Controlling Correlations in Sliced Latin Hypercube Designs

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Supplementary Material

S1 Proof of Proposition 1

Proof. By definition, $\mathbf{b}^1, \ldots, \mathbf{b}^{t+1}$ are mutually uncorrelated. Thus,

$$\mathbf{x}^\ell \leftarrow \texttt{takeout}(\mathbf{b}^1, \ldots, \mathbf{b}^{t+1}, \mathbf{x}^\ell) \tag{S1.1}$$

in Algorithm 2 is equivalent to taking all the following steps consecutively

$$\mathbf{x}^\ell \leftarrow \texttt{takeout}(\mathbf{b}^1, \mathbf{x}^\ell)$$

$$\ldots$$

$$\mathbf{x}^\ell \leftarrow \texttt{takeout}(\mathbf{b}^{t+1}, \mathbf{x}^\ell).$$

The above $t+1$ updates are equivalent to

$$\mathbf{x}^\ell \leftarrow \mathbf{x}^\ell - .5 - \sum_{s=1}^{t+1} (\mathbf{b}^s - \bar{\mathbf{b}}^s)\rho(\mathbf{b}^s, \mathbf{x}^\ell)\sigma(\mathbf{x}^\ell)/\sigma(\mathbf{b}^s), \tag{S1.2}$$

where $\rho(\mathbf{b}^s, \mathbf{x}^\ell)$ is the sample correlation between $\mathbf{b}^s$ and $\mathbf{x}^\ell$, and $\sigma(\mathbf{x}^\ell)$ and $\sigma(\mathbf{b}^s)$ are the sample standard deviation in $\mathbf{x}^\ell$ and $\mathbf{b}^s$, respectively.

As $\mathbf{b}^s_{(r)} = \bar{\mathbf{x}}^k_{(r)}$ if $s \neq r$ and $\mathbf{b}^s_{(r)} = \mathbf{x}^k_{(r)}$ otherwise for $s = 1, \ldots, t$, note that

$$\rho(\mathbf{b}^s, \mathbf{x}^\ell)\sigma(\mathbf{x}^\ell)/\sigma(\mathbf{b}^s) = \rho(\mathbf{x}^k_{(s)}, \mathbf{x}^\ell_{(s)})\sigma(\mathbf{x}^\ell_{(s)})/\sigma(\mathbf{x}^k_{(s)}), \tag{S1.3}$$

and

$$\mathbf{b}^s_{(r)} - \bar{\mathbf{b}}^s_{(r)} = \begin{cases} \mathbf{x}^k_{(r)} - \bar{\mathbf{x}}^k_{(r)}, & \text{if } s = r; \\ 0, & \text{otherwise.} \end{cases} \tag{S1.4}$$

It is straightforward to show

$$\rho(\mathbf{b}^{t+1}, \mathbf{x}^\ell)\sigma(\mathbf{x}^\ell)/\sigma(\mathbf{b}^{t+1}) = 1 \tag{S1.5}$$

and

$$\mathbf{b}^{t+1}_{(r)} - \bar{\mathbf{b}}^{t+1}_{(r)} = \bar{\mathbf{x}}^\ell_{(r)} - .5 \tag{S1.6}$$

for $r = 1, \ldots, t$.

By substituting (S1.3)-(S1.6) into (S1.2), we have for each r, (S1.1) is equivalent to

$$\mathbf{x}^\ell_{(r)} \leftarrow \mathbf{x}^\ell_{(r)} - \bar{\mathbf{x}}^\ell_{(r)} - (\mathbf{x}^k_{(r)} - \bar{\mathbf{x}}^k_{(r)})\rho(\mathbf{x}^k_{(r)}, \mathbf{x}^\ell_{(r)})\sigma(\mathbf{x}^\ell_{(r)})/\sigma(\mathbf{x}^k_{(r)}),$$

which is the outcome of $\mathbf{x}^\ell_{(r)} \leftarrow \texttt{takeout}(\mathbf{x}^k_{(r)}, \mathbf{x}^\ell_{(r)})$. $\square$

S2 Example of using Algorithm 2

Let $n = 5$, $p = 2$ and $t = 2$. In the first step, generate a random sliced Latin hypercube design as

$$\mathbf{X} = \left(\begin{array}{ccccc|ccccc} .85 & .55 & .15 & .25 & .75 & .65 & .95 & .45 & .35 & .05 \\ .45 & .75 & .95 & .15 & .25 & .85 & .65 & .35 & .55 & .05 \end{array} \right)^T$$

with $\mathbf{X}_{(1)}$ and $\mathbf{X}_{(2)}$ separated by the vertical line. The corresponding $\mathbf{A}$ and $\mathbf{\Theta}$ are

$$\mathbf{A} = \left(\begin{array}{ccccc|ccccc} 5 & 3 & 1 & 2 & 4 & 4 & 5 & 3 & 2 & 1 \\ 3 & 4 & 5 & 1 & 2 & 5 & 4 & 2 & 3 & 1 \end{array} \right)^T$$

and

$$\boldsymbol{\theta} = \left(\begin{array}{ccccc|ccccc} 1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 1 \end{array} \right)^T.$$

In the first forward step, $\mathbf{b}^1$, $\mathbf{b}^2$ and $\mathbf{b}^3$ are given as

$$\mathbf{b}^1 = (.45 \quad .75 \quad .95 \quad .15 \quad .25 \mid .51 \quad .51 \quad .51 \quad .51 \quad .51)^T$$

$$\mathbf{b}^2 = (.49 \quad .49 \quad .49 \quad .49 \quad .49 \mid .85 \quad .65 \quad .35 \quad .55 \quad .05)^T$$

$$\mathbf{b}^3 = (.01 \quad .01 \quad .01 \quad .01 \quad .01 \mid -.01 \quad -.01 \quad -.01 \quad -.01 \quad -.01)^T$$

where $.51 = \bar{\mathbf{x}}^2_{(1)}$, $.49 = \bar{\mathbf{x}}^2_{(2)}$, $-0.1 = .5 - \bar{\mathbf{x}}^2_{(1)}$ and $0.1 = .5 - \bar{\mathbf{x}}^2_{(2)}$, respectively.

The residual vector $\mathbf{x}^1$ after taking out $\mathbf{b}^1$, $\mathbf{b}^2$ and $\mathbf{b}^3$ is set to be

$$(.3217 \quad .113 \quad -.226 \quad -.370 \quad .161 \mid -.152 \quad .3215 \quad .081 \quad -.192 \quad -.059)^T,$$

before the `rank` function.

Notice that if we directly rank $\mathbf{x}^1$, we have

$$\texttt{rank}(\mathbf{x}^1) = (10 \quad 7 \quad 2 \quad 1 \quad 8 \,|\, 4 \quad 9 \quad 6 \quad 3 \quad 5)^T$$

which cannot produce a sliced Latin hypercube design. Instead, we should first rank $\mathbf{x}^1_{(1)}$ and $\mathbf{x}^1_{(2)}$ within each slice to obtain new $\mathbf{a}^1$ as

$$\mathbf{a}^1 = (5 \quad 3 \quad 2 \quad 1 \quad 4 \,|\, 2 \quad 5 \quad 4 \quad 1 \quad 3)^T.$$

For $j = 1$, we find the 4th and 9th values in $\mathbf{a}^1$ are 1. As a result, $\mathbf{x}^1(j) = (-.370 \quad -.192)^T$ and $\boldsymbol{\theta}^1(j) = \texttt{rank}(\mathbf{x}^1(j)) = (1 \quad 2)^T$. Repeating the same procedure for $j = 2, \ldots, 5$ to update $\boldsymbol{\theta}^1$ as

$$\boldsymbol{\theta}^1 = (2 \quad 2 \quad 1 \quad 1 \quad 2 \,|\, 2 \quad 1 \quad 1 \quad 2 \quad 1)^T.$$

The backward procedure can be carried out in the same way to update $\mathbf{x}^2$. The updated $\mathbf{X}$ after a complete alteration is given by

$$\mathbf{X} = \left(\begin{array}{ccccc|ccccc} .95 & .55 & .25 & .05 & .75 & .35 & .85 & .65 & .15 & .45 \\ .45 & .75 & .95 & .15 & .25 & .85 & .65 & .35 & .55 & .05 \end{array}\right)^T.$$

In this example, $\mathbf{x}^2$ cannot be updated as the algorithm converges after the first forward procedure.