VARIABLE SELECTION IN PARTLY LINEAR REGRESSION MODEL WITH DIVERGING DIMENSIONS FOR RIGHT CENSORED DATA
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Supplementary Material

We first describe the following results, which is Lemma 1 of Huang and Ma (2010). Let \( \tau = (\tau_1, \ldots, \tau_n)^T \) and \( \xi_n = \max_{1 \leq j \leq p} |\xi_j| \). Suppose that conditions (A2) and (A3) hold. Then
\[
E(\xi_n) \leq C_1 \sqrt{\log(p)} \left( \sqrt{2C_2 n \log(p)} + 4 \log(2p) + C_2 n \right)^{1/2},
\]
where \( C_1, C_2 > 0 \) are constants. In particular, when \( \log(p)/n \to 0 \),
\[
E(\xi_n) = O(1) \sqrt{n \log p}.
\]

S1 Proof of Theorem 1

Examination of Theorem 1 of Zhang and Huang (2008) suggests that the normality assumption is not necessary. As a matter of fact, as long as the tail probability \( \sim \exp(-x^2) \), Theorem 1 and its proof in Zhang and Huang (2008) holds. Part (a) of our Theorem 1 thus follows.

Under assumption (A1), \( \min_{j \in A_1} |\beta_{0j}| > b_1 > 0 \) for a constant \( b_1 \). Thus, if part (c) of Theorem 1 holds, then part (b) follows. Proof of part (c) proceeds as follows. The Lasso estimate satisfies
\[
||\tilde{Y} - \tilde{X}\tilde{\beta}||^2 + 2\lambda_n \sum_j |\tilde{\beta}_j| \leq ||\tilde{Y} - \tilde{X}\beta_0||^2 + 2\lambda_n \sum_j |\beta_{0j}|,
\]
which leads to
\[
||\tilde{Y} - \tilde{X}\tilde{\beta}||^2 + 2\lambda_n \sum_{j \in A_1} |\tilde{\beta}_j| \leq ||\tilde{Y} - \tilde{X}\beta_0||^2 + 2\lambda_n \sum_{j \in A_1} |\beta_{0j}|.
\]
Thus, we have
\[
||\tilde{X}(\tilde{\beta} - \beta_0)||^2 - 2\tau^T \tilde{X}(\tilde{\beta} - \beta_0) \leq 2\lambda_n \sum_{j \in A_1} |\tilde{\beta}_j - \beta_{0j}|.
\]
We note that
\[ \sum_{j \in A_1} |\tilde{\beta}_j - \beta_{0j}| \leq \sqrt{|A_1|} ||\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1}||, \]
where \( \tilde{\beta}_{A_1 \cup \hat{A}_1} = \{\tilde{\beta}_j : j \in A_1 \cup \hat{A}_1\} \) and \( \beta_{0A_1 \cup \hat{A}_1} = \{\beta_{0j} : j \in A_1 \cup \hat{A}_1\} \). Combining the above equations, we have
\[
||\hat{X}_{A_1 \cup \hat{A}_1} (\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1})||^2 - 2\tau^T (\hat{X}_{A_1 \cup \hat{A}_1} (\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1})) \\
\leq 2\lambda_n \sqrt{|A_1|} ||\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1}||.
\]
Define \( \tau^* = \hat{X}_{A_1 \cup \hat{A}_1} (\hat{X}_{A_1 \cup \hat{A}_1}^T \hat{X}_{A_1 \cup \hat{A}_1})^{-1} \hat{X}_{A_1 \cup \hat{A}_1}^T \tau \). From the Cauchy-Schwarz inequality, we have
\[
|2\tau^T (\hat{X}_{A_1 \cup \hat{A}_1} (\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1}))| \leq 2||\tau^*||^2 + \frac{1}{2} ||\hat{X}_{A_1 \cup \hat{A}_1} (\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1})||^2.
\]
Combining the above equations,
\[
||\hat{X}_{A_1 \cup \hat{A}_1} (\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1})||^2 \leq 4||\tau^*||^2 + 4\lambda_n \sqrt{|A_1|} \times ||\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1}||.
\]
Under assumption (A4),
\[
||\hat{X}_{A_1 \cup \hat{A}_1} (\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1})||^2 \geq nc_* ||\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1}||^2.
\]
Combining the above two equations, we have
\[
nc_* ||\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1}||^2 \leq 4||\tau^*||^2 + \frac{16\lambda_n^2 |A_1|}{2nc_*} + \frac{1}{2} nc_* ||\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1}||^2.
\]
It follows that
\[
||\tilde{\beta}_{A_1 \cup \hat{A}_1} - \beta_{0A_1 \cup \hat{A}_1}||^2 \leq 8||\tau^*||^2 \frac{1}{nc_*} + \frac{16\lambda_n^2 |A_1|}{n^2 c_*^2}.
\]
(S1.1)
Under the SRC, we also have
\[
||\tau^*||^2 \leq \frac{||\hat{X}_{A_1 \cup \hat{A}_1} \tau||^2}{nc_*} \leq \frac{\max_{B: |B| \leq p_1} ||\hat{X}_B \tau||^2}{nc_*}.
\]
We also have
\[
\max_{B: |B| \leq p_1} ||\hat{X}_B \tau||^2 \leq p_1 \max_j |\hat{X}_j^T \tau|.
\]
Applying the result described in the beginning of this section,
\[
\max_j |\hat{X}_j^T \tau| = O(n \log(p)).
\]
Thus,
\[
||\tau^*||^2 = O\left(\frac{p_1^* \log(p)}{c_*}\right).
\]
(S1.2)
Part (c) follows from equations (S1.1) and (S1.2).
S2 Proofs

By the Karush-Kuhn-Tucker condition, \( \hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_p)^T \) is the adaptive Lasso estimate if

\[
\begin{align*}
\{X_j^T(\hat{Y} - \hat{X}\hat{\beta}) &= \lambda_n v_j, \text{sign}(\hat{\beta}_j), \quad \hat{\beta}_j \neq 0 \\
|X_j^T(\hat{Y} - \hat{X}\hat{\beta})| &\leq \lambda_n v_j, \quad \hat{\beta}_j = 0
\end{align*}
\]  
(S2.1)

and the vectors \( \{\hat{X}_j : j \in \hat{A}_1\} \) are linearly independent. Define \( \hat{s}_1 = (v_j \text{sign}(\beta_{0j}), j \in A_1)^T \), \( \hat{X}_{A_1} = (\hat{X}_j, j \in A_1) \), and \( \beta_{0A_1} = (\beta_{0j}, j \in A_1)^T \). Define

\[
\hat{\beta}_{A_1} = (\hat{X}_{A_1}^T\hat{X}_{A_1})^{-1}(\hat{X}_{A_1}^T\hat{Y} - \lambda_n \hat{s}_1) = \beta_{0A_1} + (\hat{X}_{A_1}^T\hat{X}_{A_1}/n)^{-1}(\hat{X}_{A_1}^T\tau - \lambda_n \hat{s}_1)/n. 
\]  
(S2.2)

If \( \text{sign}(\hat{\beta}_{A_1}) = \text{sign}(\beta_{0A_1}) \), then (S2.1) holds for \( \hat{\beta} = (\hat{\beta}_{A_1}, 0^T)^T \). Since \( \hat{X}\hat{\beta} = \hat{X}_{A_1}\hat{\beta}_{A_1}^T \), we have

\[
\text{sign}(\hat{\beta}) = \text{sign}(\beta_0) \quad \text{if} \quad \begin{cases} 
\text{sign}(\hat{\beta}_{A_1}) = \text{sign}(\beta_{0A_1}) \\
|X_j^T(\hat{Y} - \hat{X}_{A_1}\hat{\beta}_{A_1})| \leq \lambda_n v_j, \forall j \notin A_1.
\end{cases} 
\]  
(S2.3)

Define \( H_n = I - \hat{X}_{A_1}(\hat{X}_{A_1}^T\hat{X}_{A_1})^{-1}\hat{X}_{A_1}^T \). From the definition of \( \hat{\beta}_{A_1} \),

\[
\hat{Y} - \hat{X}_{A_1}\hat{\beta}_{A_1} = \tau - \hat{X}_{A_1}(\hat{\beta}_{A_1} - \beta_{0A_1}) = H_n\tau + \hat{X}_{A_1}(\hat{X}_{A_1}^T\hat{X}_{A_1})^{-1}\hat{s}_1\lambda_n.
\]

Thus, following (S2.3),

\[
\text{sign}(\hat{\beta}) = \text{sign}(\hat{\beta}_0) \quad \text{if} \quad \begin{cases} 
\text{sign}(\beta_{0j})(\beta_{0j} - \hat{\beta}_j) \leq |\beta_{0j}|, \quad \forall j \in A_1 \\
|X_j^T(H_n\tau + \hat{X}_{A_1}(\hat{X}_{A_1}^T\hat{X}_{A_1})^{-1}\hat{s}_1\lambda_n)| < \lambda_n v_j, \quad \forall j \notin A_1.
\end{cases} 
\]  
(S2.4)

Combining equations (S2.2) and (S2.4),

\[
P \left\{ \text{sign}(\hat{\beta}) \neq \text{sign}(\beta_0) \right\} \leq P \left\{ \left| e_j^T(\hat{X}_{A_1}^T\hat{X}_{A_1})^{-1}\hat{X}_{A_1}^T\tau \right| \geq |\beta_{0j}|/2 \text{ for some } j \in A_1 \right\} + P \left\{ \left| e_j^T(\hat{X}_{A_1}^T\hat{X}_{A_1})^{-1}\hat{s}_1|\lambda_n/n \geq |\beta_{0j}|/2 \text{ for some } j \in A_1 \right\} \\
+ P \left\{ \left| X_j^TH_n\tau \right| \geq \lambda_n v_j/2 \text{ for some } j \notin A_1 \right\} + P \left\{ \left| X_j^T\hat{X}_{A_1}(\hat{X}_{A_1}^T\hat{X}_{A_1})^{-1}\hat{s}_1 \right| \geq v_j/2 \text{ for some } j \notin A_1 \right\},
\]

where \( e_j \) is the unit vector in the direction of the \( j \)-th coordinate. Following Huang et al. (2008), it can be proved that each of the above four probabilities converges to zero.