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Inside Views

Birth of the Threshold Time Series Model

Prologue

In this short note prepared for the theme volume on Threshold Models and New Developments in Time Series[1], I shall start with an account of how the threshold time series model was born and finish with some thoughts on the future directions of nonlinear time series analysis, with some random comments interspersed in between. The style is autobiographical and non-technical.

From the beginning of time series analysis, modeling was dominated by the assumption of linearity. This situation lasted until almost the end of the 1970s. In fact, before 1980, hardly any standard time series textbooks covered nonlinear time series models.

The Year 1977

In the annals of nonlinear time series modeling, I think the first year to remember is 1977. At an Ordinary Meeting of the Royal Statistical Society meeting in that year, Professor (now Sir) David Cox remarked, "all the models for the lynx data considered by Dr. Tong and by Mr. Campbell and Professor Walker are time reversible, …there is a fairly clear evidence from the data of irreversibility....a more likely explanation is the presence of nonlinearity" (See Tong (1977a, p. 453)). At the same meeting, Dr. Granville Tunnicliffe Wilson asked, "would we not prefer a model which....would exhibit stable periodic deterministic behaviour -- a limit cycle? Such limit cycles cannot arise from linear models" (p. 455). As no systematic study of nonlinear time series modeling existed at that time, he concluded pessimistically, "even if we are able to propose a wide class of nonlinear models to be used in fitting cyclical series,

[1] I am grateful to the University of Hong Kong for continuous support leading to the present note.

the problems of identifying, in the sense of Box and Jenkins, a suitable model are enormous"(p. 456).

The Challenge

As events unfolded, this 1977 Ordinary Meeting sparked some extraordinary developments (just as public schools in Britain are not so public, ordinary meetings of the RSS are not so ordinary). The above remarks openly challenged time series analysts to propose a wider class of practically useful nonlinear time series models, to gain a deep understanding of their probabilistic structure, to develop statistical identification/estimation of these models, and to address the general issue of nonlinear forecasting.

To develop useful nonlinear time series models was a daunting task indeed. Where should we start? For, any model which is not linear is nonlinear. To make a good choice we often have to rely on our value judgment, which is often influenced by the philosophy we subscribe to, the culture we have inherited and the taste we have developed. Of course, luck can sometimes come into the picture too.

Philosophy

To take up the challenge, I decided around 1977-78 that I would focus on cyclical animal population and river flow data. I saw at least two main advantages in doing so. First, it is important that the developed nonlinear time series models should be capable of offering insight into the underlying dynamics of the data. In this respect, the deterministic theory of dynamical systems should provide inspiration. Indeed, the reference to "limit cycles" by Granville mentioned above made a deep impression on me. Second, it was sound to have specific data sets in mind for quickly and constantly checking if the methodology under development was headed in the right direction. There is no doubt that I subscribe to the philosophy of the inseparability of theory and practice.

Non-linear Oscillations

Like many statisticians of my generation, I was ill-equipped mathematically because what I had received was predominantly an education in linear mathematics --I was badly taught! This meant I had to teach myself a new subject from scratch, and I started to read (rather slowly) the books by Minorsky (1962) and Andronov and Khaikin (1949). The original text of the latter was in Russian, which I could not read (and still cannot). Luckily, quite by chance, I got hold of a Chinese translation. The copy I acquired was a castaway that arrived in the UK from Shanghai during the turmoil known as the Cultural Revolution. Ironically, I have benefited culturally from the revolution! I should also mention my sense of admiration for Professor Peter Whittle when I saw his reference to exactly the same book in his celebrated paper on the analysis of the seiche record (Whittle (1954)). He noted an arithmetic relationship among the peaks in the power spectrum, explained that this must be the consequence of nonlinearity, and suggested a piecewise linear differential equation model. Of course, I only discovered this gem when writing my Springer Lecture Notes in 1983. Peter seemed (perhaps pleasantly) surprised when he saw my reference to this work because he said, “you know, Howell, you must be the only person who has cited this model of mine.”

The Penny Drops

During late 1977 and early 1978, I played around a bit with bilinear time series models after listening to a talk by the Swedish control engineer, Professor Karl J. Aström. I obtained some early results but decided that the approach was not to my taste and abandoned it. Essentially, I could not reconcile the role of the unobservable innovation, used artificially in the univariate bilinear time series models, with the control variable, cited widely in the original control engineering literature. Then one day in 1977, as I was mowing my lawn, the penny dropped: piecewise-linearity was the way! This approach could represent the different phases, increasing and decreasing, in an animal population and the impact of the melting of ice/snow on river flow. Phase transition is, of course, a fundamentally nonlinear phenomenon. Perhaps I was subconsciously reverting to the strategy of “*divide and rule*”, which has been so deeply ingrained in both Chinese and English cultures. The curious thing was that I got this idea before reaching the piecewise differential equation bits in Andronov and Khaikin. Would I have had the same idea had I read them first? In fact, while intoxicated by piecewise linearization, I thought I had also invented piecewise linear differential equations. Luckily, that only lasted for a very short time because on turning over the pages, I could see the full glory of these differential equations expounded by Andronov and Khaikin. Clearly I was born at least 40 years too late!

Pride and Prejudice

The threshold idea was thus conceived in 1977 and I recorded it in my contributions to the discussion of a paper by Tony Lawrance in 1977 (Tong (1977b)) and a NATO ASI series in 1978. However, to put the idea into practice meant a huge amount of computer experimentation. I say “huge” because we were in the late 1970s when computers were much slower than they are now. Luckily, a couple of my research students, P. K. Wong and K. S. Lim, were keen to help. I can still remember the joy of seeing the first limit cycle produced by what is now called a SETAR (self-exciting threshold autoregressive) model. Actually, this came in a round-about way. I asked Lim to do some multi-step forecast with a SETAR model via simulation. She misunderstood me and showed the result obtained by recursion of the SETAR model after deleting the innovation, that is, the skeleton in the terminology I introduced later. So, my first glimpse of a SETAR-generated limit cycle was due to my research student carrying out the wrong task. Now, I call that luck!

By the later part of 1978, I had a paper on threshold autoregression written up and submitted to a prestigious journal in the US. As usual with that journal, the review seemed to take ages. When it finally came back, it was basically positive but revision was needed. Alas, by the time I re-submitted the revised paper, there seemed to have been some changes in the editorial board. I cannot remember exactly what happened but the letter of rejection was signed by a different editor and the tone was discouraging. Dejected? Perhaps, but not for long, because I thought I could always try a better platform, namely, a discussion paper read to the Royal Statistical Society. This I did, and the paper was accepted for reading. I read the paper, *Threshold autoregression, limit cycles and cyclical data*, to the RSS on 19th March 1980.

The paper did not attain instant acclaim, although I think there was a “let-us-wait-and-see” welcome. Looking back at my work, I could have polished the paper more. I think the main reason for the hesitant reaction was that the idea was rather new, although its form was deceptively simple. There were still so many rough edges to smooth out (e.g. How to choose the threshold variables? Can the regime switching be continuous rather than discontinuous?), so many unresolved theoretical issues (e.g. What are the sampling properties of the parameter estimates? How to test for linearity within the context of SETAR? How to obtain theoretical multi-step forecast formula?), and so many more data-analytic techniques to develop. In any case, I was spurred on to smooth out the rough edges and to forge an even stronger link between (statistical) nonlinear time series and (deterministic) nonlinear dynamical systems, including chaos. I often collaborated with my students and others. Tong (1990; 1995) and Chan and Tong (2001) give a good summary of our results. Since its publication, the 1980 paper has attracted a great deal of attention and is my most frequently cited paper. What is most pleasing is the fact that many brilliant and mostly younger colleagues have been attracted to the threshold models; their input has gone a long way towards resolving many of the above mentioned issues and beyond.

What Next?

The threshold model as introduced in Tong and Lim (1980) is more general than the SETAR model. This theme issue further shows that the threshold model is still full of vitality and, like its linear predecessor (i.e. Udny Yule's linear AR model), chances are that it might stay around indefinitely. Still, where shall we go next in the wider context of nonlinear time series analysis? As I have said in my book (Tong (1990, p. 345)), he who forecasts does not know. So with this disclaimer, here I go. First, nonlinear time series modeling to-date has focused on the steady state, hence ergodicity/stationarity. The transient state has often been ignored. Nonlinear dynamics tells us that a nonlinear dynamical system can reside in the neighborhood of an equilibrium state for a certain period of time, which can be quite short or quite long, before jumping to another. (Perhaps MCMC enthusiasts can take note!) This prompts me into suggesting that there can be interaction between nonstationarity and nonlinearity, especially if all that we have are the observed data. Can we always tell them apart? Should we unscramble the omelette? If so, how? Next, multiple nonlinear time series analysis is an important area.

It is heartening to see some developments in this volume and elsewhere, but I think much more is waiting out there for us to explore. I do not need to reiterate the importance of multiple time series, linear or nonlinear, in practical applications. Of course, the multi-dimensional world is much richer than the unidimensional one. It is clear that some dimensional reduction is absolutely essential in order to ameliorate the curse of dimensionality. How to best visualize a high dimensional object is not unrelated to the choice of appropriate generalized coordinates in dynamics. It seems to me that the semi-parametric framework is a good candidate, and that there have been some encouraging developments, including at least one paper in this theme volume, but much more needs to be done. There might also be points of contact with the machine learning community.

Last but not least, spatial-temporal data abound. They require spatial-temporal models. There have been some worthwhile developments, including some reported in this volume. One ultimate goal could be some nonlinear/nonstationary spatial-temporal models. Essentially what we want is a discrete time analogue of a stochastic partial differential equation.

Epilogue

Nowadays, seeing that the threshold autoregressive models and the threshold idea have been so successfully applied to many practical problems in diverse fields such as ecology, econometrics, economics, finance, actuarial science, hydrology and many others I think that the efforts have been all worthwhile. Those models are also firmly established in the literature, including textbooks. When I see people using terms or acronyms such as STAR, DTARCH, threshold-ARCH, threshold unit-root test, threshold co-integration, Markov regime-switching (under a different name, for example, in Tong and Lim (1980, p.285)), and the amazing number of citations produced by a scholar.google.com search of these names and their cousins, I cannot help but smile and say to myself, “I bet not many of them know that they are using a US reject!”

References

Andronov, A. A. and Khaikin, S. E. (1949). *Theory of Oscillations.* (first published in Russian in 1937, and translated and adapted by S. Lefschetz), Princeton Univ. Press, Princeton, NJ.

Chan, K. S. and Tong, H. (2001). *Chaos: A Statistical Perspective.* Springer-Verlag, New York.

Minorsky, N. (1962). *Non-Linear Oscillations.* Van Nostrand, Princeton, NJ.

Tong, H. (1977a). Some comments on the Canadian lynx data (with discussion). *J. Roy. Statist. Soc. Ser. A* **140**, 432-436 and 448-468.

Tong, H. (1977b). Contribution to the discussion of the paper entitled “Stochastic modelling of riverflow time series” by A. J. Lawrance and N. T. Kottegoda. *J. Roy. Statist. Soc. Ser. A*, 34-35.

Tong, H. (1978). *On a threshold model.* In *Pattern Recognition and Signal Processing* (Edited by C. H. Chen), NATO ASI Series E: Applied Sec. No. 29, Sijthoff & Noordhoff, Amsterdam, 575-586.

Tong, H. (1983). *Threshold Models in Non-Linear Time Series Analysis.* Lecture Notes in Statistics No. 21, Springer-Verlag, New York.

Tong, H. (1990). *Non-Linear Time Series: A Dynamical System Approach.* Oxford Univ. Press, Oxford.

Tong, H. (1995). A personal overview of nonlinear time series analysis from a chaos perspective (with discussions). *Scand. J. Statist.* **22**, 399-421.

Tong, H. and Lim, K. S. (1980). Threshold autoregression, limit cycles and cyclical data (with discussion). *J. Roy. Statist. Soc. Ser. B* **42**, 245-292.

Whittle, P. (1954). The statistical analysis of a seiche record. *J. Marine Res.* (Sears Foundation) **13**, 76-100.

— Howell Tong

In the enlightened year of 1970, Howell Tong was appointed to a lectureship at the University of Manchester Institute of Science and Technology shortly after he started his Ph.D. program. He received his Ph.D. in 1972 under the supervision of Maurice Priestley, thus making him a student of a student of Maurice Bartlett. He stayed at UMIST until 1982, when he took up the Founding Chair of Statistics at the Chinese University of Hong Kong. In 1986, he returned to the UK, as the first Chinese to hold a Chair of Statistics in the history of the UK, by accepting the Chair at the University of Kent at Canterbury. He stayed there until 1997, when he went to the University of Hong Kong, first as Distinguished Visiting Professor, and then as a Chair Professor of Statistics (and sometimes as a Pro-Vice-Chancellor and the Founding Dean of the Graduate School). He was appointed to his Chair at the London School of Economics in 1999. He has written three books (one with K. S. Chan) and (with collaborators) over 145 papers in Statistics, Ecology, Actuarial Science, Control Engineering, Reliability, Meteorology, Water Engineering, Engineering Mathematics and Mathematical Education. He is a Foreign Member of the Norwegian Academy of Science and Letters, a member of the ISI, a Fellow of IMS and an Honorary Fellow of the Institute of Actuaries (UK). He won a Chinese National Natural Science Prize (Class II) in 2000. He enjoys working with colleagues or students younger and brighter than himself. Having been involved right from the beginning of nonlinear time series modeling in the late 1970s, he is delighted to see that the threshold time series models he created have become an important standard approach and percolated into Econometrics, Ecology and other fields. He enjoys traveling, good food and walking with his wife, admiring theatre sets created by his talented daughter, learning through photographs about the many far-flung places visited by his son and daughter and some solitary reading of things non-statistical.