A COPULA-MODEL BASED SEMIPARAMETRIC INTERACTION TEST UNDER THE CASE-CONTROL DESIGN

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Supplementary Material

The consistency and the asymptotic normality of the pseudo-MLE is established in S1, and a proof of the second equation in (5.2) is given in S2.

S1 Consistency and asymptotic normality of pseudo-MLE

We provide the proof for the case that the marginal cumulative distribution functions $F_X$ and $F_Y$ are estimated with the empirical distribution functions. We focus on continuous-continuous scenario since the other two scenarios can be dealt with similarly. Let $\eta = (\beta, \gamma, \xi, \theta)$ and $\hat{\eta} = (\hat{\beta}, \hat{\gamma}, \hat{\xi}, \hat{\theta})$. Let the true values of $\eta$, $F_X$, and $F_Y$ be $\eta_0$, $F_{0X}$, and $F_{0Y}$, respectively. Denote

$$c_{\theta}(u, v; \theta) = \frac{\partial c(u, v; \theta)}{\partial \theta}, \quad c_X(u, v; \theta) = \frac{\partial c(u, v; \theta)}{\partial u}, \quad c_Y(u, v; \theta) = \frac{\partial c(u, v; \theta)}{\partial v},$$

$$c_{\theta\theta}(u, v; \theta) = \frac{\partial^2 c(u, v; \theta)}{\partial \theta^2}, \quad c_{\theta X}(u, v; \theta) = \frac{\partial^2 c(u, v; \theta)}{\partial \theta \partial u}, \quad c_{\theta Y}(u, v; \theta) = \frac{\partial^2 c(u, v; \theta)}{\partial \theta \partial v}.$$

In the following we provide a proof of large sample properties for the pseudo-MLE of $\eta$, to this end, we assume the following conditions hold:

(c1) The logistic model and copula model hold true and $c(u, v; \theta)$ is identifiable in $\theta$.

(c2) The sample size $n = n_0 + n_1$ goes to infinity with $n_1/n_0$ fixed at a value $\in (0, 1)$.

(c3) Some regularity conditions on the copula function $c(u, v; \theta)$ and the marginal distributions of $X$ and $Y$ are satisfied.

First mimic Gong and Samaniego (1981) we show that there exists a local maxima, say $\hat{\eta}$, of the pseudo-likelihood function that is consistent for $\eta_0$. 
Under the regular condition (c3), we have that \( \hat{F}_X \) and \( \hat{F}_Y \) are strongly consistent for \( F_{0X} \) and \( F_{0Y} \), respectively, uniformly in \( t \). This implies that under further regular condition \( n^{-1}l_n(\eta, F_X, F_Y) \) converges to \( \tilde{l}(\eta_0, F_{0X}, F_{0Y}) = n^{-1}El_n(\eta, F_{0X}, F_{0Y}) \) uniformly in \( \eta \). Under condition (c1), for any \( \epsilon > 0 \), there exist \( \epsilon' > 0 \) such that \( \tilde{l}(\eta_0, F_{0X}, F_{0Y}) > \tilde{l}(\eta, F_{0X}, F_{0Y}) + \epsilon' \) for \( \eta \in O_\epsilon \), where \( O_\epsilon = \{ ||\eta - \eta_0|| < \epsilon \} \). Therefore, for any \( \delta, \epsilon > 0 \), there exists a sufficiently large \( N = N(\delta, \epsilon) \) such that for any \( n > N \), \( \Pr\{l_n(\eta, \hat{F}_X, \hat{F}_Y) < l_n(\eta_0, \hat{F}_X, \hat{F}_Y) \} \) for any \( \eta \in O_\epsilon \} > 1-\delta \). This shows that, with probability tending to 1, there exists in \( O_\delta \) a local maximizer of \( l_n(\eta, \hat{F}_X, \hat{F}_Y) \). The existence of a consistent pseudo-MLE is established.

Next we show that \( n^{1/2}(\hat{\eta} - \eta_0) \) converges in distribution to a multivariate normal distribution.

Write

\[
l_n(\eta, F_X, F_Y) = \sum_{i=1}^{n_0} \log c(F_X(x_{0i}), F_Y(y_{0i}); \theta) + \sum_{i=1}^{n_1} \left\{ \log c(F_X(x_{0i}), F_Y(y_{0i}); \theta) + (\beta x_{1i} + \gamma y_{1i} + \xi x_{1i} y_{1i}) - \alpha(\eta, F_X, F_Y) \right\},
\]

where

\[
\alpha(\eta, F_X, F_Y) = \log \left\{ \int \int c(F_X(x), F_Y(y); \theta) \exp(\beta x + \gamma y + \xi xy) dF_X(x) dF_Y(y) \right\}.
\]

The derivative of \( l_n(\eta, F_X, F_Y) \) with respect to \( \eta \) is

\[
l_{n,\eta}(\eta, F_X, F_Y) = \sum_{i=1}^{n_0} \left( \frac{\partial c(F_X(x_{0i}), F_Y(y_{0i}); \theta)}{\partial \eta} \right) + \sum_{i=1}^{n_1} \left\{ \left( \frac{\partial c(F_X(x_{0i}), F_Y(y_{0i}); \theta)}{\partial \eta} \right) - \frac{\partial \alpha(\eta, F_X, F_Y)}{\partial \eta} \right\} = E_1(\eta, F_X, F_Y).
\]

where

\[
\alpha(\eta, F_X, F_Y) = \frac{\partial \alpha(\eta, F_X, F_Y)}{\partial \eta} = \frac{E_1(\eta, F_X, F_Y)}{E_0(\eta, F_X, F_Y)}
\]

with

\[
E_0(\eta, F_X, F_Y) = \int \int c(F_X(x), F_Y(y); \theta) \exp(\beta x + \gamma y + \xi xy) dF_X(x) dF_Y(y)
\]

and

\[
E_1(\eta, F_X, F_Y) = \int \int \begin{pmatrix} c(F_X(x), F_Y(y); \theta) \\ \alpha(F_X(x), F_Y(y); \theta) \end{pmatrix} \exp(\beta x + \gamma y + \xi xy) dF_X(x) dF_Y(y).
\]
Let $l_{n,\eta}(\eta, F_X, F_Y) = \partial^2 l_n(\eta, F_X, F_Y)/\partial \eta \partial \eta'$, then Taylor’s expansion gives

$$0 = l_{n,\eta}(\hat{\eta}, \hat{F}_X, \hat{F}_Y) = l_{1,\eta}(\eta_0, \hat{F}_X, \hat{F}_Y) + l_{n,\eta}(\eta^*, \hat{F}_X, \hat{F}_Y)(\hat{\eta} - \eta_0)$$

or

$$\hat{\eta} = \eta_0 - l_{n,\eta}(\eta^*, \hat{F}_X, \hat{F}_Y)^{-1} l_{n,\eta}(\eta_0, \hat{F}_X, \hat{F}_Y), \quad (S1.2)$$

where $\eta^*$ lies between $\eta_0$ and $\hat{\eta}$. Under the regular condition (c3),

$$n^{-1} \left| l_{n,\eta}(\hat{\eta}^*, \hat{F}_X, \hat{F}_Y) - l_{n,\eta}(\eta_0, F_{0X}, F_{0Y}) \right| \to 0 \text{ in probability as } n \to \infty.$$ 

By the Law of Large Number,

$$-n^{-1} l_{n,\eta}(\eta_0, F_{0X}, F_{0Y}) \to \Sigma_V(\eta_0, F_{0X}, F_{0Y})$$

where

$$s_0(\eta, F_X, F_Y) = E \left[ \frac{c\theta(F_X(x_{01}), F_Y(y_{01}); \theta)}{c(F_X(x_{01}), F_Y(y_{01}); \theta)} - \frac{c_2^2(F_X(x_{01}), F_Y(y_{01}); \theta)}{c_2(F_X(x_{01}), F_Y(y_{01}); \theta)} \right],$$

$$s_1(\eta, F_X, F_Y) = E \left[ \frac{c\theta(F_X(x_{11}), F_Y(y_{11}); \theta)}{c(F_X(x_{11}), F_Y(y_{11}); \theta)} - \frac{c_2^2(F_X(x_{11}), F_Y(y_{11}); \theta)}{c_2(F_X(x_{11}), F_Y(y_{11}); \theta)} \right],$$

$$\alpha_{\eta\eta}(\eta, F_X, F_Y) = \frac{E_2(\eta, F_X, F_Y)}{E_0(\eta, F_X, F_Y)} \frac{\{E_1(\eta, F_X, F_Y)\} \{E_1(\eta, F_X, F_Y)\}_T}{E_0^2(\eta, F_X, F_Y)}$$

with

$$E_2(\eta, F_X, F_Y) = \int \int \tilde{c}(\theta, F_X, F_Y)(x, y) \exp(\beta x + \gamma y + \xi xy) dF_X(x) dF_Y(y)$$

and

$$\tilde{c}(\theta, F_X, F_Y)(x, y)$$

$$= \begin{pmatrix}
  c_{\theta}(F_X(x), F_Y(y); \theta) & x c_{\theta}(F_X(x), F_Y(y); \theta) & y c_{\theta}(F_X(x), F_Y(y); \theta) & x y c_{\theta}(F_X(x), F_Y(y); \theta) \\
  x c_{\theta}(F_X(x), F_Y(y); \theta) & x^2 c_{\theta}(F_X(x), F_Y(y); \theta) & y c_{\theta}(F_X(x), F_Y(y); \theta) & x^2 y c_{\theta}(F_X(x), F_Y(y); \theta) \\
  y c_{\theta}(F_X(x), F_Y(y); \theta) & x y c_{\theta}(F_X(x), F_Y(y); \theta) & y^2 c_{\theta}(F_X(x), F_Y(y); \theta) & y^2 x c_{\theta}(F_X(x), F_Y(y); \theta) \\
  x y c_{\theta}(F_X(x), F_Y(y); \theta) & x^2 y c_{\theta}(F_X(x), F_Y(y); \theta) & y^2 x c_{\theta}(F_X(x), F_Y(y); \theta) & x^2 y^2 c_{\theta}(F_X(x), F_Y(y); \theta)
\end{pmatrix}.$$ 

Combining (S1.2) and (S1.3) we have

$$n^{1/2}(\hat{\eta} - \eta_0) = \Sigma^{-1}_{V}(\eta_0, F_{0X}, F_{0Y}) \left\{ n^{-1/2} l_{n,\eta}(\eta_0, \hat{F}_X, \hat{F}_Y) \right\} \{1 + o_p(1)\}. \quad (S1.4)$$

In the following we derive the limit distribution of $n^{-1/2} l_{n,\eta}(\eta_0, \hat{F}_X, \hat{F}_Y)$.

Let $F_{0_{U_0}}(x, y)$ and $F_{1_{U_1}}(x, y)$ denote the empirical joint distribution functions of $(x_{01}, y_{01})$ and $(x_{11}, y_{11})$, respectively, and $F_{0}(x, y)$ and $F_{1}(x, y)$ denote the joint distributions of $(x_{01}, y_{01})$ and $(x_{11}, y_{11})$, respectively.
Noticing that $E\{l_{n, \eta}(\eta_0, F_{0X}, F_{0Y})\} = 0$ or equivalently

$$
\pi_0 \int \left( \frac{\tilde{e}(\theta_0, F_{0X}, F_{0Y})(x, y)}{x, y} \right) dF_0(x, y) + \pi_1 \int \left( \frac{\tilde{e}(\theta_0, F_{0X}, F_{0Y})(x, y)}{x, y} \right) dF_1(x, y)
$$

we have that

$$
n^{-1/2}l_{n, \eta}(\eta_0, \hat{F}_X, \hat{F}_Y) = \pi_0^{1/2} \int \left( n_0^{1/2} \tilde{e}(\theta_0, \hat{F}_X, \hat{F}_Y)(x, y) \right) dF_{0no}(x, y)
$$

$$
+ \pi_1^{1/2} \int \left( n_1^{1/2} \tilde{e}(\theta_0, \hat{F}_X, \hat{F}_Y)(x, y) \right) dF_{1n_1}(x, y)
$$

$$
- \pi_1 \left\{ n_1^{1/2} \alpha_\eta(\eta_0, \hat{F}_X, \hat{F}_Y) \right\}
$$

$$
= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7,
$$

(S1.5)

where

$$
T_1 = \pi_0^{1/2} \int \left( n_0^{1/2} \tilde{e}(\theta_0, \hat{F}_X, \hat{F}_Y)(x, y) \right) dF_0(x, y),
$$

$$
T_2 = \pi_1 \pi_0^{-1/2} \int \left( n_0^{1/2} \tilde{e}(\theta_0, \hat{F}_X, \hat{F}_Y)(x, y) \right) dF_1(x, y),
$$

$$
T_3 = -\pi_1 \pi_0^{-1/2} \left[ n_0^{1/2} \left\{ \alpha_\eta(\eta_0, \hat{F}_X, \hat{F}_Y) - \alpha_\eta(\eta_0, F_{0X}, F_{0Y}) \right\} \right],
$$

$$
T_4 = \pi_0^{1/2} \int \left( \tilde{e}(\theta_0, F_{0X}, F_{0Y})(x, y) \right) \left\{ n_0^{1/2} d(F_{0no} - F_0)(x, y) \right\},
$$

$$
T_5 = \pi_1^{1/2} \int \left( \tilde{e}(\theta_0, F_{0X}, F_{0Y})(x, y) \right) \left\{ n_1^{1/2} d(F_{1n_1} - F_1)(x, y) \right\},
$$

$$
T_6 = \pi_0^{1/2} \int \left( \tilde{e}(\theta_0, F_{0X}, F_{0Y})(x, y) \right) \left\{ n_0^{1/2} d(F_{0no} - F_0)(x, y) \right\},
$$

$$
T_7 = \pi_1^{1/2} \int \left( \tilde{e}(\theta_0, F_{0X}, F_{0Y})(x, y) \right) \left\{ n_1^{1/2} d(F_{1n_1} - F_1)(x, y) \right\}.
$$

Since $n_0^{1/2}(F_{0no} - F_0)(x, y) \rightarrow O_p(1)$, $n_1^{1/2}(F_{1n_1} - F_1)(x, y) \rightarrow O_p(1)$, $\hat{F}_X \rightarrow F_{0X}$, $\hat{F}_Y \rightarrow F_{0Y}$, $T_6$ and $T_7$ converges to 0 in probability under the regular condition (c3).

To prove the asymptotic normality of $n^{-1/2}l_{n, \eta}(\eta_0, \hat{F}_X, \hat{F}_Y)$, we shall show that $T_1$, $T_2$, $T_3$, $T_4$, and $T_5$ can be written as summations of i.i.d. random vectors.

Obviously, both $T_4$ and $T_5$ are summations of i.i.d. random vectors.
By von Mises expansion and integration by parts, the first element of $T_1$ is approximated by

$$\pi_0^{1/2} \left[ \int \hat{c}_X(\theta_0, F_{0X}, F_{0Y})(x, y) \{n_0^{1/2}(\hat{F}_X(x) - F_{0X}(x))\} dF_0(x, y) + \int \hat{c}_Y(\theta_0, F_{0X}, F_{0Y})(x, y) \{n_0^{1/2}(\hat{F}_Y(y) - F_{0X}(y))\} dF_0(x, y) \right]$$

where

$$\hat{c}_X(\theta_0, F_X, F_Y)(x, y) = \frac{c_{0X}(F_X(x), F_Y(y); \theta)}{c(F_X(x), F_Y(y); \theta)} - \frac{c_0(F_X(x), F_Y(y); \theta)c_X(F_X(x), F_Y(y); \theta)}{c^2(F_X(x), F_Y(y); \theta)},$$

$$\hat{c}_Y(\theta_0, F_X, F_Y)(x, y) = \frac{c_{0Y}(F_X(x), F_Y(y); \theta)}{c(F_X(x), F_Y(y); \theta)} - \frac{c_0(F_X(x), F_Y(y); \theta)c_Y(F_X(x), F_Y(y); \theta)}{c^2(F_X(x), F_Y(y); \theta)},$$

$$G_{1X}(x; \theta_0, F_{0X}, F_{0Y}) = \int_{-\infty}^x \left\{ \int \hat{c}_X(\theta_0, F_{0X}, F_{0Y})(u, y) f_0(u, y) dy \right\} du,$$

$$G_{1Y}(y; \theta_0, F_{0X}, F_{0Y}) = \int_{-\infty}^y \left\{ \int \hat{c}_Y(\theta_0, F_{0X}, F_{0Y})(x, v) f_0(x, v) dx \right\} dv,$$

with $f_0(x, y) = \partial^2 F_0(x, y)/\partial x \partial y$. It is seen from (S1.6) that $T_1$ is approximated by the summation of i.i.d. random vectors. Similarly, the first element of $T_2$ can also be approximated by the summation of i.i.d. random vectors:

$$\pi_1 \pi_0^{-1/2} \left[ \int G_{2X}(x; \theta_0, F_{0X}, F_{0Y}) d\{n_0^{1/2}(\hat{F}_X - F_{0X})\}(x) + \int G_{2Y}(y; \theta_0, F_{0X}, F_{0Y}) d\{n_0^{1/2}(\hat{F}_Y - F_{0Y})\}(y) \right],$$

where

$$G_{2X}(x; \theta_0, F_{0X}, F_{0Y}) = \int_{-\infty}^x \left\{ \int \hat{c}_X(\theta_0, F_{0X}, F_{0Y})(u, y) \exp(\beta_0 x + \gamma_0 y + \xi_0 xy) f_0(u, y) dy \right\} du,$$

$$G_{2Y}(y; \theta_0, F_{0X}, F_{0Y}) = \int_{-\infty}^y \left\{ \int \hat{c}_Y(\theta_0, F_{0X}, F_{0Y})(x, v) \exp(\beta_0 x + \gamma_0 y + \xi_0 xy) f_0(x, v) dx \right\} dv.$$
vectors. Applying von Mises expansion, we get

\[ n_0^{1/2} \left[ \alpha_y(\eta_0, \hat{F}_X, \hat{F}_Y) - \alpha_y(\eta_0, F_{0X}, F_{0Y}) \right] = \frac{W_{1X}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_X)}{E_0(\eta_0, F_{0X}, F_{0Y})} - \frac{E_1(\eta_0, F_{0X}, F_{0Y}) W_{0X}(\eta_0, F_{0X}, F_{0X}, \hat{F}_X)}{E_0^2(\eta_0, F_{0X}, F_{0Y})} + \frac{W_{1Y}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_Y)}{E_0(\eta_0, F_{0X}, F_{0Y})} - \frac{E_1(\eta_0, F_{0X}, F_{0Y}) W_{0Y}(\eta_0, F_{0X}, F_{0X}, \hat{F}_Y)}{E_0^2(\eta_0, F_{0X}, F_{0Y})} + o_p(1), \]

(S1.8)

Here

\[ W_{1X}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_X) = \int \int e_{1X}(\eta_0, F_{0X}, F_{0Y})(x, y) \{ n_0^{1/2} (\hat{F}_X(x) - F_{0X}(x)) \} dF_{0X}(x) dF_{0Y}(y) \]

\[ + \int \int e_1(\eta_0, F_{0X}, F_{0Y})(x, y) \{ n_0^{1/2} d(\hat{F}_X - F_{0X})(x) \} dF_{0Y}(y), \]

\[ W_{0X}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_X) = \int \int e_{0X}(\eta_0, F_{0X}, F_{0Y})(x, y) \{ n_0^{1/2} (\hat{F}_X(x) - F_{0X}(x)) \} dF_{0X}(x) dF_{0Y}(y) \]

\[ + \int \int e_0(\eta_0, F_{0X}, F_{0Y})(x, y) \{ n_0^{1/2} d(\hat{F}_X - F_{0X})(x) \} dF_{0Y}(y), \]

\[ W_{1Y}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_Y) = \int \int e_{1Y}(\eta_0, F_{0X}, F_{0Y})(x, y) \{ n_0^{1/2} (\hat{F}_Y(y) - F_{0Y}(y)) \} dF_{0X}(x) dF_{0Y}(y) \]

\[ + \int \int e_1(\eta_0, F_{0X}, F_{0Y})(x, y) dF_{0X}(x) \{ n_0^{1/2} d(\hat{F}_Y - F_{0Y})(y) \}, \]

\[ W_{0Y}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_Y) = \int \int e_{0Y}(\eta_0, F_{0X}, F_{0Y})(x, y) \{ n_0^{1/2} (\hat{F}_Y(y) - F_{0Y}(y)) \} dF_{0X}(x) dF_{0Y}(y) \]

\[ + \int \int e_0(\eta_0, F_{0X}, F_{0Y})(x, y) dF_{0X}(x) \{ n_0^{1/2} d(\hat{F}_Y - F_{0Y})(y) \}, \]

with

\[ e_0(\eta, F_X, F_Y)(x, y) = c(F_X(x), F_Y(y); \theta) \exp(\beta x + \gamma y + \xi xy), \]

\[ e_{0X}(\eta, F_X, F_Y)(x, y) = c_X(F_X(x), F_Y(y); \theta) \exp(\beta x + \gamma y + \xi xy), \]

\[ e_{0Y}(\eta, F_X, F_Y)(x, y) = c_Y(F_X(x), F_Y(y); \theta) \exp(\beta x + \gamma y + \xi xy), \]

\[ e_1(\eta, F_X, F_Y)(x, y) = \left( \begin{array}{c} e_{0}(F_X(x), F_Y(y); \theta) \\ x e_{0}(F_X(x), F_Y(y); \theta) \\ y e_{0}(F_X(x), F_Y(y); \theta) \\ x y e_{0}(F_X(x), F_Y(y); \theta) \end{array} \right) \exp(\beta x + \gamma y + \xi xy), \]
Using integration by parts, we have
\[
ee_{1X}(\eta, F_X, F_Y)(x, y) = \begin{pmatrix} e_{0X}(F_X(x), F_Y(y); \theta) \\ x_X(F_X(x), F_Y(y); \theta) \\ y_X(F_X(x), F_Y(y); \theta) \\ x_y(F_X(x), F_Y(y); \theta) \
\end{pmatrix} \exp(\beta x + \gamma y + \xi xy),
\]
\[
ee_{1Y}(\eta, F_X, F_Y)(x, y) = \begin{pmatrix} e_{0Y}(F_X(x), F_Y(y); \theta) \\ x_Y(F_X(x), F_Y(y); \theta) \\ y_Y(F_X(x), F_Y(y); \theta) \\ x_yY(F_X(x), F_Y(y); \theta) \
\end{pmatrix} \exp(\beta x + \gamma y + \xi xy).
\]

Using integration by parts, we have
\[
W_{1X}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_X) = \int H_{1X}(x; \eta_0, F_{0X}, F_{0Y})\{n_0^{1/2}d(\hat{F}_X - F_{0X})(x)\} \quad (S1.9)
\]
with
\[
H_{1X}(x; \eta, F_X, F_Y) = \int_{-\infty}^{x} \left\{ \int e_{1X}(\eta, F_X, F_Y)(u, y)dF_Y(y) \right\} dF_X(u)
\]
\[
+ \int e_{1}(\eta, F_X, F_Y)(x, y)dF_Y(y),
\]
\[
W_{1Y}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_Y) = \int H_{1Y}(y; \eta_0, F_{0X}, F_{0Y})\{n_0^{1/2}d(\hat{F}_Y - F_{0Y})(y)\} \quad (S1.10)
\]
with
\[
H_{1Y}(y; \eta, F_X, F_Y) = \int_{-\infty}^{y} \left\{ \int e_{1Y}(\eta, F_X, F_Y)(x, v)dF_X(x) \right\} dF_Y(v)
\]
\[
+ \int e_{1}(\eta, F_X, F_Y)(x, y)dF_X(x),
\]
\[
W_{0X}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_X) = \int H_{0X}(x; \eta_0, F_{0X}, F_{0Y})\{n_0^{1/2}d(\hat{F}_X - F_{0X})(x)\} \quad (S1.11)
\]
with
\[
H_{0X}(x; \eta, F_X, F_Y) = \int_{-\infty}^{x} \left\{ \int e_{0X}(\eta, F_X, F_Y)(u, y)dF_Y(y) \right\} dF_X(u)
\]
\[
+ \int e_{0}(\eta, F_X, F_Y)(x, y)dF_Y(y),
\]
\[
W_{0Y}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_Y) = \int H_{0Y}(y; \eta_0, F_{0X}, F_{0Y})\{n_0^{1/2}d(\hat{F}_Y - F_{0Y})(y)\} \quad (S1.12)
\]
with
\[
H_{0Y}(y; \eta, F_X, F_Y) = \int_{-\infty}^{y} \left\{ \int e_{0Y}(\eta, F_X, F_Y)(x, v)dF_X(x) \right\} dF_Y(v)
\]
\[
+ \int e_{0}(\eta, F_X, F_Y)(x, y)dF_X(x).
\]
It is seen that (S1.9), (S1.10), (S1.11), and (S1.12) are summations of i.i.d. random variables, so is $T_3$ from (S1.8).

From (S1.6), (S1.7), and (S1.8), we have the following asymptotic expression:

\[
n^{-1/2} l_{n, n}(\eta_0, \hat{F}_X, \hat{F}_Y) = \int \left[ \pi_0^{1/2} G_{1X}(x; \eta_0, F_{0X}, F_{0Y}) + \pi_1 \pi_0^{-1/2} \left\{ G_{2X}(x; \eta_0, F_{0X}, F_{0Y}) - H_{1X}(x; \eta_0, F_{0X}, F_{0Y}) \right\} \right.
\]

\[
\left. \times \{ \int n_0^{1/2} d(\hat{F}_X - F_{0X})(x) \} \right] + \int \left[ \pi_0^{1/2} G_{1Y}(y; \eta_0, F_{0X}, F_{0Y}) + \pi_1 \pi_0^{-1/2} \left\{ G_{2Y}(y; \eta_0, F_{0X}, F_{0Y}) - H_{1Y}(y; \eta_0, F_{0X}, F_{0Y}) \right\} \right.
\]

\[
\left. \times \{ \int n_0^{1/2} d(\hat{F}_Y - F_{0Y})(y) \} \right] + \int \pi_0^{1/2} \left( \frac{x y}{0 0} \right) \{ n_0^{1/2} d(\hat{F}_{0n} - F_0)(x, y) \}
\]

\[
\left. + \int \pi_1^{1/2} \left( \frac{x y}{y x} \right) \{ n_1^{1/2} d(F_{1n} - F_1)(x, y) \} + o_p(1) \right].
\]

We can see that $n^{-1/2} l_{n, n}(\eta_0, \hat{F}_X, \hat{F}_Y)$ is approximated by the summation of several i.i.d. random vector summations, thus it is asymptotically normally distributed with expectation 0 and a variance-covariance, say $\Sigma_S(\eta_0, F_{0X}, F_{0Y})$, which is quite complicated because the first two terms are correlated with the later two terms and it does not have a close form.

Now the limit distribution of $n^{1/2}(\hat{\eta} - \eta_0)$ is the multivariate normal with expectation 0 and variance-covariance $\Sigma^{-1}_{S}(\eta_0, F_{0X}, F_{0Y})\Sigma^{-1}_{S}(\eta_0, F_{0X}, F_{0Y})\Sigma^{-1}_{S}(\eta_0, F_{0X}, F_{0Y})$.

S2 Derivation of the second equation in (5.2)

The Gaussian copula function is

\[
C(u, v; \theta) = \Phi_{\theta}(\Phi^{-1}(u), \Phi^{-1}(v)).
\]

The derivative of $C(u, v; \theta)$ with respect to $v$ is

\[
C_2(u, v; \theta) = \Phi_{\theta}^{(2)}(\Phi^{-1}(u), \Phi^{-1}(v)) \frac{\partial \Phi^{-1}(v)}{\partial v}.
\]

Here

\[
\Phi_{\theta}^{(2)}(x, y) = \frac{\partial \Phi_{\theta}(x, y)}{\partial y} = \Phi \left\{ \frac{x - \theta y}{(1 - \theta^2)^{1/2}} \right\} \phi(y),
\]
which is the marginal density function of $Y$ at $y$ times the cumulative distribution function of $X$ given $Y = y$ at $x$. Therefore,

$$
C_2(u, v; \theta) = \Phi \left\{ \frac{\Phi^{-1}(u) - \theta \Phi^{-1}(v)}{(1 - \theta^2)^{1/2}} \right\} \phi(\Phi^{-1}(v)) \frac{\partial \Phi^{-1}(v)}{\partial v}
$$

$$
= \Phi \left\{ \frac{\Phi^{-1}(u) - \theta \Phi^{-1}(v)}{(1 - \theta^2)^{1/2}} \right\}.
$$