Multivariate spatial nonparametric modelling via kernel processes mixing

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SUPPLEMENTARY MATERIAL

A.1

Properly defined process prior

Kolmogorov existence theorem

We need to prove that the collection of finite-dimensional distributions introduced in (1) define a stochastic process $Y(s)$. We use the two Kolmogorov consistency conditions (symmetry under permutation, and dimensional consistency) to show that (1) defines a proper random process for $Y$.

**Proposition 1.** The collection of finite-dimensional distributions introduced in (1) properly define a stochastic process $Y(s)$. We use the two Kolmogorov consistency conditions: symmetry under permutation, and dimensional consistency, to define properly a random process for $Y$.

**Proof of Proposition 1:**

Symmetry under permutation.

Let $p_{i_j} = p(Y(s_j) = X(\phi_{i(s_j)})$, where $\phi_{i(s_j)}$ is the centering knot of the kernel $i(s_j)$ in the representation of $F_{s_j}(Y)$ in (1), and let $\phi_{i_j}$ be an abbreviation for $\phi_{i(s_j)}$. Then, $p_{i_1,...,i_n}$ determine the site-specific joint selection probabilities. If $\pi(1),...,\pi(n)$ is any permutation
of \( \{1, \ldots, n\} \), then we have

\[
p_{\pi(1), \ldots, \pi(n)} = p(Y(s_{\pi(1)}) = X(\phi_{\pi(i_1)}), \ldots, Y(s_{\pi(n)}) = X(\phi_{\pi(i_n)}))
\]

\[
= p(Y(s_1) = X(\phi_{i_1}), \ldots, Y(s_n) = X(\phi_{i_n})) = p_{i_1, \ldots, i_n},
\] (12)

since the observations are conditionally independent. Then,

\[
p(Y(s_1) \in A_1, \ldots, Y(s_n) \in A_n)
\]

\[
= \sum_{i_1, \ldots, i_n} p(Y(s_1) = X(\phi_{i_1}), \ldots, Y(s_n) = X(\phi_{i_n})) \delta X(\phi_{i_1})(A_1) \ldots \delta X(\phi_{i_n})(A_n)
\]

\[
= \sum_{i_1, \ldots, i_n} p_{i_1, \ldots, i_n} \delta X(\phi_{i_1})(A_1) \ldots \delta X(\phi_{i_n})(A_n)
\]

\[
= \sum_{i_1, \ldots, i_n} p_{\pi(1), \ldots, \pi(n)} \delta X(\phi_{\pi(1)})(A_{\pi(1)}) \ldots \delta X(\phi_{\pi(n)})(A_{\pi(n)})
\]

\[
= p(Y(s_{\pi(1)}) \in A_{\pi(1)}, \ldots, Y(s_{\pi(n)}) \in A_{\pi(n)}),
\]

and, the symmetry under permutation condition holds.

**Dimensional consistency.**

\[
p(Y(s_1) \in (A_1), \ldots, Y(s_k) \in \mathcal{R}, \ldots, Y(s_n) \in (A_n))
\]

\[
= \sum_{(i_1, \ldots, i_n) \in \{1, 2, \ldots\}^n} p_{i_1, \ldots, i_n} \delta X(\phi_{i_1})(A_1) \ldots \delta X(\phi_{i_k})(\mathcal{R}) \ldots \delta X(\phi_{i_n})(A_n)
\]

\[
= \sum_{(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n) \in \{1, 2, \ldots\}^{n-1}} \delta X(\phi_{i_1})(A_1) \ldots \delta X(\phi_{i_{k-1}})(A_{k-1}) \delta X(\phi_{i_{k+1}})(A_{k+1})
\]

\[
\cdots \delta X(\phi_{i_n})(A_n) \sum_{j=1}^{\infty} p_{i_1, \ldots, i_{k-1}, j, i_{k+1}, \ldots, i_n}
\]

\[
= p(Y(s_1) \in (A_1), \ldots, Y(s_{k-1}) \in A_{k-1}, Y(s_{k+1}) \in A_{k+1}, \ldots, Y(s_n) \in (A_n)).
\] (13)

In (13), we need

\[
p_{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n} = \sum_{j=1}^{\infty} p_{i_1, \ldots, i_{k-1}, j, i_{k+1}, \ldots, i_n}
\]

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which holds by Fubini Theorem and the fact that \( X \) is a properly defined Gaussian process.

A.2

Proof of Theorem 1.

The covariance function \( C \) of the underlying process \( X \) has a first order derivative, \( C' \). We introduce a Taylor expansion for \( C \) with a Lagrange remainder term,

\[
C(|\phi_{i_1} - \phi_{i_2}|) = C(|s - s'|) + C'(\psi_{i_1, i_2})\varepsilon_{i_1, i_2},
\]

where \( \varepsilon_{i_1, i_2} = (|\phi_{i_1} - \phi_{i_2}| - |s - s'|) \) and \( \psi_{i_1, i_2} \) is in between \( |s - s'| \) and \( |\phi_{i_1} - \phi_{i_2}| \).

Assuming that \( s \) and \( s' \) lie on the support of the kernels \( K_{i_1} \) and \( K_{i_2} \), respectively, i.e. \( |\phi_{i_1} - s| < \epsilon_{i_1} \) and \( |\phi_{i_2} - s'| < \epsilon_{i_2} \). We have,

\[
\varepsilon_{i_1, i_2} \leq ||\phi_{i_1} - \phi_{i_2}| - |s - s'|| \leq |(\phi_{i_1} - \phi_{i_2}) - (s - s')| \leq \epsilon_{i_1} + \epsilon_{i_2} \leq 2\epsilon,
\]

and,

\[
\varepsilon_{i_1, i_2} \geq -||\phi_{i_1} - \phi_{i_2}| - |s - s'|| \geq -|(|\phi_{i_1} - \phi_{i_2}| - (s - s')| \geq -(\epsilon_{i_1} + \epsilon_{i_2}) \geq -2\epsilon,
\]

Thus, \(-2\epsilon \leq \varepsilon_{i_1, i_2} \leq 2\epsilon\).

Let \( p(s) \) be the potentially infinite vector with all the probabilities masses \( p_{i}(s) \) in \( F_{s}(Y) \). The conditional covariance of the data process \( Y \) is written in terms of the covariance \( C \) of \( X \),

\[
\text{cov}(Y(s), Y(s')|p(s), p(s'), C) = \sum_{i_1, i_2} p_{i_1}(s)p_{i_2}(s')C(|\phi_{i_1} - \phi_{i_2}|),
\]

since the kernels all have compact support, the expression above is the same as

\[
\sum_{i_1, i_2:|\phi_{i_1} - s| < \epsilon_{i_1}, |\phi_{i_2} - s'| < \epsilon_{i_2}} p_{i_1}(s)p_{i_2}(s')C(|\phi_{i_1} - \phi_{i_2}|).
\]

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Using the Taylor approximation in (14), the $\text{cov}(Y(s), Y(s')|p(s), p(s'), C)$ can be written

$$C(|s - s'|) \left[ \sum_{i_1} p_{i_1}(s) \right] \left[ \sum_{i_2} p_{i_2}(s') \right] + \sum_{i_1, i_2: |\phi_{i_1} - s| < \epsilon_{i_1}, |\phi_{i_2} - s'| < \epsilon_{i_2}} p_{i_1}(s)p_{i_2}(s')C'(\psi_{i_1, i_2}) \varepsilon_{i_1, i_2}.$$ 

Since $C'$ is nonnegative and $\varepsilon_{i_1, i_2} \in (-2\epsilon, 2\epsilon)$, we have that for $J_{i_1, i_2} = \{(i_1, i_2); |\phi_{i_1} - s| < \epsilon_{i_1}, |\phi_{i_2} - s'| < \epsilon_{i_2}\}$,

$$-2\epsilon \sum_{i_1, i_2 \in J_{i_1, i_2}} p_{i_1}(s)p_{i_2}(s')C'(\psi_{i_1, i_2}) \leq \sum_{(i_1, i_2) \in J_{i_1, i_2}} p_{i_1}(s)p_{i_2}(s')C'(\psi_{i_1, i_2}) \varepsilon_{i_1, i_2} \leq 2\epsilon \sum_{(i_1, i_2) \in J_{i_1, i_2}} p_{i_1}(s)p_{i_2}(s')C'(\psi_{i_1, i_2}),$$

where,

$$2\epsilon \sum_{i_1, i_2 \in J_{i_1, i_2}} p_{i_1}(s)p_{i_2}(s')C'(\psi_{i_1, i_2}) \rightarrow_{\epsilon \rightarrow 0} 0,$$

because $C'$ is bounded and the sum of probability masses is always bounded by 1. The sum of probability masses would be always bounded by 1, because by proposition 1 the prior process is properly defined.

Thus, we obtain

$$\text{cov}(Y(s), Y(s')|p(s), p(s'), C) \rightarrow_{\epsilon \rightarrow 0} C(|s - s'|) \left[ \sum_{i_1} p_{i_1}(s) \right] \left[ \sum_{i_2} p_{i_2}(s') \right] = C(|s - s'|).$$

Therefore, the conditional covariance of the data process $Y$ approximates the covariance $C$ of the underlying process $X$ as the bandwiths of the kernel functions go to zero.

**A.3**

*Proof of Theorem 2.*

The cross-covariance function $C_{1,2}(s, s')$ of the underlying process $X = (X_1, X_2)$ has first order partial derivatives $\delta C_{1,2}(s, s')/\delta s$ and $\delta C_{1,2}(s, s')/\delta s'$. We introduce a Taylor expansion
for $C_{1,2}$ with a Lagrange remainder term,

$$C_{1,2}(\phi_i, \phi_{i_2}) = C_{1,2}(s, s') + (\phi_i - s) \left[ \frac{\delta C_{1,2}(s, s')}{\delta s} \right]_{(s, s')} = (\psi_{i_1, \psi_{i_2}}) + (\phi_i - s) \left[ \frac{\delta C_{1,2}(s, s')}{\delta s} \right]_{(s, s')} = (\psi_{i_1, \psi_{i_2}}),$$

where $\psi_{i_1}$ is in between $s$ and $\phi_i$, and $\psi_{i_2}$ is in between $s'$ and $\phi_{i_2}$. We follow the same steps as in Theorem 1, to bound the first order term of the Taylor expansion, and obtain that

$$\text{cov}(Y_1(s), Y_2(s')|p_1(s), p_2(s'), C_{1,2}) \xrightarrow{\epsilon \to 0} C_{1,2}(s, s'),$$

where $p_1(s)$ and $p_2(s')$ are the potentially infinite dimensional vectors with the all probability masses in the spatial stick-breaking prior processes $F_s(Y_1)$ and $F_{s'}(Y_2)$ respectively.

### A.4

**Proof of Theorem 3.**

Let $\psi(t, s)$ be the characteristic function of $Y(s)$. Then,

$$\psi(t_1, s) - \psi(t_2, s) = E_Y[\exp\{itY(s_1)\}] - E_Y[\exp\{itY(s_2)\}]$$

$$= E_X \left\{ \sum_j p_j(s_1) \exp\{itX(\phi_j)\} \right\} - E_X \left\{ \sum_j p_j(s_2) \exp\{itX(\phi_j)\} \right\}$$

$$= E_X \left\{ \sum_j (p_j(s_1) - p_j(s_2)) \exp\{itX(\phi_j)\} \right\} \xrightarrow{|s_1 - s_2| \to 0} 0. \quad (16)$$

Then, $F_{s_1}(Y)$ converges to $F_{s_2}(Y)$ for any locations $s_1, s_2$, as long as $|s_1 - s_2| \to 0$.

### A.5

**Proof of Theorem 4.**

The probability masses $p_i(s)$ in (1) are $p_i(s) = V_i K_i(s) \prod_{j=1}^{i-1} (1 - V_j K_j(s))$. Since the bandwiths $\epsilon_i$ converge uniformly to zero, then, $p_i(s) \to 1$, as $|\phi_i - s| \to 0$, where $\phi_i$ is the
knot of kernel $K_i$. This holds because $\sum_j p_j(s) = 1$ a.s. (since the process $Y$ is properly defined).

Assume now $|s_1 - s_2| \to 0$, we need to prove that $Y(s_1)$ converges a.s. to $Y(s_2)$.

Let $\phi_1$ and $\phi_2$ satisfy,

$$|\phi_1 - s_1| \to 0, \quad \text{and} \quad |\phi_2 - s_2| \to 0.$$  \hspace{1cm} (17)

Thus, we obtain that with probability 1, $Y(s_1)$ converges to $X(\phi_1)$, and $Y(s_2)$ to $X(\phi_2)$.

Since $|s_1 - s_2| \to 0$, and $|\phi_1 - \phi_2| \leq |\phi_1 - s_1| + |\phi_2 - s_2| + |s_1 - s_2|$. Then, by (17)

$$|\phi_1 - \phi_2| \to 0.$$  \hspace{1cm} (18)

We have,

$$|Y(s_1) - Y(s_2)| \leq |Y(s_1) - X(\phi_1)| + |Y(s_2) - X(\phi_2)| + |X(\phi_1) - X(\phi_2)|,$$

where $|Y(s_1) - X(\phi_1)| \to 0$ a.s., as $|s_1 - \phi_1| \to 0$; $|Y(s_2) - X(\phi_2)| \to 0$ a.s., as $|s_2 - \phi_2| \to 0$; and since $X$ is a.s. continuous, $|X(\phi_1) - X(\phi_2)| \to 0$ a.s., as $|\phi_1 - \phi_2| \to 0$ (which holds by 18).

Therefore, $|Y(s_1) - Y(s_2)| \to 0$ a.s.