Semi-parametric prediction intervals in small areas when auxiliary data are measured with error

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Supplementary Material

S1 Conditional distribution of T

We have

$$
\begin{aligned}
f_{T\,|\,Q,W,Y}(t\,|\,q,w,y) &= \int f_{T\,|\,Q,W,X,Y}(t\,|\,q,w,x,y)\, f_{X\,|\,Q,W,Y}(x\,|\,q,w,y)\,dx \\
&= \int f_{T\,|\,Q,X,Y}(t\,|\,q,x,y)\, f_{X\,|\,Q,W,Y}(x\,|\,q,w,y)\,dx\,. \qquad \text{(S1.1)}
\end{aligned}
$$

Then, using basic properties of conditional densities, we note that

$$f_{T\,|\,Q,X,Y}(t\,|\,q,x,y) = f_\epsilon(y-t)\, f_V(t-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q)\big/ f_{V+\epsilon}(y-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q)\,,$$

$$f_{X\,|\,Q,W,Y}(x\,|\,q,w,y) = \frac{f_{V+\epsilon}(y-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q)\, f_X(x)\, f_U(w-x)\, f_Q(q)}{f_{Q,W,Y}(q,w,y)}\,,$$

$$f_{Q,W,Y}(q,w,y) = f_Q(q)\int f_{V+\epsilon}\big(y-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q\big)\, f_U(w-x)\, f_X(x)\, dx\,.$$

Hence,

$$f_{T\,|\,Q,X,Y}(t\,|\,q,x,y)\, f_{X\,|\,Q,W,Y}(x\,|\,q,w,y) = \frac{f_\epsilon(y-t)\, f_V(t-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q)\, f_X(x)\, f_U(w-x)}{\int f_{V+\epsilon}(y-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q) f_U(w-x)\, f_X(x)\, dx}\,. \tag{S1.2}$$

Combining (S1.1) and (S1.2), and recalling that ϵ has a symmetric distribution, we deduce that

$$f_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \frac{f_\epsilon(t-y)\int f_V(t-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q)\, f_X(x)\, f_U(w-x)\, dx}{\int f_{V+\epsilon}(y-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q)\, f_U(w-x)\, f_X(x)\, dx}\,. \tag{S1.3}$$

S2 Estimating the unknown parameters in (2.2)

Let $\sigma_U^2 = \mathrm{var}(U)$, $\sigma_W^2 = \mathrm{var}(W)$ and $\sigma_X^2 = \mathrm{var}(X)$. We can estimate the unknown parameters using standard approaches employed in classical measurement error linear models (see e.g. Fuller, 2009 and Buonaccorsi, 2010). Like there, since $\sigma_W^2 = \sigma_X^2 + \sigma_U^2$ and σ_U^2 is known, we start by estimating σ_X^2 by $\hat\sigma_X^2 = \max\big(0, \hat\sigma_W^2 - \sigma_U^2\big)$, where $\hat\sigma_W^2 = n^{-1}\sum_{j=1}^n (W_j-\bar W)^2$ and $\bar W = n^{-1}\sum_j W_j$. Then, letting $Z_j = (1, W_j, Q_j^{\mathrm{T}})^{\mathrm{T}}$ and $\mathbf{Z} = (Z_1,\ldots,Z_n)^{\mathrm{T}}$, and defining the $(p+2)\times(p+2)$ matrix $\Sigma_U = (\Sigma_{U,i,j})_{i,j=1,\ldots,p+2}$

to be zero everywhere except for the (2,2)th component, which is equal to σ_U^2, we take $\widehat{M} = n^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{Z} - \Sigma_U$. Then, letting $\bar{Y} = n^{-1}\sum_j Y_j$, $T_{WY} = n^{-1}\sum_{j=1}^n W_j\,Y_j$, $T_{QY} = n^{-1}\sum_{j=1}^n Q_j\,Y_j$, and assuming that $\det \widehat{M} > 0$, we estimate β_0, β_1 and β_2 by

$$\left(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2^{\mathrm{T}}\right)^{\mathrm{T}} = \widehat{M}^{-1}\left(\bar{Y}, T_{WY}, T_{QY}^{\mathrm{T}}\right)^{\mathrm{T}}. \qquad \text{(S2.1)}$$

Finally, to estimate σ_V^2, let $\bar{\tau} = n^{-1}\sum_j \tau_j$ and $\hat{\sigma}_Y^2 = n^{-1}\sum_{j=1}^n (Y_j - \bar{Y})^2$. It follows from (2.1) that $\mathrm{var}(Y_j) = \beta_1^2\,\sigma_X^2 + \beta_2^{\mathrm{T}}\,\Sigma_Q\,\beta_2 + \sigma_V^2 + \tau_j$, which suggests using

$$\hat{\sigma}_V^2 = \max\left\{0, \hat{\sigma}_Y^2 - \hat{\beta}_1^2\,\hat{\sigma}_X^2 - \hat{\beta}_2^{\mathrm{T}}\,\widehat{\Sigma}_Q\,\hat{\beta}_2 - \bar{\tau}\right\}. \qquad \text{(S2.2)}$$

In our numerical examples in Section 4, our sample sizes are small, and in that case, Fuller (2009) and Buonaccorsi (2010) noted that, although it is a covariance matrix, the matrix $\widehat{M}$ is not always invertible. To overcome this difficulty, we apply to it the same correction as in page 121 of Buonaccorsi (2010). A similar problem arises with $\hat{\sigma}_V^2$, and we overcome it by applying the bagging technique described in Section 2.2 of Delaigle and Hall (2011).

The next theorem establishes root-n consistency of the estimators $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\sigma}_V^2$, defined at (S2.1) and (S2.2). The proof follows the arguments in Fuller (2009) and thus is omitted.

Theorem 1. *If the random quantities Q, U, V and X all have finite fourth moments, if $M = E\big\{(1, X, Q^{\mathrm{T}})^{\mathrm{T}}(1, X, Q^{\mathrm{T}})\big\}$ is nonsingular and $\sigma_V^2\,\sigma_X^2 \neq 0$, then $\hat{\beta}_0 - \beta_0$, $\hat{\beta}_1 - \beta_1$, $\|\hat{\beta}_2 - \beta_2\|$ and $\hat{\sigma}_V^2 - \sigma_V^2$ all equal $O_p(n^{-1/2})$ as n increases. Moreover, as $n \to \infty$ we* *have*

$$n^{1/2}\Big\{\big(\hat\beta_0,\hat\beta_1,\hat\beta_2^{\mathrm{T}}\big)^{\mathrm{T}}-\big(\beta_0,\beta_1,\beta_2^{\mathrm{T}}\big)^{\mathrm{T}}\Big\}\xrightarrow{D}N(0,\Sigma)\,,$$

where, using the notation $\tau^*=\lim_{n\to\infty}\bar\tau$ *and* $\sigma^2_{\mathrm{err}}=\tau^*+\sigma_V^2+\beta_1^2\sigma_U^2$*,*

$$\Sigma=\sigma^2_{\mathrm{err}}M^{-1}+\big\{\beta_1^2\mathrm{var}(U^2)+(\tau^*+\sigma_V^2)\sigma_U^2\big\}M^{-1}\begin{pmatrix}0&0&0_{1\times p}\\ 0&1&0_{1\times p}\\ 0&0&0_{p\times p}\end{pmatrix}M^{-1}\,.$$

S3 Discussion of the conditions in Section 3.1

It can be proved from the definition of χ, and the first assumption in (3.1)(ii), that ρ_j and ρ_j' are both bounded on any compact interval. If $\phi_U(t)$ is asymptotic to a constant multiple of t^{-2r} as $|t|\to\infty$, as it would be if (for example) the distribution of U were that of an r-fold convolution of Laplace-distributed random variables, then (3.1)(iv) is readily proved. When (3.1) holds, integrations by parts (see Appendix S5) can be used to prove that, as $|t|\to\infty$,

$$\rho_1(t)=\beta(t)^{-1}\left[\cos(tw)\,s_k+\frac{\sin t}{t}\,\big\{\Psi'_{kr}(w-)-\Psi'_{kr}(w+)\big\}\right]+O\big(t^{-2}\big)\,,\quad \text{(S3.1)}$$

$$\rho_2(t)=\beta(t)^{-1}\left[\sin(tw)\,s_k-\frac{\cos t}{t}\,\big\{\Psi'_{kr}(w-)-\Psi'_{kr}(w+)\big\}\right]+O\big(t^{-2}\big)\,,\quad \text{(S3.2)}$$

and so $|\rho_j|$ is bounded on $\mathbb{R}$. Moreover, in the Laplace case, (S3.1) and (S3.2) continue to hold if both sides of each equation are differentiated naively with respect to t. Therefore, in this case, $|\rho_j'|$ is bounded on $\mathbb{R}$, establishing the last part of (3.1)(ii).

Also, (3.1)(i) holds if the distribution of U is an r-fold convolution of Laplace distributions.

S4 Theorem 2

The methods used to derive Theorem 1 can be employed to show that, under the same conditions, all partial derivatives of $\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$ with respect to t converge at the same rate to the respective derivatives of $F_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$. In particular, if for each integer $r \geq 0$ we define

$$\begin{aligned} \widehat{F}^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) &= \left(\frac{\partial}{\partial t}\right)^r \widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y)\,, \\ F^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) &= \left(\frac{\partial}{\partial t}\right)^r F_{T\,|\,Q,W,Y}(t\,|\,q,w,y)\,, \end{aligned}$$

then the following result holds.

Theorem 2. *Assume the conditions imposed in Theorem 1, and that* (3.1)–(3.3) *and* (3.5) *hold, and let* $r \geq 0$ *be an integer. Then: (i) For each real* t *and* y*, and each* $q \in \mathrm{I\!R}^p$*,*

$$\widehat{F}^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) - F^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \begin{cases} O_p\big\{(nh)^{-1/2} + h^\ell\big\} & \textit{if } w = 0 \\ O_p\big(n^{-1/2} + h^\ell\big) & \textit{if } w \neq 0\,; \end{cases} \tag{S4.1}$$

and (ii) For each $\eta > 0$,

$$\widehat{F}^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) - F^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \begin{cases} O_p\Big\{ \big(n^{1-\eta}h\big)^{-1/2} + h^{\ell} \Big\} & \text{if } w = 0 \\ O_p\big(n^{-(1-\eta)/2} + h^{\ell}\big) & \text{if } w \neq 0\,, \end{cases}$$

uniformly in t, q *and* y *in any compact subsets of their respective domains, where in the case* $w = 0$ *we ask in addition that* $n^{1-\eta}h \to \infty$.

The methods employed to establish these results are similar to those used to derive Theorem 1. The reason the convergence rates of estimators of the distribution function derivatives $F^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$ do not depend on r is that the derivatives have the same form as the original function estimators. For example, if we define

$$\Psi^{(r)}_k(t,y,q,w) = \left(\frac{\partial}{\partial t}\right)^r \Psi_k(t,y,q,w)\,, \quad \widehat{\Psi}^{(r)}_k(t,y,q,w) = \left(\frac{\partial}{\partial t}\right)^r \widehat{\Psi}_k(t,y,q,w)\,,$$

then it can be proved that $\widehat{\Psi}^{(r)}_k(t,y,q,w) = \Psi^{(r)}_k(t,y,q,w) + O_p\{(nh)^{-1/2} + h^{\ell}\}$ for each (t,y,q,w), each $r \geq 0$ and $k = 1,2$. Therefore, using standard formulae for derivatives, such as

$$\widehat{F}^{(2)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \frac{\widehat{\Psi}'_1(t,y,q,w)\,\widehat{\Psi}_2(t,y,q,w) - \widehat{\Psi}_1(t,y,q,w)\,\widehat{\Psi}'_2(t,y,q,w)}{\widehat{\Psi}_2(t,y,q,w)^2}$$

(compare (2.7)), it can be proved that (S4.1) holds.

S5 Proof of (S3.1) and (S3.2)

Define

$$\begin{aligned}\gamma_r(t) &= \int \Psi_{kr}(x)\left(\frac{\partial}{\partial x}\,e^{itx}\right)dx = -\int e^{itx}\,d\Psi_{kr}(x)\\ &= -\left\{e^{itw}\,s_k + \left(\int_{-\infty}^{w-} + \int_{w+}^{\infty}\right) e^{itx}\,\Psi'_{kr}(x)\,dx\right\} = -\left\{e^{itw}\,s_k + \delta_r(t)\right\}\end{aligned}$$

where, in view of (3.1)(i), the function δ_r satisfies $\sup_{-\infty<t<\infty}|\delta_r(t)| < \infty$. Recall that $\chi_1 = \Re\,\chi$ and $\chi_2 = \Im\,\chi$, and put $\gamma_{r1} = \Re\,\gamma_r$, $\gamma_{r2} = \Im\,\gamma_r$, $\alpha_1(t) = \cos(tw) + \Re\,\delta_r(t)$ and $\alpha_2(t) = \sin(tw) + \Im\,\delta_r(t)$. In this notation,

$$\rho_j(t) = \frac{\chi_j(t)}{\phi_U(t)} = -\frac{\gamma_{rj}(t)}{t^{2r}\,\phi_U(t)} = \frac{\alpha_j(t)}{\beta(t)}\,. \tag{S5.1}$$

Using (3.1)(i) it can be shown that

$$\begin{aligned}-\gamma_r(t) &= e^{itw}\,s_k + \frac{1}{it}\left(\int_{-\infty}^{w-} + \int_{w+}^{\infty}\right)\Psi'_{kr}(x)\left(\frac{\partial}{\partial x}\,e^{itx}\right)dx\\ &= e^{itw}\,s_k + (it)^{-1}\,e^{itw}\left\{\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right\}\\ &\qquad -\frac{1}{it}\left(\int_{-\infty}^{w-} + \int_{w+}^{\infty}\right)\Psi''_{kr}(x)\;e^{itx}\,dx\\ &= e^{itw}\,s_k + (it)^{-1}\,e^{itw}\left\{\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right\}\\ &\qquad -\frac{1}{(it)^2}\left(\int_{-\infty}^{w-} + \int_{w+}^{\infty}\right)\Psi''_{kr}(x)\left(\frac{\partial}{\partial x}\,e^{itx}\right)dx\\ &= e^{itw}\,s_k + (it)^{-1}\,e^{itw}\left\{\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right\} + O\big(t^{-2}\big)\,.\end{aligned}$$

Hence, the functions α_1 and α_2 can be written as

$$\alpha_1(t) = \cos(tw)\,s_k + \frac{\sin t}{t}\left\{\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right\} + O\big(t^{-2}\big)\,, \tag{S5.2}$$

$$\alpha_2(t) \;=\; \sin(tw)\, s_k - \frac{\cos t}{t}\, \big\{\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\big\} + O\big(t^{-2}\big)\,, \qquad \text{(S5.3)}$$

where the remainders are of that order as $|t| \to \infty$; and more simply, $|\rho_1|$ and $|\rho_2|$ are bounded uniformly on $\mathbb{R}$. The desired results (S3.1) and (S3.2) follow from (S5.2) and (S5.3), respectively.

S6 Proof of (6.11)

Recall that χ_j, and hence also $\rho_j = \phi_j/\phi_U$, depends on k, which equals 1 or 2, and that $\phi_{W0} = \Re\,\phi_W$ or $\Im\,\phi_W$. Therefore $R_1(h)$, at (6.10), depends on j_1, j_2 and k. In each step the quantities $B_1, B_2, \ldots$ denote generic constants.

Step 1: Difference between R_1 and R_2; see (S6.1). Define

$$R_2(h) \;=\; \frac{1}{h} \int_{t_1\,:\,h<|t_1|<1} \rho_{j_1}(t_1/h)\, \phi_K(t_1)\, dt_1 \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t - t_1/h)\}\, \phi_K(ht - t_1)\, dt\,.$$

Then,

$$|R_1(h) - R_2(h)| \le \frac{B_1}{h} \int_{-h}^{h} |\phi_K(t_1)|\, dt_1 \int_{-\infty}^{\infty} |\phi_{W0}(t)|\, dt \le \frac{B_2}{h} \int_{-h}^{h} dt_1 = 2\, B_2\,. \quad \text{(S6.1)}$$

Step 2: Difference between R_2 and R_3; see (S6.3). In view of (S5.1) to (S5.3) in Appendix S5 we can write

$$\rho_j(t) = \beta(t)^{-1} \Big[\mathrm{cs}_{j1}(tw)\, s_k + (-1)^{j+1}\, \frac{\mathrm{cs}_{j2}\, t}{t}\, \big\{\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\big\}\Big] + O\big(t^{-2}\big)\,, \quad \text{(S6.2)}$$

where $(\mathrm{cs}_{j1}, \mathrm{cs}_{j2}) = (\cos, \sin)$ or $(\sin, \cos)$ according as $j = 1$ or 2, respectively. In this notation, define

$$\begin{aligned} R_3(h) &= \frac{1}{h}\int_{t_1\,:\,h<|t_1|<1} \beta(t_1/h)^{-1}\Big[\mathrm{cs}_{j_1 1}(t_1 w/h)\, s_k \\ &\qquad + (-1)^{j_1+1}\,\frac{\mathrm{cs}_{j_1 2}\,(t_1/h)}{t_1/h}\,\big\{\Psi'_{kr}(w-)-\Psi'_{kr}(w+)\big\}\Big]\,\phi_K(t_1)\,dt_1 \\ &\qquad \times \int \phi_{W0}(t)\,\rho_{j_2}\{\pm(t-t_1/h)\}\,\phi_K(ht-t_1)\,dt\,. \end{aligned}$$

Then,

$$|R_2(h)-R_3(h)| \le \frac{B_3}{h}\int_h^1 (t_1/h)^{-2}\,dt_1 \int_{-\infty}^{\infty}|\phi_{W0}(t)|\,dt \le B_4\,h\int_h^1 t_1^{-2}\,dt_1 \le B_4\,. \tag{S6.3}$$

Step 3: Difference between R_3 and R_4; see (S6.5). For b_1 as in (3.1), define

$$\begin{aligned} R_4(h) &= \frac{1}{h}\int_{t_1\,:\,h<|t_1|<1} \Big[\beta(t_1/h)^{-1}\,\mathrm{cs}_{j_1 1}(t_1 w/h)\, s_k \\ &\qquad + (-1)^{j_1+1}\,b_1^{-1}\,\frac{\mathrm{cs}_{j_1 2}\,(t_1/h)}{t_1/h}\,\big\{\Psi'_{kr}(w-)-\Psi'_{kr}(w+)\big\}\Big]\,\phi_K(t_1)\,dt_1 \\ &\qquad \times \int \phi_{W0}(t)\,\rho_{j_2}\{\pm(t-t_1/h)\}\,\phi_K(ht-t_1)\,dt\,. \end{aligned}$$

Now,

$$\big|\beta(t)^{-1}-b_1^{-1}\big| \le B_5\,(1+|t|)^{-b_2} \tag{S6.4}$$

for all $|t|>1$, where $B_5>0$ is a constant. See (3.1)(iv). Hence,

$$\begin{aligned} |R_3(h)-R_4(h)| &\le \frac{B_5}{h}\int_{t_1\,:\,h<|t_1|<1} (1+|t_1/h|)^{-b_2}\,|t_1/h|^{-1}\,\big|\Psi'_{kr}(w-)-\Psi'_{kr}(w+)\big| \\ &\qquad \times |\phi_K(t_1)|\,dt_1 \int \big|\phi_{W0}(t)\,\rho_{j_2}\{\pm(t-t_1/h)\}\,\phi_K(ht-t_1)\big|\,dt \\ &\le \frac{B_6}{h}\int_h^1 (t_1/h)^{-(1+b_2)}\,dt_1 \le B_7\,. \end{aligned} \tag{S6.5}$$

Step 4: Difference between R_4 and R_5; see (S6.11). Using (3.1)(ii), (S3.1), (S3.2), (S6.2) and (S6.4) it can be proved that, for constants $B_8, B_9 > 0$, and for all $|t| > 1$,

$$\big|\rho_j(t) - b_1^{-1}\,\mathrm{cs}_{j1}(tw)\, s_k\big| \le B_8\,(1+|t|)^{-B_9}\,. \tag{S6.6}$$

Let

$$\begin{aligned} R_5(h) &= \frac{s_k}{h}\int_{t_1\,:\,h<|t_1|<1} \beta(t_1/h)^{-1}\,\mathrm{cs}_{j_1 1}(t_1 w/h)\,\phi_K(t_1)\,dt_1 \\ &\qquad\qquad \times \int \phi_{W0}(t)\,\rho_{j_2}\{\pm(t-t_1/h)\}\,\phi_K(ht-t_1)\,dt\,. \end{aligned}$$

Then,

$$\begin{aligned} \big|b_1\{R_4(h)-R_5(h)\}\big| &= \frac{1}{h}\,\big|\Psi'_{kr}(w-)-\Psi'_{kr}(w+)\big|\,\bigg|b_1\int_{t_1\,:\,h<|t_1|<1} \frac{\mathrm{cs}_{j_1 2}\,(t_1/h)}{\beta(t_1/h)\,t_1/h}\,\phi_K(t_1)\,dt_1 \\ &\qquad\qquad \times \int \phi_{W0}(t)\,\rho_{j_2}\{\pm(t-t_1/h)\}\,\phi_K(ht-t_1)\,dt\bigg| \\ &\le h^{-1}\,\big|\Psi'_{kr}(w-)-\Psi'_{kr}(w+)\big|\,\{S_1(h)+S_2(h)\}\,, \end{aligned} \tag{S6.7}$$

where, in view of (3.1)(ii), (3.1)(iii), (3.1)(iv), (S3.1), (S3.2) and (S6.6),

$$\begin{aligned} S_1(h) &= b_1^{-1}\,\bigg|s_k\int_{t_1\,:\,h<|t_1|<1} \frac{\mathrm{cs}_{j_1 2}\,(t_1/h)}{t_1/h}\,\phi_K(t_1)\,dt_1 \\ &\qquad\qquad \times \int \phi_{W0}(t)\,\mathrm{cs}_{j_2 1}\{\pm(t-t_1/h)\}\,\phi_K(ht-t_1)\,dt\bigg|\,, \end{aligned} \tag{S6.8}$$

$$\begin{aligned} S_2(h) &= B_{10}\int_{t_1\,:\,h<|t_1|<1} |t_1/h|^{-1}\,|\phi_K(t_1)|\,dt_1 \int |\phi_{W0}(t)|\,(1+|t-t_1/h|)^{-B_9}\,dt \\ &\le B_{10}B_{11}\,h\int_{t_1\,:\,h<|t_1|<1} |t_1|^{-1}\,|\phi_K(t_1)|\,dt_1 \int (1+|t|)^{-B_{13}}\,(1+|t_1/h|)^{-B_{12}}\,dt \\ &\le B_{14}\,h^{1+B_{12}}\int_{t_1\,:\,h<|t_1|<1} |t_1|^{-1-B_{12}}\;dt_1 \le B_{15}\,h\,. \end{aligned} \tag{S6.9}$$

Here we have used the fact that there exist constants B_{11}, $B_{12} > 0$ and $B_{13} > 1$ so that, for all t and all t_1,

$$(1+|t|)^{-C_2}(1+|t-t_1/h|)^{-B_9} \le B_{11}(1+|t|)^{-B_{13}}(1+|t_1/h|)^{-B_{12}}.$$

We claim that

$$S_1(h) \le B_{18}\, h\,. \tag{S6.10}$$

To appreciate why, assume for the sake of definiteness that $j_1 = j_2 = 1$. Then, $\mathrm{cs}_{j_1 2} = \sin$ and $\mathrm{cs}_{j_2 1} = \cos$, and so

$$\mathrm{cs}_{j_2 1}\{\pm(t-t_1/h)\} = \cos(t)\,\cos(t_1/h) \mp \sin(t)\,\sin(t_1/h)\,,$$

whence by (S6.8),

$$\begin{aligned} &b_1\, S_1(h) \\ &\quad = \Bigg| \int_{t_1\,:\,h<|t_1|<1} \frac{\sin(t_1/h)\,\cos(t_1/h)}{t_1/h}\,\phi_K(t_1)\,dt_1 \int \phi_{W0}(t)\,\cos(t)\,\phi_K(ht-t_1)\,dt \\ &\qquad\qquad - \int_{t_1\,:\,h<|t_1|<1} \frac{\sin(t_1/h)\,\sin(t_1/h)}{t_1/h}\,\phi_K(t_1)\,dt_1 \int \phi_{W0}(t)\,\sin(t)\,\phi_K(ht-t_1)\,dt \Bigg|\,. \end{aligned}$$

The two terms on the right-hand side can be bounded using similar arguments. In either case the integral over $h < |t_1| < 1$ is broken up into two parts, addressing respectively $h < t_1 < 1$ and $-1 < t_1 < -h$. We illustrate by treating the first term on the right-hand side, and the first of the two integrals, which we multiply here by $2/h$:

$$\frac{2}{h}\left| \int_h^1 \frac{\sin(t_1/h)\,\cos(t_1/h)}{t_1/h}\,\phi_K(t_1)\,dt_1 \int \phi_{W0}(t)\,\cos(t)\,\phi_K(ht-t_1)\,dt \right|$$

$$\begin{aligned}
&= \frac{1}{h}\left|\int_h^1 \frac{\sin(2t_1/h)}{t_1/h}\,\phi_K(t_1)\,dt_1\int \phi_{W0}(t)\,\cos(t)\,\phi_K(ht-t_1)\,dt\right| \\
&= \left|\int_h^1 \phi_K(t_1)\left\{\frac{\partial}{\partial t_1}\,\xi_1(t_1/h)\right\}dt_1\int \phi_{W0}(t)\,\cos(t)\,\phi_K(ht-t_1)\,dt\right| \\
&\leq B_{19}+\int_h^1 |\phi_K'(t_1)\,\xi_1(t_1/h)|\,dt_1\int |\phi_{W0}(t)\,\phi_K(ht-t_1)|\,dt \\
&\qquad +\int_h^1 |\phi_K(t_1)\,\xi_1(t_1/h)|\,dt_1\int |\phi_{W0}(t)\,\phi_K'(ht-t_1)|\,dt\leq B_{20}\,,
\end{aligned}$$

where we have defined

$$\xi_1(u)=\int_1^u \frac{\sin(2v)}{v}\,dv$$

and we have used the fact that $|\phi_K|$, $|\phi_K'|$ and $|\phi_{W0}|$ are integrable, and $|\phi_K|$, $|\phi_K'|$ and $|\xi_1|$ are uniformly bounded (see (3.1)(ii) and (3.1)(iii)). This proves (S6.10).

Combining (S6.7), (S6.9) and (S6.10) we deduce that

$$|R_5(h)-R_6(h)|\leq B_{21}\,. \tag{S6.11}$$

Step 5: Bound for R_6; see (S6.13). First we treat the case where $w\neq 0$. There, defining

$$\xi_2(u)=\int_0^u \mathrm{cs}_{j_11}(v)\,dv\,,$$

we have:

$$\begin{aligned}
R_6(h) &= s_k\int_{t_1\,:\,1<|t_1|<1/h}\beta(t_1)^{-1}\,\mathrm{cs}_{j_11}(t_1w)\,\phi_K(ht_1)\,dt_1 \\
&\qquad\times\int \phi_{W0}(t)\,\rho_{j_2}\{\pm(t-t_1)\}\,\phi_K(ht-ht_1)\,dt \\
&= \frac{s_k}{w}\int_{t_1\,:\,1<|t_1|<1/h}\beta(t_1)^{-1}\,\phi_K(ht_1)\left\{\frac{\partial}{\partial t_1}\,\xi_2(t_1w)\right\}dt_1
\end{aligned}$$

$$
\begin{aligned}
&\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K(ht-ht_1)\, dt \\
&= s_k\, w^{-1}\, \{R_{61}(h) + \ldots + R_{64}(h)\} + O(1)\,, \qquad \text{(S6.12)}
\end{aligned}
$$

where

$$
\begin{aligned}
R_{61}(h) &= \int_{t_1\,:\,1<|t_1|<1/h} \beta'(t_1)\, \beta(t_1)^{-2}\, \phi_K(ht_1)\, \xi_2(t_1 w)\, dt_1 \\
&\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K(ht-ht_1)\, dt\,, \\
R_{62}(h) &= -h \int_{t_1\,:\,1<|t_1|<1/h} \beta(t_1)^{-1}\, \phi_K'(ht_1)\, \xi_2(t_1 w)\, dt_1 \\
&\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K(ht-ht_1)\, dt \\
&= -\int_{t_1\,:\,h<|t_1|<1} \beta(t_1/h)^{-1}\, \phi_K'(t_1)\, \xi_2(t_1 w/h)\, dt_1 \\
&\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1/h)\}\, \phi_K(ht-t_1)\, dt\,, \\
R_{63}(h) &= h \int_{t_1\,:\,1<|t_1|<1/h} \beta(t_1)^{-1}\, \phi_K(ht_1)\, \xi_2(t_1 w)\, dt_1 \\
&\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K'(ht-ht_1)\, dt\,, \\
R_{64}(h) &= \pm \int_{t_1\,:\,1<|t_1|<1/h} \beta(t_1)^{-1}\, \phi_K(ht_1)\, \xi_2(t_1 w)\, dt_1 \\
&\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}'\{\pm(t-t_1)\}\, \phi_K(ht-ht_1)\, dt\,,
\end{aligned}
$$

and the term represented by $O(1)$ is equal to

$$
\begin{aligned}
&\frac{s_k}{w} \left[\beta(t_1)^{-1}\, \phi_K(ht_1)\, \xi_2(t_1 w)\, dt_1 \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K(ht-ht_1)\, dt\right]_1^{1/h} \\
&+ \frac{s_k}{w} \left[\beta(t_1)^{-1}\, \phi_K(ht_1)\, \xi_2(t_1 w)\, dt_1 \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K(ht-ht_1)\, dt\right]_{-1/h}^{-1}.
\end{aligned}
$$

It can be proved from (3.1), the fact that $|\xi_2|$ and each $|\rho_j'|$ is bounded, and the fact that $|\phi_K|$, $|\phi_K'|$ and $|\phi_{W0}|$ are bounded and integrable, that $R_{6\ell}(h) = O(1)$ for $\ell = 1, \ldots, 4$. This result and (S6.12) imply that, when $w \neq 0$,

$$R_6(h) = O(1)\,. \tag{S6.13}$$

When $w = 0$, $\mathrm{cs}_{j_1 1}(t_1 w/h) \equiv 1$ or 0 according as $j_1 = 1$ or 2, respectively, and so $R_6(h) = 0$ if $j_1 = 2$, whereas if $j_1 = 1$,

$$\begin{aligned} h\, s_k^{-1}\, R_6(h) &= \int_{t_1\,:\,h<|t_1|<1} \beta(t_1/h)^{-1}\, \phi_K(t_1)\, dt_1 \\ &\qquad\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t - t_1/h)\}\, \phi_K(ht - t_1)\, dt \\ &= s_k\, b_1^{-2} \int_{-1}^{1} |\phi_K(t_1)|^2\, dt_1 \cdot \int \phi_{W0}(t)\, dt + o(1)\,, \end{aligned}$$

where the last identity holds if $j_2 = 1$; whereas if $j_1 = 1$ and $j_2 = 2$, $R_6(h) = o(1)$. Now, ϕ_{W0} denotes either $\Re\,\phi_W$ when $k = 1$, or $\Im\,\phi_W$ when $k = 2$, and so, since $\int \phi_W = 2\pi\, f_W(0)$, then $\int \phi_{W0} = 2\pi\, f_W(0)$ when $k = 1$ and equals 0 when $k = 2$. Moreover, $\int |\phi_K|^2 = 2\pi \int K^2$. Therefore, when $w = 0$,

$$R_6(h) = \begin{cases} (2\pi)^2\, s_k^2\, (b_1^2 h)^{-1}\, (\int K^2)\, f_W(0) + o(h^{-1}) & \text{if } j_1 = j_2 = k = 1\,, \\ o(h^{-1}) & \text{otherwise}\,. \end{cases} \tag{S6.14}$$

Result (6.11) follows from (S6.1), (S6.3), (S6.5), (S6.11), (S6.13) and (S6.14), which hold in the cases $w \neq 0$ and $w = 0$ respectively.

S7 Proof of Theorem 2

We treat only the case where $w = 0$. Write $\widehat{F}(t)$ and $F(t)$ for $\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$ and $F_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$, respectively. It can be proved from Theorem 2 in Appendix S4 that, if the conditions of Theorem 2 hold, then for each $r \geq 1$,

$$F(t_\alpha) = \alpha = \widehat{F}(\hat{t}_\alpha) = \widehat{F}(t_\alpha) + \sum_{j=1}^{r} \frac{(\hat{t}_\alpha - t_\alpha)^j}{j!}\, \widehat{F}^{(j)}(t_\alpha) + O_p\Big(\big|\hat{t}_\alpha - t_\alpha\big|^{r+1}\Big)\,,$$

where, in the case of part (i) of the theorem, the remainder is of the stated order for each fixed q, w, y and $\alpha \in (0,1)$, and, in the case of part (ii), the remainder is of that order uniformly in q and y in compact sets, and $\alpha \in [\alpha_1, \alpha_2]$. It is straightforward to show that $\widehat{F}(t_\alpha) - F(t_\alpha) = o_p(1)$ and $\widehat{F}'(t_\alpha) - F'(t_\alpha) = o_p(1)$, where, here and immediately below, the remainders are interpreted as in the previous sentence, and therefore it can be proved in succession that $\hat{t}_\alpha - t_\alpha = O_p\{|\widehat{F}(t_\alpha) - F(t_\alpha)|\} = o_p(1)$,

$$\hat{t}_\alpha - t_\alpha = -\{1 + o_p(1)\}\, \frac{\widehat{F}(t_\alpha) - F(t_\alpha)}{F'(t_\alpha)}$$

and

$$\hat{t}_\alpha - t_\alpha = -\frac{\widehat{F}(t_\alpha) - F(t_\alpha)}{F'(t_\alpha)} + \begin{cases} O_p\{(nh)^{-1} + h^{2\ell}\} & \text{for part (i)} \\ O_p\{(n^{1-\eta}h)^{-1} + h^{2\ell}\} & \text{for part (ii),} \end{cases} \tag{S7.1}$$

where $\eta > 0$ is arbitrarily small. Parts (i) and (ii) of Theorem 2 follow from (S7.1) and parts (i) and (ii), respectively, of Theorem 1.

S8 Proof of Theorem 3

We treat only the case where $w = 0$. Let F and $\widehat{F}$ be as in the proof of Theorem 2. Note that, as established in Theorem 2, each derivative $\widehat{F}^{(r)}$ converges to the respective $F^{(r)}$ at the same rate, $O_p\{(nh)^{-1/2} + h^\ell\}$ for each q, w and y, or $O_p\{(n^{1-\eta}h)^{-1/2} + h^\ell\}$ uniformly on compacts. Therefore, by Taylor expansion,

$$\alpha = F(t_\alpha) = \widehat{F}(\hat{t}_\alpha) = \widehat{F}(t_\alpha) + \big(\hat{t}_\alpha - t_\alpha\big)\,\widehat{F}'(t_\alpha) + \tfrac{1}{2}\big(\hat{t}_\alpha - t_\alpha\big)^2\,\widehat{F}''(t_\alpha) + \ldots\,, \quad \text{(S8.1)}$$

where, here and in (S8.2) below, it can be proved from Theorem 2 that the remainder "..." denotes a sum of successive terms of respective sizes $\{(nh)^{-1/2} + h^\ell\}^j$, for $j \geq 3$, and equals $O_p[\{(nh)^{-1/2} + h^\ell\}^{r+1}]$ (or $O_p[\{(n^{1-\eta}h)^{-1/2} + h^\ell\}^{r+1}]$ in a uniform sense) if the last included term is that involving $(\hat{t}_\alpha - t_\alpha)^r$.

In a slight abuse of previous notation, write $\Psi_k(t)$ and $\widehat{\Psi}_k(t)$ for $\Psi_k(t, y, q, w)$ and $\widehat{\Psi}_k(t, y, q, w)$, respectively, and define $\Delta_k = \widehat{\Psi}_k - \Psi_k$. Recall from (2.5) and (2.7) that

$$\begin{aligned}\widehat{F} &= \frac{\Psi_1 + \Delta_1}{\Psi_2 + \Delta_2} = \Psi_2^{-1}\,(\Psi_1 + \Delta_1)\,\big(1 - \Psi_2^{-1}\,\Delta_2 + \Psi_2^{-2}\,\Delta_2^2 - \ldots\big) \\ &= F + \big(\Psi_2^{-1}\,\Delta_1 - \Psi_2^{-2}\,\Psi_1\,\Delta_2\big) + \big(\Psi_2^{-3}\,\Psi_1\,\Delta_2^2 - \Psi_2^{-2}\,\Delta_1\,\Delta_2\big) + \ldots\,. \quad \text{(S8.2)}\end{aligned}$$

The advantage of working with this expanded form of $\widehat{F}$ is that it does not involve a random denominator. Write $\widehat{F}_r$ for the version of (S8.2) when the expansion on the right-hand side is terminated after terms of size $\{(nh)^{-1/2} + h^\ell\}^r$. For example,

$$\widehat{F}_2 = F + \big(\Psi_2^{-1}\,\Delta_1 - \Psi_2^{-2}\,\Psi_1\,\Delta_2\big) + \big(\Psi_2^{-3}\,\Psi_1\,\Delta_2^2 - \Psi_2^{-2}\,\Delta_1\,\Delta_2\big)\,. \quad \text{(S8.3)}$$

Since (T, Q, W, Y) is independent of the data $\{(Q_j, W_j, Y_j),\ 1 \le j \le n\}$, then, conditionally on Q, W, Y,

$$\begin{aligned}
&F_0(\alpha \,|\, q, w, y)\\
&\quad \equiv P\Big(T \le \hat t_\alpha \,\Big|\, Q = q, W = w, Y = y\Big) = E\big\{F\big(\hat t_\alpha\big)\big\}\\
&\quad = E\Big[\Big\{F(t_\alpha) + \big(\hat t_\alpha - t_\alpha\big)\,F'(t_\alpha) + \tfrac{1}{2}\,\big(\hat t_\alpha - t_\alpha\big)^2\,F''(t_\alpha)\Big\}\,I(\mathcal{E})\Big]\\
&\quad +O\big\{\delta^3 + P\big(\widetilde{\mathcal{E}}\big)\big\}\,, \qquad\qquad (S8.4)
\end{aligned}$$

where $\mathcal{E}$ represents the event that $|\hat t_\alpha - t_\alpha| \le \delta$, $\widetilde{\mathcal{E}}$ denotes the complement of $\mathcal{E}$, and $\delta = \delta(n)$ is a positive sequence decreasing to 0 as $n \to \infty$. Here and below, all expected values are taken conditionally on Q, W, Y. Furthermore, this expansion at (S8.4) holds uniformly in t, q and y in any compact subsets of their respective domains, and in $\alpha \in [\alpha_1, \alpha_2]$ for any $0 < \alpha_1 < \alpha_2 < 1$.

Recall from (S8.1) that $F(t_\alpha) = \widehat F(\hat t_\alpha) = \alpha$. Using this result, and Taylor-expanding as at (S8.1), we deduce that

$$\begin{aligned}
&E\Big[\Big\{\big(\hat t_\alpha - t_\alpha\big)\,F'(t_\alpha) + \tfrac{1}{2}\,\big(\hat t_\alpha - t_\alpha\big)^2\,F''(t_\alpha)\Big\}\,I(\mathcal{E})\Big]\\
&\quad = E\Big[\Big\{\big(\hat t_\alpha - t_\alpha\big)\,\widehat F_2'(t_\alpha) + \tfrac{1}{2}\,\big(\hat t_\alpha - t_\alpha\big)^2\,\widehat F_2''(t_\alpha)\Big\}\,I(\mathcal{E})\Big]\\
&\qquad -E\Big(\Big[\big(\hat t_\alpha - t_\alpha\big)\,\big\{\widehat F_2'(t_\alpha) - F'(t_\alpha)\big\} + \tfrac{1}{2}\,\big(\hat t_\alpha - t_\alpha\big)^2\,\big\{\widehat F_2''(t_\alpha) - F''(t_\alpha)\big\}\Big]\,I(\mathcal{E})\Big)\\
&\quad = -\alpha - E\Big[\big\{\widehat F_2(t_\alpha) - F(t_\alpha)\big\}\Big]\,I(\mathcal{E}) + E\big\{\widehat F_2(\hat t_\alpha)\,I(\mathcal{E})\big\} + O\big\{\delta^3 + P\big(\widetilde{\mathcal{E}}\big)\big\}\\
&\qquad -E\Big(\Big[\big(\hat t_\alpha - t_\alpha\big)\,\big\{\widehat F_2'(t_\alpha) - F'(t_\alpha)\big\} + \tfrac{1}{2}\,\big(\hat t_\alpha - t_\alpha\big)^2\,\big\{\widehat F_2''(t_\alpha) - F''(t_\alpha)\big\}\Big]\,I(\mathcal{E})\Big)\\
&\quad = -E\Big[\big\{\widehat F_2(t_\alpha) - F(t_\alpha)\big\}\,I(\mathcal{E})\Big] + O\big\{\delta^3 + P\big(\widetilde{\mathcal{E}}\big)\big\}
\end{aligned}$$

$$-E\biggl(\Bigl[\bigl(\hat t_\alpha - t_\alpha\bigr)\bigl\{\widehat F_2'(t_\alpha) - F'(t_\alpha)\bigr\} + \tfrac12\bigl(\hat t_\alpha - t_\alpha\bigr)^2\bigl\{\widehat F_2''(t_\alpha) - F''(t_\alpha)\bigr\}\Bigr]\, I(\mathcal{E})\biggr).$$

Hence, by (S8.4),

$$\begin{aligned} F_0(\alpha \,|\, q,w,y) =& \alpha - E\Bigl[\bigl\{\widehat F_2(t_\alpha) - F(t_\alpha)\bigr\}\, I(\mathcal{E})\Bigr] - E\biggl(\Bigl[\bigl(\hat t_\alpha - t_\alpha\bigr)\bigl\{\widehat F_2'(t_\alpha) - F'(t_\alpha)\bigr\} \\ &+ \tfrac12\bigl(\hat t_\alpha - t_\alpha\bigr)^2\bigl\{\widehat F_2''(t_\alpha) - F''(t_\alpha)\bigr\}\Bigr]\, I(\mathcal{E})\biggr) + O\bigl\{\delta^3 + P\bigl(\widetilde{\mathcal{E}}\bigr)\bigr\}, \quad \text{(S8.5)} \end{aligned}$$

where this identity holds uniformly in q and y in any compact subsets of their respective domains, and in $\alpha \in [\alpha_1, \alpha_2]$ for any $0 < \alpha_1 < \alpha_2 < 1$.

A modification of the Taylor-expansion argument leading to Theorem 2 (see e.g. (S7.1)) can be used to show that

$$\begin{aligned} &E\Bigl[\bigl(\hat t_\alpha - t_\alpha\bigr)\bigl\{\widehat F_2'(t_\alpha) - F'(t_\alpha)\bigr\}\, I(\mathcal{E})\Bigr] \\ &\quad = -F'(t_\alpha)^{-1}\, E\Bigl[\bigl\{\widehat F_2(t_\alpha) - F(t_\alpha)\bigr\}\bigl\{\widehat F_2'(t_\alpha) - F'(t_\alpha)\bigr\}\Bigr] + O\bigl\{\delta^3 + P\bigl(\widetilde{\mathcal{E}}\bigr)\bigr\} \\ &\quad = -F'(t_\alpha)^{-1}\, E\Bigl[\bigl\{\widehat F_1(t_\alpha) - F(t_\alpha)\bigr\}\bigl\{\widehat F_1'(t_\alpha) - F'(t_\alpha)\bigr\}\Bigr] + O\bigl\{\delta^3 + P\bigl(\widetilde{\mathcal{E}}\bigr)\bigr\} + o\bigl(\delta_1^2\bigr), \end{aligned} \quad \text{(S8.6)}$$

where $\delta_1 = (nh)^{-1/2} + h^\ell$. Similarly but more simply,

$$E\Bigl[\bigl(\hat t_\alpha - t_\alpha\bigr)^2\bigl\{\widehat F_2''(t_\alpha) - F''(t_\alpha)\bigr\}\, I(\mathcal{E})\Bigr] = O\bigl\{\delta^3 + P\bigl(\widetilde{\mathcal{E}}\bigr)\bigr\} + o\bigl(\delta_1^2\bigr). \quad \text{(S8.7)}$$

Combining (S8.5)–(S8.7) we deduce that

$$\begin{aligned} F_0(\alpha \,|\, q,w,y) &= \alpha - E\bigl\{\widehat F_2(t_\alpha) - F(t_\alpha)\bigr\} \\ &\quad + F'(t_\alpha)^{-1}\, E\Bigl[\bigl\{\widehat F_1(t_\alpha) - F(t_\alpha)\bigr\}\bigl\{\widehat F_1'(t_\alpha) - F'(t_\alpha)\bigr\}\Bigr] \end{aligned}$$

$$+O\big\{\delta^3 + P\big(\widetilde{\mathcal{E}}\big)\big\} + o\big(\delta_1^2\big)\,, \tag{S8.8}$$

uniformly in the sense described below (S8.5).

Define $\widetilde{\Psi}_k(s,y,q,w) = \int \psi_k(s,y,q,w,x)\,\hat{f}_X(x)\,dx$, where ψ_k is as at (2.6), and recall that $\Delta_k = \widehat{\Psi}_k - \Psi_k$, that $\widehat{\Psi}_k(s,y,q,w) = \int \hat{\psi}_k(s,y,q,w,x)\,\hat{f}_X(x)\,dx$, and that $\hat{\psi}_k$ is given by (2.8). It can be proved from these definitions that

$$\widehat{\Psi}_k = \widetilde{\Psi}_k + O_p\big(n^{-1/2}\big)\,, \quad E\big(\widehat{\Psi}_k\big) = E\big(\widetilde{\Psi}_k\big) + O\big(n^{-1}\big)\,, \tag{S8.9}$$

$$\begin{aligned}
E\big\{\widetilde{\Psi}_k(s,y,q,w)\big\} &= \int \psi_k(s,y,q,w,x)\,E\big\{\hat{f}_X(x)\big\}\,dx \\
&= \int\int \psi_k(s,y,q,w,x+hu)\,K(u)\,f_X(x)\,du\,dx = \int \lambda_k(hu\,|\,s,y,q,w)\,K(u)\,du \\
&= \lambda_k(0\,|\,s,y,q,w) + O\big(h^\ell\big) = \Psi_k(s,y,q,w) + O\big(h^\ell\big)\,,
\end{aligned} \tag{S8.10}$$

in a uniform sense. (Recall that λ_k was defined at (3.3).) For example, in (S8.9) uniformity means that $\sup|\widehat{\Psi}(s,y,q,w) - \widetilde{\Psi}(s,y,q,w)| = O_p(n^{-1/2})$ and $\sup|E(\widehat{\Psi}_k) - E(\widetilde{\Psi}_k)| = O(n^{-1})$, where in each case the supremum is taken over s, y and q in any compact subsets of their respective domains. To derive the last identity in (S8.10) we used (3.3) and (3.5).

Note that

$$\begin{aligned}
&E\Big\{\big|\widehat{\Psi}_k(s,y,q,w) - \widetilde{\Psi}_k(s,y,q,w)\big|^2\Big\} \\
&\quad= E\bigg[\int \Big\{\hat{\psi}_k(s,y,q,w,x) - \psi_k(s,y,q,w,x)\Big\}\,\hat{f}_X(x)\,dx\bigg]^2, \\
&\quad\le \bigg[\int E\Big\{\hat{\psi}_k(s,y,q,w,x) - \psi_k(s,y,q,w,x)\Big\}^2\,dx\bigg]\int E\big\{\hat{f}_X(x)^2\big\}\,dx
\end{aligned}$$

$$=O\left[n^a \int E\left\{\hat{\psi}_k(s,y,q,w,x)-\psi_k(s,y,q,w,x)\right\}^2 dx\right], \tag{S8.11}$$

uniformly in the sense described in the previous paragraph. To obtain the last identity in (S8.11) we used the fact that, by (3.6)(a), $\int E\{\hat{f}_X(x)^2\}\,dx = O(n^a)$ for a constant $a \geq 0$. Let $D_0, \ldots, D_3$ denote the respective quantities $|\hat{\beta}_0 - \beta_0|$, $|\hat{\beta}_1 - \beta_1|$, $\|\hat{\beta}_2 - \beta_2\|$ and $|\hat{\sigma}_V^2 - \sigma_V^2|$. If

$$\max_{0\leq j\leq 3} P\big(D_j > n^{-(1-a_1)/2}\big) = O\big(n^{-(1-a_2)}\big)\,, \tag{S8.12}$$

where $0 < a_1, a_2 < 1$, then it can be proved by Taylor expansion that

$$\int E\left\{\hat{\psi}_k(s,y,q,w,x)-\psi_k(s,y,q,w,x)\right\}^2 dx = O\big(n^{\max(a_1,a_2)-1}\big)\,.$$

Therefore, by (S8.11),

$$E\left\{\big|\widehat{\Psi}_k(s,y,q,w)-\widetilde{\Psi}_k(s,y,q,w)\big|^2\right\} = O\big(n^{a+\max(a_1,a_2)-1}\big)\,,$$

uniformly in the sense described in the previous paragraph. Hence, provided that

$$n^{a+\max(a_1,a_2)}h = O(1)\,, \tag{S8.13}$$

we have:

$$E\left\{\big|\widehat{\Psi}_k(s,y,q,w)-\widetilde{\Psi}_k(s,y,q,w)\big|^2\right\} = O\big\{(nh)^{-1}\big\}\,, \tag{S8.14}$$

again uniformly. Suppose that, as asserted in (3.6)(b), $n^{a+\varepsilon}\,h = O(1)$ for some $\varepsilon > 0$. By assuming enough finite moments of Q, U, V and X (here we are invoking (3.6)(c)) we can ensure that (S8.12) holds for a_1, a_2 in the range $0 < a_1, a_2 \leq \varepsilon$. In this case (S8.13), and hence also (S8.14), follow from the property $n^{a+\varepsilon}\,h = O(1)$ in (3.6).

Define $\widetilde{\Delta}_k = \widetilde{\Psi}_k - \Psi_k$, and let $\widetilde{\Psi}'_k$ and Ψ'_k be the derivatives of $\widetilde{\Psi}_k$ and Ψ_k with respect to s, so that $\widetilde{\Delta}'_k = \widetilde{\Psi}'_k - \Psi'_k$. Using this notation, and combining (S8.3), the second part of (S8.9), (S8.10) and (S8.14), we deduce that

$$\begin{aligned} E\big(\widehat{F}_2\big) - F &= \Psi_2^{-3}\,\Psi_1\,E\big(\widetilde{\Delta}_2^2\big) - \Psi_2^{-2}\,E\big(\widetilde{\Delta}_1\,\widetilde{\Delta}_2\big) + O\big\{(nh)^{-1} + h^\ell\big\} \\ &= O\big\{(nh)^{-1} + h^\ell\big\}\,, \qquad \text{(S8.15)} \\ E\big\{\big(\widehat{F}_1 - F\big)\big(\widehat{F}'_1 - F'\big)\big\} &= E\Big\{\big(\Psi_2^{-1}\,\widetilde{\Delta}_1 - \Psi_2^{-2}\,\Psi_1\,\widetilde{\Delta}_2\big)\big(\Psi_2^{-1}\,\widetilde{\Delta}_1 - \Psi_2^{-2}\,\Psi_1\,\widetilde{\Delta}_2\big)'\Big\} \\ &\quad + O\big\{(nh)^{-1} + h^\ell\big\} \\ &= O\big\{(nh)^{-1} + h^\ell\big\}\,, \qquad \text{(S8.16)} \end{aligned}$$

where in each case the functions on the left-hand side are evaluated at t_α, and the last identities are derived using standard calculations. Hence, by (S8.8), and again in the uniform sense prescribed two paragraphs above,

$$\begin{aligned} F_0(\alpha\,|\,q,w,y) - \alpha &= O\Big\{(nh)^{-1} + h^\ell + \delta^3 + P\big(\widetilde{\mathcal{E}}\big)\Big\} + o\big(\delta_1^2\big) \\ &= O\Big\{(nh)^{-1} + h^\ell + \delta^3 + P\big(\widetilde{\mathcal{E}}\big)\Big\}\,. \qquad \text{(S8.17)} \end{aligned}$$

We know from Theorem 2 that $\hat{t}_\alpha - t_\alpha = O_p\{(nh)^{-1/2} + h^\ell\}$, and so if we define $\delta = \{(nh)^{-1/2} + h^\ell\}\,n^\eta$, where $\eta > 0$ is chosen so small that $\{(nh)^{-1/2}\,n^\eta\}^3 = O\{(nh)^{-1}\}$, then we shall have $\delta^3 = O\{(nh)^{-1/2} + h^\ell\}$. Moreover, Markov's inequality can be used to prove that $P(\widetilde{\mathcal{E}}) = O\{(nh)^{-1/2} + h^\ell\}$. Hence, by (S8.17), $F_0(\alpha\,|\,q,w,y) - \alpha = O\{(nh)^{-1/2} + h^\ell\}$, uniformly in q and y in compact subsets of their respective domains. This result is equivalent to (3.8).

Finally we sketch a proof of the variant of Theorem 3 discussed immediately below that theorem. If in Theorem 3 we assume (3.4) then the far right-hand side of (S8.10) can be refined to $\Psi_k(s,y,q,w) + c_k\, h^\ell + o(h^\ell)$, where c_k is a constant, and therefore (S8.10) becomes

$$E\big\{\widetilde{\Psi}_k(s,y,q,w)\big\} = \Psi_k(s,y,q,w) + c_k\, h^\ell + o\big(h^\ell\big)\,. \tag{S8.18}$$

Furthermore, if in (3.6)(a) we replace $O(h^a)$ by $o(h^a)$, so that $\int E(\hat{f}_X)^2 = o(n^a)$, then (S8.13) holds with the right-hand side replaced by $o(1)$, and so (S8.14) becomes

$$E\Big\{\big|\widehat{\Psi}_k(s,y,q,w) - \widetilde{\Psi}_k(s,y,q,w)\big|^2\Big\} = o\big\{(nh)^{-1}\big\}\,. \tag{S8.19}$$

Using (S8.18) and (S8.19) instead of (S8.10) and (S8.14), respectively, the strings of identities at (S8.15) and (S8.16) can be refined to

$$\begin{aligned}
E\big(\widehat{F}_2\big) - F &= \Psi_2^{-3}\,\Psi_1\,E\big(\widetilde{\Delta}_2^2\big) - \Psi_2^{-2}\,E\big(\widetilde{\Delta}_2\,\widetilde{\Delta}_2\big) + o\big\{(nh)^{-1} + h^\ell\big\} \\
&= d_1\,(nh)^{-1} + d_2\,h^\ell + o\big\{(nh)^{-1} + h^\ell\big\}\,, && \text{(S8.20)} \\
E\big\{\big(\widehat{F}_1 - F\big)\,\big(\widehat{F}_1' - F'\big)\big\} &= E\Big\{\big(\Psi_2^{-1}\,\widetilde{\Delta}_1 - \Psi_2^{-2}\,\Psi_1\,\widetilde{\Delta}_2\big)\,\big(\Psi_2^{-1}\,\widetilde{\Delta}_1 - \Psi_2^{-2}\,\Psi_1\,\widetilde{\Delta}_2\big)'\Big\} \\
&\qquad + o\big\{(nh)^{-1} + h^\ell\big\} \\
&= d_3\,(nh)^{-1} + o\big\{(nh)^{-1} + h^\ell\big\}\,, && \text{(S8.21)}
\end{aligned}$$

where d_1, d_2 and d_3 are constants and, as in (S8.15) and (S8.16), the functions on the left-hand sides are evaluated at t_α. The remainder $O\{\delta^3 + P(\widetilde{\mathcal{E}})\} + o(\delta_1^2)$ on the right-hand side of (S8.8) equals $o\{(nh)^{-1} + h^\ell\}$ if δ is chosen appropriately, and so, on substituting (S8.20) and (S8.21) into (S8.8), we obtain:

$$F_0(\alpha \mid q, w, y) = \alpha + \big\{F'(t_\alpha)^{-1}\, d_3 - d_1\big\}\, (nh)^{-1} - d_2\, h^\ell + o\big\{(nh)^{-1} + h^\ell\big\}\,.$$

This is the version of (3.8) discussed in the paragraph immediately below Theorem 3.