NONPARAMETRIC ESTIMATOR OF FALSE DISCOVERY RATE BASED ON BERNŠTEIN POLYNOMIALS

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Abstract: Under a local dependence assumption about the p-values, an estimator of the proportion \( \pi_0 \) of true null hypotheses, having a closed-form expression, is derived based on Bernšteín polynomial density estimation. A nonparametric estimator of false discovery rate (FDR) is then obtained. These estimators are proved to be consistent, asymptotically unbiased, and normal. Confidence intervals for \( \pi_0 \) and the FDR are also given. The usefulness of the proposed method is demonstrated through simulations and its application to a microarray dataset.

Key words and phrases: Bernšteín polynomials, bioinformatics, density estimation, false discovery rate, local dependence, microarray, mixture model, multiple comparison.

1. Introduction

Statistical significance in multiple comparison problems has attracted the attention of many authors. The false discovery rate (FDR), first introduced by Benjamini and Hochberg (1995), is one measure of this statistical significance. Storey (2002a) and Storey and Tibshirani (2003) introduced the positive false discovery rate (pFDR) and proposed procedures for estimating FDR and pFDR, with applications to DNA microarrays, under the assumptions that the test statistics of the hypotheses are independent and dependent, respectively.

Let \( T \) be the test statistic for hypothesis \( H \). Denote the null and alternative hypotheses by \( H = 0 \) and \( H = 1 \), respectively. So \( \pi_0 \equiv \Pr(H = 0) \) is the proportion of true null hypotheses, and \( F_j(t) \equiv \Pr(T \leq t \mid H = j) \), \( j = 0, 1 \), are the null and alternative distribution of \( T \), respectively. Consider testing \( n \) hypotheses, \( H_1, \ldots, H_n \), with test statistics \( T_1, \ldots, T_n \). For each \( i \), denote the null and alternative hypotheses by \( H_i = 0 \) and \( H_i = 1 \), respectively. Assume \( \Pr(H_i = 0) = \pi_0 \) and \( \Pr(T_i \leq t \mid H_i = j) = F_j(t) \), \( j = 0, 1 \), for all \( i \). The set of observed values, \( t_1, \ldots, t_n \), of the test statistics \( T_1, \ldots, T_n \) is treated as a sample from the mixture distribution of \( T \):

\[
F(t) = \pi_0 F_0(t) + (1 - \pi_0) F_1(t).
\]
Let $\Gamma$ be the common rejection region for all the tests. The notion of false non-discovery rate (FNR) was introduced by (Genovese and Wasserman (2002)). The following Bayesian interpretation of the pFDR and the positive false non-discovery rate (pFNR) can be found in (Storey (2002a, 2003)):

$$pFDR = \Pr(H = 0 \mid T \in \Gamma) = \frac{\pi_0 \Pr_{F_0}(T \in \Gamma)}{\Pr_F(T \in \Gamma)},$$

$$pFNR = \Pr(H = 1 \mid T \not\in \Gamma) = 1 - \frac{\pi_0 \Pr_{F_0}(T \not\in \Gamma)}{\Pr_F(T \not\in \Gamma)}.$$

The terms $\Pr_F(T \in \Gamma)$ and $\Pr_F(T \not\in \Gamma)$ above can be estimated from the data. The probabilities $\Pr_{F_0}(T \in \Gamma)$ and $\Pr_{F_0}(T \not\in \Gamma)$ can be obtained from the null distribution which usually is known or can be estimated in some way, such as by using resampling methods. If $\pi_0$ can be estimated based on $t_1, \ldots, t_n$, then pFDR and pFNR are estimable. Allison et al. (2002) also used these quantities, and modeled the distribution of the $p$-values from microarray data analysis by a finite Beta mixture distribution. Note that the Type I error rate for each single test is $\Pr(T \in \Gamma \mid H = 0)$ and the family-wise-error-rate (FWER) is $\Pr\{\bigcup_{i=1}^n (T_i \in \Gamma, H_i = 0)\}$ (see, for example, Westfall and Young (1993) and Ge, Dudoit and Speed (2003)).

The simplest situation occurs when we know the parametric forms of both the null and alternative distributions, $F_0$ and $F_1$. In this case, we can fit a parametric mixture model to the observed test statistics (Guan, Wu and Zhao (2004)). Simulation studies have shown that the model-based approach can significantly improve pFDR and FDR estimation if the parametric model is correct.

In most applications, two other scenarios are more likely to occur. The first has the null distribution, or at least its large sample approximation, of the test statistics as known, while the alternative distribution is unknown. The other is the more difficult situation in which neither the null nor the alternative distributions are known. In this case, methods such as permutation procedures can be used to estimate the null distribution of the test statistics.

This paper assumes that both $F_0$ and $F_1$ are continuous, and that $F_0$ is known or can be estimated in some way. We use $p$-values as the test statistics and, in this case, $F_0(t) = t, 0 \leq t \leq 1$. In the rest of the paper, $F_1$ is assumed to be continuous on $[0, 1]$ and we let $t_1, \ldots, t_n$ represent the $p$-values of the $n$ tests. In terms of densities, the mixture model (1.1) with $F_0(t) = t$ can be written as

$$f(t) = \pi_0 + (1 - \pi_0)f_1(t).$$

(1.2)

In this case, if the common rejection region is $\{p \leq p_0\}$, then

$$pFDR = \frac{p_0 \pi_0}{F(p_0)} \quad \text{and} \quad pFNR = 1 - \frac{(1 - p_0) \pi_0}{1 - F(p_0)}.$$
Therefore, the key to the estimation of these quantities is the estimation of \( \pi_0 \). Write \( b = \min_{t \in [0, 1]} f_1(t) \). Clearly, \( 0 \leq b < 1 \). In order that (1.2), as a nonparametric model, is identifiable, one has to assume that \( b = 0 \); otherwise, for any \( a \in [0, b] \), \( \pi_0^* = \pi_0 + a(1 - \pi_0) \) and \( f_1^*(t) = \{ f_1(t) - a \}/(1 - a) \) satisfy the model \( f(t) = \pi_0^* + (1 - \pi_0^*)f_1^*(t) \). Furthermore, the density \( f_1 \) is assumed to be continuous on \([0, 1]\) with \( b = f_1(1) = 0 \), so that \( f(1) = \pi_0 \) \cite{WuGuanZhao2006}. Therefore, if \( \hat{f} \) is a density estimate, \( \hat{f}(1) \) is an estimate of \( \pi_0 \).

The most commonly used kernel density estimate is subject to boundary effects at 0 and 1. In order to minimize the boundary effect of kernel density estimation for distribution with bounded support, one has to make a boundary correction \cite{Jones1993}. A Bernstein polynomial density estimate seems convenient for estimating \( f(1) \), and has a closed-form expression. Let \( \hat{F} \) denote the empirical distribution of \( t_1, \ldots, t_n \). \cite{StoreyTibshirani2003} proposed using \( \hat{\pi}_0 = \hat{g}(1) \) to estimate \( \pi_0 \), with \( \hat{g} \) being the fitted spline to the data \( \{ \hat{\pi}_0(\lambda) : \lambda = 0.01, 0.02, \ldots, 0.95 \} \) where \( \hat{\pi}_0(\lambda) = [1 - \hat{F}(\lambda)]/(1 - \lambda) \). Based on nonparametric maximum likelihood estimation of the density of \( p \)-values, with restriction to convex decreasing densities, \cite{LangaasLindqvistFerkingstad2005} proposed another smoothing method and showed that their method outperforms some existing estimators with respect to root-mean-squared error.

There are several approaches to FDR estimation. Among many others, \cite{Efronetal2001}, \cite{EfronTibshirani2002}, and \cite{Efron2003} proposed the empirical Bayes method, which also uses the model (1.2). \cite{Guanetal2004} proposed a method that assumed parametric forms of \( f_0 \) and \( f_1 \) in (1.2). Readers are referred to \cite{Wuetal2006} for an extensive comparison among these methods. Although not explored directly in \cite{Allisonetal2002}, FDR could also be estimated by their method based on a mixture model of Beta distributions.

Since any continuous function on \([0, 1]\) can be uniformly approximated with Bernstein polynomials \cite{Bernstein1912}, \cite{Vitali1975} proposed using them to estimate an unknown density function. \cite{Tenbusch1994} extended this method to multidimensional situation. The rates of convergence of the posterior distribution for a Bernstein polynomial prior were obtained by \cite{Ghosal2001}. The Bernstein polynomial and the \( k \)-th order Bernstein expansion of a function \( g(t) \) are defined as

\[
B_{j,k}(t) = \binom{k}{j} t^j (1 - t)^{k-j}, \quad \mathbb{B}_k g(t) = \sum_{j=0}^{k} g\left(\frac{j}{k}\right) B_{j,k}(t).
\]

One can estimate \( F \) and \( f \) by

\[
\hat{F}_k(t) = \mathbb{B}_k \hat{F}(t) = \sum_{j=0}^{k} \hat{F}\left(\frac{j}{k}\right) B_{j,k}(t), \quad \hat{f}_k(t) = \mathbb{B}_{k-1} \hat{f}(t) = \sum_{j=0}^{k-1} \hat{f}\left(\frac{j}{k-1}\right) B_{j,k-1}(t),
\]
respectively, where
\[
\hat{f}\left(\frac{j}{k}-1\right) = k\left\{\hat{F}\left(\frac{j+1}{k}\right) - \hat{F}\left(\frac{j}{k}\right)\right\}, \quad \text{for } j = 0, \ldots, k-1.
\] (1.3)

If \(k\) is chosen proportional to \(n^{2/5}\), then, for each fixed \(t \in (0,1)\), the mean square error of \(\hat{f}_k(t)\) is proportional to \(n^{-4/5}\) [Vitale (1975)].

We propose a nonparametric method based on Bernstein polynomial density estimation. Simulation study and an application to a microarray dataset are carried out in Section 4. The proofs of the main results are given in the Appendix.

2. Estimators of \(\pi_0\) and FDR and Asymptotic Results

Albeit Bayesian interpretations of pFDR and pFNR have been used, it is convenient to work directly with the test statistics of the hypotheses. With a properly chosen \(1 \leq r < k\), one can estimate \(\pi_0\) by
\[
\tilde{\pi}_0 = \frac{1}{r} \sum_{l=1}^{r} \hat{f}_k\left(1 - \frac{l}{k}\right).
\] (2.1)

If \(r = 1\) then \(\tilde{\pi}_0 = \hat{f}_k\left(1 - 1/k\right) \approx f(1) = \pi_0\) for large \(k\). On average, this estimator has smaller variance for larger \(r > 1\), but for larger \(r\), bias increases. Of course, the magnitude of the bias depends on \(k\) and \(f\) as well. Later in this paper, we develop a method to choose \(r\) and \(k\) to balance bias-variance trade-off by minimizing a partial mean square error of \(\tilde{\pi}_0\).

The following assumptions are needed for the asymptotic results about \(\tilde{\pi}_0\):

Assumption 1. The test statistics \(T_1, \ldots, T_n\) satisfy the local dependence (LD1) of Chen and Shao (2005): for each \(T_i\), except for \(n_i\) statistics \(T_{i_1}, \ldots, T_{i_{n_i}}\), all other \(T_j\)'s are independent of \(T_i\). There exists an \(m\) independent of \(n\) so that \(\bar{n} = n^{-1} \sum_{i=1}^{n} n_i \leq m\). This is a generalization of \(m\)-dependence.

Assumption 2. The partial derivative \(f_{uv}(s,t) = \frac{\partial^2 F_{uv}(s,t)}{\partial s \partial t}\) of the joint distribution function \(F_{uv}(s,t)\) of each pair \((T_u, T_v)\) is uniformly bounded by a constant, independent of \((u,v)\).

Assumption 1 is usually satisfied for gene expression data since in the whole genome, each gene is likely to have interactions only with a limited number of other genes. A Glivenko-Cantelli lemma of Yu (1993) for dependent sequences ensures that Assumption 1 satisfies the weak dependence assumption made by Storey et al. (2004). More discussion on the dependence issue in the estimation of FDR can also be found in Langaas et al. (2003). Efron (2006) discusses the effect of correlation on the null distribution and FDR.
**Theorem 1.** Suppose \( f(t) \) is continuously differentiable on \((0, 1]\) with a bounded derivative, and that \( f(1) = \pi_0 \). Then for each fixed \( r \), \( \tilde{\pi}_0 \) is an asymptotically unbiased estimator of \( \pi_0 \). Moreover, as \( k, n \to \infty \),

\[
|E(\tilde{\pi}_0) - \pi_0| = \mathcal{O}(k^{-1}).
\]

(2.2)

If Assumptions 1 and 2 hold, then for each fixed \( r \),

\[
\lim_{k, n \to \infty} \frac{n \text{Var}(\tilde{\pi}_0)}{kh_k(r)} = \pi_0
\]

(2.3)

where, for each \( r \geq 1 \),

\[
h_k(r) \equiv \sum_{j=0}^{k-1} \left\{ \frac{1}{r} \sum_{i=1}^{r} B_{j, k-i} \left( 1 - \frac{i}{k} \right) \right\}^2,
\]

(2.4)

\[
h(r) \equiv \lim_{k \to \infty} h_k(r) = \sum_{j=0}^{\infty} \left( \frac{1}{r} \sum_{l=1}^{j} \frac{l!}{j!} e^{-l} \right)^2 = \mathcal{O}(r^{-\frac{3}{2}}).
\]

(2.5)

Furthermore, if \( k \) is of order \( n^{1/3} \), then

\[
E(\tilde{\pi}_0 - \pi_0)^2 = \mathcal{O}(n^{-\frac{2}{3}}).
\]

(2.6)

**Remark 2.1.** It should be noted that if the assumption \( f_1(1) = 0 \) is violated then \( \tilde{\pi}_0 \) is approximately conservative, i.e., \( E(\tilde{\pi}_0) > \pi_0 \) for large \( k \) and \( n \). The Storey and Tibshirani (2003) estimate \( \hat{\pi}_0(\lambda) \) has the same property.

**Remark 2.2.** It is easy to see that

\[
h(r) = \frac{1}{r^2} \sum_{l=1}^{r} I_0(2l) e^{-2l} + \frac{2}{r^2} \sum_{1 \leq i < j \leq r} I_0(2\sqrt{ij}) e^{-i-j},
\]

(2.7)

where \( I_0(x) \) is the modified first kind Bessel function \( I_\nu(x) \) with \( \nu = 0 \):

\[
I_0(x) = \sum_{j=0}^{\infty} \frac{(\frac{x}{2})^{2j}}{(j!)^2}.
\]

Clearly, \( h(r) \leq h(1) = 0.3085083 \).

**Remark 2.3.** The assumption that \( f'(1) \) is bounded can be violated in some cases. For example, let the test statistic \( T \) be \( N(0, 1) \) under \( H_0 \) and \( N(\mu, 1) \) under \( H_A \), with \( \mu > 0 \). The distribution function of the p-value of the one-sided test is \( F(t) = \pi_0 t + (1 - \pi_0) \{ 1 - \Phi[\Phi^{-1}(1 - t) - \mu] \} \), where \( \Phi \) is the distribution
function of $N(0, 1)$. Write $\varphi(t) = \Phi'(t)$. The density function and its derivative are, respectively,

$$
f(t) = F'(t) = \pi_0 + \sqrt{2\pi}(1 - \pi_0)\varphi(\mu)e^{\Phi^{-1}(1-t)},$$
$$f'(t) = -2\pi\mu(1 - \pi_0)\varphi(\mu)e^{\Phi^{-1}(1-t)[\mu + \Phi^{-1}(1-t)]}.
$$

Thus $\lim_{t \to 1^-} f(t) = \pi_0$ and $\lim_{t \to 0^+} f'(t) = \lim_{t \to 1^-} f'(t) = -\infty$, so $f'$ is not bounded.

In the proof of Theorem 2.1, it is shown that $\tilde{\pi}_0$ is a sum of locally dependent random variables, the following asymptotic normality of $\tilde{\pi}_0$ is a consequence of the recent result of Chen and Shao (2005).

**Theorem 2.2.** Suppose that Assumptions 1 and 2 hold. If $f(t)$ is continuously differentiable on $(0, 1]$ with bounded derivative and $f(1) = \pi_0$, then for fixed $r$, as $k, n \to \infty$ and $k/n \to 0$,

$$
\sqrt{n}\left\{\tilde{\pi}_0 - E(\tilde{\pi}_0)\right\} \overset{d}{\to} N(0, \pi_0).
$$

(2.8)

Given a cutoff $p_0$ for the $p$-values, FDR can be estimated by

$$
\hat{p}_{FDR}(p_0) = \frac{p_0\tilde{\pi}_0}{F(p_0)}.
$$

(2.9)

From the Glivenko-Cantelli lemma of Yu (1993), Theorems 2.1 and 2.2, it follows that $p_{FDR}(p_0)$ is also consistent and asymptotically normal. One can construct a confidence interval for $\pi_0$ as follows. For a given confidence level $1 - \alpha$, let $z_{\alpha/2}$ be the upper $\alpha/2$ quantile of the standard normal distribution, so

$$
\Pr\left\{\frac{\sqrt{n}\{\tilde{\pi}_0 - \pi_0\}}{\sqrt{k\tilde{\pi}_0 h_k(r)}} < z_{\alpha/2}\right\} \approx 1 - \alpha.
$$

Therefore $\Pr\{\tilde{\pi}_{0L}(\alpha) < \pi_0 < \tilde{\pi}_{0U}(\alpha)\} \approx 1 - \alpha$, where

$$
\tilde{\pi}_{0L}(\alpha) = \tilde{\pi}_0 - z_{\alpha/2}\sqrt{\frac{k}{n}\tilde{\pi}_0 h_k(r)} \tilde{\pi}_0 \quad \text{and} \quad \tilde{\pi}_{0U}(\alpha) = \tilde{\pi}_0 + z_{\alpha/2}\sqrt{\frac{k}{n}\tilde{\pi}_0 h_k(r)} \tilde{\pi}_0.
$$

(2.10)

Based on the confidence interval for $\pi_0$ one can obtain the confidence interval $(\hat{p}_{FDR_L}, \hat{p}_{FDR_U})$ for FDR with

$$
\hat{p}_{FDR_L} = \frac{p_0\tilde{\pi}_{0L}(\alpha)}{F(p_0)}, \quad \hat{p}_{FDR_U} = \frac{p_0\tilde{\pi}_{0U}(\alpha)}{F(p_0)}.
$$

One can also replace $\hat{F}(p_0)$ by $\hat{F}_k(p_0)$. 

3. Choosing Optimal $r$ and $k$

When the sample size $n$ is large, as in microarray data analysis, Assumptions 1 and 2 assure that the contribution made by covariances to the variance of the estimator $\tilde{\pi}_0$ is bounded above by a quantity independent of $(r, k)$ (see (A.17)). In the proof of Theorem 2.1, (A.8) gives an estimate of the bias of $\tilde{\pi}_0$:

$$B(r, k) = |E(\tilde{\pi}_0) - \pi_0| \leq \sum_{i=0}^{3} R_{1i}(k, r).$$

Then one can choose $r$ and $k$ by minimizing the partial mean square error

$$pMSE(r, k) = \left\{ \sum_{i=0}^{3} R_{1i}(k, r) \right\}^2 + D(r, k),$$

where $D(r, k) = (k\pi_0/n)h_k(r)$. One can estimate $pMSE(r, k)$ by

$$\hat{pMSE}(r, k) = \left\{ \sum_{i=0}^{3} \hat{R}_{1i}(k, r) \right\}^2 + \hat{D}(r, k),$$

where $\hat{D}(r, k) = (k\tilde{\pi}_0/n)h_k(r)$,

$$k\hat{R}_{1i}(k, r) \approx \left( \frac{1}{2} \right)^{1-i} \sum_{j=0}^{k-1} \hat{b}(j, k, r) \left| \hat{f}'_k \left( \frac{j}{k-1} \right) \right| \left( \frac{j}{k-1} \right)^i, \quad i = 0, 1,$$

$$\hat{R}_{12}(k, r) \approx \sum_{j=0}^{k-1} \hat{b}(j, k, r) \left| \hat{f}'_k \left( \frac{j}{k-1} \right) \right| \left| 1 - \frac{1}{k} - \frac{j}{k-1} \right|,$$

$$\hat{R}_{13}(k, r) \approx \frac{1}{k} \left| \hat{f}'_k \left( 1 - \frac{1}{k} \right) \right|,$$

$$\hat{f}'_k(t) = B_{k-2} \hat{f}(t) = \sum_{j=0}^{k-2} \hat{f}' \left( \frac{j}{k-2} \right) B_{j,k-2}(t),$$

$$\hat{f}' \left( \frac{j}{k-2} \right) = (k-1) \left\{ \hat{f}_k \left( \frac{j+1}{k-1} \right) - \hat{f}_k \left( \frac{j}{k-1} \right) \right\},$$

for $j = 0, \ldots, k-2$,

and $\hat{b}(j, k, r)$ is defined by (A.2). The optimal $\hat{r}$ and $\hat{k}$ satisfy

$$\hat{pMSE}(\hat{r}, \hat{k}) = \min\{\hat{pMSE}(s, t), \quad 1 \leq s < t < n\}.$$

Intuitively, the larger the number $k$ of bins, the larger the variance of $\tilde{\pi}_0$. On the other hand, increasing the number $r$ in (2.1) can reduce the variance of $\tilde{\pi}_0$. Based
on \( (2.41) \) and \( (2.49) \), \( (k\pi_0/n)h_k(r) \) is an applicable measure of the dependence of variance of \( \tilde{\pi}_0 \) upon \( r \) and \( k \). The upper bound estimate \( (3.1) \) is obtained by applying the triangle inequality. Thus \( (3.2) \) is suitable for finding optimal \( r \) and \( k \). The R package, \texttt{nFDR}, which implements the method of this paper is available on CRAN (the Comprehensive R Archive Network).

4. Simulation Studies and Application to Microarray Data

Comparison Study: In this simulation, we took \( n = 1,000, \pi_0 = 0.25, 0.50, 0.75, \) and 0.95, and \( B = 500 \) sets of \( p \)-values \( p_1, \ldots, p_n \) were simulated with \( p_i \) uniform(0, 1) or Beta(1, 6). The proportion \( \pi_0 \) of true null hypotheses is estimated in four different ways: (1) \( \tilde{\pi}_0^s \) is based on \( (r^*, k^*) \), where \( (r^*, k^*) \) is the minimizer of \( \text{pMSE}(r, k) \); (2) \( \tilde{\pi}_0 \) is based on \( (\hat{r}, \hat{k}) \); (3) \( \tilde{\pi}_0^s \) is estimated by function \texttt{convest(} \) of R package \texttt{limma}, which implements the convex decreasing density method of \textit{Langaug et al.} \( (2005) \); and (4) \( \hat{\pi}_0^q \) is estimated by the R package \texttt{qvalue} using a default setting that implements the smooth method described in \textit{Storey and Tibshirani} \( (2003) \) (see also \textit{Storey} \( (2003) \), \textit{Storey}, \textit{Taylor} and \textit{Siegmund} \( (2004) \)). Simulation shows that the \( \tilde{\pi}_0^s \)’s based on \( (\hat{r}, \hat{k}) \) have a variation close to, but with a slightly smaller bias than the ones based on \( (r^*, k^*) \). Thus the selected \( \hat{r} \) and \( \hat{k} \) performed well. The smooth method of \textit{Storey} and \textit{Tibshirani} \( (2003) \) has a larger variation, and a larger bias, than the proposed method. Except for differences in the biases, the proposed method has variation similar to the convex decreasing density method.

Impact of Dependence: In this simulation, \( n = 3,000, m = 10 \) and \( B = 500 \). First, two-sample gene expression data \( \{x_{ij}, y_{ij} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, N \} \), \( N = 10 \), were generated in three different ways. For each gene \( i \), the null hypothesis was \( H_i : \mu_x = \mu_y \). Then \( B \) sets of \( p \)-values \( p_i \) of two-sample \( t \)-test with the same variances were calculated based on \( x_{i1}, \ldots, x_{in} \) and \( y_{i1}, \ldots, y_{in} \).

1. \textit{Independence}: \( x_{ij} = \mu_{ij}, y_{ij} = \mu'_{ij} + 3I\{i \leq (1 - \pi_0)n\} \), with \( \mu_{ij} \)’s and \( \mu'_{ij} \)’s i.i.d. \( N(0, 1) \).

2. \textit{Dependence Case 1}: \( x_{ij} = (-1)^j \mu_{ij} + \varepsilon_{ij}, y_{ij} = (-1)^i \mu'_{ij} + \varepsilon'_{ij} + 2I\{i \leq (1 - \pi_0)n\}, j = 1, \ldots, N, i = (v - 1)m + 1, \ldots, (v - 1)m + m, v = 1, \ldots, n/m, \) where \( \varepsilon_{ij} \)’s and \( \varepsilon'_{ij} \)’s are i.i.d. \( N(0, 0.04^2) \) and, for each \( v \), \( \mu_{ij} \)’s and \( \mu'_{ij} \)’s were i.i.d. \( N(0, 1) \). In this case, the correlation between \( p \)-values for each pairs of genes in a group of \( m \) was about \( \pm 0.9983 \) by simulation. This dependence is similar to but has larger correlation than the dependence simulation of \textit{Storey et al.} \( (2004) \).

3. \textit{Dependence Case 2}: \( x_{ij} = \mu_{ij} + \varepsilon_j, y_{ij} = \mu'_{ij} + \varepsilon'_{ij} + 2I\{i \leq (1 - \pi_0)n\}, \) \( j = 1, \ldots, N, i = 1, \ldots, n, \) where \( \varepsilon_j \)’s and \( \varepsilon'_{ij} \)’s were i.i.d. \( N(0, 0.25^2) \). In this case, the correlation between \( p \)-values for each pairs of genes was about \( 0.0581 \) by simulation. This is the so-called “general dependence” of \textit{Storey} \( (2002b) \).
Figure 1. Simulation results for Beta distributed $p$-values. The true values of $\pi_0$ are 0.25, 0.50, 0.75, and 0.95.

The simulated $\tilde{\pi}_0$'s are summarized in Figure 2. The biases and standard deviations of the simulated $\tilde{\pi}_0$'s and the estimated coverage probabilities of the 95% confidence intervals for $\pi_0$ are given in Table 1 (The results for $\pi_0 = 0.05$ are not shown in Figure 2). The above simulation studies show that the performance of the method is satisfactory for most cases. When dependence is present and $\pi_0$ is close to 1, the variance of $\tilde{\pi}_0$ may be underestimated so that the coverage is less than the nominal one. In applications to microarray data analysis, this can be overcome by eliminating many obvious non-significant and irrelevant genes using data preprocessing and filtering.

**Leukemia Data:** In large-scale microarray data analysis, there are usually thousands or tens of thousands of genes involved. It is practical to assume that genes in the same pathway have similar expression profiles and affect the system function in a synergistic way. The number of genes in a pathway is usually relatively small compared to the total number of genes in the data. The researchers are usually interested in identifying differentially expressed genes using certain types of tests. For each gene, a value of a test statistic is calculated based on sample
Figure 2. Simulation results for independent and dependent $p$-values. The true values of $\pi_0$ are $0.25$, $0.50$, $0.75$ and $0.95$.

Table 1. Biases and standard deviations of the simulated $\tilde{\pi}_0$’s presented in Figure 2, and simulated coverage probabilities of 95% CI’s.

<table>
<thead>
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<th>$\pi_0$</th>
<th>0.05</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>0.95</th>
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<td>0.00096</td>
<td>-0.00077</td>
<td>-0.00230</td>
</tr>
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<td>0.00045</td>
<td>-0.00044</td>
<td>-0.00079</td>
<td>-0.00123</td>
</tr>
<tr>
<td>Dependence 2</td>
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Table 1. Biases and standard deviations of the simulated $\tilde{\pi}_0$’s presented in Figure 2, and simulated coverage probabilities of 95% CI’s.
observations of the expression levels. Test statistics, such as \( p \)-values generated in microarray data analysis, seem to satisfy the assumptions of this paper.

The leukemia gene expression dataset was reported in Golub et al. (1999). In this study there were \( N_1 = 47 \) patients with Acute Lymphoblastic Leukemia (ALL), and \( N_2 = 25 \) patients with Acute Myeloid Leukemia (AML). The mRNA levels of 7,129 genes were measured for these \( N = 72 \) samples. The same procedures as in Wu et al. (2006) were used to preprocess genes and calculate two sample \( t \)-test statistics. Permutations were used to obtain \( p \)-values for \( n = 3,571 \) remaining genes after data preprocessing and filtering. The histogram of the \( p \)-values (not shown here) indicates that the mixture model (12) is valid, that the assumptions of the paper are not violated. Based on the expression data, correlation tests for the \( n(n - 1)/2 \) pairs of genes, using a Bonferroni adjusted FWER of 0.05, give an estimate of \( \bar{n} = 6.529 \). The method of this paper results in \((\hat{r}, \hat{k}) = (18, 107)\) and \( \hat{\pi}_0 = 0.449 \), which is low because many obvious non-significant and irrelevant genes have been eliminated by data preprocessing and filtering.

Based on the simulation study and Theorem 2.2, data preprocessing and filtering are recommended, if possible, to have a smaller \( \pi_0 \) and thus a smaller variance of \( \tilde{\pi}_0 \). The corresponding confidence interval is \((0.399, 0.498)\). Figure 3 shows the pFDR estimations and the 95\% confidence intervals. For example, if \( p_0 = 1.808 \times 10^{-3} \) is a cutoff of the \( p \)-values, then there are about 600 genes which have smaller \( p \)-values and are claimed to be differentially expressed. The corresponding pFDR is \( 4.944 \times 10^{-3} \).

[Figure 3. pFDR Estimation for the Golub et al. (1999) data.]
Appendix

Proof of Theorem 2.1. From \( \hat{F}(t) = n^{-1} \sum_{i=1}^{n} I(T_i \leq t) \), it is easy to see that

\[
\hat{\pi}_0 = \frac{1}{n} \sum_{i=1}^{n} Y_{ki},
\]

(A.1)

\[
Y_{ki} = k \sum_{j=0}^{k-1} b(j, k, r) I \left\{ \frac{j}{k} < T_i \leq \frac{j+1}{k} \right\}, \quad b(j, k, r) = \frac{1}{r} \sum_{l=1}^{r} B_{j, k-1} \left( 1 - \frac{l}{k} \right).
\]

(A.2)

In the proof, the following results are useful. If \( |g_j| \leq M \), then for \( i \geq 1 \),

\[
\sum_{j=0}^{k-1} \tilde{b}(j, k, r) g_j = O(1),
\]

(A.3)

and for \( v \geq i \geq 0 \) and \( v \geq 1 \),

\[
\sum_{j=0}^{v} B_{j, v}(t) j^{[i]} = v^{[i]} t^i,
\]

(A.4)

where \( j^{[i]} = j(j-1) \cdots (j-i+1) \) if \( j \geq i \); and is 0 otherwise. Define \( j^{[0]} = 1 \) for \( j > 0 \). Clearly, for each fixed \( k \), \( Y_{k1}, \ldots, Y_{kn} \) are identically distributed random variables with finite mean and variance given by, respectively,

\[
\mu_k \equiv \mathbb{E}(Y_{ki}) = k \sum_{j=0}^{k-1} \tilde{b}(j, k, r) \Delta_{kj},
\]

(A.5)

\[
\sigma^2_k \equiv \text{Var}(Y_{ki}) = k^2 \sum_{j=0}^{k-1} \tilde{b}^2(j, k, r) \Delta_{kj} - \left\{ \mathbb{E}(Y_{ki}) \right\}^2,
\]

(A.6)

where \( \Delta_{kj} = F[(j+1)/k] - F(j/k) \). Since \( f(1) = \pi_0 \), \( \mathbb{E}(\hat{\pi}_0) = \mathbb{E}(Y_{k1}) \) and \( \sum_{j=0}^{k-1} \tilde{b}(j, k, r) = 1 \), the absolute bias is

\[
B(r, k) \equiv |\mathbb{E}(\hat{\pi}_0) - \pi_0| = |\mathbb{E}(Y_{k1}) - f(1) | = \left| \sum_{j=0}^{k-1} \tilde{b}(j, k, r) [k \Delta_{kj} - f(1)] \right|.
\]

Taylor expansions imply that the existence of \( \xi_{0j} \in (j/k, (j+1)/k) \), \( \xi_{1j} \in (j/k, j/(k-1)) \), \( \xi_{2j} \in (j/(k-1), 1-1/k) \) and \( \xi_{3j} \in (1-1/k, 1) \) such that

\[
k \Delta_{kj} = f \left( \frac{j}{k} \right) + \frac{1}{2k} f'(\xi_{0j}) = f \left( \frac{j}{k-1} \right) + \frac{1}{2k} f'(\xi_{0j}) - f'(\xi_{1j}) \frac{j}{k(k-1)}
\]
\[ f(1) + \frac{1}{2k} f'(\xi_{0j}) - f'(\xi_{1j}) \frac{j}{k(k-1)} - f'(\xi_{2j}) \left( 1 - \frac{1}{k} - \frac{j}{k-1} \right) - f'(\xi_{3j}) \frac{1}{k}. \]  

(A.7)

Therefore

\[ B(r, k) \leq \sum_{i=0}^{3} R_{1i}(k, r), \]  

(A.8)

where the \( R_{1i}(k, r), i = 0, 1, 2, 3, \) are defined below. Since \( tf'(t) \) is bounded,

\[ kR_{1i}(k, r) \equiv \left( \frac{1}{2} \right)^{1-i} \sum_{j=0}^{k-1} b(j, k, r) |f'(\xi_{ij})| \left( \frac{j}{k-1} \right)^i \]

\[ \approx \left( \frac{1}{2} \right)^{1-i} \sum_{j=0}^{k-1} b(j, k, r) \left| f' \left( \frac{j}{k-1} \right) \right| \left( \frac{j}{k-1} \right)^i = \mathcal{O}(1), \quad i = 0, 1. \]  

(A.9)

The Cauchy-Schwarz inequality implies

\[ R_{12}^2(k, r) \equiv \left\{ \sum_{j=0}^{k-1} b(j, k, r) |f'(\xi_{2j})| \left| 1 - \frac{1}{k} - \frac{j}{k-1} \right| \right\}^2 \]

\[ \leq \sum_{j=0}^{k-1} b(j, k, r) |f'(\xi_{2j})|^2 \sum_{j=0}^{k-1} b(j, k, r) \left( 1 - \frac{1}{k} - \frac{j}{k-1} \right)^2. \]  

(A.10)

It follows from (A.9) that

\[ \sum_{j=0}^{k-1} b(j, k, r) \left( 1 - \frac{1}{k} - \frac{j}{k-1} \right)^2 = \frac{1}{r} \sum_{i=1}^{r} \sum_{j=0}^{k-1} B_{j, k-1} \left( 1 - \frac{l}{k} \right) \left( 1 - \frac{1}{k} - \frac{j}{k-1} \right)^2 \]

\[ = \frac{1}{3k^2} (r^2 + 2 - \frac{r^2 - 1}{k-1}). \]  

(A.11)

It follows from (A.3), (A.10) and (A.11) that

\[ R_{12}(k, r) = \mathcal{O}(k^{-1}), \]  

(A.12)

\[ R_{13}(k, r) \equiv \sum_{j=0}^{k-1} b(j, k, r) \frac{1}{k} |f'(\xi_{3j})| = \frac{1}{k} |f'(\xi_{3})| \approx \frac{1}{k} |f' \left( 1 - \frac{1}{k} \right)|. \]  

(A.13)

Combining (A.9) through (A.13) proves (2.2). For \( u \neq v, \)

\[ \text{Cov} (Y_{ku}, Y_{kv}) = k^2 \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} b(i, k, r) b(j, k, r) \left\{ \Delta_{kij}^{(uv)} - \Delta_{ki} \Delta_{kj} \right\}, \]  

(A.14)
where
\[
\Delta_{kij}^{(uv)} = \Pr\left(\frac{i}{k} < T_u \leq \frac{i+1}{k}, \frac{j}{k} < T_v \leq \frac{j+1}{k}\right)
= F_{uv}\left(\frac{i+1}{k}, \frac{j}{k}\right) - F_{uv}\left(\frac{i+1}{k}, \frac{j+1}{k}\right) - F_{uv}\left(\frac{i}{k}, \frac{j}{k}\right) + F_{uv}\left(\frac{i}{k}, \frac{j+1}{k}\right).
\] (A.15)

By the Mean Value Theorem we have that, for some \(\tau_{ki} \in (i/k, (i+1)/k)\), \(\zeta_i^{(uv)} \in (i/k, (i+1)/k)\), and \(\eta_j^{(uv)} \in (j/k, (j+1)/k)\),
\[
k\Delta_{ki} = f(\tau_{ki}), \quad k^2\Delta_{kij}^{(uv)} = f_{uv}\left(\zeta_i^{(uv)}, \eta_j^{(uv)}\right).
\] (A.16)

Since \(f\) and \(f_{uv}\) are bounded, it follows from (A.14) and (A.16) that there exists a constant \(C\) such that for \(u \neq v\)
\[
\left|\text{Cov}\left(Y_{ku}, Y_{kv}\right)\right| \leq \max_{ij} \left\{f_{uv}(\zeta_i^{(uv)}, \eta_j^{(uv)}) + f(\tau_{ki})f(\tau_{kj})\right\}\left[\sum_{i=0}^{k-1} b(i, k, r)\right]^2 \leq C,
\]
\[
\left|\text{Var}\left(\tilde{\pi}_0\right) - \frac{1}{n}\sigma_k^2\right| = \frac{1}{n^2}\left[\sum_{u \neq v} \text{Cov}\left(Y_{ku}, Y_{kv}\right)\right] \leq \frac{(\bar{n} - 1)C}{n} \leq \frac{m - 1}{n}C.
\] (A.17)

From (A.13) it follows that
\[
kR_{2i}(k, r) \equiv \sum_{j=0}^{k-1} b^2(j, k, r)|f'(\xi_{ij})\left(\frac{j}{k-1}\right)^i = \mathcal{O}(1), \quad i = 0, 1.
\] (A.18)

Another application of the Cauchy-Schwarz inequality gives
\[
R_{22}^2(k, r) \equiv \left\{\sum_{j=0}^{k-1} b^2(j, k, r)|f'(\xi_{2j})|\left|1 - \frac{1}{k} - \frac{j}{k-1}\right|\right\}^2
\leq \sum_{j=0}^{k-1} b^2(j, k, r)|f'(\xi_{2j})|^2 \sum_{j=0}^{k-1} b(j, k, r)\left(1 - \frac{1}{k} - \frac{j}{k-1}\right)^2
= \frac{1}{3k^2}\left\{\sum_{j=0}^{k-1} b^2(j, k, r)|f'(\xi_{2j})|^2\right\}\left(r^2 + 2 - \frac{r^2 - 1}{k-1}\right) = \mathcal{O}(r^2k^{-2}).
\] (A.19)

\[
R_{23}(k, r) \equiv \frac{1}{k}|f'(\xi_3)|h_k(r) = \mathcal{O}(k^{-1}).
\] (A.20)

From these, it follows that
\[
\left|k \sum_{j=0}^{k-1} b^2(j, k, r)\Delta_{kj} - f(1)h_k(r)\right| = \mathcal{O}(k^{-1}).
\] (A.21)
This, (2.5), (A.6), and (A.17) imply
\[ \text{Var} \left( \frac{Y_{ki}}{kh_k(r)} \right) = \pi_0 - \mu_k \left( \frac{Y_{ki}}{kh_k(r)} \right) + O(k^{-2}) = \pi_0 + O(k^{-1}), \]

\[ \frac{n \text{Var} \left( \hat{\pi}_0 \right)}{kh_k(r)} = \text{Var} \left( \frac{Y_{ki}}{kh_k(r)} \right) + O(k^{-1}). \]

Consequently
\[ \lim_{k,n \to \infty} \frac{n \text{Var} \left( \hat{\pi}_0 \right)}{kh_k(r)} = \lim_{k \to \infty} \text{Var} \left( \frac{Y_{ki}}{kh_k(r)} \right) = \pi_0, \quad (A.22) \]

\[ E \left( \hat{\pi}_0 - \pi_0 \right)^2 = O(k^{-2}) + O \left( \frac{k}{n} \right). \]

If \( k \) is of order \( n^{1/3} \), then (2.6) follows.

Let \( X_1, \ldots, X_l, Y_1, \ldots, Y_l \) be iid Poisson r.v.'s with mean 1. Then
\[ I_0(2l)e^{-2l} = \sum_{j=0}^{\infty} \left( \frac{j!}{l^j} e^{-l} \right)^2 = \Pr \left\{ \sum_{i=1}^{l} X_i - \sum_{i=1}^{l} Y_i = 0 \right\}. \quad (A.23) \]

The Local Limit Theorem (see Petrov (1975, pp.187-188)) ensures that
\[ \lim_{l \to \infty} \sqrt{l} \Pr \left\{ \sum_{i=1}^{l} (X_i - Y_i) = 0 \right\} = \frac{1}{2\sqrt{\pi}}. \]

From this it follows that there are constants \( 0 < C_1 < C_2 \) such that
\[ \frac{C_1}{\sqrt{l}} \leq I_0(2l)e^{-2l} \leq \frac{C_2}{\sqrt{l}}, \quad \text{for } l \geq 1, \quad (A.24) \]

\[ I_0(2\sqrt{ij})e^{-ij} = I_0(2\sqrt{ij})e^{-2\sqrt{ij} e^{-\left( \sqrt{\frac{ij}{r}} \right)}} \begin{cases} \geq C_1(ij)^{-\frac{1}{r}} e^{-\left( \sqrt{\frac{ij}{r}} \right)}; \\ \leq C_2(ij)^{-\frac{1}{r}} e^{-\left( \sqrt{\frac{ij}{r}} \right)} \end{cases}. \quad (A.25) \]

Combining (2.7), (A.23), and (A.25), one obtains
\[ h(r) \leq C_2 \left\{ \frac{1}{r^2} \sum_{l=1}^{r} l^{-\frac{1}{r}} + \frac{2}{r^2} \sum_{1 \leq i < j \leq r} e^{-\left( \sqrt{\frac{ij}{r}} \right)} (ij)^{-\frac{1}{r}} \right\} \leq 2C_2 \left\{ \int_{0}^{r} t^{-\frac{1}{r}} dt + \frac{4}{\sqrt{r}} \int_{0}^{\sqrt{r}} dv \int_{0}^{v} e^{-\left( \sqrt{u} - v \right)^2 du} \right\} \leq C'_2 r^{-\frac{2}{r}}. \]

Similarly, \( h(r) \geq C'_2 r^{-3/2} \). The proof of Theorem 2.1 is complete.

**Proof of Theorem 2.2.** Let
\[ \xi_i = \frac{Y_{ki} - E \left( \hat{\pi}_0 \right)}{n \sqrt{\text{Var} \left( \hat{\pi}_0 \right)}}, \quad i = 1, 2, \ldots, n. \]
Then $\xi_i$ has mean zero and $W = \sum_{i=1}^{n} \xi_i$ has variance one. By Assumption 1, $\xi_1, \ldots, \xi_n$ are also LD1 random variables. For each $i$, let $\eta_i$ be the sum of all the random variables $\xi_{i1}, \ldots, \xi_{in_i}$ that are not independent of $\xi_i$. By Theorem 3.4 of Chen and Shao (2005), we have

$$\sup_x |\Pr(W \leq x) - \Phi(x)| \leq 2\delta^*_n,$$  

(A.26)

where

$$\delta = 4E \left[ \sum_{i=1}^{n} \{ \xi_i \eta_i - E(\xi_i \eta_i) \} + \sum_{i=1}^{n} E(\xi_i \eta_i^2) \right] \equiv \delta_1 + \delta_2,$$

$$\delta_1 \equiv 4E \left[ \sum_{i=1}^{n} \{ \xi_i \eta_i - E(\xi_i \eta_i) \} \right] = 4E \left[ \sum_{i=1}^{n} \sum_{j=1}^{n_i} \{ \xi_i \xi_j - E(\xi_i \xi_j) \} \right]$$

$$\leq \frac{4}{n^2 \Var(\hat{\pi}_0)} \left\{ E \left| \sum_{i=1}^{n} \sum_{j=1}^{n_i} [Y_{ki}Y_{kj} - E(Y_{ki}Y_{kj})] \right| + E \left| \sum_{i=1}^{n} \sum_{j=1}^{n_i} E(Y_{ki})[Y_{kj} - E(Y_{ki})] \right| + E \left| \sum_{i=1}^{n} \sum_{j=1}^{n_i} E(Y_{ki})[Y_{kj} - E(Y_{ki})] \right| \right\}$$

$$\equiv \delta_{11} + \delta_{12} + \delta_{13}.$$

It is easy to see that there exists $C_3$ such that

$$E \left[ Y_{ki}Y_{kj} - E(Y_{ki}Y_{kj}) \right]$$

$$\leq k^2 \sum_{u=0}^{k-1} \sum_{v=0}^{k-1} \bar{b}(u, k, r)\bar{b}(v, k, r)E \left| I \left\{ \frac{u}{k} < T_i \leq \frac{u+1}{k}, \frac{v}{k} < T_j \leq \frac{v+1}{k} \right\} - \Delta_{kuv}^{(ij)} \right|$$

$$= 2k^2 \sum_{u=0}^{k-1} \sum_{v=0}^{k-1} \bar{b}(u, k, r)\bar{b}(v, k, r)\Delta_{kuv}^{(ij)}(1 - \Delta_{kuv}^{(ij)}) \leq C_3.$$

Therefore

$$\delta_{11} \equiv \frac{4}{n^2 \Var(\hat{\pi}_0)} E \left[ \sum_{i=1}^{n} \sum_{j=1}^{n_i} [Y_{ki}Y_{kj} - E(Y_{ki}Y_{kj})] \right]$$

$$\leq \frac{4C_3m}{n \Var(\hat{\pi}_0)} = O \left( \frac{1}{kh_k(r)} \right) = O \left( \frac{r^2}{k} \right).$$

(A.27)

Similarly there exists $C_4$ such that, for any $j$,

$$E \left[ |Y_{kj} - E(Y_{kj})| \right] \leq k \sum_{u=0}^{k-1} \bar{b}(u, k, r)E \left| I \left\{ \frac{u}{k} < T_j \leq \frac{u+1}{k} \right\} - \Delta_{ku} \right|$$
NONPARAMETRIC ESTIMATOR OF FDR

\[ = 2k \sum_{u=0}^{k-1} b(u, k, r) \Delta_{ku} (1 - \Delta_{ku}) \leq C_4. \]

Thus
\[
\delta_{12} = \frac{4}{n^2 \text{Var}(\tilde{\pi}_0)} \sum_{i=1}^{n_n} \sum_{j=1}^{n_i} E(Y_{ki}) \left| Y_{ki} - E(Y_{ki}) \right|
\leq 4C_4m \mu_k \frac{1}{n \text{Var}(\tilde{\pi}_0)} = O\left(\frac{r^2}{k} \right).
\] (A.28)

Similarly \( \delta_{13} = O\left(\frac{r^{3/2}}{k} \right). \) So \( \delta_1 = O\left(\frac{r^{3/2}}{k} \right). \)
\[
\delta_2 = \frac{n}{n^3 \text{Var}(\tilde{\pi}_0)} \sum_{i=1}^{n_n} \sum_{j=1}^{n_i} E\left( \left| Y_{ki} - \mu_k \right| \left| Y_{kj} - \mu_k \right|^2 \right)
\leq \frac{km^2 \text{Var}(Y_{ki})}{n^2 \text{Var}(\tilde{\pi}_0)} = r^{\frac{3}{2}} O\left(\sqrt{k/n} \right).
\]

Thus, by (A.26), for fixed \( r \) as \( k, n \to \infty \) and \( k/n \to 0, \) \( W \xrightarrow{d} N(0, 1). \) The asymptotic normality \( \xrightarrow{d} \) follows from this and (A.28).

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