# Statistics and Machine Learning
# Homework1

Due on October 7, 2005

Exercise 1: **(a)** Solve

$$\min_{x \in R^2} \frac{1}{2} x^T \begin{bmatrix} 1 & 0 \\ 0 & 900 \end{bmatrix} x$$

using the *steep descent with exact line search*. You are welcome to copy the MATLAB code from my slides. Start your code with the initial point $x_0 = [1000\ 1]^T$. Stop until $\|x_{n+1} - x_n\|_2 < 10^{-8}$. Report your solution and the number of iteration.

**Ans:** We consider solving a unconstrained quadratic programming problem. That is,

$$\min_{x \in R^n} f(x) = \frac{1}{2} x' Q x + p' x.$$

Let $g_n$ be the gradient of $f(x)$ at $x_n$ and

$$h(\lambda) = f(x_n + \lambda(-g_n)) = \frac{1}{2}(x_n - \lambda g_n)' Q (x_n - \lambda g_n) + p'(x_n - \lambda g_n).$$

Find $\lambda^*$ such that $\dfrac{dh(\lambda)}{d\lambda} = 0$. We have $\lambda^* = \dfrac{g_n' g_n}{g_n' Q g_n}$.

```
function [x, f_value, iter] = grdlines(Q,p, x0, esp)
%
% min 0.5*x'Q*x+p'x
% Solving unconstrained minimization via
% steep descent with exact line search
%
%The stopping criterion:
% Either the ||gradient||_2^2 ,10^-12
```

```
%
%or    ||x_n+1 -x_n||_2<esp
flag =1; iter = 0; while flag > esp
          grad = Q*x0+p;
          temp1 = grad'*grad;
          if temp1 < 10^-12
              flag = esp
          else
              stepsize = temp1/(grad'*Q*grad);
              x1 = x0 - stepsize*grad;
              flag = norm(x1-x0);
              x0=x1;
          end;
          iter = iter+1;
 end;
 x = x0;
 f_value = 0.5*x'*Q*x+p'*x;
```

**(b)** Implement the Newton's method for minimizing a quadratic function $f(x) = \frac{1}{2}x^TQx + p^Tx$ in MATLAB code. Apply your code to solve the minimization problem in **(a)**.

**Ans:**

```
function [x, f_value, iter] = newtonqp(Q,p, x0, esp)
%
% min 0.5*x'Q*x+p'x
% Solving unconstrained QP via
% Newton's method
%
%The stopping criterion:
% Either the ||gradient||_2^2 ,10^-12
%
%or    ||x_n+1 -x_n||_2<esp
flag =1; iter = 0; while flag > esp
          grad = Q*x0+p;
          temp1 = grad'*grad;
          if temp1 < 10^-12
              flag = esp
          else
              %d=inv(Q)*grad;
```

```
            d=x0+inv(Q)*p;
            x1 = x0 - d;
            flag = norm(x1-x0);
            x0=x1;
        end;
        iter = iter+1;
end;
x = x0;
f_value = 0.5*x'*Q*x+p'*x;
```

Exercise 2: Find an approximate solution using MATLAB to the following system by minimizing $\|Ax - b\|_p$ for $p = 1, 2, \infty$. Write down both the approximate solution, and the value of the $\|Ax - b\|_p$. Draw the solution points in $R^2$ and the four equations being solved.

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 2 \\ 2x_1 & - & x_2 & = & -2 \\ x_1 & + & x_2 & = & 3 \\ 4x_1 & - & x_2 & = & -4 \end{array}$$

Ans: (a) $\|Ax - b\|_1$:

```
function [x, residual, one_error]=oneapprox(A,b)
%
%Input A: mXn matrix
%         b: m-vector
%
%Solve the problem by LP
%Output: the approximate solution of Ax=b
%            one_error =||Ax-b||_1
%
[m,n]=size(A); obj_p=[zeros(n,1); ones(m,1)];
H=[A -eye(m);-A -eye(m)]; h=[b;-b];
[sol, one_error]=linprog(obj_p,H,h);
x=sol(1:n); residual=sol((n+1):(m+n));
```

We have $x^* = [-0.6667, 1.333]'$ and $\|Ax^* - b\|_1 = 3$.

(b) $\|Ax-b\|_2$**:** This problem is equivalent to

$$\min_{x \in R^2} \frac{1}{2}\|Ax-b\|_2^2 \Leftrightarrow \min_{x \in R^2} \frac{1}{2} x'A'Ax - b'Ax.$$

Hence, can use the code given in Exercise 1 (b). Please note that the objective function value returned by the code is *not* $\|Ax - b\|_2$. We have $x^* = [-0.4552, 1.6621]'$ and $\|Ax^* - b\|_2 = 2.1367$. Of course, you can solve the *normal equation*, $x^* = (A'A)^{-1}A'b$ directly.

(c) $\|Ax-b\|_\infty$**:**

```
function [x, inf_error,residual ]=infapprox(A,b)
%
%Input A: mXn matrix
%         b: m-vector
%
%Solve the problem by LP
%Output: the approximate solution of Ax=b
%              inf_error =||Ax-b||_inf
%
 [m,n]=size(A); obj_p=[zeros(n,1); 1];
 H=[A -ones(m,1);-A -ones(m,1)]; h=[b;-b];
 [sol, one_error]=linprog(obj_p,H,h); x=sol(1:n);
 inf_error=sol((n+1)); residual=A*x-b;
```

We have $x^* = [-0.2, 1.8]'$ and $\|Ax^* - b\|_\infty = 1.4$.