METHODOLOGY AND CONVERGENCE RATES FOR FUNCTIONAL TIME SERIES REGRESSION

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Supplementary Material

This supplement contains the proof Theorem 1. This is developed in the next section, while a separate Appendix contains several auxiliary results that are required in the main proof (namely Lemmas 1-6 and Propositions 1-7), but which are stated separately for tidiness.

7. Proof of Theorem 1

In the interest of tidiness, we introduce some additional notational conventions and facts here that will be made frequent use of in the forthcoming lemmas and propositions.

- For fixed $\omega \in [0, 2\pi]$, define u_s to be an element of $\{\nu_s + 2k\pi : k \in \mathbb{Z}\}$ such that $|\omega - u_s| \leq \pi$. By this definition u_s is well-defined and

$$f^{XX}_{\nu_s} = f^{XX}_{u_s}; \qquad W^{(T)}\big(\omega - \nu_s\big) = W^{(T)}\big(\omega - u_s\big),$$

  since f^{XX}_{ω} and $W^{(T)}$ are periodic with period 2π.

- $h_T(t) = 1_{[0,T-1]}$.

- Since $B_T = T^{-\gamma}$ and $\gamma > (\alpha-1)/(\alpha+2\beta)$, we have

$$T^{-1}B_T\zeta_T^{-2} = T^{2\alpha/(\alpha+2\beta)}T^{-\gamma}T^{-1} = T^{-(2\beta-\alpha)/(\alpha+2\beta)}T^{-\gamma} = T^{-(2\beta-1)/(\alpha+2\beta)}T^{(\alpha-1)/(\alpha+2\beta)}T^{-\gamma}$$
$$= O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big).$$

  The fact that the above quantities are all equal and of order $O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big)$ is highlighted here for tidiness, as it will be made use of in several steps of the proof, without repeating an explicit calculation.

We first recall that

$$\mathbb{E}\left\{\sum_t \|\!|\mathscr{B}_t - \widehat{\mathscr{B}}_t\|\!|_2^2\right\} = \mathbb{E}\left\{\int_0^{2\pi} \|\!|\mathscr{Q}_\omega - \widehat{\mathscr{Q}}_\omega\|\!|_2^2\right\} d\omega = \int_0^{2\pi} \mathbb{E}\|\!|\mathscr{Q}_\omega - \widehat{\mathscr{Q}}_\omega\|\!|_2^2 d\omega.$$

Hence, we first need to find the Hilbert-Schmidt norm by applying part C of Lemma 1 and then take the integral over ω. Let

$$\widetilde{\mathscr{Q}}_\omega := \mathscr{F}_\omega^{YX}\big(\mathscr{F}_\omega^{XX} + \zeta_T\mathscr{I}\big)^{-1}.$$

Using the triangle inequality,

$$\|\!|\mathscr{Q}_\omega - \widehat{\mathscr{Q}}_{\omega,T}\|\!|_2^2 \le 2\|\!|\widetilde{\mathscr{Q}}_\omega - \mathscr{Q}_\omega\|\!|_2^2 + 2\|\!|\widetilde{\mathscr{Q}}_\omega - \widehat{\mathscr{Q}}_{\omega,T}\|\!|_2^2.$$

Recall that,

$$\mathscr{Q}_\omega = \Big\{\sum_{i,j} a_{ij}^\omega \varphi_i^\omega \otimes \overline{\varphi_j^\omega}\Big\}\Big\{\sum_j \frac{1}{\lambda_j^\omega}\varphi_j^\omega \otimes \overline{\varphi_j^\omega}\Big\} = \sum_j\sum_i \frac{a_{ij}^\omega}{\lambda_j^\omega}\varphi_i^\omega \otimes \overline{\varphi_j^\omega} = \sum_j\sum_i b_{ij}^\omega \varphi_i^\omega \otimes \overline{\varphi_j^\omega}$$

and note that $\widetilde{\mathscr{Q}}_\omega$ has the series representation

$$\widetilde{\mathscr{Q}}_\omega = \Big\{\sum_{i,j} a_{ij}^\omega \varphi_i^\omega \otimes \overline{\varphi_j^\omega}\Big\}\Big\{\sum_j \frac{1}{\lambda_j^\omega + \zeta_T}\varphi_j^\omega \otimes \overline{\varphi_j^\omega}\Big\} = \sum_j\sum_i \frac{a_{ij}^\omega}{\lambda_j^\omega + \zeta_T}\varphi_i^\omega \otimes \overline{\varphi_j^\omega}.$$

Thus,

$$\widetilde{\mathscr{Q}}_\omega - \mathscr{Q}_\omega = \sum_{i,j} \frac{a_{ij}^\omega \zeta_T}{\lambda_j^\omega(\lambda_j^\omega + \zeta_T)}\varphi_i^\omega \otimes \overline{\varphi_j^\omega}$$

$$\|\widetilde{\mathscr{Q}}_\omega - \mathscr{Q}_\omega\|_2^2 = \sum_{i,j} \frac{\left|a_{ij}^\omega\right|^2}{(\lambda_j^\omega)^2} \times \frac{\zeta_T^2}{(\lambda_j^\omega + \zeta_T)^2} = \sum_j \left\{ \sum_i \left|b_{ij}^\omega\right|^2 \right\} \times \frac{\zeta_T^2}{(\lambda_j^\omega + \zeta_T)^2} \leq O(1) \times \sum_j j^{-2\beta} \frac{\zeta_T^2}{(\lambda_j^\omega + \zeta_T)^2}$$
$$= O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big).$$

The last inequality comes from (9) and assumption (B1). We next decompose $\widetilde{\mathscr{Q}}_\omega - \widehat{\mathscr{Q}}_{\omega,T}$ as

$$\begin{aligned}
\widetilde{\mathscr{Q}}_\omega - \widehat{\mathscr{Q}}_{\omega,T} &= \mathscr{F}_\omega^{YX}\left[\mathscr{F}_\omega^{XX} + \zeta_T \mathscr{I}\right]^{-1} - \widehat{\mathscr{F}}_{\omega,T}^{YX}\Big[\widehat{\mathscr{F}}_{\omega,T}^{XX} + \zeta_T \mathscr{I}\Big]^{-1} \\
&= \Big(\mathscr{F}_\omega^{YX} - \widehat{\mathscr{F}}_{\omega,T}^{YX}\Big)\Big[\mathscr{F}_\omega^{XX} + \zeta_T \mathscr{I}\Big]^{-1} + \Big(\widehat{\mathscr{F}}_{\omega,T}^{YX} - \mathscr{F}_\omega^{YX}\Big)\Big(\left[\mathscr{F}_\omega^{XX} + \zeta_T \mathscr{I}\right]^{-1} - \Big[\widehat{\mathscr{F}}_{\omega,T}^{XX} + \zeta_T \mathscr{I}\Big]^{-1}\Big) + \\
&\quad \mathscr{F}_\omega^{YX}\Big(\left[\mathscr{F}_\omega^{XX} + \zeta_T \mathscr{I}\right]^{-1} - \Big[\widehat{\mathscr{F}}_{\omega,T}^{XX} + \zeta_T \mathscr{I}\Big]^{-1}\Big) \\
&= \mathscr{S}_1 + \mathscr{S}_2 + \mathscr{S}_3.
\end{aligned}$$

The remainder of the proof will deal with constructing upper bounds for the three terms $\{\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3\}$. To this aim, the strategy will be to:

1. Apply part C of Lemma 1 with the orthogonal basis $\left\{\varphi_i^\omega\right\}$ in order to to reduce the problem to calculations involving integrals of kernel functions.

2. Use Propositions 3 and 5 to break these integrals down into manageable terms.

3. Apply Lemma 2 to determine the required upper bound for each of these terms.

We organise this process into separate subsections for each of the terms $\{\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3\}$, starting with $\mathscr{S}_3$, then moving on to $\mathscr{S}_1$ and finally $\mathscr{S}_2$.

Bounding $\mathscr{S}_3$

Let $\Delta = \widehat{\mathscr{F}}_{\omega,T}^{XX} - \mathscr{F}_\omega^{XX}$. For simplicity, also write

$$V = \mathscr{F}_\omega^{XX}; \qquad \widehat{V} = \widehat{\mathscr{F}}_{\omega,T}^{XX}; \qquad V^+ = \left[\mathscr{F}_\omega^{XX} + \zeta_T \mathscr{I}\right]^{-1}; \qquad \widehat{V}^+ = \Big[\widehat{\mathscr{F}}_{\omega,T}^{XX} + \zeta_T \mathscr{I}\Big]^{-1}.$$

Using the identities

$$\widehat{V}^+ - V^+ = \widehat{V}^+ \Delta V^+ \Rightarrow \widehat{V}^+\big[I + \Delta V^+\big] = V^+ \Rightarrow \widehat{V}^+ = V^+\big[I + \Delta V^+\big]^{-1},$$

we obtain

$$\widehat{V}^+ - V^+ = V^+ \Delta \widehat{V}^+ = V^+ \Delta V^+ \big[I + \Delta V^+\big]^{-1}.$$

Thus, $\mathscr{S}_3 = \mathscr{F}^{YX}_\omega V^+ \Delta V^+ \big[I + \Delta V^+\big]^{-1}$. By Proposition 6, $\mathbb{E}\big\|\!\big|\Delta\big\|\!\big|_2^2 = O\big(B_T^{-1} T^{-1}\big)$ uniformly over ω, and $\big\|V^+\big\| \leq 1/\zeta_T$. Hence, on the event G_T from Proposition 7

$$\big\|\!\big|\Delta \zeta_T^{-1}\big\|\!\big|_2^2 \lesssim T^{2\alpha/(\alpha+2\beta)} T^{-1} T^{\gamma+2\delta} = T^{-(2\beta-\alpha)/(\alpha+2\beta)} T^{\gamma+2\delta} = o(1),$$

since $\gamma+2\delta < (2\beta-\alpha)/(\alpha+2\beta)$. Thus $\big\|\!\big|\big(I+\Delta V^+\big)^{-1}\big\|\!\big|_2$ is uniformly bounded on the even G_T. Hence, on G_T,

$$\big\|\!\big|\mathscr{S}_3\big\|\!\big|_2^2 \leq O(1) \big\|\!\big|\mathscr{F}^{YX}_\omega V^+ \Delta V^+\big\|\!\big|_2^2 \leq O(1) \big\|\!\big|\mathscr{F}^{YX}_\omega V^+\big\|\!\big|_2^2 \times \big\|\!\big|\Delta V^+\big\|\!\big|_2^2.$$

The first factor of the right hand side is bounded by

$$\sum_{i,j} \frac{\big|a^\omega_{ij}\big|^2}{(\lambda^\omega_j + \zeta_T)^2} = \sum_j \Big\{ \sum_i \frac{\big|a^\omega_{ij}\big|^2}{(\lambda^\omega_j + \zeta_T)^2} \Big\} \leq O(1) \times \sum_j j^{-2\beta} = O(1).$$

Upon expanding $\Delta = \sum_{ij} \Delta_{ij} \varphi^\omega_i \otimes \overline{\varphi^\omega_j}$, we obtain

$$\Delta\big[\mathscr{F}^{XX}_\omega + \zeta_T \mathscr{I}\big]^{-1} = \Big(\sum_{ij} \Delta_{ij} \varphi^\omega_i \otimes \overline{\varphi^\omega_j}\Big)\Big(\sum_j \frac{1}{\lambda^\omega_j + \zeta_T} \varphi^\omega_j \otimes \overline{\varphi^\omega_j}\Big) = \sum_{i,j} \frac{\Delta_{ij}}{\lambda^\omega_j + \zeta_T} \varphi^\omega_i \otimes \overline{\varphi^\omega_j}.$$

It follows that

$$\mathbb{E}\big\|\!\big|\Delta\big[\mathscr{F}^{XX}_\omega + \zeta_T \mathscr{I}\big]^{-1}\big\|\!\big|_2^2 = \sum_{i,j} \frac{\mathbb{E}\big|\Delta_{ij}\big|^2}{(\lambda^\omega_j + \zeta_T)^2}. \tag{1}$$

Letting κ be the integral kernel of Δ, another way to express $\big|\Delta_{ij}\big|^2$ is via Lemma 1, yielding

$$\mathbb{E}\big|\Delta_{ij}\big|^2 = \int_{[0,1]^4} \mathbb{E}\Big[\kappa(\tau_1,\sigma_1)\overline{\kappa(\tau_2,\sigma_2)}\Big] \times \overline{\varphi^\omega_i(\tau_1)} \varphi^\omega_j(\sigma_1) \varphi^\omega_i(\tau_2) \overline{\varphi^\omega_j(\sigma_2)} d\tau_1 d\sigma_1 d\tau_2 d\sigma_2.$$

By proposition 5, $\mathbb{E}\Big[\kappa(\tau_1,\sigma_1)\overline{\kappa(\tau_2,\sigma_2)}\Big]$ can be decomposed as

$$O\big(T^{-1}B_T^{-1}\big) \times \Big\{ f^{XX}_\omega(\tau_1,\tau_2) f^{XX}_{-\omega}(\sigma_1,\sigma_2) + 1_{I_T}(\omega) f^{XX}_\omega(\tau_1,\sigma_2) f^{XX}_{-\omega}(\tau_2,\sigma_1) \Big\} +$$

$$O\big(T^{-1}\big) \times 1_{I_T}(\omega)\Big\{f^{XX}_{\omega}(\tau_1,\sigma_2)f^{XX,(1)}_{-\omega}(\sigma_1,\tau_2) + f^{XX}_{-\omega}(\sigma_1,\tau_2)f^{XX,(1)}_{\omega}(\tau_1,\sigma_2)\Big\}+$$
$$O\big(T^{-1}B_T\big) \times \big\{\vartheta_1(\tau_1,\tau_2)\odot\vartheta_2(\sigma_1,\sigma_2) + 1_{I_T}(\omega)\vartheta_3(\tau_1,\sigma_2)\odot\vartheta_4(\sigma_1,\sigma_2\big\}+$$
$$\frac{1}{T^2}\sum_{r,s=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)p^{(T)}_{r,s}(\tau_1,\sigma_1,\tau_2,\sigma_2).$$

We now bound each term of $\mathbb{E}|\Delta_{ij}|^2$, before summing over i and j as in (1), and then taking the integral over ω.

Bounding the term $O\big(T^{-1}B_T^{-1}\big)f^{XX}_{\omega}(\tau_1,\tau_2)f^{XX}_{-\omega}(\sigma_1,\sigma_2)$.

Write

$$\begin{aligned}
&\int_{[0,1]^4} f^{XX}_{\omega}(\tau_1,\tau_2)\overline{f^{XX}_{\omega}(\sigma_1,\sigma_2)} \times \overline{\varphi^{\omega}_i(\tau_1)}\varphi^{\omega}_j(\sigma_1)\varphi^{\omega}_i(\tau_2)\overline{\varphi^{\omega}_j(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2\\
&\qquad= \int_{[0,1]^2} f^{XX}_{\omega}(\tau_1,\tau_2)\overline{\varphi^{\omega}_i(\tau_1)}\varphi^{\omega}_i(\tau_2)d\tau_1 d\tau_2 \times \int_{[0,1]^2}\overline{f^{XX}_{\omega}(\sigma_1,\sigma_2)}\varphi^{\omega}_j(\sigma_1)\overline{\varphi^{\omega}_j(\sigma_2)}d\sigma_1 d\sigma_2\\
&\qquad= \lambda^{\omega}_i\lambda^{\omega}_j.
\end{aligned}$$

Taking the sum over i and j as in (1), by Lemma 4, we obtain

$$O\big(T^{-1}B_T^{-1}\big)\sum_{i,j}\frac{\lambda^{\omega}_i\lambda^{\omega}_j}{(\lambda_j+\zeta_T)^2} = O\big(T^{-1}B_T^{-1}\big)\sum_i \lambda^{\omega}_i \sum_j \frac{\lambda^{\omega}_j}{(\lambda^{\omega}_j+\zeta_T)^2} = O\big(B_T^{-1}\big)\times O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big).$$

Bounding the term $O\big(T^{-1}B_T^{-1}\big)1_{I_T}(\omega)f^{XX}_{\omega}(\tau_1,\sigma_2)f^{XX}_{-\omega}(\tau_2,\sigma_1)$.

Write

$$\begin{aligned}
&\int_{[0,1]^4} f^{XX}_{\omega}(\tau_1,\sigma_2)\overline{f^{XX}_{\omega}(\tau_2,\sigma_1)} \times \overline{\varphi^{\omega}_i(\tau_1)}\varphi^{\omega}_j(\sigma_1)\varphi^{\omega}_i(\tau_2)\overline{\varphi^{\omega}_j(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2\\
&\qquad= \int_{[0,1]^2} f^{XX}_{\omega}(\tau_1,\sigma_2)\overline{\varphi^{\omega}_i(\tau_1)\varphi^{\omega}_j(\sigma_2)}d\tau_1 d\sigma_2 \times \int_{[0,1]^2}\overline{f^{XX}_{\omega}(\sigma_1,\tau_2)}\varphi^{\omega}_j(\sigma_1)\varphi^{\omega}_i(\tau_2)d\sigma_1 d\tau_2\\
&\qquad= \lambda^{\omega}_i\int_{[0,1]}\overline{\varphi^{\omega}_i(\sigma_2)\varphi^{\omega}_j(\sigma_2)}d\sigma_2 \times \lambda^{\omega}_j\int_{[0,1]}\varphi^{\omega}_j(\tau_2)\varphi^{\omega}_i(\tau_2)d\tau_2.
\end{aligned}$$

This is dominated by $\lambda^{\omega}_i\lambda^{\omega}_j$ (the same argument as in the previous part has been applied here).

Bounding the term $O\big(T^{-1}\big) \times 1_{I_T}(\omega) f_{\omega}^{XX}(\tau_1, \sigma_2) f_{-\omega}^{XX,(1)}(\sigma_1, \tau_2)$.

Note that $\|\!|\mathscr{F}_{\omega}^{XX}|\!\|_1$ and $\|\!|\mathscr{F}_{\omega}^{XX,(1)}|\!\|_1$ are uniformly bounded. Applying Lemma 2, we obtain the bound

$$O(T^{-1}) 1_{I_T}(\omega) \zeta_T^{-2}.$$

Note that the length of I_T is of order B_T. Taking the integral over ω, then we get $O\Big(T^{-1}\zeta_T^{-2} B_T\Big) = O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big)$.

Bounding the term $O\big(T^{-1}\big) 1_{I_T}(\omega) \times f_{\omega}^{XX,(1)}(\tau_1, \sigma_2) f_{-\omega}^{XX}(\sigma_1, \tau_2)$.

For this term, we apply Lemma 2 as in the previous paragraph.

Bounding the term $O\big(T^{-1}B_T\big) \vartheta_1(\tau_1, \tau_2) \odot \vartheta_2(\sigma_1, \sigma_2)$.

Applying Lemma 2, we obtain $O\big(T^{-1}B_T\big) \times \zeta_T^{-2} = O\big(T^{-(2\beta-1)/(\alpha+2\beta)}\big)$.

Bounding the term $O(T^{-1}B_T) \vartheta_3(\tau_1, \sigma_2) \odot \vartheta_4(\sigma_1, \tau_2)$.

Applying Lemma 2 we obtain $O\big(T^{-1}B_T\zeta_T^{-2}\big) = O\big(T^{-(2\beta-1)/(\alpha+2\beta)}\big)$.

Bounding the term $\frac{1}{T^2}\sum_{r,s=0}^{T-1} W^{(T)}(\omega - \nu_s) W^{(T)}(\omega - \nu_r) p_{r,s}^{(T)}(\tau_1, \sigma_1, \tau_2, \sigma_2)$.

Applying Lemma 2 part (C) and Proposition 3 part (C), we have

$$T^{-1}\zeta_T^{-2} = \frac{1}{B_T} O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big).$$

In summary, we have upper bounded $\mathscr{S}_3$ as required.

Bounding $\mathscr{S}_1$

Recall that

$$\widehat{\mathscr{F}}^{YX}_{\omega,T} = \frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\mathscr{P}^{YX}_{\nu_s,T}; \qquad \mathscr{S}_1 = \left(\mathscr{F}^{YX}_{\omega} - \widehat{\mathscr{F}}^{YX}_{\omega,T}\right)\left[\mathscr{F}^{XX}_{\omega} + \zeta_T \mathscr{I}\right]^{-1}.$$

Now write

$$\begin{aligned}
R_t &:= \sum_s \mathscr{B}_{t-s} X_s \\
H_{\omega,T} &:= \frac{1}{\sqrt{T}}\left\{\sum_t h_T(t) R_t \exp\{-\mathbf{i}\omega t\}\right\} \\
L_{\omega,T} &:= H_{\omega,T} - \mathscr{F}^{B}_{\omega}\widetilde{X}_{\omega,T} \\
K_{\omega,T} &:= \frac{1}{\sqrt{T}}\left\{\sum_t h_T(t)\epsilon_t \exp\{-\mathbf{i}\omega t\}\right\}.
\end{aligned}$$

In this notation,

$$\begin{aligned}
\mathscr{F}^{B}_{\omega}\widetilde{X}_{\omega,T} &= \left\{\sum_t \mathscr{B}_t \exp\{-\mathbf{i}\omega t\}\right\}\frac{1}{\sqrt{T}}\left\{\sum_s X_s h_T(s)\exp\{-\mathbf{i}\omega s\}\right\} \\
&= \frac{1}{\sqrt{T}}\sum_{t,s}\mathscr{B}_t X_s h_T(s)\exp\{-\mathbf{i}\omega(t+s)\} \\
H_{\omega,T} &= \frac{1}{\sqrt{T}}\sum_{u,v} h_T(u+v)\mathscr{B}_u X_v \exp\{-\mathbf{i}\omega(u+v)\} \\
L_{\omega,T} &= \frac{1}{\sqrt{T}}\sum_{u,v}\{h_T(v) - h_T(u+v)\}\mathscr{B}_u X_v \exp\{-\mathbf{i}\omega(u+v)\}
\end{aligned}$$

The operator $\mathscr{P}^{YX}_{\omega,T} = \widetilde{Y}_{\omega,T}\otimes\overline{\left(\widetilde{X}_{\omega,T}\right)}$ can be decomposed as

$$\begin{aligned}
\mathscr{P}^{YX}_{\omega,T} &= \frac{1}{\sqrt{T}}\left\{\sum_t h_T(t)\left[R_t + \epsilon_t\right]\exp\{-\mathbf{i}\omega t\}\right\}\otimes\widetilde{X}_{-\omega,T} \\
&= \frac{1}{\sqrt{T}}\left\{\sum_t h_T(t)R_t\exp\{-\mathbf{i}\omega t\}\right\}\otimes\widetilde{X}_{-\omega,T} + \frac{1}{\sqrt{T}}\left\{\sum_t h_T(t)\epsilon_t\exp\{-\mathbf{i}\omega t\}\right\}\otimes\widetilde{X}_{-\omega,T} \\
&= H_{\omega,T}\otimes\widetilde{X}_{-\omega,T} + K_{\omega,T}\otimes\widetilde{X}_{-\omega,T}, \\
&= \mathscr{F}^{B}_{\omega}\widetilde{X}_{\omega,T}\otimes\widetilde{X}_{-\omega,T} + L_{\omega,T}\otimes\widetilde{X}_{-\omega,T} + K_{\omega,T}\otimes\widetilde{X}_{-\omega,T}.
\end{aligned}$$

Hence,

$$\widehat{\mathscr{F}}^{YX}_{\omega,T} = \frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\Big[\mathscr{F}^{B}_{\nu_s}\widetilde{X}_{\nu_s,T}\otimes\widetilde{X}_{-\nu_s,T} + L_{\nu_s,T}\otimes\widetilde{X}_{-\nu_s,T} + K_{\nu_s,T}\otimes\widetilde{X}_{-\nu_s,T}\Big]$$

$$= \mathscr{D}_1 + \mathscr{D}_2 + \mathscr{D}_3.$$

We can now decompose $\mathscr{S}_1$ based on the $\mathscr{D}_i$,

$$\mathscr{S}_1 = \Big(\mathscr{F}^{YX}_{\omega} - \widehat{\mathscr{F}}^{YX}_{\omega,T}\Big)\big[\mathscr{F}^{XX}_{\omega} + \zeta_T\mathscr{I}\big]^{-1}$$

$$= \Big(\mathscr{F}^{YX}_{\omega} - \mathscr{D}_1\Big)\big[\mathscr{F}^{XX}_{\omega} + \zeta_T\mathscr{I}\big]^{-1} + \mathscr{D}_2\big[\mathscr{F}^{XX}_{\omega} + \zeta_T\mathscr{I}\big]^{-1} + \mathscr{D}_3\big[\mathscr{F}^{XX}_{\omega} + \zeta_T\mathscr{I}\big]^{-1}$$

$$= \mathscr{S}_{11} + \mathscr{S}_{12} + \mathscr{S}_{13}.$$

By the Cauchy-Schwarz inequality

$$\left\vert\!\left\vert\!\left\vert \mathscr{S}_1 \right\vert\!\right\vert\!\right\vert_2^2 \leq 3\left\vert\!\left\vert\!\left\vert \mathscr{S}_{11} \right\vert\!\right\vert\!\right\vert_2^2 + 3\left\vert\!\left\vert\!\left\vert \mathscr{S}_{12} \right\vert\!\right\vert\!\right\vert_2^2 + 3\left\vert\!\left\vert\!\left\vert \mathscr{S}_{13} \right\vert\!\right\vert\!\right\vert_2^2.$$

We focus on each of the three terms in the following paragraphs.

Bounding $\mathscr{S}_{13}$.

Let D_3 be the integral kernel of $\mathscr{D}_3$. Then, similar to (1),

$$\mathbb{E}\left\vert\!\left\vert\!\left\vert \mathscr{S}_{13} \right\vert\!\right\vert\!\right\vert_2^2 = \sum_{i,j}\frac{1}{(\lambda^{\omega}_j+\zeta_T)^2}\int_{[0,1]^2}\mathbb{E}\Big[D_3(\tau_1,\sigma_1)\overline{D_3(\tau_2,\sigma_2)}\Big]\overline{\varphi^{\omega}_i(\tau_1)}\varphi^{\omega}_j(\sigma_1)\varphi^{\omega}_i(\tau_2)\overline{\varphi^{\omega}_j(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2. \tag{2}$$

We need to work on the expectation inside the integral. First,

$$\overline{D_3(\tau_2,\sigma_2)} = \sum_{r=0}^{T-1} W^{(T)}(\omega-\nu_r)\overline{K_{\nu_r,T}(\tau_2)\widetilde{X}_{-\nu_r,T}(\sigma_2)} = \sum_{r=0}^{T-1} W^{(T)}(\omega-\nu_r)K_{-\nu_r,T}(\tau_2)\widetilde{X}_{\nu_r,T}(\sigma_2)$$

By independence of X and ϵ,

$$\mathbb{E}\big(K_{\nu_s,T}(\tau_1)\widetilde{X}_{-\nu_s,T}(\sigma_1)K_{-\nu_r,T}(\tau_2)\widetilde{X}_{\nu_r,T}(\sigma_2)\big) = \mathbb{E}\big[K_{\nu_s,T}(\tau_1)K_{-\nu_r,T}(\tau_2)\big]\times\mathbb{E}\big[\widetilde{X}_{-\nu_s,T}(\sigma_1)\widetilde{X}_{\nu_r,T}(\sigma_2)\big].$$

Let q_ω be the spectral density function of $\{\epsilon_t\}$. By independence of the $\{\epsilon_t\}$, we have $q_\omega = q_0$. Apply Proposition 1 to the sequence $\{\epsilon_t\}$ to obtain

$$\begin{aligned}
&\mathbb{E}\Big[D_3(\tau_1,\sigma_1)\times\overline{D_3(\tau_2,\sigma_2)}\Big] = \\
&\frac{1}{T^2}\sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)\mathbb{E}\big[K_{\nu_s,T}(\tau_1)K_{-\nu_r,T}(\tau_2)\big]\times\mathbb{E}\big[\widetilde{X}_{\nu_r,T}(\sigma_2)\widetilde{X}_{-\nu_s,T}(\sigma_1)\big] \\
&= \frac{1}{T^2}\sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)\Big\{\delta_{rs}q_{\nu_s}(\tau_1,\tau_2)+\frac{1}{T}\vartheta_{1,\nu_r,\nu_s}(\tau_1,\tau_2)\Big\}\times \\
&\qquad\qquad \Big\{\delta_{rs}f^{XX}_{\nu_s}(\sigma_1,\sigma_2)+\frac{1}{T}\vartheta_{2,\nu_r,\nu_s}(\sigma_1,\sigma_2)\Big\} \\
&= \frac{1}{T^2}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_s)q_{\nu_s}(\tau_1,\tau_2)f^{XX}_{\nu_s}(\sigma_1,\sigma_2)+ \\
&\frac{1}{T^3}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_s)\times q_{\nu_s}(\tau_1,\tau_2)\vartheta_{2,\nu_r,\nu_s}(\sigma_1,\sigma_2)+ \\
&\frac{1}{T^3}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_s)\times \vartheta_{1,\nu_r,\nu_s}(\tau_1,\tau_2)f^{XX}_{\nu_s}(\sigma_1,\sigma_2)+ \\
&\frac{1}{T^4}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)\times \vartheta_{1,\nu_r,\nu_s}(\tau_1,\tau_2)\vartheta_{2,\nu_r,\nu_s}(\sigma_1,\sigma_2) \\
&= \frac{1}{T^2}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_s)q_{\nu_s}(\tau_1,\tau_2)f^{XX}_{\nu_s}(\sigma_1,\sigma_2)+\frac{1}{T^2B_T^2}\vartheta_{1,q}(\tau_1,\tau_2)\odot\vartheta_{2,q}(\sigma_1,\sigma_2).
\end{aligned} \tag{3}$$

Replacing ν_s by u_s, and using Taylor's expansion for f_{u_s} as in Lemma 3, we obtain

$$f^{XX}_{u_s}(\sigma_1,\sigma_2) = f^{XX}_{\omega}(\sigma_1,\sigma_2)+\frac{1}{1!}(u_s-\omega)f^{XX,(1)}_{\omega}(\sigma_1,\sigma_2)+\frac{1}{2!}(u_s-\omega)^2 g_{2,u_s,\omega}(\sigma_1,\sigma_2).$$

Thus, by Lemma 5, the first term of right hand side of (3) becomes

$$\begin{aligned}
\frac{1}{T^2}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega-u_s)q_0(\tau_1,\tau_2)f^{XX}_{\omega}(\sigma_1,\sigma_2)+ \\
\frac{1}{T^2}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega-u_s)\times(u_s-\omega)\times q_0(\tau_1,\tau_2)f^{XX,(1)}_{\omega}(\sigma_1,\sigma_2)+ \\
\frac{1}{2T^2}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega-u_s)\times(u_s-\omega)^2 q_0(\tau_1,\tau_2)g_{2,u_s,\omega}(\sigma_1,\sigma_2) \\
= O\big(T^{-1}B_T^{-1}\big)\times q_0(\tau_1,\tau_2)f^{XX}_{\omega}(\sigma_1,\sigma_2)+O\big(T^{-1}B_T\big)\times\vartheta_{1,g}(\tau_1,\tau_2)\odot\vartheta_{2,g}(\sigma_1,\sigma_2).
\end{aligned}$$

For each i and j we compute

$$\int_{[0,1]^4} q_0(\tau_1,\tau_2) f_\omega^{XX}(\sigma_1,\sigma_2)\overline{\varphi_i^\omega(\tau_1)}\varphi_j^\omega(\sigma_1)\varphi_i^\omega(\tau_2)\overline{\varphi_j^\omega(\sigma_2)} d\tau_1 d\sigma_1 d\tau_2 d\sigma_2$$

$$= \Big\langle q_0(\tau_1,\tau_2), \overline{\varphi_i^\omega(\tau_1)}\varphi_i^\omega(\tau_2)\Big\rangle \times \Big\langle f_\omega^{XX}(\sigma_1,\sigma_2), \varphi_j^\omega(\sigma_1)\overline{\varphi_j^\omega(\sigma_2)}\Big\rangle$$

$$= q_{0,ii} \times \lambda_j.$$

So, taking the sum over i and j as in (2), we deduce that

$$O\big(T^{-1}B_T^{-1}\big)\sum_{i,j}\frac{q_{0,ii}^\omega \lambda_j}{(\lambda_j+\zeta_T)^2} = O\big(T^{-1}B_T^{-1}\big)\Big\{\sum_i q_{0,ii}\Big\}\times\Big\{\sum_j \frac{\lambda_j}{(\lambda_j+\zeta_T)^2}\Big\} = B_T^{-1} O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big).$$

For $\vartheta_{1,q}(\tau_1,\tau_2)\odot\vartheta_{2,q}(\sigma_1,\sigma_2)$ and $\vartheta_{1,g}(\tau_1,\tau_2)\odot\vartheta_{2,g}(\sigma_1,\sigma_2)$, we now apply Lemma 2, and we obtain $O\big(T^{-1}B_T\big)\zeta_T^{-2} = O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big)$

Bounding $\mathscr{S}_{12}$

Let D_2 be the integral kernel of $\mathscr{D}_2$ and expand $\mathscr{D}_2 = \sum_{i,j} D_{2,ij}\varphi_i^\omega \otimes \overline{\varphi_j^\omega}$. Similarly with (1),

$$\mathbb{E}|\!|\!|\mathscr{S}_{12}|\!|\!|_2^2 = \sum_{i,j}\frac{\mathbb{E}D_{2,ij}^2}{(\lambda_j^\omega+\zeta_T)^2} \le \frac{1}{\zeta_T^2}\sum_{i,j}\mathbb{E}D_{2,ij}^2$$

$$D_2(\tau_1,\sigma_1)\overline{D_2(\tau_2,\sigma_2)} = \sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)L_{\nu_s,T}(\tau_1)\widetilde{X}_{-\nu_s,T}(\sigma_1)L_{-\nu_r,T}(\tau_2)\widetilde{X}_{\nu_r,T}(\sigma_2).$$

For simplicity, denote

$$U_s = L_{\nu_s,T}(\tau_1); \qquad V_{-s} = \widetilde{X}_{-\nu_s,T}(\sigma_1); \qquad U_{-r} = L_{-\nu_r,T}(\tau_2); \qquad V_r = \widetilde{X}_{\nu_r,T}(\sigma_2).$$

Using the fourth order cumulant equation, we have

$$\mathbb{E}\big(U_sV_{-s}U_{-r}V_r\big) = \mathbb{E}\big(U_sU_{-r}\big)\mathbb{E}\big(V_{-s}V_r\big) + \mathbb{E}\big(U_sV_{-s}\big)\mathbb{E}\big(U_{-r}V_r\big) + \mathbb{E}\big(U_sV_r\big)\mathbb{E}\big(U_{-r}V_{-s}\big)$$
$$+ \quad \mathrm{cum}\big(U_s, V_r, U_{-s}, V_{-r}\big).$$

We first estimate $\mathbb{E}\big(U_sV_{-s}\big)\mathbb{E}\big(U_{-r}V_r\big) + \mathbb{E}\big(U_sV_r\big)\mathbb{E}\big(U_{-r}V_{-s}\big)$. For each term,

$$\mathbb{E}\Big(L_{\omega_1,T}\otimes\widetilde{X}_{\omega_2,T}\Big) = \frac{1}{T}\sum_{u,v,t}\big\{h_T(v)-h_T(u+v)\big\}\times h_T(t)\times\mathscr{B}_u\mathbb{E}\Big(X_v\otimes X_t\Big)\exp\big\{-\mathbf{i}(\omega_1(u+v)+\omega_2 t)\big\}$$

$$= \frac{1}{T} \sum_{u,v,t} \big\{ h_T(v) - h_T(u+v) \big\} \times h_T(t) \times \mathscr{B}_u \mathscr{R}_{t-v} \exp\big\{ - \mathbf{i}(\omega_1(u+v) + \omega_2 t) \big\} \tag{4}$$

For each u there are at most $2|u|$ different values of v such that $h_T(v) - h_T(u+v) = \pm 1$. With t ranging from 0 to $T-1$, the multiplicity of the term $\mathscr{B}_u \mathscr{R}_w$ in (4) is no more than $2\ |u|$. Therefore,

$$\begin{aligned} \big\|\big|\mathbb{E}\big(L_{\omega_1,T} \otimes \widetilde{X}_{\omega_2,T}\big)\big\|\big|_1 &\le \frac{1}{T} \sum_{u,w} 2\big|u\big| \times \big\|\big|\mathscr{B}_u \mathscr{R}_w\big\|\big|_1 \le \frac{2}{T} \sum_{u,w} \big|u\big| \times \big\|\big|\mathscr{B}_u\big\|\big|_2 \times \big\|\big|\mathscr{R}_w\big\|\big|_2 \\ &\le \frac{2}{T} \Big\{ \sum_u |u| \times \big\|\big|\mathscr{B}_u\big\|\big|_2 \Big\} \times \Big\{ \sum_w \big\|\big|\mathscr{R}_w\big\|\big|_2 \Big\} = O(T^{-1}). \end{aligned}$$

And, consequently,

$$\frac{1}{T^2} \sum_{s,r=0}^{T-1} W^{(T)}(\omega - \nu_s) W^{(T)}(\omega - \nu_r) \mathbb{E}(U_s V_r) \mathbb{E}(U_{-r} V_{-s}) = T^{-2} B_T^{-2} \vartheta_{1,uv}(\tau_1, \sigma_2) \odot \vartheta_{2,uv}(\sigma_1, \tau_2).$$

Applying Lemma 2, we obtain an upper bound of the same order. Similarly, we have

$$\frac{1}{T^2} \sum_{s,r=0}^{T-1} W^{(T)}(\omega - \nu_s) W^{(T)}(\omega - \nu_r) \mathbb{E}(U_s V_{-s}) \mathbb{E}(U_{-r} V_r) = T^{-2} B_T^{-2} \vartheta_{3,uv}(\tau_1, \tau_2) \odot \vartheta_{4,uv}(\sigma_1, \sigma_2).$$

Again, Lemma 2 yields an upper bound of the same order.

For $\mathbb{E}(U_s U_{-r}) \mathbb{E}(V_r V_{-s})$, we expand $\mathbb{E}\big(L_{\omega_1,T} \otimes L_{\omega_2,T}\big)$

$$\begin{aligned} \mathbb{E}\big(L_{\omega_1,T} \otimes L_{\omega_2,T}\big) = \frac{1}{T} \sum_{u_1,v_1,u_2,v_2} & \big\{ h_T(v_1) - h_T(u_1+v_1) \big\} \times \big\{ h_T(v_2) - h_T(u_2+v_2) \big\} \\ & \times \exp\big\{ - \mathbf{i}\big(\omega_1(u_1+v_1) + \omega_2(u_2+v_2)\big) \big\} \times \mathscr{B}_{u_1} \mathscr{R}_{v_1 - v_2} \mathscr{B}^*_{u_2}. \end{aligned}$$

For u_1 and u_2 fixed, there are at most $2|u_1|$ and $2|u_2|$ values of v_1 and v_2, respectively, such that $h_T(v_1) - h_T(u_1+v_1), h_T(v_2) - h_T(u_2+v_2) = \pm 1$. Therefore,

$$\begin{aligned} \big\|\big|\mathbb{E}\big(L_{\omega_1,T} \otimes L_{\omega_2,T}\big)\big\|\big|_1 &\le \frac{4}{T} \sum_{u_1,u_2} \big|u_1\big| \times \big|u_2\big| \times \sum_v \big\|\big|\mathscr{B}_{u_1} \mathscr{R}_v \mathscr{B}^*_{u_2}\big\|\big|_1 \\ &\le \frac{4}{T} \Big\{ \sum_u |u| \times \big\|\big|\mathscr{B}_u\big\|\big|_2 \Big\}^2 \times \Big\{ \sum_v \big\|\big|\mathscr{R}_v\big\|\big|_2 \Big\}. \end{aligned}$$

But Proposition 1 implies that

$$\mathbb{E}\big(\widetilde{X}^{(T)}_{-\nu_s}(\sigma_1)\widetilde{X}^{(T)}_{\nu_r}(\sigma_2)\big) = \begin{cases} O\big(T^{-1}\big)\vartheta_{\nu_r,\nu_s}(\sigma_1,\sigma_2) & \text{if} \quad r \neq s \\ f^{XX}_{\nu_s}(\sigma_1,\sigma_2) + O(T^{-1})\vartheta_{\nu_s}(\sigma_1,\sigma_2) & \text{if} \quad r = s. \end{cases}$$

Therefore,

$$\begin{aligned} &\frac{1}{T^2}\sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)\mathbb{E}(U_sU_{-r})\mathbb{E}(V_rV_{-s}) \\ &= \frac{1}{T^2}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)^2\big\{f_{\nu_s}(\sigma_1,\sigma_2) + O(T^{-1})\vartheta_{\nu_s}(\sigma_1,\sigma_2)\big\} \times \mathbb{E}(U_sU_{-s}) + \\ &\quad \frac{1}{T^2}\sum_{s\neq r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)O(T^{-1})\vartheta_{\nu_r,\nu_s}(\sigma_1,\sigma_2)\mathbb{E}(U_sU_{-r}) \\ &= T^{-2}B_T^{-2}\vartheta_{5,uv}(\tau_1,\tau_2)\odot\vartheta_{6,uv}(\sigma_1,\sigma_2). \end{aligned}$$

Applying Lemma 2, we obtain the desired rate. The last term in $\mathscr{S}_{12}$ is the cumulant term

$$\begin{aligned} &cum\Big(L^{(T)}_{\nu_s}, \widetilde{X}^{(T)}_{-\nu_s}, L^{(T)}_{-\nu_r}, \widetilde{X}^{(T)}_{\nu_r}\Big) \\ &= \frac{1}{T^2} cum\Bigg(\sum_{u_1,v_1}\big\{h_T(v_1) - h_T(u_1+v_1)\big\}\mathscr{B}_{u_1}X_{v_1}\exp\big\{-\mathbf{i}\nu_s(u_1+v_1)\big\}, \sum_{v_2} h_T(v_2)X_{v_2}\exp\big\{\mathbf{i}\nu_s v_2\big\}, \\ &\qquad \sum_{u_3,v_3}\big\{h_T(v_3) - h_T(u_3+v_3)\big\}\mathscr{B}_{u_3}X_{v_3}\exp\big\{\mathbf{i}\nu_r(u_3+v_3))\big\}, \sum_{v_4} h_T(v_4)X_{v_4}\exp\big\{-\mathbf{i}\nu_r v_4\big\}\Bigg) \end{aligned}$$

For fixed u_1 and u_3, denote

$$\begin{aligned} L(u_1,u_3) := &\sum_{v_1,v_2,v_3,v_4}\Big\{h_T(v_1) - h_T(u_1+v_1)\Big\} \times \Big\{h_T(v_3) - h_T(u_3+v_3)\Big\} \times h_T(v_2) \times h_T(v_4) \times \\ &\exp\Big\{-\mathbf{i}\big[\nu_s(u_1+v_1) - \nu_s v_2 - \nu_r(u_3+v_3) + \nu_r v_4\big]\Big\} \times \mathrm{cum}\Big(\mathscr{B}_{u_1}X_{v_1}, X_{v_2}, \mathscr{B}_{u_3}X_{v_3}, X_{v_4}\Big). \end{aligned}$$

Then,

$$cum\Big(L^{(T)}_{\nu_s}, \widetilde{X}^{(T)}_{-\nu_s}, L^{(T)}_{-\nu_r}, \widetilde{X}^{(T)}_{\nu_r}\Big) = \frac{1}{T^2}\sum_{u_1,u_3\in\mathbb{Z}} L(u_1,u_3).$$

By the multilinearity of cumulants,

$$cum\Big(\big(\mathscr{B}_{u_1}X_{v_1}\big)(\tau_1), X_{v_2}(\tau_2), \big(\mathscr{B}_{u_3}X_{v_3}\big)(\sigma_1), X_{v_4}(\sigma_2)\Big) =$$

$$cum\Big(\int_0^1 b_{u_1}(\tau_1,\varsigma_1)X_{v_1}(\varsigma_1)d\varsigma_1, X_{v_2}(\tau_2), \int_0^1 b_{u_3}(\sigma_1,\varsigma_3)X_{v_3}(\varsigma_3)d\varsigma_3, X_{v_4}(\sigma_2)\Big) =$$
$$\int_0^1 b_{u_1}(\tau_1,\varsigma_1)\int_0^1 b_{u_3}(\sigma_1,\varsigma_3)\mathrm{cum}\Big(X_{v_1}(\varsigma_1), X_{v_2}(\tau_2), X_{v_3}(\varsigma_3)d, X_{v_4}(\sigma_2)\Big)d\varsigma_1 d\varsigma_3.$$

Denote the above term by $B_{u_1}\mathrm{cum}\Big(X_{v_1}, X_{v_2}, X_{v_3}, X_{v_4}\Big)B^*_{u_3}$. Replace $v_i = t_i + v_4$ for $i = 1,2,3,4$, then $t_4 = 0$. Now the exponential factor in $L(u_1,u_3)$ is

$$\exp\Big\{-\mathbf{i}\big[\nu_s(u_1+v_1) - \nu_s v_2 - \nu_r(u_3+v_3) + \nu_s v_4\big]\Big\}$$
$$= \exp\Big\{-\mathbf{i}\big[\nu_s u_1 - \nu_r u_3\big]\Big\} \times \exp\Big\{-\mathbf{i}\big[\nu_s t_1 - \nu_s t_2 - \nu_r t_3\big]\Big\} \times \exp\Big\{-\mathbf{i}v_4 \times 0\Big\}.$$

For the h_T factor, denote

$$H_T(t_1,t_2,t_3,v_4) := \Big\{h_T(t_1+v_4) - h_T(u_1+t_1+v_4)\Big\} \times \Big\{h_T(t_3+v_4) - h_T(u_3+t_3+v_4)\Big\} \times h_T(t_2+v_4) \times h_T(v_4).$$

Then

$$L(u_1,u_3) = \sum_{t_1,t_2,t_3,v_4} H_T(t_1,t_2,t_3,v_4) \times \exp\Big\{-\mathbf{i}\big(\nu_s u_1 - \nu_r u_3\big)\Big\}$$
$$\times \exp\Big\{-\mathbf{i}\big[\nu_s t_1 - \nu_s t_2 - \nu_r t_3\big]\Big\} \times B_{u_1}\mathrm{cum}\Big(X_{t_1}, X_{t_2}, X_{t_3}, X_0\Big)B^*_{u_3}.$$

Note that the number of v_4 such that $h_T(v_4) = 1$ is T. Letting $\mathcal{L}(u_1,u_3)$ be the integral operator of $L(u_1,u_3)$, we have

$$\left\|\!\left|\mathcal{L}(u_1,u_3)\right|\!\right\|_1 \le T \times \sum_{t_1,t_2,t_3} \left\|\!\left|\mathscr{B}_{u_1}\right|\!\right\|_2 \left\|\!\left|cum(X_{t_1},X_{t_2},X_{t_3},X_0)\right|\!\right\|_2 \left\|\!\left|\mathscr{B}_{u_3}\right|\!\right\|_2.$$

By assumptions (B4) and (B6),

$$cum\Big(L^{(T)}_{\nu_s}, \widetilde{X}^{(T)}_{-\nu_s}, L^{(T)}_{-\nu_r}, \widetilde{X}^{(T)}_{\nu_r}\Big) = \frac{1}{T^2}\sum_{u_1,u_3} L(u_1,u_3) = O(T^{-1}).$$

Similarly to previous steps, applying Lemma 2 and Proposition 3 yields

$$O\Big(T^{-1}\zeta_T^{-2}\Big) = \frac{1}{B_T} O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big).$$

Bounding $\mathscr{S}_{11}$

The steps in this case are similar to those involved in bounding $\mathscr{S}_3$. Recall that

$$\mathscr{D}_1 - \mathscr{F}_\omega^{YX} = \frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\mathscr{F}_{\nu_s}^B \widetilde{X}_{\nu_s}^{(T)} \otimes \widetilde{X}_{-\nu_s}^{(T)} - \mathscr{F}_\omega^{YX} = \frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\mathscr{F}_{\nu_s}^B \mathscr{P}_{\nu_s,T}^{XX} - \mathscr{F}_\omega^B \mathscr{F}_\omega^{XX}.$$

We first work with $\mathbb{E}\big[(\mathscr{D}_1 - \mathscr{F}_\omega^{YX}) \otimes (\mathscr{D}_1 - \mathscr{F}_\omega^{YX})^*\big]$. We have

$$\mathbb{E}\big[(\mathscr{D}_1 - \mathscr{F}_\omega^{YX}) \otimes (\mathscr{D}_1 - \mathscr{F}_\omega^{YX})^*\big] = \mathbb{E}\big[\mathscr{D}_1 \otimes \mathscr{D}_1^*\big] - \mathbb{E}\mathscr{D}_1 \otimes \mathbb{E}\mathscr{D}_1^* + (\mathbb{E}\mathscr{D}_1 - \mathscr{F}_\omega^{YX}) \otimes (\mathbb{E}\mathscr{D}_1 - \mathscr{F}_\omega^{YX})^*.$$

To determine $\mathbb{E}\mathscr{D}_1$, we use Proposition 1

$$\mathbb{E}\Big[\frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\mathscr{F}_{\nu_s}^B \mathscr{P}_{\nu_s,T}^{XX}\Big] = \frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\mathscr{F}_{\nu_s}^B \mathbb{E}\Big[\mathscr{P}_{\nu_s,T}^{XX}\Big]$$
$$= \frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\mathscr{F}_{\nu_s}^B\Big\{\mathscr{F}_{\nu_s}^{XX} + \frac{1}{T}\mathscr{V}_{1,\nu_s}\Big\} = \frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\mathscr{F}_{\nu_s}^B \mathscr{F}_{\nu_s}^{XX} + \frac{1}{TB_T}\mathscr{V}_{1,\nu},$$

with $\left\|\!\left|\mathscr{V}_{1,\nu}\right|\!\right\|_1 = O(1)$ uniformly over $\nu \in [0,2\pi]$. Taylor expanding, we have

$$f_{u_s}^{XX} = f_\omega^{XX} + \sum_{j=1}^{p-1}\frac{(u_s-\omega)^j}{j!} f_\omega^{XX,(j)} + (u_s-\omega)^p g_{p,u_s,\omega}$$
$$f_{u_s}^{B} = f_\omega^{B} + \sum_{j=1}^{p-1}\frac{(u_s-\omega)^j}{j!} \times f_\omega^{B,(j)} + (u_s-\omega)^p g_{p,u_s,\omega}^B.$$

Thus,

$$\frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\mathscr{F}_{\nu_s}^B \mathscr{F}_{\nu_s}^{XX} = \frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\Big\{\sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\mathscr{F}_\omega^{B,(k)}\mathscr{F}_\omega^{XX,(j)} \times (u_s-\omega)^{k+j} + \mathscr{V}_{u_s,\omega} \times (u_s-\omega)^p\Big\}.$$

Note that $\left\|\!\left|\mathscr{F}_\omega^{B,(k)}\mathscr{F}_\omega^{XX,(j)}\right|\!\right\|_1 < \infty$. Using the same idea as in Proposition 2, we have

$$\frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\mathscr{F}_{\nu_s}^B \mathscr{F}_{\nu_s}^{XX} = \mathscr{F}_\omega^B \mathscr{F}_\omega^{XX} + \frac{1}{B_T T}\mathscr{V}_\omega,$$

with $\left\|\!\left|\mathscr{V}_\omega\right|\!\right\|_1 = O(1)$ uniformly over $\omega \in [0,2\pi]$. It follows that $\mathbb{E}\mathscr{D}_1 - \mathscr{F}_\omega^B\mathscr{F}_\omega^{XX} = \frac{1}{TB_T}\mathscr{V}_\omega$. Then, $\mathbb{E}\mathscr{D}_1 - \mathscr{F}_\omega^{YX}) \otimes (\mathbb{E}\mathscr{D}_1 - \mathscr{F}_\omega^{YX})^* = \frac{1}{T^2B_T^2}\mathscr{V}_\omega \otimes \mathscr{V}_\omega^*$. Finally, we apply Lemma 2, which takes care of the term $\mathbb{E}\mathscr{D}_1 - \mathscr{F}_\omega^{YX}$ Now we need to bound $\mathbb{E}\Big[D_1(\varsigma_1,\sigma_1)\overline{D_1(\varsigma_2,\sigma_2)}\Big]$. This equals

$$\frac{1}{T^2}\mathbb{E}\left[\sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)\int_{[0,1]} f_{\nu_s}^B(\varsigma_1,\tau_1)p_{\nu_s}^{(T)}(\tau_1,\sigma_1)d\tau_1 \times \int_{[0,1]} f_{-\nu_r}^B(\varsigma_2,\tau_2)p_{-\nu_r}^{(T)}(\tau_2,\sigma_2)d\tau_2\right]$$

$$= \frac{1}{T^2}\sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)\int_{[0,1]^2} f^B_{\nu_s}(\varsigma_1,\tau_1)f^B_{-\nu_r}(\varsigma_2,\tau_2)\mathbb{E}\left[p^{(T)}_{\nu_s}(\tau_1,\sigma_1)p^{(T)}_{-\nu_r}(\tau_2,\sigma_2)\right]d\tau_1 d\tau_2. \tag{5}$$

By Proposition 4,

$$\begin{aligned}
&\mathbb{E}\left[p^{(T)}_{\nu_s}(\tau_1,\sigma_1)p^{(T)}_{-\nu_r}(\tau_2,\sigma_2)\right] = \\
&\qquad \mathbb{E}\Big[p^{(T)}_{\nu_s}(\tau_1,\sigma_1)\Big]\times\mathbb{E}\Big[p^{(T)}_{-\nu_r}(\tau_2,\sigma_2)\Big]+p^{(T)}_{r,s}(\tau_1,\sigma_1,\tau_2,\sigma_2)+ \\
&\qquad \eta(\nu_r-\nu_s)f^{XX}_{\nu_s}(\tau_1,\tau_2)f^{XX}_{-\nu_s}(\sigma_1,\sigma_2)+\frac{1}{T}\eta(\nu_s-\nu_r)\vartheta_{1,\nu_s,\nu_r,f}(\tau_1,\tau_2)\odot\vartheta_{2,\nu_s,\nu_r,f}(\sigma_1,\sigma_2)+ \\
&\qquad \eta(\nu_s+\nu_r)f^{XX}_{\nu_s}(\tau_1,\sigma_2)f^{XX}_{-\nu_s}(\sigma_1,\tau_2)+\frac{1}{T}\eta(\nu_s+\nu_r)\vartheta_{3,\nu_s,\nu_r,f}(\sigma_1,\tau_2)\odot\vartheta_{4,\nu_s,\nu_r,f}(\tau_1,\sigma_2)+ \\
&\qquad \frac{1}{T^2}\vartheta_{1,\nu_s,\nu_r}(\tau_1,\sigma_1)\times\vartheta_{2,\nu_s,\nu_r}(\tau_2,\sigma_2)+\frac{1}{T^2}\vartheta_{3,\nu_s,\nu_r}(\tau_1,\sigma_2)\times\vartheta_{4,\nu_s,\nu_r}(\sigma_1,\tau_2) \\
&\qquad =\mathbb{E}\Big[p^{(T)}_{\nu_s}(\tau_1,\sigma_1)\Big]\times\mathbb{E}\Big[p^{(T)}_{-\nu_r}(\tau_2,\sigma_2)\Big]+\mathbb{G}_{s,r}.
\end{aligned} \tag{6}$$

Moreover,

$$\begin{aligned}
&\mathbb{E}\big[D_1(\varsigma_1,\sigma_1)\big]\times\mathbb{E}\big[\overline{D_1(\varsigma_2,\sigma_2)}\big] = \\
&\frac{1}{T^2}\sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)\int_{[0,1]^2} f^B_{\nu_s}(\varsigma_1,\tau_1)f^B_{-\nu_r}(\varsigma_2,\tau_2)\mathbb{E}\left[p^{(T)}_{\nu_s}(\tau_1,\sigma_1)\right]\times\mathbb{E}\left[p^{(T)}_{-\nu_r}(\tau_2,\sigma_2)\right]d\tau_1 d\tau_2.
\end{aligned} \tag{7}$$

Combining (5), (6) and (7), we obtain

$$\begin{aligned}
&\mathbb{E}\Big[D_1(\varsigma_1,\sigma_1)\overline{D_1(\varsigma_2,\sigma_2)}\Big]-\mathbb{E}D_1(\varsigma_1,\sigma_1)\mathbb{E}\overline{D_1(\varsigma_2,\sigma_2)} = \\
&\frac{1}{T^2}\sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)\int_{[0,1]^2} f^B_{\nu_s}(\varsigma_1,\tau_1)f^B_{-\nu_r}(\varsigma_2,\tau_2)\times\mathbb{G}_{s,r}d\tau_1 d\tau_2.
\end{aligned}$$

We must now consider each term resulting from the summands consituting $\mathbb{G}_{s,r}$.

First we begin with the summand $f^{XX}_{u_s}(\tau_1,\tau_2)f^{XX}_{-u_s}(\sigma_1,\sigma_2)$ which contributes

$$\frac{1}{T^2}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega-u_s)\int_{[0,1]^2} f^B_{u_s}(\varsigma_1,\tau_1)f^B_{-u_s}(\varsigma_2,\tau_2)f^{XX}_{u_s}(\tau_1,\tau_2)f^{XX}_{-u_s}(\sigma_1,\sigma_2)d\tau_1 d\tau_2.$$

Taylor expanding yields

$$f^B_{u_s}=f^B_\omega+(u_s-\omega)f^{B,(1)}_\omega+(u_s-\omega)^2 g^B_{2,u_s,\omega}$$

$$f_{u_s}^{XX} = f_\omega^{XX} + (u_s-\omega) f_\omega^{XX,(1)} + (u_s-\omega)^2 g_{2,u_s,\omega}.$$

Then,

$$f_{u_s}^B f_{-u_s}^B f_{u_s}^{XX} f_{-u_s}^{XX} = f_\omega^B f_{-\omega}^B f_\omega f_{-\omega} + (u_s-\omega)\Big\{ f_\omega^{B,(1)} f_{-\omega}^B f_\omega^{XX} f_{-\omega}^{XX} + f_\omega^B f_{-\omega}^{B,(1)} f_\omega^{XX} f_{-\omega}^{XX} + \\ f_\omega^B f_{-\omega}^B f_\omega^{XX,(1)} f_{-\omega}^{XX} + f_\omega^B f_{-\omega}^B f_\omega^{XX} f_{-\omega}^{XX,(1)} \Big\} + (u_s-\omega)^2 \vartheta_{1,u_s} \odot \vartheta_{2,u_s}.$$

These further terms are treated individually in the following bullet points:

- $f_\omega^B f_{-\omega}^B f_\omega^{XX} f_{-\omega}^{XX}$: Recall that

$$f_\omega^B(\varsigma,\tau) = \sum_{k,l} b_{kl}^\omega \varphi_k^\omega(\varsigma) \overline{\varphi_l^\omega(\tau)}.$$

Then,

$$\frac{1}{T^2}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)^2 \int_{[0,1]^2} f_\omega^B(\varsigma_1,\tau_1) f_{-\omega}^B(\varsigma_2,\tau_2) f_\omega^{XX}(\tau_1,\tau_2) f_{-\omega}^{XX}(\sigma_1,\sigma_2) d\tau_1 d\tau_2$$
$$= \frac{O(1)}{TB_T}\int_{[0,1]^2} \Big\{\sum_{k,l} b_{kl}^\omega \varphi_k^\omega(\varsigma_1)\overline{\varphi_l^\omega(\tau_1)}\Big\}\Big\{\sum_l \lambda_l^\omega \varphi_l^\omega(\tau_1)\overline{\varphi_l^\omega(\tau_2)}\Big\}\Big\{\sum_{m,l} \overline{b_{ml}^\omega} \varphi_l^\omega(\tau_2)\overline{\varphi_m^\omega(\varsigma_2)}\Big\} \times f_{-\omega}^{XX}(\sigma_1,\sigma_2) d\tau_1 d\tau_2$$
$$= \frac{O(1)}{TB_T}\Big\{\sum_{k,l,m} b_{kl}^\omega \overline{b_{ml}^\omega} \lambda_l^\omega \varphi_k^\omega(\varsigma_1)\overline{\varphi_m^\omega(\varsigma_2)}\Big\} \times f_{-\omega}^{XX}(\sigma_1,\sigma_2).$$

Multiplying by $\overline{\varphi_i^\omega(\varsigma_1)}\varphi_i^\omega(\varsigma_2)\varphi_j^\omega(\sigma_1)\overline{\varphi_j^\omega(\sigma_2)}$ and then integrating, we have

$$\int_{[0,1]^2} \Big\{\sum_{k,m,l} b_{kl}^\omega \overline{b_{ml}^\omega}\lambda_l^\omega \varphi_k^\omega(\varsigma_1)\overline{\varphi_m^\omega(\varsigma_2)}\Big\}\overline{\varphi_i^\omega(\varsigma_1)}\varphi_i^\omega(\varsigma_2) d\varsigma_1 d\varsigma_2 \int_{[0,1]^2} f_{-\omega}^{XX}(\sigma_1,\sigma_2)\varphi_j^\omega(\sigma_1)\overline{\varphi_j^\omega(\sigma_2)} d\sigma_1 d\sigma_2$$
$$= \sum_l |b_{il}^\omega|^2 \lambda_l^\omega \times \lambda_j^\omega.$$

As in (1), dividing by $(\lambda_j+\zeta_T)^2$ and taking the sum over i and j yields

$$\frac{1}{TB_T}\sum_{i,j}\sum_l \frac{|b_{il}^\omega|^2\lambda_l^\omega \times \lambda_j^\omega}{(\lambda_j^\omega+\zeta_T)^2} = \sum_{i,l}|b_{il}^\omega|^2\lambda_l^\omega \frac{1}{B_T T}\sum_j \frac{\lambda_j^\omega}{(\lambda_j^\omega+\zeta_T)^2} \le \sum_l l^{-2\beta-\alpha}\frac{O(1)}{B_T} T^{-(2\beta-1)/(2\beta+\alpha)}$$
$$= \frac{O(1)}{B_T} T^{-(2\beta-1)/(2\beta+\alpha)}.$$

- $f_\omega^{B,(1)} f_{-\omega}^B f_\omega^{XX} f_{-\omega}^{XX} + f_\omega^B f_{-\omega}^{B,(1)} f_\omega^{XX} f_{-\omega}^{XX} + f_\omega^B f_{-\omega}^B f_\omega^{XX,(1)} f_{-\omega}^{XX} + f_\omega^B f_{-\omega}^B f_\omega^{XX} f_{-\omega}^{XX,(1)}$: Recall the result in Lemma 5, which states that

$$\frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)^2 \times (\omega-u_s) = O\big(T^{-1}B_T^{-2}\big).$$

Note that

$$\int_{[0,1]^2} f_\omega^{B,(1)}(\varsigma_1,\tau_1) f_{-\omega}^{B}(\varsigma_2,\tau_2) f_\omega^{XX}(\tau_1,\tau_2) f_{-\omega}^{XX}(\sigma_1,\sigma_2) d\tau_1 d\tau_2$$

can be written in the form of $\vartheta_1(\varsigma_1,\varsigma_2)\times\vartheta_2(\sigma_1,\sigma_2)$ so that their corresponding operators $\mathscr{V}_1$,$\mathscr{V}_2$ have finite nuclear norm. Now we may apply Lemma 2.

- $(u_s-\omega)^2\vartheta_{1,u_s}\odot\vartheta_{2,u_s}$: Recall the result in Lemma 5, stating that

$$\frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)^2 \times (\omega-u_s)^2 = O\big(B_T\big).$$

It follows that $\frac{1}{T^2}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)^2(\omega-u_s)^2 \times \vartheta_{1,u_s}(\varsigma_1,\varsigma_2)\odot\vartheta_{2,u_s}(\sigma_1,\sigma_2)$ has the form $O(T^{-1}B_T)\vartheta_{1,u}(\varsigma_1,\varsigma_2)\odot\vartheta_{2,u}(\sigma_1,\sigma_2)$. We may now apply Lemma 2 as in the previous part.

This concludes our treatment of the summand $f_{u_s}(\tau_1,\tau_2)f_{-u_s}(\sigma_1,\sigma_2)$ in $\mathbb{G}_{s,r}$.

We move on to the summand $f_{u_s}^{XX}(\tau_1,\sigma_2)f_{-u_s}^{XX}(\sigma_1,\tau_2)$ in $\mathbb{G}_{s,r}$. This contributes the term

$$\frac{1}{T^2}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega+u_s)\int_{[0,1]^2} f_{u_s}^{B}(\varsigma_1,\tau_1)f_{-u_s}^{B}(\varsigma_2,\tau_2)f_{u_s}(\tau_1,\sigma_2)f_{-u_s}(\sigma_1,\tau_2)d\tau_1 d\tau_2$$

We apply the same process as with the previous term $f_{u_s}^{XX}(\tau_1,\tau_2)f_{-u_s}^{XX}(\sigma_1,\sigma_2)$ in $\mathbb{G}_{s,r}$. This is done in the following bullet points:

- $f_\omega^B f_{-\omega}^B f_\omega^{XX} f_{-\omega}^{XX}$: We start with the integral

$$\begin{aligned}\int_{[0,1]} f_\omega^{B}(\varsigma_1,\tau_1) f_\omega^{XX}(\tau_1,\sigma_2)d\tau_1 &= \int_{[0,1]}\Big\{\sum_{k,l} b_{kl}^\omega\varphi_k^\omega(\varsigma_1)\overline{\varphi_l^\omega(\tau_1)}\Big\}\Big\{\sum_l \lambda_l^\omega\varphi_l^\omega(\tau_1)\overline{\varphi_l^\omega(\sigma_2)}\Big\}d\tau_1\\ &= \sum_{k,l} b_{kl}^\omega\lambda_l^\omega\varphi_k^\omega(\varsigma_1)\overline{\varphi_l^\omega(\sigma_2)}\end{aligned}$$

$$\begin{aligned}\int_{[0,1]} f_{-\omega}^{B}(\varsigma_2,\tau_2) f_{-\omega}^{XX}(\sigma_1,\tau_2)d\tau_2 &= \int_{[0,1]}\Big\{\sum_{u,v} \overline{b_{uv}^\omega}\,\overline{\varphi_u^\omega(\varsigma_2)}\varphi_v^\omega(\tau_2)\Big\}\Big\{\sum_v \lambda_v^\omega\overline{\varphi_v^\omega(\sigma_1)}\varphi_v^\omega(\tau_2)\Big\}d\tau_2\\ &= \sum_{u,v} \overline{b_{uv}^\omega}\lambda_{v'}^\omega\overline{\varphi_u^\omega(\varsigma_2)\varphi_{v'}^\omega(\sigma_1)}\end{aligned}$$

Then,

$$\int_{[0,1]^2} f^B_\omega(\varsigma_1,\tau_1) f^B_{-\omega}(\varsigma_2,\tau_2) f^{XX}_\omega(\tau_1,\sigma_2) f^{XX}_{-\omega}(\sigma_1,\tau_2) d\tau_1 d\tau_2 =$$
$$\Big\{ \sum_{k,l} b^\omega_{kl}\lambda^\omega_l \varphi^\omega_k(\varsigma_1)\overline{\varphi^\omega_l(\sigma_2)}\Big\} \times \Big\{ \sum_{u,v} \overline{b^\omega_{uv}}\lambda^\omega_{v'} \overline{\varphi^\omega_u(\varsigma_2)\varphi^\omega_{v'}(\sigma_1)}\Big\}.$$

We multiply by $\overline{\varphi^\omega_i(\varsigma_1)}\varphi^\omega_j(\sigma_1)\varphi^\omega_i(\varsigma_2)\overline{\varphi^\omega_j(\sigma_2)}$ and integrate to obtain

$$b^\omega_{ij'}\lambda^\omega_{j'}\overline{b^\omega_{ij'}}\lambda^\omega_j.$$

Then,

$$\frac{1}{TB_T}\sum_{i,j}\frac{\left|b^\omega_{ij'}\right|^2\lambda^\omega_{j'}\lambda^\omega_j}{(\lambda^\omega_j+\zeta_T)^2} = \frac{1}{TB_T}\sum_j \frac{\sum_i |b^\omega_{ij'}|^2\lambda^\omega_{j'}\lambda^\omega_j}{(\lambda^\omega_j+\zeta_T)^2} = \frac{1}{TB_T}\sum_j\frac{(j')^{-2\beta-\alpha}\lambda^\omega_j}{(\lambda^\omega_j+\zeta_T)^2}$$
$$= \frac{O(1)}{B_T} O\big(T^{-(2\beta-1)/(\alpha+2\beta)}\big).$$

- $f^{B,(1)}_\omega f^B_{-\omega} f^{XX}_\omega f^{XX}_{-\omega} + f^B_\omega f^{B,(1)}_{-\omega} f^{XX}_\omega f^{XX}_{-\omega} + f^B_\omega f^B_{-\omega} f^{XX,(1)}_\omega f^{XX}_{-\omega} + f^B_\omega f^B_{-\omega} f^{XX}_\omega f^{XX,(1)}_{-\omega}$: Note that

$$\int_{[0,1]^2} f^{B,(1)}_\omega(\varsigma_1,\tau_1) f^B_{-\omega}(\varsigma_2,\tau_2) f^{XX}_\omega(\tau_1,\sigma_2) f^{XX}_{-\omega}(\sigma_1,\tau_2) d\tau_1 d\tau_2$$

can be rewritten via a form $\vartheta_1(\varsigma_1,\sigma_2)\times\vartheta_2(\sigma_1,\varsigma_2)$ with their correspoding operators have finite Schatten 1-norm $\left\|\!\left|\mathscr{V}_1\right|\!\right\|_1, \left\|\!\left|\mathscr{V}_2\right|\!\right\|_1 < C$. Applying Lemma 5,

$$\frac{1}{T^2}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega+u_s)(\omega-u_s) = 1_{I_T}(\omega)O(T^{-1}),$$

and then we obtain the bound

$$1_{I_T}(\omega)O\big(T^{-1}\big)\vartheta_1(\varsigma_1,\sigma_2)\vartheta_2(\sigma_1,\varsigma_2).$$

We multiply this by $\overline{\varphi^\omega_i(\varsigma_1)}\varphi^\omega_j(\sigma_1)\varphi^\omega_i(\varsigma_2)\overline{\varphi^\omega_j(\sigma_2)}$ and integrate. Applying Lemma 2, we obtain a bound of order $O\big(T^{-1}\zeta_T^{-2}\big)$. Now we integrate over $\omega\in I_T$, obtaining an integral of order

$$O\big(T^{-1}\zeta_T^{-2}B_T\big) = O\big(T^{-(2\beta-1)/(\alpha+2\beta)}\big).$$

• $(u_s-\omega)^2\vartheta_{1,u_s}\odot\vartheta_{2,u_s}$: The same argument is applied here, using Lemma 2 and Lemma 5.

Now we move on to the summand $\frac{1}{T}\eta(\nu_s-\nu_r)\vartheta_{1,\nu_s,\nu_r,f}(\tau_1,\tau_2)\odot\vartheta_{2,\nu_s,\nu_r,f}(\sigma_1,\sigma_2)$ of $\mathbb{G}_{s,r}$. Note that

$$\frac{1}{T^3}\sum_{s=0}^{T-1}W^{(T)}(\omega-\nu_s)^2 f_{\nu_s}f^B_{-\nu_s}\vartheta_{1,\nu_s,\nu_r,f}\odot\vartheta_{2,\nu_s,\nu_r,f}=\frac{1}{T^2}\times\frac{1}{T}\sum_{s=0}^{T-1}W^{(T)}(\omega-\nu_s)^2\vartheta_{1,\nu_s,b}\odot\vartheta_{2,\nu_s,b}.$$

and the latter is $O\big(T^{-2}B_T^{-1}\big)$, uniformly over s. Similarly to our treatment of $\mathscr{S}_3$, we may apply Lemma 2.

The same argument can be applied to the summand $\frac{1}{T}\eta(\nu_s+\nu_r)\vartheta_{3,\nu_s,\nu_r,f}(\sigma_1,\tau_2)\odot\vartheta_{4,\nu_s,\nu_r,f}(\tau_1,\sigma_2)$ of $\mathbb{G}_{s,r}$.

We thus move on to the summands $\vartheta_{1\nu_s,\nu_r}\times\vartheta_{2,\nu_s,\nu_r}$ and $\vartheta_{3,\nu_s,\nu_r}\times\vartheta_{4,\nu_s,\nu_r}$ of $\mathbb{G}_{s,r}$. The quantity

$$\frac{1}{T^4}\sum_{s,r=0}^{T-1}W^{(T)}(\omega-\nu_s)^2\big[\vartheta_{1,\nu_s,\nu_r}\times\vartheta_{2,\nu_s,\nu_r}+\vartheta_{3,\nu_s,\nu_r}\times\vartheta_{4,\nu_s,\nu_r}\big].$$

is uniformly of order $O\big(T^{-2}B_T^{-1}\big)$. Similarly with the etimation of $\mathscr{S}_3$ we apply Lemma 2.

Finally, we turn to the summand $p^{(T)}_{r,s}(\tau_1,\sigma_1,\tau_2,\sigma_2)$ of $\mathbb{G}_{s,r}$. We need to bound

$$\frac{1}{T^2}\sum_{s=0}^{T-1}W^{(T)}(\omega-u_s)W^{(T)}(\omega-u_r)\int_{[0,1]^2}f^B_{u_s}(\varsigma_1,\tau_1)f^B_{-u_r}(\varsigma_2,\tau_2)p^{(T)}_{r,s}(\tau_1,\sigma_1,\tau_2,\sigma_2)d\tau_1 d\tau_2. \quad (8)$$

Let $\mathscr{F}^B_{u_s}\mathscr{P}^{(T)}_{r,s}\mathscr{F}^B_{-u_r}$ be the operator corresponding to the kernels in the integrand. Then,

$$\big\|\!\big|\mathscr{F}^B_{u_s}\mathscr{P}^{(T)}_{r,s}\mathscr{F}^B_{-u_r}\big\|\!\big|_1\le\big\|\!\big|\mathscr{F}^B_{u_s}\big\|\!\big|_2\times\big\|\!\big|\mathscr{P}^{(T)}_{r,s}\big\|\!\big|_2\times\big\|\!\big|\mathscr{F}^B_{u_s}\big\|\!\big|_2=O(T^{-1}).$$

Applying Lemma 2 and Proposition 3, we obtain a bound of order $O(T^{-1}\zeta_T^{-2})=\frac{1}{B_T}O\big(T^{(2\beta-1)/(\alpha+2\beta)}\big)$.

Bounding $\mathscr{S}_2$

$$\mathscr{S}_2:=\Big(\widehat{\mathscr{F}}^{YX}_{\omega,T}-\mathscr{F}^{YX}_{\omega}\Big)\Big(\big[\mathscr{F}^{XX}_{\omega}+\zeta_T\mathscr{I}\big]^{-1}-\Big[\widehat{\mathscr{F}}^{XX}_{\omega}+\zeta_T\mathscr{I}\Big]^{-1}\Big)$$

$$= \Big(\widehat{\mathscr{F}}^{YX}_{\omega,T} - \mathscr{F}^{YX}_{\omega}\Big)\Big(\mathscr{F}^{XX}_{\omega} + \zeta_T \mathscr{I}\Big)^{-1}\Delta\Big(\mathscr{F}^{XX}_{\omega} + \zeta_T \mathscr{I}\Big)^{-1}\Big(\mathscr{I} + \Delta\big[\mathscr{F}^{XX}_{\omega} + \zeta_T \mathscr{I}\big]^{-1}\Big)^{-1}$$
$$= \mathscr{S}_1\Delta\Big(\mathscr{F}^{XX}_{\omega} + \zeta_T \mathscr{I}\Big)^{-1}\Big(\mathscr{I} + \Delta\big[\mathscr{F}^{XX}_{\omega} + \zeta_T \mathscr{I}\big]^{-1}\Big)^{-1}.$$

The product of the third (second) and fourth (third) terms has finite nuclear norm by our treatment of $\mathscr{S}_3$. The last term has finite nuclear norm on the set G_T. Then, we have the order of $T^{-(2\beta-1)/(\alpha+2\beta)}$ on G_T in the Hilbert-Schimdt norm.

In conclusion, all terms have been shown to be bounded above by at most $\frac{1}{B_T}O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big)$ on the set G_T, and the proof is complete.

8. Appendix

This Appendix contains the statements and proofs of several auxiliary results (namely Lemmas 1-6 and Propositions 1-7) that are required in the proof of Theorem 1.

Lemma 1. *Let $\{\varphi_i\}$ be a complete orthonormal system of functions in $L^2\big([0,1]^2,\mathbb{C}\big)$ and $\phi(\tau,\sigma)$ be a random bivariate function in $L^2\big([0,1]^2,\mathbb{C}\big)$ with induced integral operator Φ, then*

$$(A)\qquad \phi(\tau,\sigma) = \sum_{i,j}\phi_{ij}\varphi_i(\tau)\overline{\varphi_j(\sigma)}, a.s.,$$

$$(B)\qquad \mathbb{E}\big|\phi_{ij}\big|^2 = \int_{[0,1]^2}\mathbb{E}\big[\phi(\tau_1,\sigma_1)\overline{\phi(\tau_2,\sigma_2)}\big]\times\overline{\varphi_i(\tau_1)}\varphi_j(\sigma_1)\varphi_i(\tau_2)\overline{\varphi_j(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2$$

$$(C)\qquad \mathbb{E}|||\Phi|||_2^2 = \sum_{i,j}\int_{[0,1]^2}\mathbb{E}\big[\phi(\tau_1,\sigma_1)\overline{\phi(\tau_2,\sigma_2)}\big]\times\overline{\varphi_i(\tau_1)}\varphi_j(\sigma_1)\varphi_i(\tau_2)\overline{\varphi_j(\sigma_2)}d\tau_1 d\tau_2 d\sigma_1 d\sigma_2.$$

Proof. Since $\{\varphi_i\}$ is a complete orthogonal system, $\phi \in L_2$ and

$$\phi_{ij} = \int_{[0,1]^2}\phi(\tau,\sigma)\overline{\varphi_i(\tau)}\varphi_j(\sigma)d\tau d\sigma,$$

so that part (A) is proved. We have

$$\big|\phi_{ij}\big|^2 = \int_{[0,1]^2}\phi(\tau_1,\sigma_1)\overline{\varphi_i(\tau_1)}\varphi_j(\sigma_1)d\tau_1 d\sigma_1\int_{[0,1]^2}\overline{\phi(\tau_2,\sigma_2)}\varphi_i(\tau_2)\overline{\varphi_j(\sigma_2)}d\tau_2 d\sigma_2$$

$$= \int_{[0,1]^4} \phi(\tau_1,\sigma_1)\overline{\phi(\tau_2,\sigma_2)} \times \overline{\varphi_i(\tau_1)}\varphi_j(\sigma_1)\varphi_i(\tau_2)\overline{\varphi_j(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2$$

$$\mathbb{E}\big|\phi_{ij}\big|^2 = \int_{[0,1]^4} \mathbb{E}\Big[\phi(\tau_1,\sigma_1)\overline{\phi(\tau_2,\sigma_2)}\Big] \times \overline{\varphi_i(\tau_1)}\varphi_j(\sigma_1)\varphi_i(\tau_2)\overline{\varphi_j(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2,$$

This proves part (B). Now, by definition,

$$\mathbb{E}\|\!|\Phi\|\!|_2^2 = \mathbb{E}\sum_{i,j}\big|\phi_{ij}\big|^2 = \sum_{i,j}\int_{[0,1]^4} \mathbb{E}\big\{\phi(\tau_1,\sigma_1)\overline{\phi(\tau_2,\sigma_2)}\big\} \times \overline{\varphi_i(\tau_1)}\varphi_i(\tau_2)\varphi_j(\sigma_1)\overline{\varphi_j(\sigma_2)}d\tau_1 d\tau_2 d\sigma_1 d\sigma_2,$$

and

$$\int_{[0,1]^2} \phi(\tau,\sigma)\overline{\phi(\tau,\sigma)}d\tau d\sigma = \int_{[0,1]^2} \Big\{\sum_{i,j}\phi_{ij}\overline{\varphi_i(\tau)}\varphi_j(\sigma)\Big\} \times \Big\{\sum_{k,l}\overline{\phi_{kl}}\varphi_k(\tau)\overline{\varphi_l(\sigma)}\Big\}d\tau d\sigma$$

$$= \int_{[0,1]^2} \sum_{i,j,k,l}\phi_{ij}\overline{\phi_{kl}} \times \overline{\varphi_i(\tau)}\varphi_k(\tau)\varphi_j(\sigma)\overline{\varphi_l(\sigma)}d\tau d\sigma = \sum_{i,j}\big|\phi_{ij}\big|^2,$$

proving part (C). □

Lemma 2. *Let $\{\varphi_i\}$ be a complete orthonormal basis in $L^2([0,1],\mathbb{C})$ that is closed under conjugation (i.e. satisfying the condition $\{\varphi_i : i = 1,2,\dots\} = \{\overline{\varphi_i} : i = 1,2,\dots\}$). Let $\xi_1,\xi_2 \in L^2([0,1]^2,\mathbb{C})$ and $\xi_3 \in L^2([0,1]^4,\mathbb{C})$. Let $\mathscr{U}_i$ be the induced operator of ξ_i for $i = 1,2,3$. Then*

$$(A)\quad \sum_{i,j}\left|\int_{[0,1]^4} \xi_1(\tau_1,\tau_2)\xi_2(\sigma_1,\sigma_2) \times \overline{\varphi_i(\tau_1)}\varphi_j(\sigma_1)\varphi_i(\tau_2)\overline{\varphi_j(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2\right| \le \|\!|\mathscr{U}_1\|\!|_1 \|\!|\mathscr{U}_2\|\!|_1$$

$$(B)\quad \sum_{i,j}\left|\int_{[0,1]^4} \xi_1(\tau_1,\sigma_2)\xi_2(\sigma_1,\tau_2) \times \overline{\varphi_i(\tau_1)}\varphi_j(\sigma_1)\varphi_i(\tau_2)\overline{\varphi_j(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2\right| \le \|\!|\mathscr{U}_1\|\!|_2^2 + \|\!|\mathscr{U}_2\|\!|_2^2$$

$$(C)\quad \sum_{i,j}\left|\int_{[0,1]^4} \xi_3(\tau_1,\sigma_1,\tau_2,\sigma_2) \times \overline{\varphi_i(\tau_1)}\varphi_j(\sigma_1)\varphi_i(\tau_2)\overline{\varphi_j(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2\right| \le \|\!|\mathscr{U}_3\|\!|_1.$$

Proof. Let $\xi_k(\tau,\sigma) = \sum_{i,j}\xi_{k,ij}\varphi_i(\tau)\overline{\varphi_j(\sigma)}$ for $k = 1,2$.

(A) We start with

$$\int_{[0,1]^4} \xi_1(\tau_1,\tau_2) \times \xi_2(\sigma_1,\sigma_2) \times \overline{\varphi_i(\tau_1)}\varphi_j(\sigma_1)\varphi_i(\tau_2)\overline{\varphi_j(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2$$

$$= \int_{[0,1]^2} \xi_1(\tau_1,\tau_2)\overline{\varphi_i(\tau_1)}\varphi_i(\tau_2)d\tau_1 d\tau_2 \times \int_{[0,1]^2} \xi_2(\sigma_1,\sigma_2)\varphi_j(\sigma_1)\overline{\varphi_j(\sigma_2)}d\sigma_1 d\sigma_2$$

$$= \xi_{1,ii} \times \xi_{2,jj}.$$

Then

$$\sum_{i,j} \left|\xi_{1,ii} \times \xi_{2,jj}\right| \leq \sum_i \left|\xi_{1,ii}\right| \times \sum_j \left|\xi_{2,jj}\right| \leq |||\mathscr{U}_1|||_1 |||\mathscr{U}_2|||_1.$$

(B) Recall that for each i, there exists only one i' such that $\overline{\varphi_i} = \varphi_{i'}$. So,

$$\int_{[0,1]^4} \xi_1(\tau_1, \sigma_2) \times \xi_2(\sigma_1, \tau_2) \times \overline{\varphi_i(\tau_1)}\varphi_j(\sigma_1)\varphi_i(\tau_2)\overline{\varphi_j(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2$$
$$= \int_{[0,1]^2} \xi_1(\tau_1, \sigma_2)\overline{\varphi_i(\tau_1)\varphi_j(\sigma_2)}d\tau_1 d\sigma_2 \times \int_{[0,1]^2} \xi_2(\sigma_1, \tau_2)\varphi_j(\sigma_1)\varphi_i(\tau_2)d\sigma_1 d\tau_2$$
$$= \int_{[0,1]^2} \sum_{k,l} \xi_{1,kl}\varphi_k(\tau_1)\overline{\varphi_l(\sigma_2)\varphi_i(\tau_1)\varphi_j(\sigma_2)}d\tau_1 d\sigma_2 \times \int_{[0,1]^2} \sum_{u,v} \xi_{2,uv}\varphi_u(\sigma_1)\overline{\varphi_v(\tau_2)}\varphi_j(\sigma_1)\varphi_i(\tau_2)d\sigma_1 d\tau_2$$
$$= \xi_{1,ij'} \times \xi_{2,j'i}.$$

Taking the sum over i, j now yields

$$\sum_{i,j} \left|\xi_{1,ij'} \times \xi_{2,j'i}\right| \leq \sum_{i,j} \left\{\left|\xi_{1,ij}\right|^2 + \left|\xi_{2,ij}\right|^2\right\} = |||\mathscr{U}_1|||_2^2 + |||\mathscr{U}_2|||_2^2.$$

(C) Since the φ_i is a complete orthonormal basis in $L^2([0,1],\mathbb{C})$, the collection $\{\overline{\varphi_i}\varphi_j : i, j\}$ is a complete orthonormal basis in $L^2([0,1]^2,\mathbb{C})$. Writing $\overline{\varphi_i}\varphi_j = \varphi_{ij}$ we have

$$\int_{[0,1]^4} \xi_3(\tau_1, \sigma_1, \tau_2, \sigma_2) \times \overline{\varphi_i(\tau_1)}\varphi_j(\sigma_1)\varphi_i(\tau_2)\overline{\varphi_j(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2 = \langle \mathscr{U}_3\varphi_{ij}, \varphi_{ij}\rangle.$$

Taking the absolute value and summing over i, j, we get the upper bound $|||\mathscr{U}_3|||_1$. $\square$

Lemma 3. *The spectral density of X and the cross-spectral density of $\{X, Y\}$ have the form*

$$f_\nu^{XX} = f_{\nu,\Re}^{XX} + \mathbf{i} f_{\nu,\Im}^{XX}$$

$$f_\nu^{B} = f_{\nu,\Re}^{B} + \mathbf{i} f_{\nu,\Im}^{B}$$

and assume that they satisfy condition (B5) and (B4). Furthermore, for $\nu, \omega, \alpha \in [0, 2\pi]$, they admit the Taylor expansions

$$f_\nu^{XX} = f_\omega^{XX} + \sum_{j=1}^{p-1} \frac{(\nu-\omega)^j}{j!} f_\omega^{XX,(j)} + \left(\nu - \omega\right)^p g_{p,\nu,\omega}$$

$$f_\nu^B = f_\omega^B + \sum_{j=1}^{p-1} \frac{(\nu-\omega)^j}{j!} f_\omega^{B,(j)} + (\nu-\omega)^p g_{p,\nu,\omega}^B,$$

where $f_\omega^{XX,(j)} = \frac{\partial^j f_\alpha^{XX}}{\partial \alpha^j}\Big|_{\alpha=\omega}$, $f_\omega^{B,(j)} = \frac{\partial^j f_\alpha^{B}}{\partial \alpha^j}\Big|_{\alpha=\omega}$, *and*

$$(\nu-\omega)^p g_{p,\nu,\omega} = \int_\omega^\nu \frac{(\nu-\zeta)^p}{p!} f_{\zeta,\Re}^{XX,(p+1)} d\zeta + \mathbf{i}\int_\omega^\nu \frac{(\nu-\zeta)^p}{p!} f_{\zeta,\Im}^{XX,(p+1)} d\zeta$$

$$(\nu-\omega)^p g_{p,\nu,\omega}^B = \int_\omega^\nu \frac{(\nu-\zeta)^p}{p!} f_{\zeta,\Re}^{B,(p+1)} d\zeta + \mathbf{i}\int_\omega^\nu \frac{(\nu-\zeta)^p}{p!} f_{\zeta,\Im}^{B,(p+1)} d\zeta$$

$$f_\omega^{XX,(p+1)} = f_{\zeta,\Re}^{XX,(p+1)} + \mathbf{i} f_{\zeta,\Im}^{XX,(p+1)}$$

$$f_\omega^{B,(p+1)} = f_{\zeta,\Re}^{B,(p+1)} + \mathbf{i} f_{\zeta,\Im}^{B,(p+1)}.$$

Finally, there exists a constant C *that does not depend on* ν, ω, α *and such that* $\left\|\!\left|\mathscr{F}_\omega^{XX,(j)}\right|\!\right\|_1$, $\left\|\!\left|\mathscr{F}_\omega^{B,(j)}\right|\!\right\|_1$, $\left\|\!\left|\mathscr{G}_{p,\nu,\omega}\right|\!\right\|_1$, $\left\|\!\left|\mathscr{G}_{p,\nu,\omega}^B\right|\!\right\|_1$, $\left\|\!\left|\mathscr{F}_\omega^\epsilon\right|\!\right\|_1$ *are uniformly bounded by* C*. Here* $\mathscr{F}_\omega^{XX,(j)}$, $\mathscr{F}_\omega^{B,(j)}$, $\mathscr{G}_{p,\nu,\omega}$, $\mathscr{G}_{p,\nu,\omega}^B$ *and* $\mathscr{F}_\omega^\epsilon$ *are the operators induced by the kernels* $f_\omega^{XX,(j)}, f_\omega^{B,(j)}, g_{p,\nu,\omega}, g_{p,\nu,\omega}^B$ *and* f_ω^ϵ.

Proof. Recall that

$$f_\omega^{XX}(\tau,\sigma) = \sum_{t\in\mathbb{Z}} e^{-\mathbf{i}t\omega} r_t^X(\tau,\sigma).$$

Since $f_\omega(\tau,\sigma)$ is a complex-valued function, $f_\omega^{XX}(\tau,\sigma) = f_{\omega,\Re}^{XX}(\tau,\sigma) + \mathbf{i} f_{\omega,\Im}^{XX}(\tau,\sigma)$. Using a Taylor expansion of the functions $f_{\omega,\Re}^{XX}(\tau,\sigma)$ and $f_{\omega,\Im}^{XX}(\tau,\sigma)$, we have

$$f_{\nu,\Re}^{XX}(\tau,\sigma) = f_{\omega,\Re}^{XX}(\tau,\sigma) + \sum_{j=1}^{p-1} \frac{(\nu-\omega)^j}{j!} f_{\omega,\Re}^{XX,(j)} + \int_\omega^\nu \frac{(\nu-\zeta)^p}{p!} f_{\zeta,\Re}^{XX,(p+1)} d\zeta$$

$$f_{\nu,\Im}^{XX}(\tau,\sigma) = f_{\omega,\Im}^{XX}(\tau,\sigma) + \sum_{j=1}^{p-1} \frac{(\nu-\omega)^j}{j!} f_{\omega,\Im}^{XX,(j)} + \int_\omega^\nu \frac{(\nu-\zeta)^p}{p!} f_{\zeta,\Im}^{XX,(p+1)} d\zeta.$$

Then

$$f_\nu^{XX} = f_\omega^{XX} + \sum_{j=1}^{p-1} \frac{(\nu-\omega)^j}{j!} f_\omega^{XX,(j)} + \int_\omega^\nu \frac{(\nu-\zeta)^p}{p!} f_{\zeta,\Re}^{XX,(p+1)} d\zeta + \mathbf{i}\int_\omega^\nu \frac{(\nu-\zeta)^p}{p!} f_{\zeta,\Im}^{XX,(p+1)} d\zeta.$$

Under condition (B5),

$$f_\omega^{XX,(j)}(\tau,\sigma) = \sum_{t\in\mathbb{Z}\setminus\{0\}} e^{-\mathbf{i}t\omega} t^j r_t^X(\tau,\sigma)$$

$$f_{\zeta,\Re}^{XX,(p+1)}(\tau,\sigma)=\begin{cases}\sum_{t\in\mathbb{Z}\setminus\{0\}}(-1)^{(p+1)/2}\cos(-t\zeta)(-t)^{p+1}r_t^X(\tau,\sigma) & \text{for } p \text{ odd}\\ \sum_{t\in\mathbb{Z}\setminus\{0\}}(-1)^{(p+2)/2}\sin(-t\zeta)(-t)^{p+1}r_t^X(\tau,\sigma) & \text{for } p \text{ even.}\end{cases}$$

Under condition (B5), for p odd,

$$\begin{aligned}\int_\omega^\nu \frac{(\nu-\zeta)^p}{p!}f_{\zeta,\Re}^{XX,(p+1)}(\tau,\sigma)d\zeta &= \int_\omega^\nu \sum_{t\in\mathbb{Z}\setminus\{0\}}\frac{(\nu-\zeta)^p}{p!}(-1)^{(p+1)/2}\cos(-t\zeta)(-t)^{p+1}r_t^X(\tau,\sigma)d\zeta\\ &= \int_\omega^\nu \sum_{t\in\mathbb{Z}\setminus\{0\}}(-1)^{(p+1)/2}\frac{(\nu-\zeta)^p}{p!}\frac{\cos(-t\zeta)}{t^4}\Big\{(-t)^{p+5}r_t^X(\tau,\sigma)\Big\}d\zeta\\ &= \sum_{t\in\mathbb{Z}\setminus\{0\}}\int_\omega^\nu (-1)^{(p+1)/2}\frac{(\nu-\zeta)^p}{p!}\frac{\cos(-t\zeta)}{t^4}d\zeta\times\Big\{(-t)^{p+5}r_t^X(\tau,\sigma)\Big\}.\end{aligned}$$

Under condition (B4), it follows that

$$|||\mathscr{F}_\omega^{XX,(j)}|||_1 \le \sum_{t\in\mathbb{Z}\setminus\{0\}}|t|^j|||\mathscr{R}_t^X|||_1$$

Given a system of complete orthonormal functions $\{e_n\}$

$$\sum_{n\in\mathbb{N}}\left|\int_{[0,1]^2}\int_\omega^\nu \frac{(\nu-\zeta)^p}{p!}f_{\zeta,\Re}^{XX,(p+1)}(\tau,\sigma)d\zeta\times e_n(\tau)\overline{e_n(\sigma)}d\tau d\sigma\right|=$$

$$=\sum_{n\in\mathbb{N}}\left|\sum_{t\in\mathbb{Z}\setminus\{0\}}\int_\omega^\nu \frac{(\nu-\zeta)^p}{p!}\frac{\cos(-t\zeta)}{t^4}d\zeta\times\Big\{t^{p+5}\int_{[0,1]^2}r_t^X(\tau,\sigma)e_n(\tau)\overline{e_n(\sigma)}d\tau d\sigma\Big\}\right|.$$

Denoting $\int_{[0,1]^2}r_t^X(\tau,\sigma)e_n(\tau)\overline{e_n(\sigma)}d\tau d\sigma = r_{t,n}^X$, the above term is bounded above by

$$\sum_{n\in\mathbb{N}}\sum_{t\in\mathbb{Z}}|t|^{p+5}\times|r_{t,n}|\int_\omega^\nu\frac{|\nu-\zeta|^p}{p!}d\zeta = O(1)\times|\nu-\omega|^p\sum_{t\in\mathbb{Z}}|t|^{p+5}|||\mathscr{R}_t|||_1 = O(1)|\nu-\omega|^p.$$

Since the induced operator of $\int_\omega^\nu \frac{(\nu-\zeta)^p}{p!}f_{\zeta,\Re}^{XX,(p+1)}(\tau,\sigma)d\zeta$ is the limit of a sequence of compact operators, it is a compact operator.

Following the proof of Theorem 1.27 in Zhu (2007), it follows that the induced operator of $\int_\omega^\nu \frac{(\nu-\zeta)^p}{p!}f_{\zeta,\Re}^{XX,(p+1)}d\zeta$ has nucelar norm of order $|\nu-\omega|^p$. Similar steps yield the same result for p even. We obtain the same result for the induced operator of $\int_\omega^\nu \frac{(\nu-\zeta)^p}{p!}f_{\zeta,\Im}^{XX,(p+1)}d\zeta$. Then $|||\mathscr{G}_{p,\nu,\omega}|||_1$ is uniformly bounded. The same method of proof is applied to $\mathscr{F}_\omega^{B,(j)}$, $\mathscr{G}_{\nu,\omega}^B$ and $\mathscr{F}_\omega^\epsilon$. $\square$

Lemma 4. *Let α and β be two positive numbers as in assumption (B1). Let $\lambda_j = j^{-\alpha}, b_j = j^{-\beta}; \zeta_T = T^{-\alpha/(\alpha+2\beta)}$ then*

$$(A) \qquad \sum_{j=1}^{\infty} \zeta_T^2 \frac{b_j^2}{(\lambda_j+\zeta_T)^2} = O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big) \tag{9}$$

$$(B) \qquad \sum_{j=1}^{\infty} \frac{1}{T} \frac{\lambda_j}{(\lambda_j+\zeta_T)^2} = O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big) \tag{10}$$

$$(C) \qquad \sum_{j=1}^{\infty} \frac{\lambda_j^2}{(\lambda_j+\zeta_T)^2} = O\Big(T^{1/(\alpha+2\beta)}\Big). \tag{11}$$

Proof. We use the following facts in the proof

$$\sum_{i=1}^{T} i^t \asymp \begin{cases} T^{t+1} & t > -1 \\ \log n & t = -1 \\ C & t < -1 \end{cases}$$

$$\sum_{i=T+1}^{\infty} i^{-t} \asymp \quad T^{-t+1} \qquad t > 1.$$

(A) Let $J = \Big\lceil T^{1/(\alpha+2\beta)} \Big\rceil$. Since $2\beta > 1$ and $2\beta - 2\alpha < 1$, using the above results we may write

$$\begin{aligned}
\sum_{j=1}^{\infty} \zeta_T^2 \frac{\lambda_j^2}{(\lambda_j+\zeta_T)^2} \asymp \sum_{j=1}^{\infty} \zeta_T^2 \frac{j^{-2\beta}}{(j^{-\alpha}+\zeta_T)^2} &\le \sum_{j\le J} j^{-2\beta+2\alpha} \times \zeta_T^2 + \sum_{j>J} j^{-2\beta} \asymp J^{-2\beta+2\alpha+1}\zeta_T^2 + J^{-2\beta+1} \\
&\asymp T^{(-2\beta+2\alpha+1)/(\alpha+2\beta)} T^{-2\alpha/(\alpha+2\beta)} + T^{-(2\beta-1)/(\alpha+2\beta)} \\
&= O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big).
\end{aligned}$$

(B) For $\alpha > 1$,

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{1}{T} \frac{\lambda_j}{(\lambda_j+\zeta_T)^2} &= \sum_{j\le J} \frac{1}{T} \frac{\lambda_j}{(\lambda_j+\zeta_T)^2} + \sum_{j>J} \frac{1}{T} \frac{\lambda_j}{(\lambda_j+\zeta_T)^2} \le \sum_{j\le J} \frac{1}{T} \frac{1}{\lambda_j} + \sum_{j>J} \frac{1}{T} \frac{\lambda_j}{\zeta_T^2} \\
&\asymp \frac{1}{T} \sum_{j\le J} j^{\alpha} + \frac{1}{T} T^{-2\alpha/(\alpha+2\beta)} \sum_{j>J} j^{-\alpha} \asymp \frac{1}{T} J^{\alpha+1} + T^{-(2\beta-\alpha)(\alpha+2\beta)} J^{-\alpha+1} \\
&= T^{-1} T^{-(\alpha+1)/(\alpha+2\beta)} + T^{-(2\beta-\alpha)/(\alpha+2\beta)} T^{-(\alpha-1)/(\alpha+2\beta)} \\
&= O\Big(T^{-(2\beta-1)/(\alpha+2\beta)}\Big).
\end{aligned}$$

(C) For $J = T^{1/(\alpha+2\beta)}$

$$\sum_{j=1}^{\infty} \frac{\lambda_j^2}{(\lambda_j+\zeta_T)^2} = \sum_{j\le J} \frac{\lambda_j^2}{(\lambda_j+\zeta_T)^2} + \sum_{j>J} \frac{\lambda_j^2}{(\lambda_j+\zeta_T)^2} \le J + \sum_{j>J} \zeta_T^{-2} j^{-2\alpha} = T^{1/(\alpha+2\beta)} + T^{2\alpha/(\alpha+2\beta)} J^{-2\alpha+1}$$
$$\asymp T^{1/(\alpha+2\beta)} + T^{2\alpha/(\alpha+2\beta)} T^{(-2\alpha+1)/(\alpha+2\beta)}$$
$$\asymp T^{1/(\alpha+2\beta)}.$$

□

Lemma 5. *For a fixed $\omega \in [0,2\pi]$ and a non-negative integer k,*

$$(A) \qquad \frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)\times(\omega-u_s)^j = \delta_{0j} + O\big(T^{-1}B_T^{-1}\big) \qquad \text{for} \quad 0\le j<p$$

$$(B) \qquad \frac{1}{T}\sum_{s=0}^{T-1} \Big|W^{(T)}(\omega-u_s)\Big| \times \Big|\omega-u_s\Big|^p = O\big(B_T^p\big) + O\big(T^{-1}B_T^{-1}\big).$$

$$(C) \qquad \frac{1}{T}\sum_{s=0}^{T-1} \Big\{W^{(T)}(\omega-u_s)\Big\}^2 \times(\omega-u_s)^k = \begin{cases} C_k B_T^{k-1} + O\big(T^{-1}B_T^{-2}\big) & k \quad \text{even} \\ O\big(T^{-1}B_T^{-2}\big) & k \quad \text{odd}\end{cases}$$

$$(D) \qquad \frac{1}{T}\sum_{s=0}^{T-1} \Big|W^{(T)}(\omega-u_s)W^{(T)}(\omega+u_s)\times(\omega-u_s)^j\Big| = 1_{I_T}(\omega)O\big(B_T^{j-1}\big),$$

where δ_{0j} is Kronecker symbol , $0 \le C_k \le \int_{\mathbb{R}} W(\alpha)^2\alpha^k d\alpha$ and $I_T = \big[0,B_T\big]\cup\big[\pi-B_T,\pi+B_T\big]\cup\big[2\pi-B_T,2\pi\big]$.

Proof. The proof uses the results of Panaretos and Tavakoli (2013) on the total variation $V_a^b(h)$ of a function $h:[a,b]\to\mathbb{C}$. We first have the following results. For any positive integers ℓ and k and $x\in[-\pi\le\pi]$,

$$V_0^{2\pi}\Big(\big\{W^{(T)}\big\}^\ell x^k\Big) \le 4\big\|W^{(T)}\big\|_\infty \times(2\pi)^k \times V_0^{2\pi}\Big(\big\{W^{(T)}\big\}^\ell\Big) = O\big(B_T^{-\ell}\big).$$

In the rest of the proof of this lemma, we frequently use Lemma 7.12 in Panaretos and Tavakoli (2013) to get an upper bound for the difference between an integral and its linear approximation based on grid of points in that interval.

(A) We have

$$\frac{1}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)\times(\omega-u_s)^j = \int_{-\pi}^{\pi} W^{(T)}(\alpha)\alpha^j d\alpha + O\Big(T^{-1}B_T^{-1}\Big)$$
$$= \delta_{0j} + O\Big(T^{-1}B_T^{-1}\Big).$$

In the second line, we replaced $\alpha=\omega-v$, then α from $-\pi$ to π. The integral equals zero, by the properties of W. This proves part (A).

(B) For this part,

$$\frac{1}{T}\sum_{s=0}^{T-1}\Big|W^{(T)}(\omega-u_s)\Big|\times\Big|\omega-u_s\Big|^p = \int_{-\pi}^{\pi}\Big|W^{(T)}(\alpha)\Big|\times|\alpha|^p d\alpha + O\big(T^{-1}B_T^{-1}\big) \le \int_{\mathbb{R}}\Big|W(\alpha)\Big|\times|\alpha|^p d\alpha\times B_T^p + O\big(T^{-1}B_T^{-1}\big)$$
$$= O\big(B_T^p\big) + O\big(T^{-1}B_T^{-1}\big).$$

(C) We have

$$\frac{1}{T}\sum_{s=0}^{T-1}\Big\{W^{(T)}(\omega-u_s)\Big\}^2\times(\omega-u_s)^k = \int_{-\pi}^{\pi}\Big\{W^{(T)}(\alpha)\Big\}^2\alpha^k d\alpha + \frac{1}{T}O\left(V_0^{2\pi}\Big(\big\{W^{(T)}\big\}^2x^k\Big)\right).$$

The second term is bounded by $O\big(T^{-1}B_T^{-2}\big)$, for odd k, while the first term vanishes. We now consider the integral term for even k. Recall that

$$W^{(T)}(\alpha) = \frac{1}{B_T}\sum_{i\in\mathbb{Z}} W(\frac{\alpha+2\pi i}{B_T}),$$

and so

$$\big\{W^{(T)}(\alpha)\big\}^2 = \Big\{\frac{1}{B_T}\sum_i W\Big(\frac{\alpha+2\pi i}{B_T}\Big)\Big\}\times\Big\{\frac{1}{B_T}\sum_j W\Big(\frac{\alpha+2\pi j}{B_T}\Big)\Big\}.$$

For $i\neq j$, $\pi>2B_T$, and W is supported on $[-1,1]$. It follows that at least one of $W\Big(\frac{\alpha+2\pi i}{B_T}\Big)$ and $W\Big(\frac{\alpha+2\pi j}{B_T}\Big)$ must be zero. Thus,

$$\big\{W^{(T)}(\alpha)\big\}^2 = \frac{1}{B_T^2}\sum_i W^2\Big(\frac{\alpha+2\pi i}{B_T}\Big).$$

Moreover $\alpha\in[-\pi,\pi]$, so for $|i|\ge 1$ we have $W\Big(\frac{\alpha+2\pi i}{B_T}\Big)=0$. Hence, for even k,

$$\int_{-\pi}^{\pi} W^{(T)}\big(\alpha\big)^2\alpha^k d\alpha = \frac{1}{B_T^2}\int_{-\pi}^{\pi} W\Big(\frac{\alpha}{B_T}\Big)^2\alpha^k d\alpha \le B_T^{k-1}\int_{\mathbb{R}} W(\alpha)^2\alpha^k d\alpha.$$

(D) For $\omega \in I_T$

$$\frac{1}{T}\sum_{s=0}^{T-1}\Big|W^{(T)}(\omega-u_s)W^{(T)}(\omega+u_s)\times(\omega-u_s)^j\Big| = \int_{-\pi}^{\pi}\Big|W^{(T)}(\alpha)W^{(T)}(2\omega-\alpha)\times\alpha^j\Big|d\alpha + O\big(T^{-1}B_T^{-2}\big).$$

Since W is supported on $[-1,1]$, $W\Big(\frac{2\omega-\alpha+2k\pi}{B_T}\Big)\neq 0$ iff $|2\omega-\alpha+2k\pi|\leq B_T$. For T sufficiently large, $\alpha\in[-\pi,\pi],\omega\in[0,2\pi]$, the inequality will hold only for $-4\leq k\leq 4$. Thus, the integral is bounded by

$$\begin{aligned}\sum_{k=-4}^{4}\int_{-\pi}^{\pi}\frac{1}{B_T^2}\Big|W\Big(\frac{\alpha}{B_T}\Big)\times W\Big(\frac{2\omega-\alpha+2k\pi}{B_T}\Big)\times\alpha^j\Big|d\alpha &= \sum_{k=-4}^{4}\int_{-\pi B_T^{-1}}^{\pi B_T^{-1}}\frac{1}{B_T^2}\Big|W(x)W\Big(\frac{2\omega-2k\pi}{B_T}-x\Big)\times B_T^{j+1}x^j\Big|dx\\ &= O\Big(B_T^{j-1}\Big).\end{aligned}$$

When $\omega\notin I_T$, since $0<\omega<2\pi$, we have $\omega\notin\cup_{k\in\mathbb{Z}}\big[k\pi-B_T,k\pi+B_T\big]$, and so $\big|k\pi-\omega\big|>B_T$. It follows that

$$\Big|\frac{\alpha}{B_T}+\frac{2\omega-\alpha+2k\pi}{B_T}\Big| = \Big|\frac{2k\pi+2\omega}{B_T}\Big|\geq 2.$$

Then, at least one of $W\Big(\frac{\alpha}{B_T}\Big)$ and $W\Big(\frac{2\omega-\alpha+2k\pi}{B_T}\Big)$ must equal 0. When $|\alpha|>B_T$, $W(\alpha/B_T)=0$. We deduce that for T large enough and $\omega\notin[0,B_T]\cup[\pi-B_T,\pi+B_T]\cup[2\pi-B_T,2\pi]$

$$W^{(T)}(\alpha)W^{(T)}(\alpha-2\omega) = \frac{1}{B_T^2}W\Big(\frac{\alpha}{B_T}\Big)\sum_{k\in\mathbb{Z}}W\Big(\frac{\alpha+2k\pi-2\omega}{B_T}\Big) = 0.$$

Thus we get zero for $\omega\notin I_T$. $\square$

Lemma 6. *Let $h_T(t)=1_{[0,T-1]}(t)$ and $\Delta^{(T)}(\omega)=\sum_{t=0}^{T-1}e^{-\mathbf{i}\omega t}$. Let*

$$f_{\omega_1,\ldots,\omega_{k-1}}(\tau_1,\ldots,\tau_k) = \frac{1}{(2\pi)^{k-1}}\sum_{t_1,\ldots,t_{k-1}=-\infty}^{\infty}\exp\Big\{-\mathbf{i}\sum_{j=1}^{k-1}\omega_j t_j\Big\}\operatorname{cum}\big(X_{t_1}(\tau_1),\ldots,X_{t_{k-1}}(\tau_{k-1}),X_0(\tau_k)\big)$$

$$\varrho_{T,k} := \varrho_T(\tau_1,\ldots,\tau_k) = \frac{1}{(2\pi)^{k-1}}\sum_{t_i\geq T}\exp\Big\{-\mathbf{i}\sum_{j=1}^{k-1}\omega_j t_j\Big\}\operatorname{cum}\big(X_{t_1}(\tau_1),\ldots,X_{t_{k-1}}(\tau_{k-1}),X_0(\tau_k)\big)$$

$$\begin{aligned}\rho_{T,k} := \rho_T(\tau_1,\ldots,\tau_k) = \sum_{|t_j|\leq T-1}&\exp\Big\{-\sum_{j=1}^{k-1}t_j\omega_j\Big\}\operatorname{cum}\big(X_{t_1}(\tau_1),\ldots,X_{t_{k-1}}(\tau_{k-1}),X_0(\tau_k)\big)\times\\ &\left[\sum_{t=0}^{T-1}\exp\Big\{-t\sum_{j=1}^{k}\omega_j\Big\}\Big\{1-h_T(t+t_1)h_T(t+t_2)\ldots h_T(t+t_{k-1})h_T(t)\Big\}\right]\end{aligned}$$

Then

$$\operatorname{cum}\Big(\widetilde{X}^{(T)}_{\omega_1},\dots,\widetilde{X}^{(T)}_{\omega_k}\Big) = \frac{(2\pi)^{k/2-1}}{T^{k/2}} f_{\omega_1,\dots,\omega_{k-1}} \times \Delta^{(T)}\Big(\sum_{j=1}^{k}\omega_j\Big) - \frac{(2\pi)^{k/2-1}}{T^{k/2}}\Delta^{(T)}\Big(\sum_{j=1}^{k}\omega_j\Big) \times \varrho_{T,k} + \frac{1}{(2\pi T)^{k/2}}\rho_{T,k}.$$

Let $\mathscr{F}_{\omega_1,\dots,\omega_{k-1}}, \mathscr{U}_{T,k}$ and $\mathscr{V}_{T,k}$ be the operators induced by $f_{\omega_1,\dots,\omega_{k-1}}, \varrho_{T,k}$ and $\rho_{T,k}$, respectively. Then

$$\big\|\mathscr{F}_{\omega_1,\dots,\omega_{k-1}}\big\|_1 \le \frac{1}{(2\pi)^{k-1}} \sum_{t_1,\dots,t_{k-1}=-\infty}^{\infty} \big\|\mathscr{R}_{t_1,\dots,t_{k-1}}\big\|_1$$

$$\big\|\mathscr{U}_{T,k}\big\|_1 \le \sum_{|t_j|\ge T} \big\|\mathscr{R}_{t_1,\dots,t_{k-1}}\big\|_1$$

$$\big\|\mathscr{V}_{T,k}\big\|_1 \le \sum_{|t_j|\le T-1} \big\|\mathscr{R}_{t_1,\dots,t_{k-1}}\big\|_1 \times \sum_{j=1}^{k-1}\big|t_j\big|.$$

Proof. Proofs of these results can be found in Panaretos and Tavakoli (2013). □

Proposition 1. *Assume assumptions (A1)-(A3) and (B1)-(B6) in Section 4 and 7 are satisfied, then*

$$\mathbb{E}p^{(T)}_{\nu_s}(\tau,\sigma) = f^{XX}_{\nu_s}(\tau,\sigma) + \frac{1}{T}\vartheta_{\nu_s}(\tau,\sigma).$$

For $\nu_r + \nu_s \neq 2\pi$,

$$\operatorname{cum}\Big(\widetilde{X}_{\nu_s}(\tau),\widetilde{X}_{\nu_r}(\sigma)\Big) = \frac{1}{T}\vartheta_{\nu_s,\nu_r}(\tau,\sigma).$$

For the corresponding integral operator, we have $\mathscr{P}^{XX}_{\nu_s,T} = \mathscr{F}^{XX}_{\nu_s} + T^{-1}\mathscr{V}_{\nu_s}$ and $\big\|\mathscr{V}_{\nu_s}\big\|_1 < C$ and $\mathbb{E}\widetilde{X}_{\nu_s}\otimes\widetilde{X}_{\nu_r} = T^{-1}\mathscr{V}_{\nu_s,\nu_r}$ with $\big\|\mathscr{V}_{\nu_s,\nu_r}\big\|_1 < C$ uniformly over s,r, where C is an universal constant.

Proof. The results follow by Proposition 2.6 in Panaretos and Tavakoli (2013) and by Lemma 6. □

Proposition 2. *Assume assumptions (A1)-(A3) and (B1)-(B6) in Section 4 and 7 are satisfied. Then,*

$$\mathbb{E}f_\omega^{(T)}(\tau,\sigma) = f_\omega^{XX}(\tau,\sigma) + \frac{1}{B_T T}\vartheta_\omega(\tau,\sigma). \tag{12}$$

For the induced operators, $\mathbb{E}\widehat{\mathscr{F}}_\omega^{XX} = \mathscr{F}_\omega^{XX} + B_T^{-1}T^{-1}\mathscr{V}_\omega$ *with* $\left\|\!\left|\mathscr{V}_\omega\right|\!\right\|_1 < C$ *for an universal constant* C.

Proof. The first statement is equivalent to proposition 3.1 in Panaretos and Tavakoli (2013). Now, by definition,

$$f_\omega^{(T)} = \frac{2\pi}{T}\sum_{s=1}^{T} W^{(T)}(\omega-\nu_s)p_{\nu_s}^{(T)}.$$

Using Proposition 1,

$$\begin{aligned}
\mathbb{E}f_\omega^{(T)}(\tau,\sigma) &= \frac{2\pi}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)f_{\nu_s}^{XX}(\tau,\sigma) + \frac{2\pi}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\frac{1}{T}\vartheta_{\nu_s}(\tau,\sigma)\\
&= \frac{2\pi}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)f_{\nu_s}^{XX}(\tau,\sigma) + \frac{1}{TB_T}\vartheta_{(1)}(\tau,\sigma),
\end{aligned}$$

with the operator $\mathscr{V}_{(1)}$ induced by $\vartheta_{(1)}$ satisfying $\left\|\!\left|\mathscr{V}_{(1)}\right|\!\right\|_1 = O(1)$ uniformly over ω. Replacing ν_s by u_s, and using a Taylor expansion,

$$f_{u_s}^{XX}(\tau,\sigma) = f_\omega^{XX}(\tau,\sigma) + \sum_{j=1}^{p-1}\frac{\big(u_s-\omega\big)^j}{j!}\frac{\partial^j f_\alpha^{XX}(\tau,\sigma)}{\partial\alpha^j}\Big|_{\alpha=\omega} + \big(u_s-\omega\big)^p g_{p,u_s,\omega}(\tau,\sigma).$$

Summing over s now gives

$$\begin{aligned}
f_\omega^{(T)}(\tau,\sigma) = \frac{2\pi}{T}\sum_{j=0}^{p-1}\Big\{\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)\times(u_s-\omega)^j\Big\}\times\frac{\partial^j f_\alpha^{XX}(\tau,\sigma)}{\partial\alpha^j}\Big|_{\alpha=\omega} + \\
\frac{2\pi}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)\times\big(\omega-u_s\big)^p g_{p,u_s,\omega}(\tau,\sigma).
\end{aligned}$$

For $0 \le j \le p-1$, using results in Lemma 5,

$$\frac{2\pi}{T}\Big\{\sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)\times(\omega-u_s)^j\Big\}\times\frac{\partial^j f_\alpha^{XX}(\tau,\sigma)}{\partial\alpha^j}\Big|_{\alpha=\omega} = \Big\{\delta_{0j} + O\big(T^{-1}B_T^{-1}\big)\Big\}\times\frac{\partial^j f_\alpha^{XX}(\tau,\sigma)}{\partial\alpha^j}\Big|_{\alpha=\omega}.$$

Taking the sum over j then yields

$$\frac{2\pi}{T}\sum_{j=0}^{p-1}\Big\{\sum_{s=0}^{T-1}W^{(T)}(\omega-u_s)\times(u_s-\omega)^j\Big\}\times f_\omega^{XX,(j)}(\tau,\sigma)=f_\omega^{XX}+O\big(T^{-1}B_T^{-1}\big)\sum_{j=1}^{p-1}f_\omega^{XX,(j)}(\tau,\sigma).$$

Finally, letting $\mathscr{G}_{p,u_s,\omega}$ be the operator induced by $g_{p,u_s,\omega}$, we have

$$\begin{aligned}\Big|\!\Big|\!\Big|\frac{2\pi}{T}\sum_{s=0}^{T-1}W^{(T)}(\omega-u_s)\times\big(\omega-u_s\big)^p\mathscr{G}_{p,u_s,\omega}\Big|\!\Big|\!\Big|_1&\le\frac{2\pi}{T}\sum_{s=0}^{T-1}\Big|W^{(T)}(\omega-u_s)\times\big(\omega-u_s\big)^p\Big|\times\sup_s|\!|\!|\mathscr{G}_{p,u_s,\omega}|\!|\!|_1\\&\le O\big(B_T^p+T^{-1}B_T^{-1}\big)=O(T^{-1}B_T^{-1}),\end{aligned}$$

by Lemma 5 and Lemma 3. Combining the above results completes the proof. □

Proposition 3. *(A) Let*

$$p_{r,s}^{(T)}=\operatorname{cum}\big(\widetilde{X}_{\nu_s}^{(T)}(\tau_1),\widetilde{X}_{-\nu_s}^{(T)}(\sigma_1),\widetilde{X}_{-\nu_r}^{(T)}(\tau_2),\widetilde{X}_{\nu_r}^{(T)}(\sigma_2)\big)$$

and $\mathscr{P}_{r,s}^{(T)}$ be its induced operator. Then,

$$|\!|\!|\mathscr{P}_{r,s}^{(T)}|\!|\!|_1=O(T^{-1}).$$

(B) Let $p_{r,s}\in L^2([0,1]^4,\mathbb{C})$ such that its associated operator $\mathscr{P}_{r,s}$ satisfies $|\!|\!|\mathscr{P}_{r,s}|\!|\!|_1<CT^{-1}$ for all $r,s=0,\dots,T-1$ and a universal constant C. Let

$$\sum_{r,s=0}^{T-1}W^{(T)}(\omega-\nu_r)W^{(T)}(\omega-\nu_s)p_{r,s}=p_\omega.$$

and $\mathscr{P}_\omega$ be the induced operator of p_ω. Then

$$\int_0^{2\pi}|\!|\!|\mathscr{P}_\omega|\!|\!|_1 d\omega=O(T).$$

Proof. (A) Using Lemma 6

$$\begin{aligned}|\!|\!|\mathscr{P}_{r,s}^{(T)}|\!|\!|_1&\le\frac{1}{T}\sum_{t_1,t_2,t_3=-\infty}^{\infty}|\!|\!|\mathscr{R}_{t_1,t_2,t_3}|\!|\!|_1+\frac{2\pi}{T}\sum_{|t_j|\ge T}|\!|\!|\mathscr{R}_{t_1,t_2,t_3}|\!|\!|_1+\frac{1}{(2\pi T)^2}\sum_{|t_j|<T-1}|\!|\!|\mathscr{R}_{t_1,t_2,t_3}|\!|\!|_1\times\sum_{j=1}^{3}|t_j|\\&\le\frac{O(1)}{T}\sum_{t_1,t_2,t_3=-\infty}^{\infty}|\!|\!|\mathscr{R}_{t_1,t_2,t_3}|\!|\!|_1=O(T^{-1}).\end{aligned}$$

(B) Recall that

$$W^{(T)}(\omega-\nu_s)=\frac{1}{B_T}\sum_{i\in\mathbb{Z}}W\Big(\frac{\omega-\nu_s+2i\pi}{B_T}\Big);\quad W^{(T)}(\omega-\nu_r)=\frac{1}{B_T}\sum_{j\in\mathbb{Z}}W\Big(\frac{\omega-\nu_r+2j\pi}{B_T}\Big).$$

Note now that $W(x)=0$ for all $|x|>1$. Hence, if $|x-y|>2$, then at least one of $W(x)$ and $W(y)$ vanishes. We have $-2\pi\le\nu_r-\nu_s\le 2\pi$, and so if

$$\nu_r-\nu_s\notin S_T:=[-2\pi,-2\pi+2B_T]\cup[-2B_T,2B_T]\cup[2\pi-2B_T,2\pi],$$

then $\big|\nu_r-\nu_s+2k\pi\big|>2B_T$ for any integer k. Thus, if $\nu_r-\nu_s\notin S_T$, for all $i,j\in\mathbb{Z}$

$$\left|\frac{\omega-\nu_s+2i\pi}{B_T}-\frac{\omega-\nu_r+2j\pi}{B_T}\right|=\left|\frac{\nu_r-\nu_s+2(i-j)\pi}{B_T}\right|>2.$$

This means that $W\Big(\frac{\omega-\nu_s+2j_s\pi}{B_T}\Big)W\Big(\frac{\omega-\nu_r+2j_r\pi}{B_T}\Big)=0$ for all $i,j\in\mathbb{Z}$, if $\nu_r-\nu_s\notin S_T$. Write

$$S_{T,s}=[-2\pi+\nu_s,-2\pi+2B_T+\nu_s]\cup[-2B_T+\nu_s,2B_T+\nu_s]\cup[2\pi-2B_T+\nu_s,2\pi+\nu_s].$$

Then $W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)=0$ for $\nu_r\notin S_{T,s}$. The number of r such that $\nu_r\in S_{T,s}$ is of order TB_T. Therefore,

$$\begin{aligned}\sum_{s,r=0}^{T-1}W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)p_{r,s}&=\sum_{s=0}^{T-1}\sum_{r\in S_{T,s}}W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)p_{r,s}\\&=\sum_{s=0}^{T-1}W^{(T)}(\omega-\nu_s)\times\sum_{r\in S_{T,s}}W^{(T)}(\omega-\nu_r)p_{r,s}.\end{aligned}$$

For fixed ν_s, let

$$I_{T,s}=\Big\{\omega:0\le\omega\le 2\pi,\quad W\big(\frac{\omega-\nu_s+2i\pi}{B_T}\big)\neq 0,\quad \text{for some}\quad i\in\mathbb{Z}\Big\}.$$

This means that if $\omega\in I_{T,s}$, then

$$-B_T\le\omega-\nu_s+2i\pi\le B_T\iff\omega\in[-B_T+\nu_s-2i\pi,B_T+\nu_s-2i\pi]$$

for some $i\in\mathbb{Z}$. The length of $[-B_T+\nu_s-2i\pi,B_T+\nu_s-2i\pi]$ is $2B_T$. For $|i|\ge 4$, $[-B_T+\nu_s-2i\pi,B_T+\nu_s-2i\pi]\cap[0,2\pi]=\emptyset$. Hence, the length of $I_{T,s}$ is of order $O(B_T)$. By the definition of $I_{T,s}$,

$$\left|W^{(T)}(\omega-\nu_s)\right| \leq 1_{I_{T,s}}(\omega)\frac{\|W\|_\infty}{B_T}.$$

The number of r such that $\nu_r \in S_{T,s}$ is of order TB_T. Thus, combining our results

$$\begin{aligned}\left\|\!\left|\sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r)\mathscr{P}_{r,s}\right|\!\right\|_1 &\leq \sum_{s=0}^{T-1} 1_{I_{T,s}}(\omega)\frac{\|W\|_\infty}{B_T}O(TB_T)\frac{\|W\|_\infty}{B_T}\sup_{r,s}\left\|\!\left|\mathscr{P}_{r,s}\right|\!\right\|_1 \\ &= \sum_{s=0}^{T-1} 1_{I_{T,s}}(\omega)\frac{O(1)}{B_T}.\end{aligned}$$

Integrating over ω and remarking that $I_{T,s}$ is of order B_T, we obtain

$$\int_0^{2\pi} \left\|\!\left|\mathscr{P}_\omega\right|\!\right\|_1 d\omega = O(T).$$

$\square$

Proposition 4. *Assume assumptions (A1)-(A3) and (B1)-(B6) in Section 4 and 7 are satisfied, then*

$$\begin{aligned}\mathbb{E}\Big[p^{(T)}_{\nu_s}(\tau_1,\sigma_1)\times p^{(T)}_{-\nu_r}(\tau_2,\sigma_2)\Big] &= \mathbb{E}\Big[p^{(T)}_{\nu_s}(\tau_1,\sigma_1)\Big]\times\mathbb{E}\Big[p^{(T)}_{-\nu_r}(\tau_2,\sigma_2)\Big] + p^{(T)}_{r,s}(\tau_1,\sigma_1,\tau_2,\sigma_2)+ \\ &\eta(\nu_r-\nu_s)f^{XX}_{\nu_s}(\tau_1,\tau_2)f^{XX}_{-\nu_s}(\sigma_1,\sigma_2)+\frac{1}{T}\eta(\nu_s-\nu_r)\vartheta_{1,\nu_s,\nu_r,f}(\tau_1,\tau_2)\odot\vartheta_{2,\nu_s,\nu_r,f}(\sigma_1,\sigma_2)+ \\ &\eta(\nu_s+\nu_r)f^{XX}_{\nu_s}(\tau_1,\sigma_2)f^{XX}_{-\nu_s}(\sigma_1,\tau_2)+\frac{1}{T}\eta(\nu_s+\nu_r)\vartheta_{3,\nu_s,\nu_r,f}(\sigma_1,\tau_2)\odot\vartheta_{4,\nu_s,\nu_r,f}(\tau_1,\sigma_2)+ \\ &\frac{1}{T^2}\vartheta_{1,\nu_s,\nu_r}(\tau_1,\sigma_1)\times\vartheta_{2,\nu_s,\nu_r}(\tau_2,\sigma_2)+\frac{1}{T^2}\vartheta_{3,\nu_s,\nu_r}(\tau_1,\sigma_2)\times\vartheta_{4,\nu_s,\nu_r}(\sigma_1,\tau_2),\end{aligned}$$

where $\eta(x)$ equals one if $x \in 2\pi\mathbb{Z}$ and zero otherwise, and $\vartheta_{i,\nu_s,\nu_r,f}\odot\vartheta_{j,\nu_s,\nu_r,f} \in Conv_C\big(L^2([0,1]^2,\mathbb{C})\times L^2([0,1]^2,\mathbb{C})\big)$ with a universal constant C.

Proof. To simplify notation, let

$$A=\widetilde{X}^{(T)}_{\nu_s}(\tau_1);\qquad B=\widetilde{X}^{(T)}_{-\nu_s}(\sigma_1);\qquad C=\widetilde{X}^{(T)}_{-\nu_r}(\tau_2);\qquad D=\widetilde{X}^{(T)}_{\nu_r}(\sigma_2).$$

We use the formula

$$\mathbb{E}\big[ABCD\big]=\mathbb{E}\big[AB\big]\times\mathbb{E}\big[CD\big]+\mathbb{E}\big[AC\big]\times\mathbb{E}\big[BD\big]+\mathbb{E}\big[AD\big]\times\mathbb{E}\big[BC\big]+\mathrm{cum}\big(A,B,C,D\big).$$

The term cum(A, B, C, D) will be denoted by $p_{r,s}^{(T)}$. Applying Proposition 2,

$$
\begin{aligned}
\mathbb{E}\big[AB\big] \times \mathbb{E}\big[CD\big] =& \mathbb{E}\Big[p_{\nu_s}^{(T)}(\tau_1,\sigma_1)\Big] \times \mathbb{E}\Big[p_{-\nu_r}^{(T)}(\tau_2,\sigma_2)\Big] \\
\mathbb{E}\big[AC\big] \times \mathbb{E}\big[BD\big] =& \Big\{\eta(\nu_s-\nu_r) f_{\nu_s}^{XX}(\tau_1,\tau_2) + \frac{1}{T}\vartheta_{1,\nu_s,\nu_r}(\tau_1,\tau_2)\Big\} \times \Big\{\eta(\nu_s-\nu_r) f_{-\nu_s}^{XX}(\sigma_1,\sigma_2) + \frac{1}{T}\vartheta_{2,\nu_s,\nu_r}(\sigma_1,\sigma_2)\Big\} \\
=& \eta(\nu_s-\nu_r) f_{\nu_s}^{XX}(\tau_1,\tau_2) f_{-\nu_s}^{XX}(\sigma_1,\sigma_2) + \frac{1}{T}\eta(\nu_s-\nu_r)\Big\{f_{\nu_s}^{XX}(\tau_1,\tau_2)\vartheta_{2,\nu_s,\nu_r}(\sigma_1,\sigma_2) + \\
& f_{-\nu_s}^{XX}(\sigma_1,\sigma_2)\vartheta_{1,\nu_s,\nu_r}(\tau_1,\tau_2)\Big\} + \frac{1}{T^2}\vartheta_{1,\nu_s,\nu_r}(\tau_1,\tau_2) \times \vartheta_{2,\nu_s,\nu_r}(\sigma_1,\sigma_2) \\
=& \eta(\nu_s-\nu_r) f_{\nu_s}^{XX}(\tau_1,\tau_2) f_{-\nu_s}^{XX}(\sigma_1,\sigma_2) + \frac{1}{T}\eta(\nu_s-\nu_r)\vartheta_{1,\nu_s,\nu_r,f}(\tau_1,\tau_2) \odot \vartheta_{2,\nu_s,\nu_r,f}(\sigma_1,\sigma_2) + \\
& + \frac{1}{T^2}\vartheta_{1,\nu_s,\nu_r}(\tau_1,\tau_2) \times \vartheta_{2,\nu_s,\nu_r}(\sigma_1,\sigma_2); \\
\mathbb{E}\big[AD\big] \times \mathbb{E}\big[BC\big] =& \Big\{\eta(\nu_s+\nu_r) f_{\nu_s}^{XX}(\tau_1,\sigma_2) + \frac{1}{T}\vartheta_{3,\nu_s,\nu_r}(\tau_1,\sigma_2)\Big\} \times \Big\{\eta(\nu_s+\nu_r) f_{-\nu_s}^{XX}(\sigma_1,\tau_2) + \frac{1}{T}\vartheta_{4,\nu_s,\nu_r}(\sigma_1,\tau_2)\Big\} \\
=& \eta(\nu_s+\nu_r) f_{\nu_s}^{XX}(\tau_1,\sigma_2) f_{-\nu_s}^{XX}(\sigma_1,\tau_2) + \frac{1}{T}\eta(\nu_s+\nu_r)\Big\{\vartheta_{4,\nu_s,\nu_r}(\sigma_1,\tau_2) f_{\nu_s}^{XX}(\tau_1,\sigma_2) + \\
& f_{-\nu_s}^{XX}(\sigma_1,\tau_2)\vartheta_{3,\nu_s,\nu_r}(\tau_1,\sigma_2)\Big\} + \frac{1}{T^2}\vartheta_{3,\nu_s,\nu_r}(\tau_1,\sigma_2) \times \vartheta_{4,\nu_s,\nu_r}(\sigma_1,\tau_2) \\
=& \eta(\nu_s+\nu_r) f_{\nu_s}^{XX}(\tau_1,\sigma_2) f_{-\nu_s}^{XX}(\sigma_1,\tau_2) + \frac{1}{T}\eta(\nu_s+\nu_r)\vartheta_{3,\nu_s,\nu_r,f}(\sigma_1,\tau_2) \odot \vartheta_{4,\nu_s,\nu_r,f}(\tau_1,\sigma_2) \\
& + \frac{1}{T^2}\vartheta_{3,\nu_s,\nu_r}(\tau_1,\sigma_2) \times \vartheta_{4,\nu_s,\nu_r}(\sigma_1,\tau_2).
\end{aligned}
$$

Combining these results completes the proof. ☐

Proposition 5. *Assume assumptions (A1)-(A3) and (B1)-(B6) in Section 4 and 7 are satisfied, then*

$$
\begin{aligned}
&\mathbb{E}\Big[\Big\{f_\omega^{(T)}(\tau_1,\sigma_1) - f_\omega(\tau_1,\sigma_1)\Big\} \times \Big\{\overline{f_\omega^{(T)}(\tau_2,\sigma_2)} - f_{-\omega}(\tau_2,\sigma_2)\Big\}\Big] = \\
&\quad O\big(T^{-1}B_T^{-1}\big) \times \Big\{f_\omega^{XX}(\tau_1,\tau_2) f_{-\omega}^{XX}(\sigma_1,\sigma_2) + 1_{I_T}(\omega) f_\omega^{XX}(\tau_1,\sigma_2) f_{-\omega}^{XX}(\tau_2,\sigma_1)\Big\} \\
&\quad + O\big(T^{-1}B_T\big)\Big\{\vartheta_1(\tau_1,\tau_2)\odot\vartheta_2(\sigma_1,\sigma_2) + 1_{I_T}(\omega)\vartheta_3(\tau_1,\sigma_2)\odot\vartheta_4(\sigma_1,\tau_2)\Big\} \\
&\quad + 1_{I_T}(\omega) \times O\big(T^{-1}\big) \times \Big\{f_\omega^{XX}(\tau_1,\sigma_2) f_{-\omega}^{XX,(1)}(\tau_2,\sigma_1) + f_{-\omega}^{XX}(\tau_2,\sigma_1) f_\omega^{XX,(1)}(\tau_1,\sigma_2)\Big\} \\
&\quad + \frac{1}{T^2}\sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s) W^{(T)}(\omega-\nu_r) p_{r,s}^{(T)}(\tau_1,\sigma_1,\tau_2,\sigma_2),
\end{aligned}
$$

where I_T is as in Lemma 5 and $\vartheta_i \odot \vartheta_j \in Conv_C\Big(L^2([0,1]^2,\mathbb{C}) \times L^2([0,1]^2,\mathbb{C})\Big)$.

Proof. We use the same notation A, B, C, D as in the proof of Proposition 4. By definition of $f_\omega^{(T)}$,

$$\mathbb{E}\left[f_\omega^{(T)}(\tau_1,\sigma_1) \times \overline{f_\omega^{(T)}(\tau_2,\sigma_2)}\right] = \left(\frac{2\pi}{T}\right)^2 \sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r) \times \mathbb{E}\Big[p_{\nu_s}^{(T)}(\tau_1,\tau_2) \times p_{-\nu_r}^{(T)}(\sigma_1,\sigma_2)\Big].$$

We use Proposition 4 to decompose $\mathbb{E}\Big[p_{\nu_s}^{(T)}(\tau_1,\tau_2)\times p_{-\nu_r}^{(T)}(\sigma_1,\sigma_2)\Big]$ and treat each part separately. Consider first $\mathbb{E}[AB] \times \mathbb{E}[CD]$, given by

$$\begin{aligned}
&\left(\frac{2\pi}{T}\right)^2 \sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r) \times \mathbb{E}\Big[p_{\nu_s}^{(T)}(\tau_1,\tau_2)\Big] \times \mathbb{E}\Big[p_{-\nu_r}^{(T)}(\sigma_1,\sigma_2)\Big] \\
&= \left\{\frac{2\pi}{T}\sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)\mathbb{E}\Big[p_{\nu_s}^{(T)}(\tau_1,\sigma_1)\Big]\right\} \times \left\{\frac{2\pi}{T}\sum_{r=0}^{T-1} W^{(T)}(\omega-\nu_r)\mathbb{E}\Big[p_{-\nu_r}^{(T)}(\tau_2,\sigma_2)\Big]\right\} \\
&= \mathbb{E}f_\omega^{(T)}(\tau_1,\sigma_1) \times \mathbb{E}\overline{f_\omega^{(T)}(\tau_2,\sigma_2)}.
\end{aligned}$$

Note that $\eta(x) = 0$ when $x \neq 2k\pi$. Next, consider $\mathbb{E}[AC] \times \mathbb{E}[BD]$ which is

$$\begin{aligned}
&\left(\frac{2\pi}{T}\right)^2 \sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r) \times \Big[\eta(\nu_s-\nu_r)f_{\nu_s}^{XX}(\tau_1,\tau_2)f_{-\nu_s}^{XX}(\sigma_1,\sigma_2) + \\
&\frac{1}{T}\eta(\nu_s-\nu_r)\vartheta_{1,\nu_s,\nu_r,f}(\tau_1,\tau_2)\odot\vartheta_{2,\nu_s,\nu_r,f}(\sigma_1,\sigma_2) + \frac{1}{T^2}\vartheta_{1,\nu_s,\nu_r}(\tau_1,\tau_2)\vartheta_{2,\nu_s,\nu_r}(\sigma_1,\sigma_2)\Big] \\
&= \left(\frac{2\pi}{T}\right)^2 \sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_s) \times f_{\nu_s}^{XX}(\tau_1,\tau_2)f_{-\nu_s}^{XX}(\sigma_1,\sigma_2) + B_T^{-2}T^{-2}\vartheta_{1,f}(\tau_1,\tau_2)\odot\vartheta_{2,f}(\sigma_1,\sigma_2),
\end{aligned}$$

where

$$\begin{aligned}
&T^{-2}B_T^{-2} \times \vartheta_{1,f}(\tau_1,\tau_2)\odot\vartheta_{2,f}(\sigma_1,\sigma_2) = \\
&\left(\frac{2\pi}{T}\right)^2 \sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_s) \times \frac{1}{T}\vartheta_{1,\nu_s,\nu_r,f}(\tau_1,\tau_2)\odot\vartheta_{2,\nu_s,\nu_r,f}(\sigma_1,\sigma_2) \\
&+ \left(\frac{2\pi}{T}\right)^2 \sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r) \times \frac{1}{T^2}\vartheta_{1,\nu_s,\nu_r}(\tau_1,\tau_2)\vartheta_{2,\nu_s,\nu_r}(\sigma_1,\sigma_2).
\end{aligned}$$

For the term containing $f_{\nu_s}^{XX}(\tau_1,\tau_2)f_{-\nu_s}^{XX}(\sigma_1,\sigma_2)$, we replace ν_s by u_s and use a Taylor expansion as in Lemma 3

$$f_{u_s}^{XX}(\tau_1,\tau_2) = f_\omega^{XX}(\tau_1,\tau_2) + \sum_{j=1}^{p-1}\frac{(u_s-\omega)^j}{j!}\frac{\partial^j f_\alpha^{XX}(\tau_1,\tau_2)}{\partial\alpha^j}\Big|_{\alpha=\omega} + (u_s-\omega)^p g_{2,u_s,\omega}(\tau_1,\tau_2)$$

$$f^{XX}_{-u_s}(\sigma_1,\sigma_2) = f^{XX}_{-\omega}(\sigma_1,\sigma_2) + \sum_{j=1}^{p-1} \frac{(\omega-u_s)^j}{j!} \frac{\partial^j f^{XX}_{-\alpha}(\sigma_1,\sigma_2)}{\partial\alpha^j}\Big|_{\alpha=\omega} + (\omega-u_s)^p g_{2,u_s,\omega}(\sigma_1,\sigma_2).$$

Their product becomes

$$f^{XX}_{u_s}(\tau_1,\tau_2) \times f^{XX}_{-u_s}(\sigma_1,\sigma_2) = f^{XX}_{\omega}(\tau_1,\tau_2) f^{XX}_{-\omega}(\sigma_1,\sigma_2) + (\omega-u_s)\Big\{ f^{XX}_{\omega}(\tau_1,\tau_2) f^{XX,(1)}_{-\omega}(\sigma_1,\sigma_2) - f^{XX}_{-\omega}(\sigma_1,\sigma_2) f^{XX,(1)}_{\omega}(\tau_1,\tau_2)\Big\} + (\omega-u_s)^2 \times \vartheta_{1,u_s,g}(\tau_1,\tau_2)\odot\vartheta_{2,u_s,g}(\sigma_1,\sigma_2).$$

Taking the sum over s and using Lemma 5, now gives

$$\Big(\frac{2\pi}{T}\Big)^2 \sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega-u_s) \times f^{XX}_{u_s}(\tau_1,\tau_2) f^{XX}_{-u_s}(\sigma_1,\sigma_2) =$$
$$\Big(\frac{2\pi}{T}\Big)^2 \sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega-u_s) \times f^{XX}_{\omega}(\tau_1,\tau_2) f^{XX}_{-\omega}(\sigma_1,\sigma_2) +$$
$$\Big(\frac{2\pi}{T}\Big)^2 \sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega-u_s) \times (\omega-u_s) \times \Big\{ f^{XX}_{\omega}(\tau_1,\tau_2) f^{XX,(1)}_{-\omega}(\sigma_1,\sigma_2) - f^{XX,(1)}_{\omega}(\tau_1,\tau_2) f^{XX}_{-\omega}(\sigma_1,\sigma_2)\Big\} +$$
$$\Big(\frac{2\pi}{T}\Big)^2 \sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega-u_s) \times (\omega-u_s)^2 \times \vartheta_{1,\nu_s,g}(\tau_1,\tau_2)\odot\vartheta_{2,\nu_s,g}(\sigma_1,\sigma_2)$$
$$= O\big(T^{-1}B_T^{-1}\big) \times f^{XX}_{\omega}(\tau_1,\tau_2) f^{XX}_{-\omega}(\sigma_1,\sigma_2) + O\big(T^{-1}B_T\big) \times \vartheta_{1,g}(\tau_1,\tau_2)\odot\vartheta_{2,g}(\sigma_1,\sigma_2).$$

Turning to $\mathbb{E}[AD] \times \mathbb{E}[BC]$, similar manipulations yield

$$\Big(\frac{2\pi}{T}\Big)^2 \sum_{s,r=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega-\nu_r) \times \Big[\eta(\nu_s+\nu_r) f^{XX}_{\nu_s}(\tau_1,\sigma_2) f^{XX}_{-\nu_s}(\sigma_1,\tau_2) + \frac{1}{T}\eta(\nu_s+\nu_r)\vartheta_{3,\nu_s,\nu_r,f}(\tau_1,\sigma_2)\odot\vartheta_{4,\nu_s,\nu_r,f}(\sigma_1,\tau_2) + \frac{1}{T^2}\vartheta_{3,\nu_s,\nu_r}(\tau_1,\sigma_2)\times\vartheta_{4,\nu_s,\nu_r}(\sigma_1,\tau_2)\Big]$$
$$= \Big(\frac{2\pi}{T}\Big)^2 \sum_{s=0}^{T-1} W^{(T)}(\omega-\nu_s)W^{(T)}(\omega+\nu_s) \times \Big[f^{XX}_{\nu_s}(\tau_1,\sigma_2) f^{XX}_{-\nu_s}(\sigma_1,\tau_2)\Big] + B_T^{-2}T^{-2}\vartheta_{3,f}(\tau_1,\sigma_2)\odot\vartheta_{4,f}(\sigma_1,\tau_2).$$

Again, replacing ν_s by u_s, using a Taylor expansion, and employing Lemma 5, we have

$$\frac{1}{T}\sum_{s=0}^{T-1} \big|W^{(T)}(\omega-u_s)\big| \times \big|W^{(T)}(\omega+u_s)\big| \times \big|\omega-u_s\big|^j = 1_{I_T}(\omega) O\big(B_T^{j-1}\big).$$

Then

$$\Big(\frac{2\pi}{T}\Big)^2 \sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega+u_s) \times f^{XX}_{u_s}(\tau_1,\sigma_2) f^{XX}_{-u_s}(\sigma_1,\tau_2) =$$

$$
\begin{aligned}
&\Big(\frac{2\pi}{T}\Big)^2 \sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega+u_s) \times f_{\omega}^{XX}(\tau_1,\sigma_2) f_{-\omega}^{XX}(\sigma_1,\tau_2) + \\
&\Big(\frac{2\pi}{T}\Big)^2 \sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega+u_s) \times (\omega-u_s) \times \big\{ f_{\omega}^{XX}(\tau_1,\sigma_2) f_{-\omega}^{XX,(1)}(\sigma_1,\tau_2) - f_{\omega}^{XX,(1)}(\tau_1,\sigma_2) f_{-\omega}^{XX}(\sigma_1,\tau_2) \big\} + \\
&\Big(\frac{2\pi}{T}\Big)^2 \sum_{s=0}^{T-1} W^{(T)}(\omega-u_s)W^{(T)}(\omega+u_s) \times (\omega-u_s)^2 \times \vartheta_{3,\nu_s,g}(\tau_1,\sigma_2) \odot \vartheta_{4,\nu_s,g}(\sigma_1,\tau_2) \\
&= 1_{I_T}(\omega) O\big(T^{-1}B_T^{-1}\big) \times f_{\omega}^{XX}(\tau_1,\sigma_2) f_{-\omega}^{XX}(\sigma_1,\tau_2) + 1_{I_T}(\omega) O\big(T^{-1}\big) \big\{ f_{\omega}^{XX}(\tau_1,\sigma_2) f_{-\omega}^{XX,(1)}(\sigma_1,\tau_2) + \\
&\qquad 1_{I_T}(\omega) f_{-\omega}^{XX}(\sigma_1,\tau_2) f_{\omega}^{XX,(1)}(\tau_1,\sigma_2) \big\} + 1_{I_T}(\omega) O\big(T^{-1}B_T\big) \times \vartheta_3(\tau_1,\sigma_2) \odot \vartheta_4(\sigma_1,\tau_2).
\end{aligned}
$$

Finally, we turn to $\operatorname{cum}(A,B,C,D)$, which consists in

$$
\frac{1}{T^2} \sum_{r,s=0}^{T-1} W^{(T)}(\omega-\nu_s) W^{(T)}(\omega-\nu_r) p_{r,s}^{(T)}(\tau_1,\sigma_1,\tau_2,\sigma_2).
$$

For random variables U and V with $\mathbb{E}U = u, \mathbb{E}v = v$ and constants a and b, it holds that

$$
\mathbb{E}\big[(U-a)\times(V-b)\big] = \mathbb{E}[UV] - av - bu + ab = \mathbb{E}\big[(U-u)(V-v)\big] + (a-u)(b-v).
$$

We use this formula with $U = f_{\omega}^{(T)}(\tau_1,\sigma_1)$, $V = f_{-\omega}^{(T)}(\tau_2,\sigma_2)$, $a = f_{\omega}(\tau_1,\sigma_1)$, and $b = f_{-\omega}(\tau_2,\sigma_2)$ to obtain

$$
\begin{aligned}
&\mathbb{E}\left[\Big\{ f_{\omega}^{(T)}(\tau_1,\sigma_1) - f_{\omega}^{XX}(\tau_1,\sigma_1) \Big\} \times \Big\{ f_{-\omega}^{(T)}(\tau_2,\sigma_2) - f_{-\omega}^{XX}(\tau_2,\sigma_2) \Big\} \right] = \\
&\qquad \mathbb{E}\left[\Big\{ f_{\omega}^{(T)}(\tau_1,\sigma_1) - \mathbb{E} f_{\omega}^{(T)}(\tau_1,\sigma_1) \Big\} \times \Big\{ f_{2\pi-\omega}^{(T)}(\tau_2,\sigma_2) - \mathbb{E} f_{2\pi-\omega}^{(T)}(\tau_2,\sigma_2) \Big\} \right] + \\
&\qquad \Big\{ \mathbb{E} f_{\omega}^{(T)}(\tau_1,\sigma_1) - f_{\omega}^{XX}(\tau_1,\sigma_1) \Big\} \times \Big\{ \mathbb{E} f_{\omega}^{(T)}(\tau_2,\sigma_2) - f_{\omega}^{XX}(\tau_2,\sigma_2) \Big\} \\
&= O\big(T^{-1}B_T^{-1}\big) \times \Big\{ f_{\omega}^{XX}(\tau_1,\tau_2) f_{-\omega}^{XX}(\sigma_1,\sigma_2) + f_{\omega}^{XX}(\tau_1,\sigma_2) f_{-\omega}^{XX}(\tau_2,\sigma_1) \Big\} + \\
&\quad O\big(T^{-1}B_T\big) \Big\{ \vartheta_1(\tau_1,\tau_2) \odot \vartheta_2(\sigma_1,\sigma_2) + \vartheta_3(\tau_1,\sigma_2) \odot \vartheta_4(\sigma_1,\tau_2) \Big\} \\
&\quad + 1_{I_T}(\omega) \times O\big(T^{-1}\big) \times \Big\{ f_{\omega}^{XX}(\tau_1,\sigma_2) f_{-\omega}^{XX,(1)}(\tau_2,\sigma_1) + f_{-\omega}^{XX}(\tau_2,\sigma_1) f_{\omega}^{XX,(1)}(\tau_1,\sigma_2) \Big\} \\
&\quad + \frac{1}{T^2} \sum_{r,s=0}^{T-1} W^{(T)}(\omega-\nu_s) W^{(T)}(\omega-\nu_r) p_{r,s}^{(T)}(\tau_1,\sigma_1,\tau_2,\sigma_2).
\end{aligned}
$$

$\square$

Proposition 6. *Assume assumptions (A1)-(A3) and (B1)-(B6) in Section 4 and 7 are satisfied, then there exists an universal constant C such that*

$$\mathbb{E}\left\|\left\|\widehat{\mathscr{F}}_{\omega,T}^{XX} - \mathscr{F}_{\omega}^{XX}\right\|\right\|_2^2 \leq C \times T^{-1}B_T^{-1}.$$

Proof of Proposition 6. By part C of Lemma 1,

$$\mathbb{E}\left\|\left\|\widehat{\mathscr{F}}_{\omega,T}^{XX} - \mathscr{F}_{\omega}^{XX}\right\|\right\|_2^2 =$$

$$\sum_{i,j} \mathbb{E}\left[\left\{f_{\omega}^{(T)}(\tau_1,\sigma_1) - f_{\omega}^{XX}(\tau_1,\sigma_1)\right\} \times \left\{f_{-\omega}^{(T)}(\tau_2,\sigma_2) - f_{-\omega}^{XX}(\tau_2,\sigma_2)\right\}\right] \overline{\varphi_i^{\omega}(\tau_1)}\varphi_j^{\omega}(\sigma_1)\varphi_i^{\omega}(\tau_2)\overline{\varphi_j^{\omega}(\sigma_2)}d\tau_1 d\sigma_1 d\tau_2 d\sigma_2.$$

We first decompose the right hand side by Proposition 5, then we apply Lemma 2 and follow the proof of Proposition 3 to obtain the upper bound $CT^{-1}B_T^{-1}$. □

Proposition 7. *Assume assumptions (A1)-(A3) and (B1)-(B6) in Section 4 and 7 are satisfied, the operator $\widehat{\mathscr{F}}_{\omega,T}^{XX} + \zeta_T \mathscr{I}$ is strictly positive definite on an event G_T satisfying $\mathbb{P}[G_T] \overset{T\to\infty}{\longrightarrow} 1$.*

Note that this proposition establishes that even if the kernel function W takes on some negative values, the ridge-estimator $\widehat{\mathscr{F}}_{\omega,T}^{XX} + \zeta_T \mathscr{I}$ will remain positive definite with high probability. Hence, we can find its inverse operator.

Proof of Proposition 7. By the result in our last proposition, there exists a constant C that does not depend on ω such that:

$$\mathbb{E}\left\|\left\|\widehat{\mathscr{F}}_{\omega,T}^{XX} - \mathscr{F}_{\omega}^{XX}\right\|\right\|_2^2 \leq C \times T^{-1}B_T^{-1}.$$

Let δ be a positive number such that $\gamma + 2\delta < \frac{2\beta-\alpha}{\alpha+2\beta}$. Define

$$G_T = \left\{\theta : \theta \in \Omega; \left\|\left\|\widehat{\mathscr{F}}_{\omega,T}^{XX} - \mathscr{F}_{\omega}^{XX}\right\|\right\|_2 \leq C^{1/2}T^{-1/2}B_T^{-1/2}T^{\delta}\right\}.$$

Then for $\delta > 0$, $\mathbb{P}(G_T) \to 1$. Let $\widehat{\lambda}_{j,T}^{\omega}$ denote the jth eigenvalue of $\widehat{\mathscr{F}}_{\omega,T}^{XX}$. Then, on the even G_T, we have

$$C^{1/2}T^{-1/2}B_T^{-1/2}T^{\delta} \geq \left\|\left\|\widehat{\mathscr{F}}_{\omega,T}^{XX} - \mathscr{F}_{\omega}^{XX}\right\|\right\|_2 \geq \left|\widehat{\lambda}_{j,T}^{\omega} - \lambda_j^{\omega}\right| \geq \lambda_j^{\omega} - \widehat{\lambda}_{j,T}^{\omega}$$

$$\widehat{\lambda}^{\omega}_{j,T} \geq -C^{1/2}T^{-1/2}B_T^{-1/2}T^{\delta}.$$

Since $B_T = T^{-\gamma}$ and $\alpha/(\alpha+2\beta) < 1/2 - \gamma/2 - \delta$, it must be that $\zeta_T > C^{1/2}T^{-1/2}T^{\gamma/2}T^{\delta} = C^{1/2}T^{-1/2}B_T^{-1/2}T^{\delta}$. It follows that

$$\zeta_T + \widehat{\lambda}^{\omega}_{j,T} \geq T^{-\alpha/(\alpha+2\beta)} - C^{1/2}B_T^{-1/2}T^{-1/2}T^{\delta} > 0,$$

on G_T. $\square$

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