Estimation of Errors-in-Variables Partially Linear Additive Models

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Supplementary Material

We introduce two additional lemmas that are frequently used in the proofs.

Lemma S.1. Assume that the conditions in Theorem 1 hold. Let f be a continuously differentiable function on $[0,1]$. Then, for $1 \leq j \leq p$ and $1 \leq k \leq d$, it holds that

$$\max_{1\leq i\leq n} \left| \int_{[0,1]^d} (X_j^i - \eta_j(\mathbf{z})) f(z_k) K_g^\star(\mathbf{z}, \mathbf{Z}^{*i})\, d\mathbf{z} \right| = O_p\left(\|f\|_\infty \cdot g^{-\beta}(n^\alpha + g^{-\beta}) \right), \tag{S.1}$$

for any arbitrarily small $\alpha > 0$.

Proof. For an additive function $\delta \in \mathcal{H}$ and a univariate function γ, define

$$T_{n,k}(x, \mathbf{u}; \delta, \gamma) = \int_{[0,1]^d} (x - \delta(\mathbf{z}))\gamma(z_k) K_g^\star(\mathbf{z}, \mathbf{u})\, d\mathbf{z}.$$

From $\eta_j(\mathbf{z}) = \sum_{\ell=1}^d \eta_{j,\ell}(z_\ell)$ and the normalization property of $K_g^\star$ that

$\int_0^1 K_g^\star(z_\ell, u)\, dz_\ell = 1$ for all $u \in [0,1]$, we get

$$\begin{aligned} T_{n,k}(X_j, \mathbf{u}; \eta_j, f) = X_j \int_0^1 f(z_k) K_g^\star(z_k, u_k)\, dz_k - \int_0^1 \eta_{j,k}(z_k) f(z_k) K_g^\star(z_k, u_k)\, dz_k \\ - \sum_{\ell \neq k}^{d} \int_0^1 \eta_{j,\ell}(z_\ell) K_g^\star(z_\ell, u_\ell)\, dz_\ell \cdot \int_0^1 f(z_k) K_g^\star(z_k, u_k)\, dz_k. \end{aligned} \tag{S.2}$$

For the first term on the right hand side of (S.2) we observe that

$$\begin{aligned} &\int_0^1 f(z_k) K_g^\star(z_k, u_k)\, dz_k \\ &\quad = \int_0^1 f(z_k) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it(z_k - u_k)} \cdot \frac{\phi_K(gt)\phi_K(gt; z_k)}{\phi_{V_k}(t)}\, dt\, dz_k \\ &\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itu_k} \frac{\phi_K(gt)^2}{\phi_{V_k}(t)} \phi_f(-t)\, dt \\ &\qquad + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itu_k} \frac{\phi_K(gt)}{\phi_{V_k}(t)} \int_{I_0^c} e^{-itz_k} \left(\phi_K(gt; z_k) - \phi_K(gt)\right) f(z_k)\, dz_k\, dt \\ &\quad =: S_{n,1}(u_k; f) + S_{n,2}(u_k; f), \end{aligned} \tag{S.3}$$

where $I_0 = [2g, 1-2g]$ and I_0^c denotes its complement in $[0,1]$. Since f is continuously differentiable, integration by parts entails that $|\phi_f(t)| \le c_0 \|f\|_\infty (1+|t|)^{-1}$ for some $c_0 > 0$. Moreover,

$$\left| \int_{I_0^c} e^{-itz_k} (\phi_K(gt; z_k) - \phi_K(gt)) f(z_k)\, dz_k \right| \le c_1 g \|f\|_\infty$$

for some constant $c_1 > 0$. It can be shown that

$$\begin{aligned}\sup_{u\in\mathbb{R}}\max_{1\le i\le n}|S_{n,1}(u;f)| &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\phi_K(gt)^2}{\phi_{V_k}(t)}\phi_f(-t)\right| dt \\ &\le (\text{const.})\|f\|_\infty\int_{-\infty}^{\infty}(1+|t|)^{\beta-1}\phi_K(gt)^2\ dt \\ &\le (\text{const.})\|f\|_\infty\cdot g^{-\beta}\int_0^{\infty}(g+t)^{\beta-1}\phi_K(t)^2\ dt \\ &\le (\text{const.})\|f\|_\infty\cdot g^{-\beta}.\end{aligned} \tag{S.4}$$

Similarly, we get

$$\begin{aligned}\sup_{u\in\mathbb{R}}\max_{1\le i\le n}|S_{n,2}(u;f)| &\le (\text{const.})\|f\|_\infty\cdot g\int_{-\infty}^{\infty}\left|\frac{\phi_K(gt)}{\phi_{V_k}(t)}\right| dt \\ &\le (\text{const.})\|f\|_\infty\cdot g^{-\beta}\int_0^{\infty}(g+t)^{\beta}\phi_K(gt)\ dt \\ &\le (\text{const.})\|f\|_\infty\cdot g^{-\beta}.\end{aligned} \tag{S.5}$$

The second term on the right hand side of (S.2) equals $S_{n,1}(u_k;\eta_{j,k}f)+S_{n,2}(u_k,\eta_{j,k}f)$. The third term is a sum of the products of $S_{n,A}(u_k;f)$ and $S_{n,B}(u_\ell;\eta_{j,\ell})$ for $A,B=1,2$. Thus, we may get similar bounds for these terms as in (S.4) and (S.5), and conclude that there exists a constant $c_2>0$ such that

$$\sup_{\mathbf{u}\in\mathbb{R}^d}|T_{n,k}(x,\mathbf{u};\eta_j,f)| \le c_2\|f\|_\infty\cdot g^{-\beta}\left(|x|+g^{-\beta}\right). \tag{S.6}$$

Since X_j are sub-Gaussian, by applying Markov inequality we may also get $\max_{1\le i\le n}|X_j^i| = O_p(n^\alpha)$ for any arbitrarily small $\alpha>0$. This with (S.6) completes the proof of Lemma S.1. $\square$

Lemma S.2. Under the conditions of Lemma S.1, there exists a constant $C > 0$ such that, for all $1 \le j \le p$ and $1 \le k \le d$,

$$\mathrm{E}\left| \int_{[0,1]^d} (X_j - \eta_j(\mathbf{z})) f(z_k) K_g^\star(\mathbf{z}, \mathbf{Z}^*) \, d\mathbf{z} \right|^2 \le C \|f\|_\infty^2 \, \tau(g;\beta)^2 \left(1 + \tau(g;\beta)^2\right).$$

Proof. Letting p_W denote the density of W for a random variable W, we get

$$\begin{aligned}
\mathrm{E}|S_{n,1}(Z_k^*; f)|^2 &\le \frac{\|p_{Z_k^*}\|_\infty}{4\pi^2} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} e^{itu} \frac{\phi_K(gt)^2}{\phi_{V_k}(t)} \phi_f(-t) \, dt \right|^2 du \\
&= \frac{\|p_{Z_k^*}\|_\infty}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\phi_K(gt)^2}{\phi_{V_k}(t)} \phi_f(-t) \right|^2 dt \\
&\le (\text{const.}) \|f\|_\infty^2 \int_{-\infty}^{\infty} (1 + |t|)^{2\beta - 2} \phi_K(gt)^4 \, dt \qquad \text{(S.7)} \\
&\le (\text{const.}) \|f\|_\infty^2 g^{1-2\beta} \int_0^{\infty} (g + |t|)^{2\beta-2} \phi_K(t)^4 \, dt \\
&= O(\|f\|_\infty^2 \, \tau(g;\beta)^2).
\end{aligned}$$

The first equality in (S.7) follows from the Plancherel identity. Similarly, we obtain

$$\begin{aligned}
\mathrm{E}|S_{n,2}(Z_k^*; f)|^2 &\le (\text{const.}) \|f\|_\infty^2 \, g^2 \int_{-\infty}^{\infty} \left| \frac{\phi_K(gt)}{\phi_{V_k}(t)} \right|^2 dt \\
&\le (\text{const.}) \|f\|_\infty^2 \, g^2 \int_0^{\infty} (1 + |t|)^{2\beta} \phi_K(gt)^4 \, dt \\
&\le (\text{const.}) \|f\|_\infty^2 \, g^{-1-2\beta} \int_0^{\infty} (g + t)^{2\beta} (1 + t)^{-2\lfloor \beta \rfloor - 2} \, dt \qquad \text{(S.8)} \\
&\le (\text{const.}) \|f\|_\infty^2 \, g^{1-2\beta} \\
&= O(\|f\|_\infty^2 \tau(g;\beta)^2).
\end{aligned}$$

Since $\eta_{j,\ell}$ are bounded, it also holds that, for $A, B = 1, 2$,

$$\mathrm{E}|S_{n,A}(Z_\ell^*; \eta_{j,\ell}) S_{n,B}(Z_k^*; f)|^2 = O(\|f\|_\infty^2 \, \tau(g;\beta)^4) \tag{S.9}$$

for all $1 \le \ell \ne k \le d$. This with (S.7) and (S.8) entails

$$\begin{aligned} \mathrm{E}|T_{n,k}(X_j, \mathbf{Z}^*; \eta_j, f)|^2 \le C_1 \cdot \mathrm{E}\left[X_j^2(|S_{n,1}(Z_k^*; f)|^2 + |S_{n,2}(Z_k^*; f)|^2)\right] \\ + C_2 \cdot \|f\|_\infty^2 \, \tau(g;\beta)^2 \left(1 + \tau(g;\beta)^2\right) \end{aligned} \tag{S.10}$$

for some C_1, $C_2 > 0$. On the other hand, since $\mathrm{E}(X_j^2 | Z_k = \cdot)$ is bounded on $[0, 1]$,

$$\begin{aligned} &\mathrm{E}\left(X_j^2 |S_{n,1}(Z_k^*; f)|^2\right) \\ &\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1 x_j^2 |S_{n,1}(u; f)|^2 p_{X_j, Z_k}(x_j, z_k) p_{V_k}(u - z_k)\, dz_k\, du\, dx_j \\ &\quad \le \|p_{Z_k^*}\|_\infty \int_{-\infty}^{\infty}\int_0^1 \mathrm{E}(X_j^2 | Z_k = z_k) \cdot |S_{n,1}(u; f)|^2 \, dz_k\, du \\ &\quad \le (\text{const.}) \|f\|_\infty^2 \, \tau(g;\beta)^2. \end{aligned} \tag{S.11}$$

Similarly, we get $\mathrm{E}\left(X_j^2 |S_{n,2}(Z_k^*; f)|^2\right) \le (\text{const.}) \|f\|_\infty^2 \, \tau(g;\beta)^2$. This with (S.10) and (S.11) establishes Lemma S.2. □

S.1 Proof of Lemma 1

We consider a class $\Delta_n \equiv \Delta_n(\kappa_1, \kappa_2)$ of differentiable functions f on $[0, 1]$ for absolute constants κ_1 and $\kappa_2 > 0$ such that

$$\sup_{0 \le u \le 1} |f(u)| \le \kappa_1 h, \quad \sup_{0 \le u \le 1} |f'(u)| \le \kappa_2. \tag{S.12}$$

From the results of Han and Park (2017) we get

$$\sup_{0\le u\le 1} |\hat{\eta}_{k,\ell}(u) - \eta_{k,\ell}(u)| = O_p\left(h + \sqrt{\frac{\log n}{nh^{1+2\beta}}}\cdot \tau(h;\beta)\right),$$

$$\sup_{0\le u\le 1} |\hat{\eta}'_{k,\ell}(u) - \eta'_{k,\ell}(u) - h\cdot \alpha^{(n)}_{k,\ell}(u)| = O_p\left(\sqrt{\frac{\log n}{nh^{3+2\beta}}}\cdot \tau(h;\beta)\right)$$

for deterministic sequences of bounded functions $\{\alpha^{(n)}_{k,\ell}\}_{n\ge 1}$. Thus, for given $1\le k\le p$ and $1\le \ell\le d$, $\hat{\eta}_{k,\ell}-\eta_{k,\ell}$ belongs to Δ_n with probability tending to 1, provided that $nh^{3+2\beta}/(\tau(h;\beta)^2\log n)$ is bounded away from zero as $n\to\infty$. Below, we prove that, for all $1\le j\le p$ and $1\le \ell\le d$,

$$\sup_{f\in\Delta_n}\left|n^{-1}\sum_{i=1}^{n} T_{n,\ell}(X^i_j,\mathbf{Z}^{*i};\eta_j,f)\right| = O\left(g^2h + n^{-1/2}hg^{-2\beta}\sqrt{\log n}\right), \quad \text{(S.13)}$$

which establishes Lemma 1.

Recall that $\tilde{K}_h(z,u) = K_h(z,\cdot) * K_h(u)$ and $\tilde{K}_h(\mathbf{z},\mathbf{u}) = \tilde{K}_h(z_1,u_1)\times\cdots\times\tilde{K}_h(z_d,u_d)$. We note that the unbiased scoring property, the second identity at (3.4), and the independence of V_j for $1\le j\le d$ conditional on $\mathbf{Z}$, implies $\mathrm{E}(K^*_g(\mathbf{z},\mathbf{Z}^*)|\mathbf{Z}) = \tilde{K}_g(\mathbf{z},\mathbf{Z})$. From this with the additivity of η_j and the normalization property of $K^\star_h$, we have

$$\begin{aligned}
&\mathrm{E}\, T_{n,k}(X_j,\mathbf{Z}^*;\eta_j,f)\\
&\quad = \int_0^1 f(u)\mathrm{E}\left[(\eta_{j,k}(Z_k)-\eta_{j,k}(u))\tilde{K}_g(u,Z_k)\right]du\\
&\qquad + \sum_{\ell\ne k}^{d}\int_{[0,1]^2} f(u)\mathrm{E}\left[(\eta_{j,\ell}(Z_\ell)-\eta_{j,\ell}(v))\tilde{K}_g(u,Z_k)\tilde{K}_g(v,Z_\ell)\right]du\,dv\\
&\quad =: (\mathrm{I}) + (\mathrm{II}).
\end{aligned}$$

Here we have used the property that $X_j - \eta_j(\mathbf{Z})$ is perpendicular to any additive function of $\mathbf{Z}$ in the sense that $\mathrm{E}(X_j - \eta_j(\mathbf{Z}))\,(q_1(Z_1) + \cdots + q_d(Z_d)) = 0$ for all square integrable univariate functions q_j. By the standard theory of kernel smoothing we get that, for $1 \le k \ne \ell \le d$,

$$
\begin{aligned}
&\mathrm{E}\left[(\eta_{j,\ell}(Z_\ell) - \eta_{j,\ell}(v))\tilde{K}_g(v, Z_\ell)\tilde{K}_g(u, Z_k)\right] \\
&\qquad = \int_{[0,1]^2} (\eta_{j,\ell}(\zeta_\ell) - \eta_{j,\ell}(v))\tilde{K}_g(v, \zeta_\ell)\tilde{K}_g(u, \zeta_k) p_{Z_k, Z_\ell}(\zeta_k, \zeta_\ell)\, d\zeta_\ell\, d\zeta_k \\
&\qquad = \int_0^1 (g \cdot \eta'_{j,\ell}(v)\mu_{1,\ell}(v) + O(g^2)) \cdot \tilde{K}_g(u, \zeta_k) \cdot (p_{Z_k, Z_\ell}(\zeta_k, v) + o(1))\, d\zeta_k \\
&\qquad = g \cdot \eta'_{j,\ell}(v)\mu_{1,\ell}(v)\mu_{0,k}(u) p_{Z_k, Z_\ell}(u, v) + O(g^2)
\end{aligned}
$$

uniformly for u and v in $[0, 1]$. Here, $\mu_{i,\ell}(v) = g^{-i} \int_0^1 (\zeta_\ell - v)^i \tilde{K}_g(v, \zeta_\ell)\, d\zeta_\ell$ for $i \ge 0$. Since $\mu_{1,\ell}(v) = 0$ for $v \in I_0$ and is bounded elsewhere in $[0, 1]$, from (S.12) we obtain

$$
(\mathrm{II}) = \sum_{\ell \ne k}^{d} \int_{[0,1]^2} f(u)\mu_{0,k}(u) \cdot g \cdot \eta'_{j,\ell}(v)\mu_{1,\ell}(v) p_{Z_k, Z_\ell}(u, v)\, du\, dv + O(g^2 h) = O(g^2 h)
$$

uniformly for $f \in \Delta_n$. Similarly, we also get $(\mathrm{I}) = O(g^2 h)$.

Now, let $\tilde{T}_{n,\ell}(X_j^i, \mathbf{Z}^{*i}; \eta_j, f) = T_{n,\ell}(X_j^i, \mathbf{Z}^{*i}; \eta_j, f) - \mathrm{E}\, T_{n,k\ell}(X_j, \mathbf{Z}^*; \eta_j, f)$. We claim

$$
\limsup_{n \to \infty} P\left(\sup_{f \in \Delta_n} \left| n^{-1} \sum_{i=1}^{n} \tilde{T}_{n,\ell}(X_j, \mathbf{Z}^*; \eta_j, f) \right| > n^{-1/2} h g^{-2\beta} \sqrt{\log n}\right) = 0, \tag{S.14}
$$

which completes the proof of Lemma 1.

To prove (S.14), we consider δ-covering sets of Δ_n, denoted by $\Delta_n(\delta)$, such that for any $f \in \Delta_n$, there exists $g \in \Delta_n(\delta)$ satisfying $\|f - g\|_\infty \leq \delta$. Specifically, we take $\delta = \kappa_1 h \cdot 2^{-N}$ for $N \geq 1$. Note that the entropy of $\Delta_n(\kappa_1 h \cdot 2^{-N})$ equals $C_0 2^N$ for some constant $C_0 > 0$. For a given $f \in \Delta_n$ and $N \geq 1$, we choose $f_N \in \Delta_n(\kappa_1 h \cdot 2^{-N})$ such that $\|f - f_N\|_\infty \leq \kappa_1 h \cdot 2^{-N}$ and set $f_0 \equiv 0$. Then, using the bound at (S.6) we get

$$|\tilde{T}_{n,\ell}(X_j, \mathbf{Z}^*; \eta_j, f) - \tilde{T}_{n,\ell}(X_j, \mathbf{Z}^*; \eta_j, f_N)| \leq C_1 \cdot 2^{-N} h g^{-\beta}(|X_j| + g^{-\beta}) \quad \text{(S.15)}$$

for some constants $C_1 > 0$. Set $\epsilon_n = n^{-1/2} h g^{-2\beta} \sqrt{\log n}$ and define

$$J_n = \min\left\{N \geq 1 : 2^{-N} \leq \frac{\epsilon_n}{2C_1 \cdot hg^{-\beta}(n^\alpha + g^{-\beta})}\right\},$$

where $\alpha > 0$ is arbitrarily small. Note that $\limsup_{n\to\infty} P(\max_{1\leq i\leq n} |X_j^i| > n^\alpha) = 0$ for any arbitrarily small α. From (S.15) it holds that

$$P\left(\sup_{f\in\Delta_n}\left|n^{-1}\sum_{i=1}^n \left(\tilde{T}_{n,\ell}(X_j^i, \mathbf{Z}^{*i}; \eta_j, f) - \tilde{T}_{n,\ell}(X_j^i, \mathbf{Z}^{*i}; \eta_j, f_{J_n})\right)\right| \leq \epsilon_n/2\right) \to 1$$

as $n \to \infty$. This basically enables us to restrict the supremum over $f \in \Delta_n$ in (S.14) to the one over $f \in \Delta_n(\kappa_1 h \cdot 2^{-J_n})$.

Now, let $\{\nu_N\}$ be a sequence of positive numbers such that $\sum_{N=1}^\infty \nu_N \leq 1$. An application of the chaining technique gives

$$\begin{aligned} &P\left(\sup_{f\in\Delta_n(\kappa_1 h\cdot 2^{-J_n})}\left|n^{-1}\sum_{i=1}^n \tilde{T}_{n,\ell}(X_j, \mathbf{Z}^*; \eta_j, f)\right| > \epsilon_n/2\right) \\ &\quad\leq \sum_{N=1}^{J_n} \exp(C_0 2^N) \qquad\qquad \text{(S.16)} \\ &\quad\times \sup{}^* P\left(\left|n^{-1}\sum_{i=1}^n \left(\tilde{T}_{n,\ell}(X_j^i, \mathbf{Z}^{*i}; \eta_j, f_N) - \tilde{T}_{n,\ell}(X_j^i, \mathbf{Z}^{*i}; \eta_j, f_{N-1})\right)\right| > \nu_N \varepsilon_n/2\right), \end{aligned}$$

where $\sup^*$ runs over all $f_N \in \Delta_n(\kappa_1 h \cdot 2^{-N})$ and $f_{N-1} \in \Delta_n(\kappa_1 h \cdot 2^{-N+1})$ satisfying $\|f_N - f_{N-1}\|_\infty \leq \kappa_1 h \cdot 2^{-N+1}$. Moreover, since there exist constants K_j and $\sigma_j > 0$ satisfying

$$\max_{1\leq i\leq n} K_j^2 \cdot \mathrm{E}\left(e^{|X_j^i|^2/K_j^2} - 1\right) \leq \sigma_j^2,$$

we may find an absolute constant $C_2 > 0$ such that, for all $f \in \Delta_n$,

$$\begin{aligned}\max_{1\leq i\leq n} K_{n,j}(f)^2 \cdot \mathrm{E}\left(e^{|T_{n,\ell}(X_j^i,\mathbf{Z}^{*i};\eta_j,f)|^2/K_{n,j}^2} - 1\right) &\leq \sigma_j^2 \cdot c_2^2\|f\|_\infty^2 \cdot g^{-2\beta}(K_j + g^{-\beta})^2 \\ &\leq C_2\|f\|_\infty^2 \cdot g^{-4\beta},\end{aligned}$$

where $K_{n,j}(f) = c_2\|f\|_\infty \cdot g^{-\beta}(K_j + g^{-\beta})$. For the above inequalities we have used

$$\begin{aligned}\sup_{\mathbf{u}\in\mathbb{R}^d} \frac{|T_{n,\ell}(x,\mathbf{u};\eta_j,f)|}{K_{n,j}(f)} &\leq \frac{c_2\|f\|_\infty \cdot g^{-\beta}\left(|x| + g^{-\beta}\right)}{c_2\|f\|_\infty \cdot g^{-\beta}(K_j + g^{-\beta})} \\ &\leq \frac{|x| + g^{-\beta}}{K_j + g^{-\beta}},\end{aligned}$$

which follows from (S.6). We take $\nu_N = 2^{-N/2}\sqrt{N}/C_3$ for sufficiently large C_3 so that $\sum_{N=1}^\infty \nu_N \leq 1$. Then, by applying an exponential inequality for sums of independent sub-Gaussian random variables we may prove that each summand on the right hand side of (S.16) is bounded by

$$\exp\left(C_0 2^N - \frac{n \cdot \epsilon_n^2 \nu_N^2/2^2}{2 \cdot C_2 \cdot 2^{-2N} h^2 g^{-4\beta}}\right) \leq \exp\left(-C_4 \cdot N \cdot \log n\right)$$

for sufficiently large n for some constant $C_4 > 0$. This proves

$$P\left(\sup_{f\in\Delta_n(\kappa_1 h\cdot 2^{-J_n})} \left| n^{-1}\sum_{i=1}^n \tilde{T}_{n,\ell}(X_j,\mathbf{Z}^*;\eta_j,f)\right| > \epsilon_n/2\right) \leq 2\exp\left(-C_4 \cdot \log n\right) \to 0$$

as $n \to \infty$ and completes the proof of Lemma 1.

S.2 Proofs of Lemma 2 and Lemma 3

Recall the definitions of $S_{n,1}$ and $S_{n,2}$ at (S.3). From the additivity of η_j and the normalization property of $K_g^\star$, we have

$$\begin{aligned}
&\mathrm{E}\left|\int_{[0,1]^d}(X_j-\eta_j(\mathbf{z}))K_g^\star(\mathbf{z},\mathbf{Z}^*)\,d\mathbf{z}\right|^2\\
&\quad=\mathrm{E}\left|X_j-\sum_{\ell=1}^d\left(S_{n,1}(Z_\ell^*;\eta_{j,\ell})+S_{n,2}(Z_\ell^*;\eta_{j,\ell})\right)\right|^2\\
&\quad\le(\text{const.})\left(\mathrm{E}X_j^2+\sum_{\ell=1}^d\left(\mathrm{E}\left|S_{n,1}(Z_\ell^*;\eta_{j,\ell})\right|^2+\mathrm{E}\left|S_{n,2}(Z_\ell^*;\eta_{j,\ell})\right|^2\right)\right)\\
&\quad=O(\tau(g;\beta)^2).
\end{aligned}$$

The last equality follows from (S.7) and (S.8). This establishes Lemma 2.

We now prove Lemma 3. From the normalization property of $K_g^\star$, we have

$$\begin{aligned}
&\int_{[0,1]^d}(X_j-\eta_j(\mathbf{z}))(X_k-\eta_k(\mathbf{z}))K_g^\star(\mathbf{z},\mathbf{Z}^*)\ d\mathbf{z}\\
&\quad=(X_j-\eta_j(\mathbf{Z}))(X_k-\eta_k(\mathbf{Z}))+(X_j-\eta_j(\mathbf{Z}))\int_{[0,1]}(\eta_k(\mathbf{Z})-\eta_k(\mathbf{z}))K_g^\star(\mathbf{z},\mathbf{Z}^*)\ d\mathbf{z}\\
&\qquad+\int_{[0,1]^d}(\eta_j(\mathbf{Z})-\eta_j(\mathbf{z}))K_g^\star(\mathbf{z},\mathbf{Z}^*)\ d\mathbf{z}\cdot(X_k-\eta_k(\mathbf{Z}))\\
&\qquad+\int_{[0,1]^d}(\eta_j(\mathbf{Z})-\eta_j(\mathbf{z}))(\eta_k(\mathbf{Z})-\eta_k(\mathbf{z}))K_g^\star(\mathbf{z},\mathbf{Z}^*)\ d\mathbf{z}\\
&\quad=:A_{jk}+B_{jk}+C_{jk}+D_{jk}.
\end{aligned}\tag{S.17}$$

Clearly $\mathrm{E}A_{jk}^2=O(1)$ since $\mathrm{E}\,|X_jX_k|^2<\infty$. Also, writing $\phi_{D_{j,\ell}}(t;u)=$

$\int_0^1 e^{itv}(\eta_{j,\ell}(u) - \eta_{j,\ell}(v))\, dv$, we note that

$$\begin{aligned}
&\int_0^1 (\eta_{k,\ell}(u) - \eta_{k,\ell}(v)) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it(v-w)} \cdot \frac{\phi_K(gt)\phi_K(gt;v)}{\phi_{V_\ell}(t)}\, dt\, dv \\
&\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itw} \frac{\phi_K(gt)^2}{\phi_{V_\ell}(t)} \phi_{D_{j,\ell}}(-t;u)\, dt \\
&\qquad + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itw} \frac{\phi_K(gt)}{\phi_{V_\ell}(t)} \int_{I_0^c} e^{-itv} \left(\phi_K(gt;v) - \phi_K(gt)\right) (\eta_{k,\ell}(u) - \eta_{k,\ell}(v))\, dv\, dt \\
&\quad =: W_{n,1}(u,w;\eta_{j,\ell}) + W_{n,2}(u,w;\eta_{j,\ell}).
\end{aligned} \tag{S.18}$$

Thus,

$$\begin{aligned}
\mathrm{E}|B_{jk}|^2 &= \mathrm{E}|C_{kj}|^2 \\
&= \mathrm{E}\left| (X_j - \eta_j(\mathbf{Z})) \sum_{\ell=1}^{d} \int_0^1 (\eta_{k,\ell}(Z_\ell) - \eta_{k,\ell}(v)) K_g^\star(v, Z_\ell^*)\ dv \right|^2 \\
&\le (\text{const.}) \sum_{\ell=1}^{d} \mathrm{E}\left((X_j^2 + 1) \left| \int_0^1 (\eta_{j,\ell}(Z_\ell) - \eta_{j,\ell}(v)) K_h^\star(v, Z_\ell^*)\ dv \right|^2 \right) \\
&\le (\text{const.}) \sum_{\ell=1}^{d} \mathrm{E}\left((X_j^2 + 1) \left| W_{n,1}(Z_\ell, Z_\ell^*; \eta_{j,\ell}) + W_{n,2}(Z_\ell, Z_\ell^*; \eta_{j,\ell}) \right|^2 \right).
\end{aligned} \tag{S.19}$$

Since $\mathrm{E}(X_j^2 | Z_\ell = \cdot)$ is bounded, following the lines in (S.11) we may prove $\mathrm{E}|B_{jk}|^2 = \mathrm{E}|C_{kj}|^2 = O(\tau_n(g;\beta)^2)$. Finally, we note that

$$\begin{aligned}
D_{jk} &= \sum_{\ell=1}^{d} (\tilde{W}_{n,1}(Z_\ell, Z_\ell^*; \eta_{j,\ell}) + \tilde{W}_{n,2}(Z_\ell, Z_\ell^*; \eta_{j,\ell})) \\
&\quad + \sum_{\ell \ne \ell'}^{d} (W_{n,1}(Z_\ell, Z_\ell^*; \eta_{j,\ell}) + W_{n,2}(Z_\ell, Z_\ell^*; \eta_{j,\ell}))(W_{n,1}(Z_{\ell'}, Z_{\ell'}^*; \eta_{j,\ell'}) + W_{n,2}(Z_{\ell'}, Z_{\ell'}^*; \eta_{j,\ell'})),
\end{aligned}$$

where $\tilde{W}_{n,1}(u,w;\eta_{j,\ell})$ and $\tilde{W}_{n,2}(u,w;\eta_{j,\ell})$ are defined as $W_{n,1}(u,w;\eta_{j,\ell})$ and $W_{n,2}(u,w;\eta_{j,\ell})$, respectively, with $(\eta_{j,\ell}(u) - \eta_{j,\ell}(v))^2$ replacing $\eta_{j,\ell}(u) - \eta_{j,\ell}(v)$ in the definition (S.18). By using similar arguments as those in (S.7)–(S.9), we get

$$\begin{aligned} \mathrm{E}|\tilde{W}_{n,A}(Z_\ell, Z_\ell^*; \eta_{j,\ell})|^2 &= O(\tau_n(g;\beta)^2) \\ \mathrm{E}|W_{n,A}(Z_\ell, Z_\ell^*; \eta_{j,\ell}) W_{n,B}(Z_{\ell'}, Z_{\ell'}^*; \eta_{j,\ell'})| &= O(\tau_n(g;\beta)^4) \end{aligned}$$

for all $A, B = 1, 2$. This with (S.19) completes the proof of Lemma 3.

S.3 Proof of Lemma 4

Proof of Lemma 4 is simpler than that of Lemma 1. In the proof of Lemma 1, we simply put $\eta_j \equiv 0$. Note that $\mathrm{E}\, T_{n,\ell}(U_j, \mathbf{Z}^*; 0, f) = 0$. Thus, instead of (S.15) we have

$$|T_{n,\ell}(U_j, \mathbf{Z}^*; 0, f) - T_{n,\ell}(U_j, \mathbf{Z}^*; 0, f_N)| \leq C_3 \cdot 2^{-N} h g^{-\beta} |U_j|$$

for some constant $C_3 > 0$. Also, we change ϵ_n to $\tilde{\epsilon}_n = n^{-1/2} h g^{-\beta} \sqrt{\log n}$ and J_n to

$$\tilde{J}_n = \min \left\{ N \geq 1 : 2^{-N} \leq \frac{\tilde{\epsilon}_n}{2C_3 \cdot h g^{-\beta} \cdot n^\alpha} \right\}$$

for an arbitrarily small $\alpha > 0$. Then, following the lines in the proof of Lemma 1, we get

$$\limsup_{n\to\infty} P\left(\sup_{f \in \Delta_n} \left| \frac{1}{n} \sum_{i=1}^n T_{n,\ell}(U_j, \mathbf{Z}^*; 0, f) \right| > \tilde{\epsilon}_n \right) = 0.$$

This completes the proof of Lemma 4.