0.4pt=0pt

SENSITIVITY ANALYSIS USING PERMUTATIONS

Shifeng Xiong, Xu He, Yuanzhen He, and Weiyan Mu

Chinese Academy of Sciences, Beijing Normal University,

and Beijing University of Civil Engineering and Architecture

Supplementary Material

S1 The Input Matrices in Section 6.1

For $n = 7$, the input values are columns of the matrix

$$\begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} + \rho \begin{pmatrix} 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 & 1 & 1+\rho \\ 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 & 1 & 1+\rho \\ 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 & 1 & 1+\rho \end{pmatrix},$$

where the parameter ρ controls the correlations between the three dimensions. If $\rho = 0$, then the correlations between the three rows are zeros. For $n = 14$, the input values are columns of

$$\begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \quad \rho \begin{pmatrix} 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 \\ 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 \\ 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 & 1 & 1+\rho & \rho+\rho^2 \end{pmatrix}.$$

S2 Proofs

Proof of Theorem 1. Let $Z_{(1)} \leqslant \cdots \leqslant Z_{(n)}$ denote the order statistics of $Z_1, \ldots, Z_n$. Under the conditions, given $(Z_{(1)}, \ldots, Z_{(n)},\ \mathbf{E}') = (z_{(1)}, \ldots, z_{(n)},\ \mathbf{e}')$, T follows the discrete distribution: the probability of $T = T(z_{i_1}, \ldots, z_{i_n};\ \mathbf{e})$ is $1/n!$ for each permutation $(i_1, \ldots, i_n) \in \mathbb{S}_n$.

Let $t_{(1)} \leqslant \cdots \leqslant t_{(n!)}$ be the non-decreasing permutation of $\{T(z_{i_1}, \ldots, z_{i_n};\ \mathbf{e})\}_{(i_1,\ldots,i_n)\in\mathbb{S}_n}$. Let $[\cdot]$ denote the floor function. If $t_{([n!\alpha])} < t_{([n!\alpha]+1)}$, then

$$\begin{aligned} & \Pr\big(q < \alpha \mid (Z_{(1)}, \ldots, Z_{(n)},\ \mathbf{E}') = (z_{(1)}, \ldots, z_{(n)},\ \mathbf{e}')\big) \\ = & \Pr\big(T < t_{([n!\alpha])} \mid (Z_{(1)}, \ldots, Z_{(n)},\ \mathbf{E}') = (z_{(1)}, \ldots, z_{(n)},\ \mathbf{e}')\big) \\ = & [n!\alpha]/n! \leqslant \alpha. \end{aligned} \tag{S2.1}$$

Otherwise, let $S = \{k = 1, \ldots, [n!\alpha] - 1 :\ t_{([k])} < t_{([n!\alpha])}\}$. Define $k_0 = 0$ for $S = \emptyset$ and $k_0 = \max S$ otherwise. We have

$$\begin{aligned} & \Pr\big(q < \alpha \mid (Z_{(1)}, \ldots, Z_{(n)},\ \mathbf{E}') \\ = & (z_{(1)}, \ldots, z_{(n)},\ \mathbf{e}')\big) = k_0/n! < [n!\alpha]/n! \leqslant \alpha. \end{aligned} \tag{S2.2}$$

By (S2.1) and (S2.2), the inequality $\Pr(q < \alpha \mid Z_{(1)}, \ldots, Z_{(n)},\ \mathbf{E}') \leqslant \alpha$ always holds. Taking expectations in the inequality, we complete the proof.

□

Proof of Lemma 1. Denote $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ and $\nu = n - \mathbf{z}_1'\mathbf{P}_2\mathbf{z}_1$. Under

H_0, $(\mathbf{I}_n - \mathbf{P})\mathbf{X} = (\mathbf{I}_n - \mathbf{P}_2)\mathbf{Z}_2 = \mathbf{0}$. Thus,

$$
\begin{aligned}
T &= \mathbf{y}'(\mathbf{I}_n - \mathbf{P}_2)\mathbf{y} - \mathbf{y}'(\mathbf{I}_n - \mathbf{P})\mathbf{y} = \boldsymbol{\varepsilon}'\mathbf{P}\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}'\mathbf{P}_2\boldsymbol{\varepsilon} \\
&= \boldsymbol{\varepsilon}'\mathbf{X}\begin{pmatrix} n & \mathbf{z}_1'\mathbf{Z}_2 \\ \mathbf{Z}_2'\mathbf{z}_1 & \mathbf{Z}_2'\mathbf{Z}_2 \end{pmatrix}^{-1}\mathbf{X}'\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}'\mathbf{P}_2\boldsymbol{\varepsilon} \\
&= \boldsymbol{\varepsilon}'\mathbf{X}\begin{pmatrix} 1/\nu & -\mathbf{z}_1'\mathbf{Z}_2(\mathbf{Z}_2'\mathbf{Z}_2)^{-1}/\nu \\ -(\mathbf{Z}_2'\mathbf{Z}_2)^{-1}\mathbf{Z}_2'\mathbf{z}_1/\nu & (\mathbf{Z}_2'\mathbf{Z}_2)^{-1} + (\mathbf{Z}_2'\mathbf{Z}_2)^{-1}\mathbf{Z}_2'\mathbf{z}_1\mathbf{z}_1'\mathbf{Z}_2(\mathbf{Z}_2'\mathbf{Z}_2)^{-1}/\nu \end{pmatrix}\mathbf{X}'\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}'\mathbf{P}_2\boldsymbol{\varepsilon} \\
&= \boldsymbol{\varepsilon}'(\mathbf{z}_1\mathbf{z}_1' + \mathbf{P}_2\mathbf{z}_1\mathbf{z}_1'\mathbf{P}_2 - \mathbf{z}_1\mathbf{z}_1'\mathbf{P}_2 - \mathbf{P}_2\mathbf{z}_1\mathbf{z}_1')\boldsymbol{\varepsilon}/\nu \\
&= \left[\mathbf{z}_1'(\mathbf{I}_n - \mathbf{P}_2)\boldsymbol{\varepsilon}\right]^2/\nu = \left[\mathbf{z}_1'(\mathbf{I}_n - \mathbf{P}_2)\mathbf{y}\right]^2/\nu,
\end{aligned}
$$

which completes the proof. $\square$

Lemma 2. *Let* $\boldsymbol{\xi} = (\xi_{n1}, \ldots, \xi_{nn})' = \mathbf{u}\mathbf{u}'\boldsymbol{\varepsilon}$ *for some* $\mathbf{u} = (u_{n1}, \ldots, u_{nn})' \in \mathbb{R}^n$ *with* $\|\mathbf{u}\| = 1$. *Under Assumption 3, we have*

(i) $\sum_{i=1}^n \xi_{ni}/n = o(1)$ *(a.s.);*

(ii) $\sum_{i=1}^n \xi_{ni}^2/n = o(1)$ *(a.s.).*

Furthermore, if Assumption 4 holds, then

(iii) $\sum_{i=1}^n |\xi_{ni}|^r/n = o(1)$ *(a.s.) for any* $r > 2$.

Proof. (i) We have $|\sum_{i=1}^n \xi_{ni}/n| = |\sum_{i=1}^n u_{ni}/n| \cdot |\mathbf{u}'\boldsymbol{\varepsilon}| \leqslant |\mathbf{u}'\boldsymbol{\varepsilon}|/\sqrt{n} \to 0$ (a.s.), where the convergence is from Chow (1966).

(ii) We have $\sum_{i=1}^n \xi_{ni}^2/n = \sum_{i=1}^n u_{ni}^2|\mathbf{u}'\boldsymbol{\varepsilon}|^2/n = |\mathbf{u}'\boldsymbol{\varepsilon}|^2/n \to 0$ (a.s.).

(iii) Note that $\sum_{i=1}^n |\xi_{ni}|^r/n = \sum_{i=1}^n |u_{ni}|^r|\mathbf{u}'\boldsymbol{\varepsilon}|^r/n \leqslant \|\mathbf{u}\|^2|\mathbf{u}'\boldsymbol{\varepsilon}|^r/n = |\mathbf{u}'\boldsymbol{\varepsilon}/n^{1/r}|^r$.

Under Assumption 4, $\mathbf{u}'\boldsymbol{\varepsilon} \sim N(0, \sigma^2)$. Let Φ denote the c.d.f. of $N(0,1)$. We have $\Pr(|\mathbf{u}'\boldsymbol{\varepsilon}/n^{1/r}| > \epsilon) = 2[1-\Phi(\epsilon\sigma n^{1/r})] < (\epsilon\sigma n^{1/r})^{-1}\exp(-\epsilon^2\sigma^2 n^{2/r}/2)$, which implies $\sum_{n=1}^{\infty}\Pr(|\mathbf{u}'\boldsymbol{\varepsilon}/n^{1/r}| > \epsilon) < \infty$. By the Borel-Cantelli lemma, $|\mathbf{u}'\boldsymbol{\varepsilon}/n^{1/r}|^r \to 0$ (a.s.), and we complete the proof. $\square$

Lemma 3. *Under Assumption 3,* $\max_{1\leqslant i\leqslant n} e_{ni}^2/n \to 0$ *(a.s.), where* e_{ni} *is defined by (13).*

Proof. We first show

$$\max_{1\leqslant i\leqslant n} \varepsilon_i^2/n \to 0 \quad \text{(a.s.)}. \tag{S2.3}$$

Note that $\varepsilon_n^2/n = \sum_{i=1}^n \varepsilon_i^2/n - n^{-1}(n-1)\sum_{i=1}^n \varepsilon_i^2/(n-1) \to 0$ (a.s.). For any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\max_{1\leqslant i\leqslant n} \varepsilon_i^2/n \leqslant \max_{1\leqslant i\leqslant n_0} \varepsilon_i^2/n + \max_{n_0<i\leqslant n} \varepsilon_i^2/n < \max_{1\leqslant i\leqslant n_0} \varepsilon_i^2/n + \epsilon$. Letting $n \to \infty$, we obtain (S2.3).

Since the rank of the idempotent matrix $\mathbf{P}_2$ is $d-1$, we can write it as $\sum_{j=1}^{d-1}\mathbf{u}_j\mathbf{u}_j'$, where $\mathbf{u}_j \in \mathbb{R}^n$ with $\|\mathbf{u}_j\| = 1$ for $j = 1, \ldots, d-1$. Therefore, $\mathbf{e}_n$ in (13) can be written as

$$\mathbf{e}_n = \boldsymbol{\varepsilon} - \sum_{j=1}^{d-1} \tilde{\mathbf{e}}_j, \tag{S2.4}$$

where $\tilde{\mathbf{e}}_j = (\tilde{e}_{j1}, \ldots, \tilde{e}_{jn})' = \mathbf{u}_j\mathbf{u}_j'\boldsymbol{\varepsilon}$.

Consequently, by (S2.3), (S2.4), and Lemma 2 (ii), we have $\max_{1\leqslant i\leqslant n} e_{ni}^2/n = \max_{1\leqslant i\leqslant n}(\varepsilon_i - \sum_{j=1}^{d-1}\tilde{e}_{ni})^2/n \leqslant d\max_{1\leqslant i\leqslant n}(\varepsilon_i^2 + \sum_{j=1}^{d-1}\tilde{e}_{ni}^2)/n \leqslant d\max_{1\leqslant i\leqslant n}\varepsilon_i^2/n +$

$d \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{d-1} \tilde{e}_{ni}^2/n \leqslant o(1) + d \sum_{j=1}^{d-1} \sum_{i=1}^{n} \tilde{e}_{ni}^2/n = o(1)$ (a.s.), which completes the proof. □

Definition 1. For a vector $\mathbf{a}_n = (a_{n1}, \ldots, a_{nn})'$, denote $\bar{a}_n = \sum_{i=1}^{n} a_{ni}/n$ and $\mu_r(\mathbf{a}_n) = \sum_{i=1}^{n}(a_{ni} - \bar{a}_n)^r/n$ for $r = 2, 3, \ldots$. We say $\mathbf{a}_n$ satisfies condition (W) if for any fixed $r = 3, 4, \ldots$, $\mu_r(\mathbf{a}_n)/\mu_2^{r/2}(\mathbf{a}_n) = O(1)$, and we say $\mathbf{a}_n$ satisfies condition (N) if for any fixed $r = 3, 4, \ldots$, $\mu_r(\mathbf{a}_n)/\mu_2^{r/2}(\mathbf{a}_n) = o(n^{r/2-1})$.

Conditions (W) and (N) (Wald and Wolfowitz (1944); Noether (1949)) are basic conditions for discussing asymptotics of permutation tests.

Lemma 4. *Under Assumption 3, $\mathbf{e}_n$ in (13) satisfies condition (N) (a.s.).*

Proof. By Lemma 2 (i),

$$\bar{e}_n = \sum_{i=1}^{n} e_{ni}/n = \sum_{i=1}^{n} \varepsilon_i/n - \sum_{j=1}^{d-1} \sum_{i=1}^{n} \tilde{e}_{ji}/n = o(1) \quad \text{(a.s.)}. \tag{S2.5}$$

We have $|\mu_2(\mathbf{e}_n) - \sigma^2| = |\sum_{i=1}^{n}(\varepsilon_i - \sum_{j=1}^{d-1} \tilde{e}_{ji} - \bar{e}_n)^2/n - \sigma^2| \leqslant |\sum_{i=1}^{n} \varepsilon_i^2/n - \sigma^2| + 2|\sum_{i=1}^{n}(\sum_{j=1}^{d-1} \tilde{e}_{ji} + \bar{e}_n)\varepsilon_i/n| + |\sum_{i=1}^{n}(\sum_{j=1}^{d-1} \tilde{e}_{ji} + \bar{e}_n)^2/n| \leqslant o(1) + 2|\sum_{i=1}^{n} \sum_{j=1}^{d-1} \tilde{e}_{ji}\varepsilon_i|/n + 2|\bar{e}_n||\sum_{i=1}^{n} \varepsilon_i|/n + (d-1)\sum_{j=1}^{d-1} \sum_{i=1}^{n} \tilde{e}_{ji}^2/n + (d-1)\bar{e}_n^2$ (a.s.). By (S2.5) and Lemma 2 (ii), $2|\bar{e}_n||\sum_{i=1}^{n} \varepsilon_i|/n + (d-1)\sum_{j=1}^{d-1} \sum_{i=1}^{n} \tilde{e}_{ji}^2/n + (d-1)\bar{e}_n^2 = o(1)$ (a.s.). By the Cauchy-Schwarz inequality and Lemma 2 (ii), $|\sum_{i=1}^{n} \sum_{j=1}^{d-1} \tilde{e}_{ji}\varepsilon_i|/n \leqslant \left[\sum_{i=1}^{n}(\sum_{j=1}^{d-1} \tilde{e}_{ji})^2 \left(\sum_{i=1}^{n} \varepsilon_i^2\right)\right]^{1/2}/n$

$\leqslant \left[\left\{(d-1)\sum_{j=1}^{d-1}\sum_{i=1}^{n}\tilde{e}_{ji}^2/n\right\}\left(\sum_{i=1}^{n}\varepsilon_i^2/n\right)\right]^{1/2} = o(1)$ (a.s.). Therefore,

$$\mu_2(\mathbf{e}_n) \to \sigma^2 \quad \text{(a.s.)}. \tag{S2.6}$$

For even $r = 2, 4, \ldots$, by the C_r inequality and Lemma 3, $\mu_r(\mathbf{e}_n) = \sum_{i=1}^{n}(e_{ni} - \bar{e}_n)^r/n \leqslant 2^{r-1}\sum_{i=1}^{n} e_{ni}^r/n + 2^{r-1}\bar{e}_n^r = 2^{r-1}\sum_{i=1}^{n}(e_{ni}^2)^{r/2-1}e_{ni}^2/n + o(1) \leqslant 2^{r-1}(\max_{1\leqslant i\leqslant n} e_{ni}^2)^{r/2-1}\sum_{i=1}^{n} e_{ni}^2/n + o(1) = o(n^{r/2-1})\sum_{i=1}^{n} e_{ni}^2/n + o(1) = o(n^{r/2-1})$ (a.s.). For odd $r = 2s+1 = 3, 5, \ldots$, by the Cauchy-Schwarz inequality, $\mu_r^2(\mathbf{e}_n) = \left(\sum_{i=1}^{n}(e_{ni} - \bar{e}_n)^r\right)^2/n^2 \leqslant \left(\sum_{i=1}^{n}(e_{ni} - \bar{e}_n)^{2s}\right)\left(\sum_{i=1}^{n}(e_{ni} - \bar{e}_n)^{2s+2}\right)/n^2 = \mu_{2s}(\mathbf{e}_n)\mu_{2s+2}(\mathbf{e}_n) = o(n^{r-2})$ (a.s.). Combining these results for $\mu_r(\mathbf{e}_n)$ and (S2.6), we completes the proof. □

Lemma 5. *Under Assumption 4,* $\mathbf{e}_n$ *in (13) satisfies condition (W) (a.s.).*

Proof. Since (S2.5) and (S2.6) hold under Assumption 3, it suffices to consider $\mu_r(\mathbf{e}_n)$. For $r > 2$, by the C_r inequality and Lemma 2 (iii)

$$\begin{aligned} |\mu_r(\mathbf{e}_n)| &\leqslant \sum_{i=1}^{n}|\varepsilon_i - \sum_{j=1}^{d-1}\tilde{e}_{ji} - \bar{e}_n|^r/n \\ &\leqslant (d+1)^{r-1}\left(\sum_{i=1}^{n}|\varepsilon_i|^r/n + \sum_{j=1}^{d-1}\sum_{i=1}^{n}|\tilde{e}_{ji}|^r/n + |\bar{e}_n|^r\right) = O(1) \quad \text{(a.s.)}, \end{aligned}$$

which completes the proof. □

Lemma 6. *Under Assumption 5,* $\mathbf{z}_1$ *satisfies condition (N); Under Assumption 6,* $\mathbf{z}_1$ *satisfies condition (W).*

Proof. Based on (9), the proof of the first statement is similar to the proof of Lemma 4. The second statement is obvious. □

Lemma 7. *Under Assumption 7,* $\boldsymbol{\eta}_n'\mathbf{P}_2\boldsymbol{\eta}_n = o_p(n)$.

Proof. For $j = 2, \ldots, d$, we have

$$\mathrm{E}(\mathbf{z}_j'\boldsymbol{\eta}_n) = \sum_{(i_1,\ldots,i_n)\in\mathbb{S}_n} (x_{i_1 1}x_{1j}+\cdots+x_{i_n 1}x_{nj})/n! = (n-1)!\sum_{i=1}^n x_{i1}\sum_{i=1}^n x_{ij}/n! = 0, \tag{S2.7}$$

and

$$\begin{aligned}
&\mathrm{Var}(\mathbf{z}_j'\boldsymbol{\eta}_n) = \sum_{(i_1,\ldots,i_n)\in\mathbb{S}_n} (x_{i_1 1}x_{1j}+\cdots+x_{i_n 1}x_{nj})^2/n! \\
= & \sum_{(i_1,\ldots,i_n)\in\mathbb{S}_n}\left(x_{i_1 1}^2 e_{1j}^2+\cdots+x_{i_n 1}^2 x_{nj}^2 + 2\sum_{l<k} x_{i_l 1}x_{i_k 1}x_{lj}x_{kj}\right)/n! \\
= & \sum_{i=1}^n x_{i1}^2\sum_{i=1}^n x_{1j}^2/n + 2\sum_{l<k}\left(x_{lj}x_{kj}\sum_{(i_1,\ldots,i_n)\in\mathbb{S}_n} x_{i_l 1}x_{i_k 1}\right)/n! \\
= & n + 2\sum_{l<k}\left(x_{lj}x_{kj}\cdot 2(n-2)!\sum_{l<k} x_{l1}x_{k1}\right)/n! \\
= & n + \left(\left(\sum_{i=1}^n x_{i1}\right)^2 - \sum_{i=1}^n x_{i1}^2\right)\left(\left(\sum_{i=1}^n x_{ij}\right)^2 - \sum_{i=1}^n x_{ij}^2\right)/\big(n(n-1)\big) \\
= & n^2/(n-1).
\end{aligned} \tag{S2.8}$$

Therefore, $\mathrm{E}\|\mathbf{Z}_2'\boldsymbol{\eta}_n/n\|^2 = \sum_{j=2}^d \mathrm{E}(\mathbf{z}_j'\boldsymbol{\eta}_n)^2/n^2 = (d-1)/(n-1) = o(1)$, which implies $\mathbf{Z}_2'\boldsymbol{\eta}_n/n = o_p(1)$. By Assumption 7, $\boldsymbol{\eta}_n'\mathbf{P}_2\boldsymbol{\eta}_n/n = (\boldsymbol{\eta}_n'\mathbf{Z}_2/n)(\mathbf{Z}_2'\mathbf{Z}_2/n)^{-1}(\mathbf{Z}_2'\boldsymbol{\eta}_n/n) = o_p(1)$. This completes the proof. □

Proof of Theorem 2. Similar to (S2.7) and (S2.8),

$$\mathrm{E}(\mathbf{e}_n'\boldsymbol{\eta}_n \mid \boldsymbol{\varepsilon}) = (n-1)!\sum_{i=1}^n x_{i1}\sum_{i=1}^n e_{ni}/n! = 0,$$

and

$$\begin{aligned}
&\mathrm{Var}(\mathbf{e}_n'\boldsymbol{\eta}_n \mid \boldsymbol{\varepsilon}) = \sum_{(i_1,\ldots,i_n)\in\mathbb{S}_n}(x_{i_11}e_{n1}+\cdots+x_{i_n1}e_{nn})^2/n! \\
= &\sum_{i=1}^n e_{ni}^2 + \left(\left(\sum_{i=1}^n x_{i1}\right)^2 - \sum_{i=1}^n x_{i1}^2\right)\left(\left(\sum_{i=1}^n e_{ni}\right)^2 - \sum_{i=1}^n e_{ni}^2\right)\big/\big(n(n-1)\big) \\
= &\; n^2\left(\sum_{i=1}^n e_{ni}^2/n - \bar{e}_n^2\right)/(n-1) = n^2\mu_2(\mathbf{e}_n)/(n-1).
\end{aligned}$$

By (S2.6), $\mathrm{Var}^*(\mathbf{e}_n'\boldsymbol{\eta}_n)/n \to \sigma^2$ (a.s.). Therefore, by Lemmas 4, 5, and 6, and the permutation central limit theorem (see, e.g. Chapter 3 of Pesarin and Salmaso (2010) or Theorem 6.6 of Chen (1999)), the distribution of $\mathbf{e}_n'\boldsymbol{\eta}_n/\sqrt{n}$ conditional on $\boldsymbol{\varepsilon}$ converges to $N(0,\sigma^2)$ (a.s.),. Combining this result and Lemma 7, we complete the proof. □

Proof of Theorem 3. Under Assumptions 3 and 8, by Theorem 2 of Eicker (1963), we have

$$\frac{\mathbf{z}_1'(\mathbf{I}_n - \mathbf{P}_2)\boldsymbol{\varepsilon}}{\sqrt{n - \mathbf{z}_1'\mathbf{P}_2\mathbf{z}_1}} \to N(0,\sigma^2) \tag{S2.9}$$

in distribution. By Lemma 1, (S2.9) implies this theorem. □

REFERENCES

Chen, X. (1999). *Advanced Mathematical Statistics* (In Chinese). University of Science and Technology of China Press.

Chow, Y. S. (1966). Some convergence theorems for independent random variables. *Ann. Math. Statist.* **37**, 1482–1493.

Eicker, F. R. (1963). Asymptotic normality and consistency of the least squares estimators for families of linear regressions. *Ann. Math. Statist.* **34**, 447–456.

Noether, G. E. (1949). On a theorem by Wald and Wolfowitz. *Ann. Math. Statist.* **20**, 455-458.

Pesarin, F. and Salmaso, L. (2010). *Permutation Tests for Complex Data: Theory, Applications and Software.* John Wiley & Sons.

Wald, A. and Wolfowitz, J. (1944). Statistical tests based on permutations of the observations. *Ann. Math. Statist.* **15**, 358–372.