2}\sum_{i=1}^N \frac{1-p_i}{p_i} e_i^{\otimes 2} + \frac{1}{N^2}\sum_{i=1}^N (1-p_i) p_i^{-2} n_B N^{-2} z_i^2 e_i^{\otimes 2} \\ &+ \frac{1}{N^2}\sum_{i\neq j} p_i^{-1} p_j^{-1} o_p(\frac{n_B}{N^2}) e_i e_j' + o_p(n_B^{-1}) \\ &= \frac{1}{N^2}\sum_{i=1}^N \frac{1-p_i}{p_i} e_i^{\otimes 2} + o_p(n_B^{-1}) = V_{poi}(\hat{\bar{e}}_p) + o_p(n_B^{-1}). \end{aligned}$$

So, (S3.8) is proved. Together with (S3.4), (S3.5) and (S3.6),

$$||V_{rej}(\hat{\bar{e}}_{HT}) - V_{poi}(\hat{\bar{e}}_p)|| = o_p(n_B^{-1}), \qquad \text{(S3.9)}$$

where $\hat{\bar{e}}_{HT} = N^{-1}\sum_{i=1}^N I_i \pi_i^{-1} e_i$.

Hence, result (4.12) in Theorem 3 can be obtained by (S3.7), (S3.9) and assumptions (C3), (C4).

S4 Proof of Theorem 4

By using the argument similar to the proof of Theorem 2, it can be shown that

$$R_n(\theta_0) = \bar{r}_N \left\{ V_{poi}(\bar{r}_N) \right\}^{-1} (\bar{r}_N)' + o_p(1), \tag{S4.1}$$

where $\bar{r}_N = \hat{Q}_1(\theta_0, 0) - S_{11}S_{21}^{-1}\hat{Q}_2(\theta_0, 0)$, and $\hat{Q}_1(\theta_0, 0), \hat{Q}_2(\theta_0, 0), S_{11}, S_{21}$ are defined in (S3.3) of the proof for Theorem 3. $\bar{r}_N = N^{-1}\sum_{i=1}^{N} r_i$, and p is the dimension of θ_0. According to (S3.8) in the proof of Theorem 3 and (S4.1), we have $R_n(\theta_0) \to^d \chi_p^2$.