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STRUCTURE FUNCTIONS FOR REGULAR s^{l-m} DESIGNS WITH MULTIPLE GROUPS OF FACTORS

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Abstract: Identities about the wordlength patterns of regular s^{l-m} designs and their complementary designs are established through a first-order differential equation satisfied by a structure function. The identities are then generalized to s^{l-m} designs with multiple groups of factors. An advantage of using the structure function and partial differential equation is that it can easily adapt to some structural constraints of designs. The application of this approach to regular blocked fractional factorial designs generates identities relating the split wordlength patterns of regular (s^{l-m}, s^r) blocked designs and their complementary blocked designs. Practical rules are proposed for selecting optimal blocking schemes in terms of their complementary designs.

Key words and phrases: Fractional factorial design, Robust parameter design, Wordtype pattern, Structure function.

1. Introduction

The s^{l-m} fractional factorial designs (or briefly s^{l-m} designs), where s is a prime or a prime power, are among the most important factorial plans in practice. Maximum resolution (Box and Hunter (1961)) and minimum aberration (Fries and Hunter (1980)) are commonly used criteria to select optimal designs. The criteria were originally proposed for 2^{l-m} designs only. Franklin (1984) extended them to s^{l-m} designs. In the past two decades, much progress has been made in understanding the properties and structure of s^{l-m} designs with minimum aberration, especially for s=2. See, among others, Chen and Wu (1991), Chen (1992), Chen, Sun and Wu (1993), Tang and Wu (1996), Suen, Chen and Wu (1997), Cheng, Steinberg and Sun (1999) and Cheng and Mukerjee (1998). Recently, the concepts of resolution and aberration have been further generalized to nonregular designs. Generalized maximum resolution and minimum aberration criteria are proposed for selecting optimal nonregular fractional factorial designs (Tang and Deng (1999) and Xu and Wu (2001)).

Tang and Wu (1996) suggested using complementary designs to characterize 2^{l-m} designs with a large number of factors. This technique has led to many interesting results and is a useful tool to unveil the intrinsic aliasing relations in fractional factorial designs. Using MacWilliams identities and Krawtchouk polynomials from coding theory, Suen, Chen and Wu (1997) obtained some general identities that relate the wordlength pattern of an s^{l-m} design and that of its complementary design. These identities were also established for nonregular designs in Xu and Wu (2001) following a similar approach in Suen, Chen and Wu (1997), and were further extended to nonregular blocked designs in Ai and Zhang (2004).

The 2^{l-m} designs with multiple groups of factors have received much attention lately. In several interesting types of designs such as blocked fractional factorial design (Sun, Wu and Chen (1997)), split-plot design (Bingham and Sitter (1999)) and robust parameter design (Wu and Zhu (2003)), factors under investigation consist of several groups whose differences should be taken into consideration in experimental planning and data analysis. For example, in a robust parameter design experiment, there are control factors and noise factors (Wu and Hamada (2000)), and factorial effects involving different combinations of control and noise factors play different roles in parameter design. Suppose there exist two groups of factors in an experiment, which are denoted as Group I and Group II and contain l_1 and l_2 factors, respectively. The fractional factorial design used to investigate these factors is denoted as $s^{(l_1+l_2)-m}$. Discriminating defining words involving different numbers of Group I and Group II factors, Zhu (2003) proposed to use wordtype matrices instead of wordlength patterns to characterize the aliasing patterns of $2^{(l_1+l_2)-m}$ designs and established the relationships between their wordtype patterns and those of their complementary designs via a structure function and a first-order partial differential equation satisfied by the structure function.

In this paper, we first extend the approach of Zhu (2003) to s^{l-m} designs, then to $s^{(l_1+l_2)-m}$ designs with multiple groups of factors, and finally apply it to the study of regular (s^{l-m}, s^r) blocked designs. This approach can easily accommodate some structural constraints of factorial designs as demonstrated by its application to blocked designs, and can handle multiple groups of factors in a unified fashion. Furthermore, the approach can be used to study the letter pattern (Draper and Mitchell (1970)) and the aliasing structure of an s^{l-m} design.

The rest of the paper is organized as follows. In Section 2, notation and basic definitions are given. Several concepts like structure index array N and structure function f are defined. Based on Tang and Wu (1996), a recursive equation for N is derived. In Section 3, a first-order partial differential equation in f will be derived. A main theorem about N and a closed form solution to the partial differential equation are obtained. In Section 4, the results obtained in Section 3 are generalized to s^{l-m} designs with multiple groups of factors. In Section 5, the theoretical results in the previous sections are employed to study regular (s^{l-m}, s^r) blocked designs. Identities relating a blocked design and its blocked complementary design are also obtained. Practical rules are proposed for selecting optimal blocked designs using the complementary design approach. Concluding remarks on the potential use of the reported results are given in Section 6.

2. Notation and Definitions

Let F_s denote a finite field with s elements and $EG(k, s)$ denote a k-dimensional vector space over F_s. Suppose $u' = (u_1, \ldots, u_k)$ and $v' = (v_1, \ldots, v_k)$ are two vectors from $EG(k, s)$, where u' is the transpose of u. Vectors in this paper are meant to be column vectors. If there exists $t \in F_s$ and $t \neq 0$ such that $u = tv$, then u and v are said to be equivalent. The set of equivalent classes forms a $(k-1)$-dimensional projective geometry over F_s and is denoted by $PG(k-1, s)$. There are $(s^k - 1)/(s - 1)$ elements (or points) in $PG(k - 1, s)$. An introduction to general projective geometry theory can be found in Hirschfeld (1979). For the applications of finite projective geometry in fractional factorial designs, see Bose (1947) and Mukerjee and Wu (2001).

An s^l full factorial design consists of all vectors in $EG(l, s)$. An s^{l-m} fractional factorial design is an s^{-m} fraction of the s^l design. There are several ways to generate s^{l-m} designs, one of which is to use projective geometry. Let $k = l - m$ and assume that $l \leq (s^k - 1)/(s - 1)$. Choose l points $\alpha_1, \ldots, \alpha_l$ from $PG(k - 1, s)$. Let G be a $k \times l$ matrix whose columns are the chosen points, that is, $G = (\alpha_1, \ldots, \alpha_l)$. The linear space spanned by the rows of G forms an s^{l-m} design, which is denoted by $\mathcal{D}$. Given a subset $\{\alpha_{r_1}, \ldots, \alpha_{r_i}\}$ of the columns of G, if there exist t_1, …, and t_i from $F_s - \{0\}$ such that $t_1\alpha_{r_1} + \cdots + t_i\alpha_{r_i} = 0$, then $w = \alpha_{r_1}^{t_1} \cdots \alpha_{r_i}^{t_i}$ is called a generalized defining word. Two generalized defining words $w = \alpha_{r_1}^{t_1} \cdots \alpha_{r_i}^{t_i}$ and $w' = \alpha_{r'_1}^{t'_1} \cdots \alpha_{r'_i}^{t'_i}$ are equivalent if there exists τ in $F_s - \{0\}$ such that $\alpha_{r_j} = \alpha_{r'_j}$ and $t_j = \tau t'_j$ for $1 \leq j \leq i$. The set of equivalent classes of generalized defining words and the identity element form the defining contrasts subgroup $\mathcal{G}$ associated with $\mathcal{D}$. Let $A_i(\mathcal{D})$ be the number of defining words in $\mathcal{G}$ that involve i different columns (points). The vector $A(\mathcal{D}) = (A_1(\mathcal{D}), \ldots, A_l(\mathcal{D}))$ is called the wordlength pattern of $\mathcal{D}$. The resolution of $\mathcal{D}$ is the smallest i such that $A_i(\mathcal{D}) > 0$. Two designs $\mathcal{D}_1$ and $\mathcal{D}_2$ with the same resolution can be further discriminated by their wordlength patterns $A(\mathcal{D}_1)$ and $A(\mathcal{D}_2)$. Let i_0 be the smallest integer i such that $A_i(\mathcal{D}_1) \neq A_i(\mathcal{D}_2)$. If $A_{i_0}(\mathcal{D}_1) < A_{i_0}(\mathcal{D}_2)$, then $\mathcal{D}_1$ is said to have less aberration than $\mathcal{D}_2$. An s^{l-m} design has *minimum aberration* if no other s^{l-m} designs have less aberration.

The points in $PG(k - 1, s) - \{\alpha_1, \ldots, \alpha_l\}$, denoted by $\{\beta_1, \ldots, \beta_{\bar{l}}\}$, can generate another design $\bar{\mathcal{D}}$, where $\bar{l} = (s^k - 1)/(s - 1) - l$. And $\bar{\mathcal{D}}$ is called the *complementary design* of $\mathcal{D}$. The defining contrasts subgroup $\bar{\mathcal{G}}$ and the wordlength pattern $\bar{A}$ for $\bar{\mathcal{D}}$ can be defined similarly. It is clear that an s^{l-m} design induces a partition of $PG(k-1,s)$, that is,

$$PG(k-1,s)=\{\alpha_1,\ldots,\alpha_l\}\cup\{\beta_1,\ldots,\beta_{\bar{l}}\}=\mathcal{D}\cup\bar{\mathcal{D}}. \tag{1}$$

For any fixed pair (i,j) with $0\le i\le l$ and $0\le j\le \bar{l}$, i points $\alpha_{r_1},\ldots,\alpha_{r_i}$ chosen from $\mathcal{D}$, and j points $\beta_{t_1},\ldots,\beta_{t_j}$ from $\bar{\mathcal{D}}$ are said to have a $[i,j]$-relation if there exist $u'=(u_1,\ldots,u_i)$ and $v'=(v_1,\ldots,v_j)$, both with nonzero coordinates, such that

$$u_1\alpha_{r_1}+\cdots+u_i\alpha_{r_i}+v_1\beta_{t_1}+\cdots+v_j\beta_{t_j}=0.$$

To indicate the dependence on u and v, this $[i,j]$-relation is called a $[i,j;u,v]$-relation. A $[i,0]$-relation corresponds to a generalized defining word for $\mathcal{D}$ and a $[0,j]$-relation corresponds to a generalized defining word for $\bar{\mathcal{D}}$. Let $N_{i,j}$ be the total number of distinct $[i,j]$-relations and N be the $(l+1)\times(\bar{l}+1)$ matrix with entries $N_{i,j}$. N is called the *structure index array* (Zhu (2003)). Clearly, $N_{i,0}=(s-1)A_i(\mathcal{D})$ and $N_{0,j}=(s-1)A_i(\bar{\mathcal{D}})$, where $1\le i\le l$ and $1\le j\le\bar{l}$. For convenience, define $N_{0,0}=1$ and $N_{i,j}=0$ when the $[i,j]$-relations are not defined.

Lemma 1. $\{N_{i,j}\}$ *satisfy the following iterative equation:*

$$\begin{aligned}&(i+1)N_{i+1,j}+(j+1)N_{i,j+1}+C_{i,j}N_{i,j}\\&\quad=(s-1)^{i+j}\binom{l}{i}\binom{\bar{l}}{j}-[(s-1)(l-i+1)N_{i-1,j}+(s-1)(\bar{l}-j+1)N_{i,j-1}],\end{aligned}\tag{2}$$

where $C_{i,j}=(s-2)i+(s-2)j+1$.

Remark. Similar equations with $C_{i,j}=1$ have been derived for 2^{l-m} designs in Tang and Wu (1996) and Zhu (2003). However, the $(s-2)i+(s-2)j$ part in $C_{i,j}$ is missing in two-level designs. This shows the major difference between the wordlength pattern of a general s^{l-m} design and that of a 2^{l-m} design.

Proof. Recall the partition in (1). Suppose i points are selected from $\mathcal{D}$ and j points from $\bar{\mathcal{D}}$ to form linear combinations with nonzero coefficients over F_s. We call them nonzero linear combinations. This results in $(s-1)^{i+j}\binom{l}{i}\binom{\bar{l}}{j}$ combinations. Suppose one of the combinations is given by

$$l(u,v)=u_1\alpha_{r_1}+\cdots+u_i\alpha_{r_i}+v_1\beta_{t_1}+\cdots+v_j\beta_{t_j},$$

where all the coordinates of $u'=(u_1,\ldots,u_i)$ and $v'=(v_1,\ldots,v_j)$ are nonzero. Let $A=\{\alpha_{r_1},\alpha_{r_2},\ldots,\alpha_{r_i}\}\subset\mathcal{D}$, and $C=\{\beta_{t_1},\beta_{t_2},\ldots,\beta_{t_j}\}\subset\bar{\mathcal{D}}$. Define $B=\{\alpha_{r_{i+1}},\ldots,\alpha_{r_l}\}=\mathcal{D}-A$ and $E=\bar{\mathcal{D}}-C=\{\beta_{t_{j+1}},\ldots,\beta_{t_{\bar{l}}}\}$. Clearly $l(u,v)$ is a vector in $EG(k,s)$. Suppose there exist $\tau_0 \in F_s - \{0\}$ and $\alpha_{r_{i_0}}$ in A such that $l(u,v) = \tau_0\alpha_{r_{i_0}}$. Then

$$l(u,v)-\tau_0\alpha_{r_{i_0}}=u_1\alpha_{r_1}+\cdots(u_{i_0}-\tau_0)\alpha_{r_{i_0}}+\cdots+u_i\alpha_{r_i}+v_1\beta_{t_1}+\cdots+v_j\beta_{t_j}=0.$$

If $u_{i_0} = \tau_0$, then $A - \{\alpha_{i_0}\}$ and C form a $[i-1, j, \tilde{u}, v]$-relation where $\tilde{u}' = (u_1, \ldots, u_{i_0-1}, u_{i_0+1}, \ldots, u_i)$. Then $l(u,v)$ is said to be a nonzero linear combination of type A. Conversely, every $[i-1, j, \tilde{u}, v]$-relation can generate $(s-1)(l-i+1)$ nonzero linear combinations of type A. Hence, the total number of type A nonzero linear combinations is equal to $(s-1)(l-i+1)N_{i-1,j}$. If $u_{i_0} \neq \tau_0$, then $l(u,v) - \tau_0\alpha_{r_{i_0}}$ is indeed a $[i, j, \hat{u}, v]$-relation, where $\hat{u} = (u_1, \ldots, u_{i_0-1}, u_{i_0} - \tau_0, u_{i_0+1}, \ldots, u_i)$. This $l(u,v)$ is said to be a nonzero linear combination of type $\hat{A}$. Conversely, every $[i, j, \hat{u}, v]$-relation can generate $(s-2)i$ nonzero linear combinations of type $\hat{A}$. In total there are $(s-2)iN_{i,j}$ nonzero linear combinations of type $\hat{A}$. Suppose there exist $\tau_0 \in F_s - \{0\}$ and $\alpha_{r_{i_0}} \in B$ such that $l(u,v) = \tau_0\alpha_{r_{i_0}}$. Then

$$l(u,v) - \tau_0\alpha_{r_{i_0}} = u_1\alpha_{r_1} + \cdots + u_i\alpha_{r_i} + (-\tau_0)\alpha_{r_{i_0}} + v_1\beta_{t_1} + \cdots + v_j\beta_{t_j} = 0.$$

Note that $l(u,v) - \tau_0\alpha_{r_{i_0}}$ is a $[i+1, j]$-relation, and the linear combination $l(u,v)$ is said to be of type B. Conversely, every $[i+1, j]$-relation can generate $(i+1)$ nonzero linear combinations of type B. There are $(i+1)N_{i+1,j}$ nonzero linear combinations of type B. In summary, the total number of nonzero linear combinations of types A, $\hat{A}$, and B is $(s-1)(l-i+1)N_{i-1,j}+(s-2)iN_{i,j}+(i+1)N_{i+1,j}$. Similarly, nonzero linear combinations $l(u,v)$ of types C, $\hat{C}$ and E can be defined, and their total number is $(s-1)(\bar{l}-j+1)N_{i,j-1}+ (s-2)jN_{i,j} +(j+1)N_{i,j+1}$. Finally, if $l(u,v) = 0$, then it is a $[i,j]$-relation and there are $N_{i,j}$ linear combinations of this type. Note that a nonzero linear combination $l(u,v)$ can belong to only one type. Summing the numbers of $l(u,v)$'s of different types, we have equation (2).

The structure index array N of an s^{l-m} design describes its structure and property. The moment generating function of N is defined as

$$f(x,y) = \sum_{i=0}^{l}\sum_{j=0}^{\bar{l}} N_{i,j}x^iy^j = 1 + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 0}} N_{i,j}x^iy^j. \tag{3}$$

We call $f(x,y)$ the *structure function* of the s^{l-m} design.

3. Main Results

In this section, we derive a first-order partial differential equation satisfied by f based on (2). The differential equation unveils an intricate relation among the $N_{i,j}$. Then an explicit expression for f is obtained by solving the equation under certain conditions.

Theorem 1. *The structure function f of an s^{l-m} design satisfies the following first-order partial differential equation*

$$[1+(s-2)x-(s-1)x^2]\frac{\partial f}{\partial x}+[1+(s-2)y-(s-1)y^2]\frac{\partial f}{\partial y}$$
$$+[1+(s-1)lx+(s-1)\bar{l}y]f-[1+(s-1)x]^l[1+(s-1)y]^{\bar{l}}=0, \qquad (4)$$

where $\bar{l}=(s^{l-m}-1)/(s-1)-l$.

Proof. Multiplying both sides of (2) by x^iy^j, summing over i,j, and rearranging terms, we have

$$\begin{aligned}
&\sum_{i,j}(s-1)^i(s-1)^j\binom{l}{i}\binom{\bar{l}}{j}x^iy^j\\
&\quad=\sum_{i,j}(s-1)(l-i+1)N_{i-1,j}x^iy^j+\sum_{i,j}(i+1)N_{i+1,j}x^iy^j\\
&\qquad+\sum_{i,j}(s-1)(\bar{l}-j+1)N_{i,j-1}x^iy^j+\sum_{i,j}(j+1)N_{i,j+1}x^iy^j\\
&\qquad+\sum_{i,j}((s-2)i+(s-2)j+1)N_{i,j}x^iy^j.
\end{aligned} \qquad (5)$$

Denote the five terms in the right-hand side of (5) by R_1, R_2, R_3, R_4 and R_5 from left to right, respectively. Let R_0 be the left-hand side of equation (5). It is clear that

$$\begin{aligned}
R_0&=[1+(s-1)x]^l[1+(s-1)y]^{\bar{l}},\\
R_1&=(s-1)l\sum_{i,j}N_{i-1,j}x^iy^j-(s-1)\sum_{i,j}(i-1)N_{i-1,j}x^iy^j\\
&=(s-1)lx\sum_{i,j}N_{i-1,j}x^{i-1}y^j-(s-1)\sum_{i,j}(i-1)N_{i-1,j}x^iy^j\\
&=(s-1)lxf-(s-1)x^2\frac{\partial f}{\partial x},\\
R_2&=\sum_{i,j}\frac{\partial}{\partial x}(N_{i+1,j}x^{i+1,j})=\frac{\partial f}{\partial x}.
\end{aligned}$$

Similarly, $R_3=(s-1)\bar{l}yf-(s-1)y^2\frac{\partial f}{\partial y}$, $R_4=\frac{\partial f}{\partial y}$, and

$$R_5=(s-2)x\sum_{i,j}iN_{i,j}x^{i-1}y^j+(s-2)y\sum_{i,j}jN_{i,j}x^iy^{j-1}+\sum_{i,j}N_{i,j}x^iy^j$$

$$= (s-2)x\frac{\partial f}{\partial x} + (s-2)y\frac{\partial f}{\partial y} + f.$$

Because $R_0 = R_1 + R_2 + R_3 + R_4 + R_5$, (4) follows.

Recall that an s^{l-m} design $\mathcal{D}$ induces the partition $PG(k-1, s) = \mathcal{D} \cup \bar{\mathcal{D}}$. The structure index array $\{N_{i,j}\}$ contains information about the aliasing within the designs $\mathcal{D}$ and $\bar{\mathcal{D}}$ as well as information about the relationship between $\mathcal{D}$ and $\bar{\mathcal{D}}$. Intuitively, it is not difficult to see that, if the wordlength pattern of either $\mathcal{D}$ or $\bar{\mathcal{D}}$ is known, that is, either $\{N_{i,0}\}$ or $\{N_{0,j}\}$ is given, the other structure indices can be determined uniquely. The following theorem validates this, and the dependence can be derived explicitly by solving (4).

Theorem 2. *Given $\{N_{0,j}\}$, there exists a unique structure function f that satisfies* (4)*. Furthermore,*

$$\begin{aligned} f(x,y) = s^{-k}&[1+(s-1)x]^{l-s^{k-1}}[1+(s-1)y]^{\bar{l}}\{[1+(s-1)x]^{s^{k-1}} - (1-x)^{s^{k-1}}] \\ &+[1+(s-1)x]^{l-s^{k-1}}(1-x)^{s^{k-1}-\bar{l}}[1+(s-2)x-(s-1)xy]^{\bar{l}} \\ &\times h((y-x)[1+(s-2)x-(s-1)xy]^{-1}), \end{aligned} \tag{6}$$

where $h(t) = \sum_j N_{0,j}t^j$.

A sketch of the proof is given in the Appendix.

The explicit connections between $N_{i,j}$ and $N_{0,j}$ can be derived by applying the Taylor expansion to the terms in (6) and comparing them to the definition of f in (3), or by calculating $\partial^{i+j}f/\partial x^i \partial y^j \mid_{x=0,y=0}$. When the parameters s, l, m, k and $\bar{l}$ are given, much simplified formulas can be obtained. General formulas similar to those in Zhu (2003) can also be obtained. An important feature of this approach is that it reveals not only how $N_{i,0}$ and $N_{0,j}$ are related to each other, but also the relationship between $N_{i,j}$ and $N_{0,j}$ with $j > 0$, which has further implications regarding the structure and properties of $\mathcal{D}$.

4. Design with Multiple Groups of Factors

As discussed in Section 1, many important designs involve multiple groups of factors. Aliasing between effects within the same groups and between the groups has different implications for design and analysis. This distinction should be considered in the choice of design. Readers are referred to Suen, Chen and Wu (1997) for discussions on blocked design, Bingham and Sitter (1999) on split-plot design, and Wu and Zhu (2003) on robust parameter design. In this section, we assume that an s^{l-m} design is employed to investigate l factors, among which l_1 factors belong to Group I and l_2 belong to Group II. Let $\mathcal{D}_{1,2}$ and $\mathcal{G}_{1,2}$ denote the design and its defining contrasts subgroup. $\mathcal{D}_{1,2}$ can also be generated by a collection of $l_1 + l_2$ points from $PG(k-1, s)$ as in Section 2. Suppose the set of points corresponding to the factors in Group I is $\mathcal{L}_1$=$\{\alpha_1, \alpha_2, \ldots, \alpha_{l_1}\}$, and the set of points corresponding to the factors in Group II is $\mathcal{L}_2$=$\{\beta_1, \beta_2, \ldots, \beta_{l_2}\}$. There are $l_3 = (s^k-1)/(s-1) - l_1 - l_2$ points remaining in $PG(k-1, s)$, which are denoted by $\mathcal{L}_3$=$\{\gamma_1, \ldots, \gamma_{l_3}\}$. Hence, $\mathcal{D}_{1,2}$ induces a three-way partition: $PG(k-1, s) = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. Note that $\mathcal{D}_{1,2}$ is the design generated by $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $\mathcal{D}_{1,3}$ denote the design generated by $\mathcal{L}_1$ and $\mathcal{L}_3$ and $\mathcal{D}_{2,3}$ by $\mathcal{L}_2$ and $\mathcal{L}_3$. Both $\mathcal{D}_{1,3}$ and $\mathcal{D}_{2,3}$ can be considered as complementary designs of $\mathcal{D}_{1,2}$. The properties and structures of these three designs depend on each other.

For any fixed triplet (i_1, i_2, i_3) with $0 \le i_j \le l_j$ for $j = 1, 2, 3$, a collection of i_j points from $\mathcal{L}_j$ for $j = 1, 2, 3$, respectively, is said to have a $[i_1, i_2, i_3]$-relation, if there exists a nonzero linear combination of them, which is equal to the 0-vector in $EG(k, s)$. Similarly, define N_{i_1,i_2,i_3} to be the total number of distinct $[i_1, i_2, i_3]$-relations. It is clear that $\{N_{i_1,i_2,0}\}$, $\{N_{0,i_2,i_3}\}$ and $\{N_{i_1,0,i_3}\}$ correspond to the generalized wordtype patterns of $\mathcal{D}_{1,2}$, $\mathcal{D}_{2,3}$ and $\mathcal{D}_{1,3}$, respectively. Again we call $\{N_{i_1,i_2,i_3}\}$ the *structure index array* associated with $\mathcal{D}_{1,2}$. The structure function f is then defined as

$$f(x_1, x_2, x_3) = \sum_{i_1=0}^{l_1} \sum_{i_2=0}^{l_2} \sum_{i_3=0}^{l_3} N_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} = 1 + \sum_{\substack{i_1+i_2+i_3 \ge 3 \\ i_1 \ge 0, i_2 \ge 0, i_3 \ge 0}} N_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}. \tag{7}$$

Following the same arguments and derivations as in Sections 2 and 3, the following results can be established.

Lemma 2. *The N_{i_1,i_2,i_3} satisfy*

$$\begin{aligned}
&(i_1+1)N_{i_1+1,i_2,i_3} + (i_2+1)N_{i_1,i_2+1,i_3} + (i_3+1)N_{i_1,i_2,i_3+1} + C_{i_1,i_2,i_3} N_{i_1,i_2,i_3} \\
&\quad = (s-1)^{i_1+i_2+i_3} \binom{l_1}{i_1} \binom{l_2}{i_2} \binom{l_3}{i_3} - [(s-1)(l_1 - i_1 + 1) N_{i_1-1,i_2,i_3} \\
&\qquad + (s-1)(l_2 - i_2 + 1) N_{i_1,i_2-1,i_3} + (s-1)(l_3 - i_3 + 1) N_{i_1,i_2,i_3-1}],
\end{aligned} \tag{8}$$

where $C_{i_1,i_2,i_3} = (s-2)i_1 + (s-2)i_2 + (s-2)i_3 + 1$.

Theorem 3. *The structure function f defined in* (7) *satisfies the first-order differential equation*

$$\sum_{j=1}^{3} [1+(s-2)x_j-(s-1)x_j^2] \frac{\partial f}{\partial x_j} + [1+\sum_{j=1}^{3}(s-1)l_j x_j] f - \prod_{j=1}^{3}[1+(s-1)x_j]^{l_j} = 0. \tag{9}$$

Theorem 4. *Given $\{N_{0,i_2,i_3}\}$, there exists a unique structure function f that satisfies* (9)*; it may be written as*

$$f(x_1, x_2, x_3) = s^{-k}[1+(s-1)x_1]^{l_1 - s^{k-1}}[1+(s-1)x_2]^{l_2}[1+(s-1)x_3]^{l_3}$$

$$\begin{aligned}
&\times\{[1+(s-1)x_1]^{s^{k-1}}-(1-x_1)^{s^{k-1}}\}\\
&+[1+(s-1)x_1]^{l_1-s^{k-1}}(1-x)^{s^{k-1}-l_2-l_3}[1+(s-2)x_1-(s-1)x_1x_2]^{l_2}\\
&\times[1+(s-2)x_1-(s-1)x_1x_3]^{l_3}h(x_1,x_2,x_3), \qquad (10)
\end{aligned}$$

where

$$\begin{aligned}
&h(x_1,x_2,x_3)\\
&=\sum_{i_2=0}^{l_2}\sum_{i_3=0}^{l_3}N_{0,i_2,i_3}\left[\frac{x_2-x_1}{1+(s-2)x_1-(s-1)x_1x_2}\right]^{i_2}\left[\frac{x_3-x_1}{1+(s-2)x_1-(s-1)x_1x_3}\right]^{i_3}.
\end{aligned}$$

The proofs of Lemma 2, Theorem 3 and Theorem 4 are similar to those of Lemma 1, Theorem 1 and Theorem 2. These results can be easily extended to more than two groups of factors. The explicit relationship between N_{i_1,i_2,i_3} and N_{0,i_2,i_3} can be obtained, but they are not reported here due to limited space. The subsets $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ in the partition of $PG(k-1,s)$ are arbitrary subsets in general. However, for some designs, these subsets may possess certain structures or satisfy certain constraints, and simplified and direct relationships can be obtained by considering these structures and constraints. In the next section, we use blocked designs to illustrate how the approach using structure function and partial differential equation can accommodate the structure in the partition induced by a blocked design and lead to identities relating the blocked design and its complementary design.

5. Regular (s^{l-m}, s^r) Blocked Design

Blocking is a commonly used strategy to eliminate systematic variations due to inhomogeneities of experimental units. Typical block factors include time, location, batch, operator and so on. In the recent literature, much attention has been given to the issue of characterization of blocked fractional factorial designs and optimal blocking schemes. See, Bisgaard (1994), Sun, Wu and Chen (1997), Sitter, Chen and Feder (1997),Mukerjee and Wu (1999),Chen and Cheng (1999), Cheng and Wu (2002) and Ai and Zhang (2004). In a blocked fractional factorial design, there are two different types of effect aliasing, the aliasing between treatment effects and the confounding between treatment effects and block effects (Wu and Hamada (2000)). Because of this complexity, it is not immediately clear whether popular criteria for regular fractional factorial designs such as maximum resolution and minimum aberration can be directly generalized to blocked designs. Useful optimality criteria should be based on a good understanding of the properties of blocked designs.

In this section, we do not intend to propose any new optimality criteria. Instead, the results from the previous sections are employed to investigate regular (s^{l-m}, s^r) blocked designs and their complementary designs. Using the MacWilliams identities from coding theory, Chen and Cheng (1999) studied $(2^{l-m}, 2^r)$ blocked designs, and Ai and Zhang (2004) studied nonregular blocked designs that cannot generally be generated by defining relations. The latter proposed a concept called blocked consulting design and generated identities that relate blocked designs and their consulting blocked designs. Although the results of Ai and Zhang (2004) are probably the best one can hope for general nonregular blocked designs, when applied to regular blocked designs the concept of blocked consulting design is not appropriate, and the identities can be complicated and redundant. The primary reason that such results are not the best possible for regular blocked designs is that they do not consider the linear structure of regular blocked designs. This will be clear when we compare the blocked consulting design and the blocked complementary design in the next paragraph.

A regular (s^{l-m}, s^r) blocked design can be viewed as an s^{l-m} design that is partitioned into s^r blocks, each with s^{k-r} experimental units, where $k = l - m$ (Mukerjee and Wu (1999)), or an $s^{(l+r)-(m+r)}$ design with l treatment factors and r block factors, such that the main effects of the treatment factors are not confounded with block main effects or interactions (Chen and Cheng (1999)). Following the second viewpoint and the discussion in Section 4, a regular (s^{l-m}, s^r) blocked design induces a three-way partition of $PG(k-1, s)$, that is,

$$PG(k-1, s) = \{\alpha_i\}_{i=1}^l \cup \{\beta_j\}_{j=1}^r \cup (PG(k-1, s) - (\{\alpha_i\}_{i=1}^l \cup \{\beta_j\}_{j=1}^r)),$$

where $\{\alpha_i\}_{i=1}^l$ correspond to the treatment factors and $\{\beta_j\}_{j=1}^r$ correspond to the block factors. Clearly $\{\alpha_i\}_{i=1}^l \cup \{\beta_j\}_{j=1}^r$ generates the original blocked design. Ai and Zhang (2004) referred to the design generated by $\{\beta_j\}_{j=1}^r \cup (PG(k-1, s) - (\{\alpha_i\}_{i=1}^l \cup \{\beta_j\}_{j=1}^r))$ as the *blocked consulting design* with $(s^k-1)/(s-1) - l - r$ treatment factors and r block factors. The blocked consulting design is not a legitimate blocked design, because some of its treatment factors are confounded with the interactions of the block factors. This explains why the identities between a blocked design and its consulting design are not the best possible in regular cases. Let $\mathcal{B}$ be the subspace spanned by $\{\beta_j\}_{j=1}^r$ in $PG(k-1, s)$, and let $\mathcal{D} = \{\alpha_i\}_{i=1}^l$. $\mathcal{B}$ contains $(s^r-1)/(s-1)$ points and must be disjoint with $\mathcal{D}$. Let $\bar{\mathcal{D}} = PG(k-1, s) - \mathcal{D} - \mathcal{B}$. Then we arrive at a more appropriate partition for studying the blocked design,

$$PG(k-1, s) = \mathcal{D} \cup \mathcal{B} \cup \bar{\mathcal{D}}. \tag{11}$$

For convenience, we let $l_1 = l$, $l_r = (s^r-1)/(s-1)$, and $l_3 = (s^k-1)/(s-1) - l_r - l_1$, the cardinalities of $\mathcal{D}$, $\mathcal{B}$, and $\bar{\mathcal{D}}$ respectively.

The partition in (11) is similar to the three-way partition in Section 4, however $\mathcal{B}$ is not just an arbitrary subset but rather a $(r-1)-$dimensional subspace of $PG(k-1,s)$. This is a unique feature of blocked fractional factorial designs. The structure indices $N_{i,j,k}$ and the structure function f can be defined in the same way as in Section 4. Note that $\{(s-1)^{-1}N_{i,0,0}\}_{i=1}^{l_1}$ and $\{(s-1)^{-1}N_{i,1,0}\}_{i=1}^{l_1}$ form the split wordlength patterns of the blocked design (Sun, Wu and Chen (1997)). $\mathcal{B}$ and $\bar{\mathcal{D}}$ generate another blocked design, which is referred to as the *blocked complementary design* of $(\mathcal{D},\mathcal{B})$ and denoted by $(\mathcal{B},\bar{\mathcal{D}})$. $(\mathcal{B},\bar{\mathcal{D}})$ was called the blocked residual design in Chen and Cheng (1999). Then $\{(s-1)^{-1}N_{0,0,k}\}_{k=1}^{l_3}$ and $\{(s-1)^{-1}N_{0,1,k}\}_{k=1}^{l_3}$ are the split wordlength patterns of $(\mathcal{B},\bar{\mathcal{D}})$. Here, only the subsets of $\{N_{i,j,k}\}$ with $j=0,1$ are relevant for blocked designs. According to Lemma 1 in Mukerjee and Wu (1999) or Lemma 2 in Chen and Cheng (1999), we have

$$N_{i,j,k}=\gamma(j)N_{i,0,k}+(s-1)^{-1}\alpha(j)N_{i,1,k}, \tag{12}$$

where $\gamma(j)$ is the number of distinct nonzero linear combinations of j points in $\mathcal{B}$ that are equal to zero, and $\alpha(j)$ is the number of distinct nonzero linear combinations of j points (or vectors) in $\mathcal{B}$ that are equivalent to a given point (or vector) in $\mathcal{B}$. Note that $\alpha(j)$ does not depend on the given point (Mukerjee and Wu (1999)). Define $\gamma(y)=\sum_{j\ge0}\gamma(j)y^j$, $\alpha(y)=\sum_{j\ge0}\alpha(j)y^j$, $f_1(x,z)=\sum_{i,k}N_{i,0,k}x^iz^k$, and $f_2(x,z)=\sum_{i,k}N_{i,1,k}x^iz^k$. Let f be the structure function based on $\{N_{i,j,k}\}$. Then we have

$$f(x,y,z)=\gamma(y)f_1(x,z)+(s-1)^{-1}\alpha(y)f_2(x,z), \tag{13}$$

according to (12). Equation (13) indicates that the structure function f possesses a simplified expression due to the fact that $\mathcal{B}$ is a $(r-1)$-dimensional subspace. Note that only $f_1(x,z)$ and $f_2(x,z)$ are relevant for a blocked design and its blocked complementary design.

Lemma 3.

(i) $$\gamma(y)=s^{-r}\left\{[1+(s-1)y]^{l_r}+(s^r-1)[1+(s-1)y]^{l_r}(1-y)^{s^{r-1}}\right\}, \tag{14}$$

(ii) $$\alpha(y)=l_r^{-1}[1+(s-1)y]^{l_r}-l_r^{-1}\gamma(y), \tag{15}$$

(iii) *$\gamma(y)$ and $\alpha(y)$ satisfy the following equations:*

$$[1+(s-2)y-(s-1)y^2]\frac{d\gamma(y)}{dy}+[1+(s-1)l_ry]\gamma(y)-[1+(s-1)y]^{l_r}=0, \tag{16}$$

$$[1+(s-2)y-(s-1)y^2]\frac{d\alpha(y)}{dy}+[1+(s-1)l_ry]\alpha(y)-(s-1)[1+(s-1)y]^{l_r}=0. \tag{17}$$

Proof. (i) is a well-known result for the weight distribution of the Hamming code $[(s^r-1)/(s-1),(s^r-1)/(s-1)-r,3]$. (ii) can be derived from Lemma 1 (b) in Mukerjee and Wu (1999). Equations (16) and (17) can be verified directly.

According to Theorem 3, $f(x,y,z)$ satisfies

$$\begin{aligned}&[1+(s-2)x-(s-1)x^2]\frac{\partial f}{\partial x}+[1+(s-2)y-(s-1)y^2]\frac{\partial f}{\partial y}\\&\quad+[1+(s-2)z-(s-1)z^2]\frac{\partial f}{\partial z}+[1+(s-1)l_1x+(s-1)l_ry+(s-1)l_3z]f\\&\quad-[1+(s-1)x]^{l_1}[1+(s-1)y]^{l_r}[1+(s-1)z]^{l_3}=0.\end{aligned}\tag{18}$$

Equation (18) holds for any structure function for two groups of factors, but it does not take into consideration that f has a simplified expression as at (13). Replacing f in (18) with its expression in (13), and further simplifying (18) with the help of Lemma 3, we derive the following theorem regarding $f_1(x,z)$ and $f_2(x,z)$.

Theorem 5. *f_1 and f_2 satisfy*

$$\begin{aligned}&[1+(s-2)x-(s-1)x^2]\frac{\partial f_1}{\partial x}+[1+(s-2)z-(s-1)z^2]\frac{\partial f_1}{\partial z}\\&\quad+[1+(s-1)l_1x+(s-1)l_3z]f_1+f_2-[1+(s-1)x]^{l_1}[1+(s-1)z]^{l_3}=0,\end{aligned}\tag{19}$$

$$\begin{aligned}&[1+(s-2)x-(s-1)x^2]\frac{\partial f_2}{\partial x}+[1+(s-2)z-(s-1)z^2]\frac{\partial f_2}{\partial z}\\&\quad+[s^r-1+(s-1)l_1x+(s-1)l_3z]f_2\\&\quad+(s^r-1)f_1-(s^r-1)[1+(s-1)x]^{l_1}[1+(s-1)z]^{l_3}=0.\end{aligned}\tag{20}$$

Using the MacWilliams identities, Chen and Cheng (1999) obtained combinatorial identities that govern the relationship between the split wordlength pattern $\{N_{i,j,0}\}_{0\le i\le l_1,0\le j\le 1}$ of a blocked 2^{l-m} design and the split wordlength pattern $\{N_{0,j,k}\}_{0\le j\le 1,0\le k\le l_3}$ of its blocked complementary design. Next we derive similar identities for blocked s^{l-m} designs based on (19) and (20). Note that $f_1(x,0)$, $f_2(x,0)$, $f_1(0,z)$ and $f_2(0,z)$ are the moment generating functions of $\{N_{i,0,0}\}$, $\{N_{i,1,0}\}$, $\{N_{0,0,k}\}$, and $\{N_{0,1,k}\}$, respectively, and $f_1(x,z)$ and $f_2(x,z)$ satisfy (19) and (20). Following similar arguments and derivations as in Sections 3 and 4, solving (19) and (20) lead to identities between $\{f_1(x,z),f_2(x,z)\}$ and $\{f_1(0,z),f_2(0,z)\}$. Without loss of generality, we assume that $\{f_1(0,z),f_2(0,z)\}$ are known in the following.

Theorem 6. *Given $f_1(0,z)$ and $f_2(0,z)$, there exist unique solutions $f_1(x,z)$ and $f_2(x,z)$ to* (19) *and* (20)*, and*

$$(s^r-1)f_1(x,z)-f_2(x,z)$$

$$
\begin{aligned}
&= [1+(s-1)x]^{l_1-(s^{k-1}-s^{r-1})}(1-x)^{(s^{k-1}-s^{r-1})-l_3}h(x,z), && (21)\\
f_2(x,z) &= (s^r-1)s^{-k}[1+(s-1)x]^{l_1-s^{k-1}}[1+(s-1)z]^{l_3}\\
&\quad \times\{[1+(s-1)x]^{s^{k-1}}-(1-x)^{s^{k-1}}\}\\
&\quad -s^{-r}[1+(s-1)x]^{l_1-s^{k-1}}(1-x)^{s^{k-1}-s^{r-1}-l_3}\\
&\quad \times\{[1+(s-1)x]^{s^{r-1}}-(1-x)^{s^{r-1}}\}h(x,z)\\
&\quad +[1+(s-1)x]^{l_1-s^{k-1}}(1-x)^{s^{k-1}-l_3}g(x,z), && (22)
\end{aligned}
$$

in which

$$
\begin{aligned}
h(x,z) &= [1+(s-2)x-(s-1)xz]^{l_3}\\
&\quad \times\{(s^r-1)f_1(0,(z-x)[1+(s-2)x-(s-1)xz]^{-1})\\
&\quad -f_2(0,(z-x)[1+(s-2)x-(s-1)xz]^{-1})\}, && (23)\\
g(x,z) &= [1+(s-2)x-(s-1)xz]^{l_3}f_2(0,(z-x)[1+(s-2)x-(s-1)xz]^{-1}). && (24)
\end{aligned}
$$

A sketch of the derivations of (21) and (22) is included in the Appendix.

Note that (21) and (22) provide exact relationships between $\{f_1(x,z), f_2(x,z)\}$ and $\{f_1(0,z), f_2(0,z)\}$. Identities between $\{f_1(\cdot,0), f_2(\cdot,0)\}$ and $\{f_1(0,\cdot), f_2(0,\cdot)\}$ can be easily obtained by setting $z=0$ in (21)–(24). In the rest of the paper, if $=_c$ instead of $=$ is used in an equation, it indicates that a function or a constant that does not depend on $\{N_{0,0,j}\}$ and $\{N_{0,1,j}\}$ may be omitted from the equation. These functions and constants can be calculated, but are omitted to save space. Setting $z=0$ in (21)–(24), we have

$$
\begin{aligned}
(s^r-1)f_1(x,0) &= [1+(s-1)x]^{l_1-(s^{k-1}-s^{r-1})}(1-x)^{(s^{k-1}-s^{r-1})-l_3}h(x,0)+f_2(x,0),\\
f_2(x,0) &=_c -s^{-r}[1+(s-1)x]^{l_1-s^{k-1}}(1-x)^{s^{k-1}-s^{r-1}-l_3}\\
&\quad \times\{[1+(s-1)x]^{s^{r-1}}-(1-x)^{s^{r-1}}\}h(x,0)\\
&\quad +[1+(s-1)x]^{l_1-s^{k-1}}(1-x)^{s^{k-1}-l_3}g(x,0),
\end{aligned}
$$

where

$$
\begin{aligned}
h(x,0) &= [1+(s-1)x]^{l_3}\{(s^r-1)f_1(0,-x[1+(s-2)x]^{-1})-f_2(0,-x[1+(s-2)x]^{-1})\}\\
&= (s^r-1)\sum_j N_{0,0,j}(-x)^j[1+(s-2)x]^{l_3-j}\\
&\quad -\sum_j N_{0,1,j}(-x)^j[1+(s-2)x]^{l_3-j},\\
g(x,0) &= [1+(s-2)x]^{l_3}f_2(0,-x[1+(s-2)x]^{-1})\\
&= \sum_j N_{0,1,j}(-x)^j[1+(s-2)x]^{l_3-j}.
\end{aligned}
$$

Because $f_1(x,0) = \sum_i N_{i,0,0}x^i$ and $f_2(x,0) = \sum_i N_{i,1,0}x^i$, exact identities between $\{N_{i,0,0},N_{i,1,0}\}$ and $\{N_{0,0,j},N_{0,1,j}\}$ can be obtained by applying the Taylor expansion to the polynomial terms in the equations above. Define P and Q by

$$P(j_4;l_3) = \sum_{j=0}^{j_4}(-1)^j(s-2)^{j_4-j}\binom{l_3-j}{j_4-j}N_{0,0,j},$$
$$Q(j_4;l_3) = \sum_{j=0}^{j_4}(-1)^j(s-2)^{j_4-j}\binom{l_3-j}{j_4-j}N_{0,1,j}.$$

Then one has

$$\begin{aligned}N_{i,1,0} =_c &\sum_{j_1+j_2+j_4=i}(-1)^{j_2}(s-1)^{j_1}\binom{l_1-s^{k-1}}{j_1}\binom{s^{k-1}-s^{r-1}-l_2}{j_2}Q(j_4;l_3)\\ &-\sum_{j_1+j_2+j_3+j_4=i}s^{-r}(-1)^{j_2+1}(s-1)^{j_1}[(s-1)^{j_3}-(-1)^{j_3}]\\ &\times\binom{l_1-s^{k-1}}{j_1}\binom{s^{k-1}-s^{r-1}-l_3}{j_2}\binom{s^{r-1}}{j_3}((s^r-1)P(j_4;l_3)-Q(j_4;l_3)),\end{aligned}\tag{25}$$

$$\begin{aligned}N_{i,0,0} =_c &(s^r-1)^{-1}N_{i,1,0}+(s^r-1)^{-1}\sum_{j_1+j_2+j_4=i}(-1)^{j_2}(s-1)^{j_1}\\ &\times\binom{l_1-s^{k-1}+s^{r-1}}{j_1}\binom{s^{k-1}-s^{r-1}-l_3}{j_2}((s^r-1)P(j_4;l_3)-Q(j_4;l_3)).\end{aligned}\tag{26}$$

Under the assumption that factorial effects with order three or higher are negligible, the split wordlength patterns with $i = 2$, 3 and 4 are of practical importance. Specifying $i = 2, 3, 4$ respectively in (25) and (26) results in the following corollary.

Corollary 1. *The following identities hold:*

$$N_{2,1,0} =_c N_{0,1,2},\tag{27}$$
$$N_{3,1,0} =_c -(s^r+2s-5)N_{0,1,2} - N_{0,1,3},\tag{28}$$
$$N_{3,0,0} =_c -N_{0,1,2} - N_{0,0,3},\tag{29}$$
$$N_{4,0,0} =_c \frac{1}{2}(s^r+5s-10)N_{0,1,2} + N_{0,1,3} + (3s-5)N_{0,0,3} + N_{0,0,4}.\tag{30}$$

We use $N^s_{i,j,k}$ to indicate the dependence on s. For $(2^{l_1-(l_1-k)}, 2^r)$ blocked designs:

$$N^2_{2,1,0} =_c N^2_{0,1,2},\tag{31}$$
$$N^2_{3,1,0} =_c -(2^r-1)N^2_{0,1,2} - N^2_{0,1,3},\tag{32}$$

$$N^2_{3,0,0} =_c -N^2_{0,1,2} - N^2_{0,0,3}, \tag{33}$$
$$N^2_{4,0,0} =_c 2^{r-1}N^2_{0,1,2} + N^2_{0,1,3} + N^2_{0,0,3} + N^2_{0,0,4}. \tag{34}$$

Equations (31), (33) and (34) were also reported in equation (19) in Chen and Cheng (1999). (However, there is an error: $\alpha_r(2)$ should be equal to $2^{r-1} - 1$, not to $2^{r-1}(2^{r-1} - 1)$.) For $(3^{l_1-(l_1-k)}, 3^r)$ blocked designs,

$$N^3_{2,1,0} =_c N^3_{0,1,2}, \tag{35}$$
$$N^3_{3,1,0} =_c -(3^r + 1)N^3_{0,1,2} - N^3_{0,1,3}, \tag{36}$$
$$N^3_{3,0,0} =_c -N^3_{0,1,2} - N^3_{0,0,3}, \tag{37}$$
$$N^3_{4,0,0} =_c \frac{1}{2}(3^r + 5)N^3_{0,1,2} + N^3_{0,1,3} + 4N^3_{0,0,3} + N^3_{0,0,4}. \tag{38}$$

Since there are two types of wordlength patterns in a regular (s^{l-m}, s^r) blocked design, it is crucial to find a criterion to rank-order the relative importance of the defining words. Three ordering criteria have been proposed (Sitter, Chen and Feder (1997), Chen and Cheng (1999) and Cheng and Wu (2002))):

$$W_{SCF} = (N_{3,0,0}, N_{2,1,0}, N_{4,0,0}, N_{3,1,0}, N_{5,0,0}, N_{4,1,0}, \dots), \tag{39}$$
$$W_{CC} = (N_{3,0,0}, N_{2,1,0}, N_{4,0,0}, N_{5,0,0}, N_{3,1,0}, N_{6,0,0}, \ldots), \tag{40}$$
$$W_{CW} = (N_{3,0,0}, N_{4,0,0}, N_{2,1,0}, N_{5,0,0}, N_{6,0,0}, N_{3,1,0}, \cdots). \tag{41}$$

The wordlength patterns can be used to define optimality criteria and discriminate blocked designs. For each of W_{SCF}, W_{CC} and W_{CW}, sequentially minimizing the ordered wordlength patterns leads to the corresponding generalized minimum aberration blocked designs. Discussions and comparison of these criteria can be found in Cheng and Wu (2002). In this paper, only W_{CW} will be considered, and the corresponding minimum aberration design is called *minimum W_{CW} aberration design*. Based on Corollary 1, some general rules for identifying minimum W_{CW} aberration (s^{l-m}, s^r) blocked designs can be established using their complementary designs, as follows.

Rule 1. A regular (s^{l-m}, s^r) blocked design $(\mathcal{D}_*, \mathcal{B}_*)$ has minimum W_{CW} aberration if

(i) $N_{0,1,2} + N_{0,0,3}$ is the maximum among all the blocked complementary designs $(\mathcal{B}, \bar{\mathcal{D}})$,

(ii) $(\mathcal{D}_*, \mathcal{B}_*)$ is the unique design satisfying (i).

Rule 2. A regular (s^{l-m}, s^r) blocked design $(\mathcal{D}_*, \mathcal{B}_*)$ has minimum W_{CW} aberration if

(i) as in Rule 1,

(ii) $(s^r+5s-10)N_{0,1,2}/2+N_{0,1,3}+(3s-5)N_{0,0,3}+N_{0,0,4}$ is the minimum among all the blocked designs satisfying (i),

(iii) $(\mathcal{D}_*, \mathcal{B}_*)$ is the unique design satisfying (i) and (ii).

Rule 3. A regular (s^{l-m}, s^r) blocked design $(\mathcal{D}_*, \mathcal{B}_*)$ has minimum W_{CW} aberration if

(i) (ii) as in Rule 2,

(iii) $N_{0,1,2}$ is the minimum among all the blocked complementary designs satisfying (i) and (ii),

(iv) $(\mathcal{D}_*, \mathcal{B}_*)$ is the unique design satisfying (i), (ii) and (iii).

Based on (25) and (26), it is not difficult to derive general rules involving higher order wordlength patterns. Taking s to be 2 or 3, Rules 1-3 can be used to construct two-level or three-level blocked designs with minimum W_{CW} aberration. The construction of some three-level blocked designs is illustrated in the following example.

Example 1. $(3^{9-6}, 3^1)$ blocked designs can be employed to investigate nine treatment factors in three blocks with 27 runs. There is only one block factor denoted by b, let $\mathcal{B} = \{b\}$. $\mathcal{D}$ consists of nine points from $PG(2,3)$ that are different from b. Since $PG(2,3)$ contains 13 points, there are three points left. Denote them by $\bar{\mathcal{D}} = \{r_1, r_2, r_3\}$. Because the blocked complementary designs $(\mathcal{B}, \bar{\mathcal{D}})$ have only four points, it is easy to verify that there exist four non-isomorphic designs:

(1) $(\mathcal{B}_1, \bar{\mathcal{D}}_1)$, where b and r_1 are independent, $r_2 = br_1$ and $r_3 = br_1^2$;
(2) $(\mathcal{B}_2, \bar{\mathcal{D}}_2)$, where b, r_1 and r_2 are independent, and $r_3 = br_1$;
(3) $(\mathcal{B}_3, \bar{\mathcal{D}}_3)$, where b, r_1 and r_2 are independent, and $r_3 = r_1r_2$;
(4) $(\mathcal{B}_4, \bar{\mathcal{D}}_4)$, where b, r_1 and r_2 are independent, and $r_3 = br_1r_2$.

The associated split wordlength patterns can also be obtained:

(1) $N_{0,0,3} = 2$, $N_{0,0,4} = 0$, $N_{0,1,2} = 6$, $N_{0,1,3} = 0$;
(2) $N_{0,0,3} = 0$, $N_{0,0,4} = 0$, $N_{0,1,2} = 2$, $N_{0,1,3} = 0$;
(3) $N_{0,0,3} = 2$, $N_{0,0,4} = 0$, $N_{0,1,2} = 0$, $N_{0,1,3} = 0$;
(4) $N_{0,0,3} = 0$, $N_{0,0,4} = 0$, $N_{0,1,2} = 0$, $N_{0,1,3} = 2$.

Note that $(\mathcal{B}_1, \bar{\mathcal{D}}_1)$ has $N_{0,1,2} + N_{0,0,3}$ equal to 8, which is the maximum among the four blocked complementary designs, and the maximum is unique. Applying Rule 1, the blocked design $(\mathcal{D}_1, \mathcal{B}_1)$ is the minimum W_{CW} aberration $(3^{9-6}, 3^1)$ blocked design; remaining designs can be further discriminated based on W_{CW}. Because $N_{0,0,2} + N_{0,1,2}$ is equal to 0 for $(\mathcal{B}_4, \bar{\mathcal{D}}_4)$ and equal to 2 for both $(\mathcal{B}_2, \bar{\mathcal{D}}_2)$ and $(\mathcal{B}_3, \bar{\mathcal{D}}_3)$, $(\mathcal{D}_4, \mathcal{B}_4)$ has the maximum W_{CW} aberration. We further compare the second term $N_{4,0,0}$ of W_{CW} in order to distinguish $(\mathcal{B}_2, \bar{\mathcal{D}}_2)$ and $(\mathcal{B}_3, \bar{\mathcal{D}}_3)$. Applying (38), $N_{4,0,0} =_c 4N_{0,1,2}+N_{0,1,3}+4N_{0,0,3}+N_{0,0,4} = 8$ for both $(\mathcal{B}_2, \bar{\mathcal{D}}_2)$ and $(\mathcal{B}_3, \bar{\mathcal{D}}_3)$. So the third term $N_{2,1,0}$ of W_{CW} has to be employed. Because $N_{2,1,0} =_c$

$N_{0,1,2} = 0$ for $(\mathcal{B}_3, \bar{\mathcal{D}}_3)$, and equal to 2 for $(\mathcal{B}_2, \bar{\mathcal{D}}_2)$, $(\mathcal{D}_3, \mathcal{B}_3)$ has less W_{CW} aberration than $(\mathcal{D}_2, \mathcal{B}_2)$. Hence these four designs can be rank-ordered using the W_{CW} aberration from the minimum to the maximum as $(\mathcal{D}_1, \mathcal{B}_1)$,$(\mathcal{D}_3, \mathcal{B}_3)$, $(\mathcal{D}_2, \mathcal{B}_2)$, $(\mathcal{D}_4, \mathcal{B}_4)$. By deleting points from $PG(2,3)$, it is straight-forward to derive the defining words for the original blocked designs. Note that $(\mathcal{D}_1, \mathcal{B}_1)$, $(\mathcal{D}_3, \mathcal{B}_3)$ and $(\mathcal{D}_2, \mathcal{B}_2)$ are listed as 9-6.1/B1.1, 9-6.2/B1.1 and 9-6.3/B1.1 in Table 4 in Cheng and Wu (2002).

We give another example to demonstrate the power of our approach to recover all the structure indices besides the split patterns of a regular blocked design. A $(2^{6-2}, 2^3)$ blocked design is used for illustration.

Example 2. Suppose six treatment factors A, B, C, D, E and F and three block factors b_1, b_2 and b_3 are involved in a $(2^{6-2}, 2^3)$ blocked design. Assume that A, B, C and D are independent in the design. The defining words for E, F and the block factors are $E = ABC$, $F = ABD$, $b_1 = AB$, $b_2 = AC$, and $b_3 = AD$. Hence the defining relation is

$$\begin{aligned} I &= ABCE = ABDF = CDEF = ABb_1 = CEb_1 = DFb_1 = ACb_2 = BEb_2 \\ &= ADb_3 = BFb_3 = AEb_1b_2 = BCb_1b_2 = BDb_1b_3 = AFb_1b_3 = CDb_2b_3 \\ &= EFb_2b_3 = CFb_1b_2b_3 = DEb_1b_2b_3 = ADEFb_2 = BCDFb_2 = ACEFb_3 \\ &= BCDEb_3 = ACDFb_1b_2 = BDEFb_1b_2 = ACDEb_1b_3 = BCEFb_1b_3 \\ &= ABCFb_2b_3 = ABDEb_2b_3 = ABCDb_1b_2b_3 = ABEFb_1b_2b_3 = ABCDEFb_1. \end{aligned}$$

The induced partition of $PG(3,2)$ is $\mathcal{D} \cup \mathcal{B} \cup \bar{\mathcal{D}}$, where $\mathcal{D} = \{A, B, C, D, E, F\}$, $\mathcal{B} = \{b_1, b_2, b_3, b_1b_2, b_1b_3, b_2b_3, b_1b_2b_3\}$, $\bar{\mathcal{D}} = \{r_1, r_2\}$, and r_1 and r_2 are the two remaining points $r_1 = ACD$ and $r_2 = BCD$. The partition is summarized in the following table.

$\mathcal{D}$						$\mathcal{B}$							$\bar{\mathcal{D}}$	
A	B	C	D	E	F	b_1	b_2	b_3	b_1b_2	b_1b_3	b_2b_3	$b_1b_2b_3$	r_1	r_2
1	0	0	0	1	1	1	1	1	0	0	0	1	1	0
0	1	0	0	1	1	1	0	0	1	1	0	1	0	1
0	0	1	0	1	0	0	1	0	1	0	1	1	1	1
0	0	0	1	0	1	0	0	1	0	1	1	1	1	1

Since $\bar{\mathcal{D}}$ contains only two points, it is very easy to derive the split wordlength patterns of $(\mathcal{B}, \bar{\mathcal{D}})$: $N_{0,0,0} = 1$, $N_{0,0,1} = 0$, $N_{0,0,2} = 0$, $N_{0,1,0} = 0$. $N_{0,1,1} = 0$ and $N_{0,1,2} = 1$. Thus $f_1(0, z) = 1$ and $f_2(0, z) = z^2$. Using Theorems 5 and 6 and some algebraic simplification, we have

$$f_1(x, z) = 1 + 3x^4 + 8x^3z + (3x^2 + x^6)z^2, \tag{42}$$

$$f_2(x,z) = 15x^2 + 12x^4 + x^6 + (12x + 32x^3 + 12x^5)z + (1 + 12x^2 + 15x^4)z^2. \quad (43)$$

Using (42) and (43) and the definition of f_1 and f_2, all the structure indices $N_{i,j,k}$ can be immediately recovered. For example, setting $z = 0$ in (42) and (43) leads to $f_1(x,0) = 1 + 3x^4$ and $f_2(x,0) = 15x^2 + 12x^4 + x^6$. The split wordlength patterns of $(\mathcal{D},\mathcal{B})$ immediately follow: $N_{3,0,0} = 3$, $N_{2,1,0} = 15$, $N_{4,1,0} = 12$, and $N_{6,1,0} = 1$. This can be easily verified by the defining relation given above. Furthermore, it is also easy to verify that $N_{3,1,1} = 32$, $N_{2,1,2} = 12$, etc. This example shows explicitly how the structure indices $N_{i,j,k}$ are determined by the two simple functions $f_1(0,z) = 1$ and $f_2(0,z) = z^2$.

6. Concluding Remarks

In Section 3, a general framework is given to relate the aliasing pattern of a design to that of its complementary design through the structure function and a first-order partial differential equation satisfied by the structure function. The results in Sections 4 and 5 demonstrate that the framework is flexible and powerful enough to accommodate special design structures like multiple groups of factors and blocking. There is ongoing work to employ this framework to study detailed properties of fractional factorial designs like letter patterns and aliasing structures. The results will be reported elsewhere.

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Appendix

Proof of Theorem 2. Because $\{N_{0,j}\}$ is given, $f(0,y) = \sum_j y^j$ is known. The existence and uniqueness of the solution to (6) with $f(0,y)$ given follow from standard results on first-order partial differential equations (John (1982)). Introduce two auxiliary variables τ and t. Let $x = x(\tau,t)$, $y = y(\tau,t)$ and $w = f(x,y) = f(x(\tau,t), y(\tau,t))$. It is well-known that equation (6) with $f(0,y)$ given is equivalent to the following system of ordinary differential equations (John (1982, Chap. 1)):

$$\frac{dx}{dt} = 1 + (s-2)x - (s-1)x^2, \quad (44)$$

$$\frac{dy}{dt} = 1 + (s-2)y - (s-1)y^2, \quad (45)$$

$$\frac{dw}{dt} = -[1 + (s-1)lx + (s-1)\bar{l}y]w + [1 + (s-1)x]^l[1 + (s-1)y]^{\bar{l}}, \quad (46)$$

with the initial conditions

$$x(\tau,0)=0, \tag{47}$$

$$y(\tau,0)=\tau, \tag{48}$$

$$w(\tau,0)=f(0,\tau)=\sum_j N_{0,j}\tau^j. \tag{49}$$

Solving (44) with (47) leads to

$$\frac{1+(s-1)x}{1-x}=e^{st} \text{ or, equivalently, } x=\frac{-1+e^{st}}{(s-1)+e^{st}}. \tag{50}$$

Similarly solving (45) with (48) leads to

$$y=\frac{-1+ce^{st}}{(s-1)+ce^{st}}, \text{ with } c=\frac{1+(s-1)\tau}{1-\tau}. \tag{51}$$

The solution of (46) under (49) is

$$\begin{aligned} w(\tau,t)=&\Big(\int_0^t[1+(s-1)x]^l[1+(s-1)y]^{\bar{l}}\exp\Big\{\int_0^t[1+(s-1)lx+(s-1)\bar{l}y]dt\Big\}dt\\ &+f(0,\tau)\Big)\exp\Big\{-\int_0^t[1+(s-1)lx+(s-1)\bar{l}y]dt\Big\}. \end{aligned} \tag{52}$$

Based on (50) and (51), (52) can be simplified to

$$\begin{aligned} w(\tau,t)=&s^{l+\bar{l}-k}c^{\bar{l}}[1+(s-1)e^{-st}]^{-l}[c+(s-1)e^{-st}]^{-\bar{l}}(1-e^{-s^kt})\\ &+s^l[c+(s-1)]^{\bar{l}}[1+(s-1)e^{-st}]^{-l}[c+(s-1)e^{-st}]^{-\bar{l}}e^{-s^kt}f(0,\tau). \end{aligned} \tag{53}$$

From (50) and (51) again, t, c and τ can be written as functions of x and y as follows:

$$\begin{aligned} e^{-st}&=[1+(s-1)x]^{-1}(1-x),\\ c&=[1+(s-1)x]^{-1}(1-x)[1+(s-1)y](1-y)^{-1},\\ \tau&=(y-x)[1+(s-2)x-(s-1)xy]^{-1}. \end{aligned}$$

Hence,

$$\begin{aligned} &w(\tau(x,y),t(x,y))\\ &\quad=s^{-k}[1+(s-1)x]^{l-s^{k-1}}[1+(s-1)y]^{\bar{l}}\{[1+(s-1)x]^{s^{k-1}}-(1-x)^{s^{k-1}}\}\\ &\qquad+[1+(s-1)x]^{l-s^{k-1}}(1-x)^{s^{k-1}-\bar{l}}[1+(s-2)x-(s-1)xy]^{\bar{l}}\\ &\qquad\times f(0,(y-x)[1+(s-2)x-(s-1)xy]^{-1}). \end{aligned} \tag{54}$$

Noting that $f(x,y) = w(\tau(x,y), t(x,y))$ concludes the proof.

Proof of Theorem 6. Define $\tilde{f}(x,z) = (s^r - 1)f_1(x,z) - f_2(x,z)$. Subtracting (20) from (19) multiplied by $s^r - 1$, we have

$$[1+(s-2)x-(s-1)x^2]\frac{\partial \tilde{f}}{\partial x}+[1+(s-2)z-(s-1)z^2]\frac{\partial \tilde{f}}{\partial z}+[(s-1)l_1x+(s-1)l_3z]\tilde{f}=0. \quad (55)$$

Equation (20) can be rewritten as

$$\begin{aligned}&[1 + (s-2)x - (s-1)x^2]\frac{\partial f_2}{\partial x} + [1 + (s-2)z - (s-1)z^2]\frac{\partial f_2}{\partial z}\\ &\quad +[s^r + (s-1)l_1x + (s-1)l_3z]f_2\\ &\quad = (s^r - 1)[1 + (s-1)x]^{l_1}[1 + (s-1)z]^{l_3} - \tilde{f}. \qquad (56)\end{aligned}$$

The system of equations (19) and (20) for f_1 and f_2 is equivalent to the system of equations (55) and (56) for $\tilde{f}$ and f_2. Because $f_1(0,z)$ and $f_2(0,z)$ are given, $\tilde{f}(0,z) = (s^r-1)f_1(0,z)$-$f_2(0,z)$ and $f_2(0,z)$ are also known. Note that equation (55) only involves $\tilde{f}$. Therefore, (55) and (56) can be solved sequentially. Solving (55) with given $\tilde{f}(0,z)$ leads to the expression of $\tilde{f}(x,z)$ in (21). (Techniques similar to the proof of Theorem 2 are used to derive the solution.) Replacing $\tilde{f}$ in (56) with its expression in (21), (56) becomes an equation f_2 only involving. Given $f_2(0,z)$, the solution of (56) is (22).

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