A nonparametric approach for partial areas under ROC curves and ordinal dominance curves

Hanfang Yang, Kun Lu and Yichuan Zhao

Renmin University of China
University of Chicago
Georgia State University

Supplementary Material

S1 Proof of Corollary 1

Proof of Corollary 1 We follow the proof of Theorem 3.10 of Liu (2006) and obtain Corollary 1. □

S2 Proof of Theorem 1

Proof of Theorem 1

We set $f(t, P_0) = I\{t > S_G^{-1}(P_0)\}$ and $f_n(t, P_0) = I\{t > S_{G,n}^{-1}(P_0)\}$. For the fixed $P_0 \in (0, 1)$,

$$
\begin{aligned}
&\int \{f_n(t, P_0) - f(t, P_0)\}^2 dG(t) \\
=&\int [I\{t > S_{G,n}^{-1}(P_0)\} - I\{t > S_G^{-1}(P_0)\}]^2 dG(t) \\
=&\int |I\{t > S_{G,n}^{-1}(P_0)\} - I\{t > S_G^{-1}(P_0)\}| dG(t) \\
=&\int [I\{S_G^{-1}(P_0) > t > S_{G,n}^{-1}(P_0)\} I\{S_G^{-1}(P_0) \geq S_{G,n}^{-1}(P_0) \\
&+ I\{S_{G,n}^{-1}(P_0) > t > S_G^{-1}(P_0)\} I\{S_G^{-1}(P_0) \leq S_{G,n}^{-1}(P_0)\}] dG(t) \\
=&\int_{S_{G,n}^{-1}(P_0)}^{S_G^{-1}(P_0)} I\{S_G^{-1}(P_0) \geq S_{G,n}^{-1}(P_0)\} dG(t) + \int_{S_G^{-1}(P_0)}^{S_{G,n}^{-1}(P_0)} I\{S_G^{-1}(P_0) \leq S_{G,n}^{-1}(P_0)\} dG(t) \\
\leq&\left|\int_{S_{G,n}^{-1}(P_0)}^{S_G^{-1}(P_0)} dG(t)\right| + \left|\int_{S_G^{-1}(P_0)}^{S_{G,n}^{-1}(P_0)} dG(t)\right| \\
=&2\left|G\{S_G^{-1}(P_0)\} - G\{S_{G,n}^{-1}(P_0)\}\right| \\
=&2\left|S_G\{S_G^{-1}(P_0)\} - S_G\{S_{G,n}^{-1}(P_0)\}\right| \\
=&2\left|[S_{G,n}^{-1}(P_0) - S_G^{-1}(P_0)] S_G^{'}\{S_G^{-1}(P_0)\}\right| + o_p(1) \\
&\xrightarrow{P} 0.
\end{aligned}
$$

Hence, $\int\{f_n(t,P_0)-f(t,P_0)\}^2 dG(t)$ converges to 0 in probability. Applying Lemma 19.24 in van der Vaart (2000) directly, we know that $\mathbb{B}_n(f_n-f)\xrightarrow{P}0$, where $\mathbb{B}_n=\sqrt{n}(\mathbb{G}_n-\mathbb{G})$, $\mathbb{G}_n f=\frac{1}{n}\sum_{j=1}^n f(Y_j)$ and $\mathbb{G}f=\int f(t)dG(t)$. Thus, we can prove that

$$\begin{aligned}
&\sqrt{n}(\mathbb{G}_n f_n-\mathbb{G}_n f-\mathbb{G}f_n+\mathbb{G}f)\\
=&\sqrt{n}\{\mathbb{G}_n(f_n-f)\}-\sqrt{n}\{\mathbb{G}(f_n-f)\}\\
=&\sqrt{n}(\mathbb{G}_n-\mathbb{G})(f_n-f)\\
=&\mathbb{B}_n(f_n-f)\\
=&o_p(1).
\end{aligned}$$

We know that

$$\mathbb{G}_n f_n=\frac{1}{n}\sum_{j=1}^n I\{Y_j>S_{G,n}^{-1}(P_0)\}=S_{G,n}\{S_{G,n}^{-1}(P_0)\}.$$

Similarly, $\mathbb{G}_n f=S_{G,n}\{S_G^{-1}(P_0)\}$, $\mathbb{G}f_n=S_G\{S_{G,n}^{-1}(P_0)\}$ and $\mathbb{G}f=S_G\{S_G^{-1}(P_0)\}$. Therefore, $\sqrt{n}[S_{G,n}\{S_{G,n}^{-1}(P_0)\}-S_{G,n}\{S_G^{-1}(P_0)\}-S_G\{S_{G,n}^{-1}(P_0)\}+P_0]=o_p(1)$. After the adjustment, we obtain

$$\sqrt{n}[S_{G,n}\{S_{G,n}^{-1}(P_0)\}-P_0]=\sqrt{n}[S_{G,n}\{S_G^{-1}(P_0)\}-P_0]+\sqrt{n}[S_G\{S_{G,n}^{-1}(P_0)\}-P_0]+o_p(1).$$

From Corollary 21.5 in van der Vaart (2000), we have the following result,

$$\begin{aligned}
\sqrt{n}[S_{G,n}^{-1}(P_0)-S_G^{-1}(P_0)]=&\sqrt{n}[G_n^{-1}(1-P_0)-G^{-1}(1-P_0)]\\
=&\sqrt{n}\frac{1-P_0-G_n\{G^{-1}(1-P_0)\}}{G'(G^{-1}(1-P_0))}+o_p(1)\\
=&\sqrt{n}\frac{P_0-S_{G,n}\{S_G^{-1}(P_0)\}}{S_G'(S_G^{-1}(P_0))}+o_p(1).
\end{aligned}$$

Moreover, applying the δ-method, we have that

$$\begin{aligned}
\sqrt{n}[S_G\{S_{G,n}^{-1}(P_0)\}-P_0]=&\sqrt{n}\int_{S_G^{-1}(P_0)}^{S_{G,n}^{-1}(P_0)}dS_G(t)\\
=&\sqrt{n}[S_{G,n}^{-1}(P_0)-S_G^{-1}(P_0)]S_G'(S_G^{-1}(P_0))+o_p(1)\\
=&\sqrt{n}\frac{P_0-S_{G,n}\{S_G^{-1}(P_0)\}}{S_G'(S_G^{-1}(P_0))}S_G'(S_G^{-1}(P_0))+o_p(1)\\
=&\sqrt{n}[P_0-S_{G,n}\{S_G^{-1}(P_0)\}]+o_p(1).
\end{aligned}$$

Hence, we get $\sqrt{n}[S_{G,n}\{S_{G,n}^{-1}(P_0)\} - P_0] = o_p(1)$. We know that

$$\begin{aligned}
&\widehat{pAUC}(P_0)\\
=&\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} I\big(X_i \geq Y_j\big) I\big\{Y_j \geq S_{G,n}^{-1}(P_0)\big\}\\
=&\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} I\big(X_i \geq Y_j \geq S_{G,n}^{-1}(P_0)\big) I\big(Y_j \geq S_{G,n}^{-1}(P_0)\big)\\
=&\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} \Big\{ I\big(Y_j \geq S_{G,n}^{-1}(P_0)\big) - I\big(Y_j \geq X_i\Big) \Big\} I\big(Y_j \geq S_{G,n}^{-1}(P_0)\big)\\
=&\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} \Big\{ I\big(Y_j \geq S_{G,n}^{-1}(P_0)\big) \Big\} - \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} I\big(Y_j \geq X_i\Big) I\big(Y_j \geq S_{G,n}^{-1}(P_0)\big)\\
=&S_{G,n}\big(S_{G,n}^{-1}(P_0)\big) - \frac{1}{n}\sum_{j=1}^{n}\frac{1}{m}\sum_{i=1}^{m} I\Big(Y_j \geq \max\big\{X_i, S_{G,n}^{-1}(P_0)\big\}\Big)\\
=&S_{G,n}\big(S_{G,n}^{-1}(P_0)\big) - \frac{1}{m}\sum_{i=1}^{m} \min\Big\{ \frac{1}{n}\sum_{j=1}^{n} I\big(Y_j \geq X_i\big), \frac{1}{n}\sum_{j=1}^{n} I\big(Y_j \geq S_{G,n}^{-1}(P_0)\big) \Big\}\\
=&S_{G,n}\big(S_{G,n}^{-1}(P_0)\big) - \frac{1}{m}\sum_{i=1}^{m} \min\Big\{ \frac{1}{n}\sum_{j=1}^{n} I\big(Y_j > X_i\big), S_{G,n}\big(S_{G,n}^{-1}(P_0)\big) \Big\}\\
=&S_{G,n}\big(S_{G,n}^{-1}(P_0)\big) - \frac{1}{m}\sum_{i=1}^{m} \min\Big\{ S_{G,n}(X_i), S_{G,n}\big(S_{G,n}^{-1}(P_0)\big) \Big\}\\
=&P_0 - \frac{1}{m}\sum_{i=1}^{m} \min\Big\{ S_{G,n}(X_i), P_0 \Big\} + o_p(n^{-1/2}).
\end{aligned}$$

Because of $S_{G,n}\big(S_{G,n}^{-1}(P_0)\big) - P_0 = o_p(n^{-1/2})$, it is clear that

$$\begin{aligned}
\widehat{pAUC}(P_0) =&\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} I\Big\{X_i \geq Y_j\Big\} I\big(Y_j \geq S_{G,n}^{-1}(P_0)\big)\\
=&P_0 - \frac{1}{m}\sum_{j=1}^{m} \min\Big\{ S_{G,n}(X_i), P_0 \Big\} + o_p(n^{-1/2})\\
=&\widetilde{pAUC}(P_0) + o_p(n^{-1/2}).
\end{aligned}$$

Under the conditions C.1-C.4, $\sqrt{m+n}(\widehat{pAUC}(P_0) - pAUC(P_0)) \xrightarrow{d} N\left(0, \frac{\sigma_3^2}{\lambda} + \frac{\sigma_4^2}{1-\lambda}\right)$. Hence, we obtain that

$$\sqrt{m+n}\left(\widetilde{pAUC}(P_0) - pAUC(P_0)\right) \xrightarrow{d} N\left(0, \frac{\sigma_3^2}{\lambda} + \frac{\sigma_4^2}{1-\lambda}\right), m, n \to \infty. \qquad \square$$

S3 Proof of Lemma 2

$$\widetilde{pAUC}_h(P_0) = \begin{cases} P_0 - \frac{1}{m-1}\sum_{i \neq h}^{m} \min\{S_{G,n}(X_i), P_0\}, & 1 \le h \le m \\ P_0 - \frac{1}{m}\sum_{i=1}^{m} \min\{S_{G,n-1,h-m}(X_i), P_0\}, & m+1 \le h \le m+n. \end{cases}$$

Then, a jackknife pseudo sample $V_h(P_0)$ from $h = 1, 2 \ldots, m$ is obtained as follows,

$$\begin{aligned} V_h(P_0) =& (m+n)\widetilde{pAUC}(P_0) - (m+n-1)\widetilde{pAUC}_h(P_0) \\ =& P_0 - \frac{m+n}{m}\sum_{i=1}^{m} \min\{S_{G,n}(X_i), P_0\} + \frac{m+n-1}{m-1}\sum_{i=1, i \neq h}^{m} \min\{S_{G,n}(X_i), P_0\} \\ =& P_0 - \left\{\frac{n+m}{m} - \frac{n+m-1}{m-1}\right\}\sum_{i=1}^{m} \min\{S_{G,n}(X_i), P_0\} - \frac{m+n-1}{m-1}\min\{S_{G,n}(x_h), P_0\} \\ =& P_0 + \frac{n}{m(m-1)}\sum_{i=1}^{m} \min\{S_{G,n}(X_i), P_0\} - \frac{m+n-1}{m-1}\min\{S_{G,n}(x_h), P_0\}, h = 1, 2, \ldots, m. \end{aligned}$$

Then, we consider the partial sum of pseudo sample,

$$\begin{aligned} & \frac{1}{m+n}\sum_{h=1}^{m} V_h(P_0) \\ =& \frac{m}{m+n}P_0 + \frac{m}{m+n} \cdot \frac{n}{m(m-1)}\sum_{i=1}^{m} \min\{S_{G,n}(X_i), P_0\} - \frac{m+n-1}{m-1} \cdot \frac{1}{m+n}\sum_{h=1}^{m} \min\{S_{G,n}(x_h), P_0\} \\ =& \frac{m}{m+n}P_0 + \left\{\frac{n}{(m+n)(m-1)} - \frac{m+n-1}{(m+n)(m-1)}\right\}\sum_{i=1}^{m} \min\{S_{G,n}(X_i), P_0\} \\ =& \frac{m}{m+n}P_0 - \frac{m}{m+n} \cdot \frac{1}{m}\sum_{i=1}^{m} \min\{S_{G,n}(X_i), P_0\} \\ =& \frac{m}{m+n}\widetilde{pAUC}(P_0). \end{aligned}$$

For $h = m+1, \dots m+n$, we know

$$\begin{aligned}
V_h(P_0) =&(m+n)\widetilde{pAUC}(P_0) - (m+n-1)\widetilde{pAUC}_h(P_0) \\
=&P_0 - \frac{m+n}{m}\sum_{i=1}^{m} \min\{S_{G,n}(X_i), P_0\} + \frac{m+n-1}{m}\sum_{i=1}^{m} \min\{S_{G,n-1,h-m}(X_i), P_0\} \\
=&P_0 - \frac{1}{m}\sum_{i=1}^{m} \min\{S_{G,n}(X_i), P_0\} \\
+&\frac{m+n-1}{m}\sum_{i=1}^{m}\Big[\min\{S_{G,n-1,h-m}(X_i), P_0\} - \min\{S_{G,n}(X_i), P_0\}\Big] \\
=&\widetilde{pAUC}(P_0) \\
+&\underbrace{\frac{m+n-1}{m}\sum_{i=1}^{m} I\big\{S_{G,n-1,h-m}(X_i) \le P_0\big\} I\big\{S_{G,n}(X_i) \le P_0\big\}\Big[S_{G,n-1,h-m}(X_i) - S_{G,n}(X_i)\Big]}_{\mathrm{I}} \\
+&\underbrace{\frac{m+n-1}{m}\sum_{i=1}^{m}\Big(S_{G,n-1,h-m}(X_i) - P_0\Big) I\big\{S_{G,n-1,h-m}(X_i) < P_0\big\} I\big\{S_{G,n}(X_i) > P_0\big\}}_{\mathrm{II}} \\
+&\underbrace{\frac{m+n-1}{m}\sum_{i=1}^{m}\big(P_0 - S_{G,n}(X_i)\big) I\big\{S_{G,n}(X_i) < P_0 < S_{G,n-1,h-m}(X_i)\big\}}_{\mathrm{III}}.
\end{aligned}$$

For $\sum\limits_{h=m+1}^{m+n} V_h(P_0)$, we consider the summation from part I

$$\begin{aligned}
&\sum_{h=m+1}^{m+n} \frac{m+n-1}{m}\sum_{i=1}^{m} I\big\{S_{G,n-1,h-m}(X_i) \le P_0\big\} I\big\{S_{G,n}(X_i) \le P_0\big\}\Big[S_{G,n-1,h-m}(X_i) - S_{G,n}(X_i)\Big] \\
=&\sum_{h=m+1}^{m+n} \frac{m+n-1}{m}\sum_{i=1}^{m} I\big\{S_{G,n}(X_i) \le P_0\big\} I\big\{S_{G,n}(X_i) \le P_0\big\}\Big[S_{G,n-1,h-m}(X_i) - S_{G,n}(X_i)\Big] \\
+&\sum_{h=m+1}^{m+n} \frac{m+n-1}{m}\sum_{i=1}^{m} \big[I\big\{S_{G,n-1,h-m}(X_i) \le P_0\big\} - I\big\{S_{G,n}(X_i) \le P_0\big\}\big]\Big[S_{G,n-1,h-m}(X_i) - S_{G,n}(X_i)\Big] \\
=&\frac{m+n-1}{m}\sum_{i=1}^{m} I\big\{S_{G,n}(X_i) \le P_0\big\} \sum_{h=m+1}^{m+n}\Big[S_{G,n-1,h-m}(X_i) - S_{G,n}(X_i)\Big] \\
+&\frac{m+n-1}{m}\sum_{h=m+1}^{m+n}\sum_{i=1}^{m} I\{S_{G,n-1,h-m}(X_i) \le P_0 < S_{G,n}(X_i)\}\Big[S_{G,n-1,h-m}(X_i) - S_{G,n}(X_i)\Big] \\
=&\frac{m+n-1}{m}\sum_{h=m+1}^{m+n}\sum_{i=1}^{m} I\{S_{G,n-1,h-m}(X_i) \le P_0 < S_{G,n}(X_i)\}\Big[S_{G,n-1,h-m}(X_i) - S_{G,n}(X_i)\Big] \\
=&O_p(1),
\end{aligned}$$

where

$$\begin{aligned}\sum_{h=m+1}^{m+n}\Big[S_{G,n-1,h-m}(X_i)-S_{G,n}(X_i)\Big]=&\sum_{h=1}^{n}S_{G,n-1,h}(X_i)-nS_{G,n}(X_i)\\=&\sum_{h=1}^{n}\frac{1}{n-1}\sum_{j\neq h}^{n}I\{Y_j>X_i\}-nS_{G,n}(X_i)\\=&0.\end{aligned}$$

For II, we know

$$\sum_{h=m+1}^{m+n}\frac{m+n-1}{m}\sum_{i=1}^{m}\Big(S_{G,n-1,h-m}(X_i)-P_0\Big)I\big\{S_{G,n-1,h-m}(X_i)<P_0<S_{G,n}(X_i)\big\}=O_p(1).$$

With the similar result for III, we consider the partial sum of pseudo sample for $h=m+1,\dots,m+n$,

$$\frac{1}{m+n}\sum_{h=m+1}^{m+n}V_h(P_0)=\frac{n}{m+n}\widetilde{pAUC}(P_0)+O_p(n^{-1}).$$

From the conditions C.1-C.4,

$$\sqrt{m+n}\left(\frac{1}{m+n}\sum_{h=m+1}^{m+n}V_h(P_0)-pAUC(P_0)\right)\xrightarrow{d}N\left(0,\frac{\sigma_3^2}{\lambda}+\frac{\sigma_4^2}{1-\lambda}\right),m,n\to\infty. \qquad \square$$

S4 Proof of Lemma 3

For the jackknife pseudo sample variance estimator, we first study V_h^2, $m+1\le h\le m+n$, which equals to

$$\begin{aligned}&\Big[P_0-\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\\&+(m+n-1)\Big\{\frac{1}{m-1}\sum_{i=1,i\neq h}^{m}\min\{S_{G,n}(X_i),P_0\}-\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\Big\}\Big]^2\\=&\left[\widetilde{pAUC}(P_0)+(m+n-1)\left\{\frac{1}{m-1}\sum_{i=1,i\neq h}^{m}\min\{S_{G,n}(X_i),P_0\}-\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\right\}\right]^2.\end{aligned}$$

Hence,

$$V_h^2(P_0)=\begin{cases}\left[\widetilde{pAUC}(P_0)+(m+n-1)\left\{\frac{1}{m-1}\sum_{i=1,i\neq h}^{m}\min\{S_{G,n}(X_i),P_0\}-\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\right\}\right]^2\\ \qquad 1\le h\le m\\ \left[\widetilde{pAUC}(P_0)+\frac{m+n-1}{m}\sum_{i=1}^{m}[\min\{S_{G,n-1,h-m}(X_i),P_0\}-\min\{S_{G,n}(X_i),P_0\}]\right]^2\\ \qquad m+1\le h\le m+n.\end{cases}$$

Then, we show that

$$
\begin{aligned}
&\frac{1}{m+n}\sum_{h=1}^{m+n} V_h^2(P_0)\\
=&\frac{1}{m+n}\sum_{h=1}^{m} V_h^2(P_0)+\frac{1}{m+n}\sum_{h=m+1}^{m+n} V_h^2(P_0)\\
=&\frac{1}{m+n}\sum_{h=1}^{m}\Big[\widetilde{pAUC}(P_0)\\
+&(m+n-1)\left\{\frac{1}{m-1}\sum_{i=1,i\neq h}^{m}\min\{S_{G,n}(X_i),P_0\}-\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\right\}\Big]^2\\
+&\frac{1}{m+n}\sum_{h=m+1}^{m+n}\left[\widetilde{pAUC}(P_0)+\frac{m+n-1}{m}\sum_{i=1}^{m}[\min\{S_{G,n-1,h-m}(X_i),P_0\}-\min\{S_{G,n}(X_i),P_0\}]\right]^2\\
=&\frac{1}{m+n}\sum_{h=1}^{m+n}\widetilde{pAUC}^2(P_0)+\frac{2(m+n-1)}{m+n}\widetilde{pAUC}(P_0)\Big\{\frac{1}{m-1}\sum_{h=1}^{m}\sum_{i=1,i\neq h}^{m}\min\{S_{G,n}(X_i),P_0\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{m}\sum_{h=1}^{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\\
&+\frac{1}{m}\sum_{h=m+1}^{m+n}\sum_{i=1}^{m}\min\{S_{G,n-1,h-m}(X_i),P_0\}-\frac{1}{m}\sum_{h=m+1}^{m+n}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\Big\}\\
&+\underbrace{\frac{(m+n-1)^2}{m+n}\sum_{h=1}^{m}\left\{\frac{1}{m-1}\sum_{i=1,i\neq h}^{m}\min\{S_{G,n}(X_i),P_0\}-\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\right\}^2}_{\mathrm{IV}}\\
&+\underbrace{\frac{(m+n-1)^2}{m+n}\sum_{h=m+1}^{m+n}\left\{\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n-1,h-m}(X_i),P_0\}-\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\right\}^2}_{\mathrm{V}}.
\end{aligned}
$$

The middle term is

$$
\begin{aligned}
&\frac{2(m+n-1)}{m+n}\widetilde{pAUC}(P_0)\Big\{\frac{1}{m-1}\sum_{h=1}^{m}\sum_{i=1,i\neq h}^{m}\min\{S_{G,n}(X_i),P_0\}-\frac{1}{m}\sum_{h=1}^{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\\
+&\frac{1}{m}\sum_{h=m+1}^{m+n}\sum_{i=1}^{m}\min\{S_{G,n-1,h-m}(X_i),P_0\}-\frac{1}{m}\sum_{h=m+1}^{m+n}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\Big\}=o_p(1).
\end{aligned}
$$

For part IV, we know

$$\frac{(m+n-1)^2}{m+n}\sum_{h=1}^{m}\left\{\frac{1}{m-1}\sum_{i=1,i\neq h}^{m}\min\{S_{G,n}(X_i),P_0\}-\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\right\}^2$$

$$=\frac{(m+n-1)^2}{m+n}\sum_{h=1}^{m}\Big\{\frac{1}{m-1}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}-\frac{1}{m-1}\min\{S_{G,n}(X_h),P_0\}$$

$$-\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\Big\}^2$$

$$=\frac{(m+n-1)^2}{m+n}\sum_{h=1}^{m}\left\{\frac{1}{m(m-1)}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\right\}^2$$

$$-\frac{2(m+n-1)^2}{m+n}\sum_{h=1}^{m}\frac{1}{m(m-1)}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\frac{1}{m-1}\min\{S_{G,n}(X_h),P_0\}$$

$$+\frac{(m+n-1)^2}{m+n}\sum_{h=1}^{m}\left\{\frac{1}{m-1}\min\{S_{G,n}(X_h),P_0\}\right\}^2$$

$$=\frac{(m+n-1)^2 m}{(m+n)(m-1)^2}\left\{\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\right\}^2$$

$$-\frac{2(m+n-1)^2 m}{(m+n)(m-1)^2}\left\{\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\right\}\left\{\frac{1}{m}\sum_{h=1}^{m}\min\{S_{G,n}(X_h),P_0\}\right\}$$

$$+\frac{(m+n-1)^2 m}{(m+n)(m-1)^2}\frac{1}{m}\sum_{h=1}^{m}\{\min\{S_{G,n}(X_h),P_0\}\}^2$$

$$=\frac{(m+n-1)^2 m}{(m+n)(m-1)^2}\left[\frac{1}{m}\sum_{h=1}^{m}(\min\{S_{G,n}(X_h),P_0\})^2-\left\{\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\right\}^2\right]$$

$$=\frac{(m+n-1)^2 m}{(m+n)(m-1)^2}\left[\frac{1}{m}\sum_{h=1}^{m}(P_0-\min\{S_{G,n}(X_h),P_0\})^2-\left\{P_0-\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\right\}^2\right]$$

$$=\frac{(m+n-1)^2 m}{(m+n)(m-1)^2}\Big[\frac{1}{m}\sum_{h=1}^{m}\left(\frac{1}{n}\sum_{j=1}^{n}I\big(X_h\geq Y_j\big)I\big\{Y_j\geq S_{G,n}^{-1}(P_0)\big\}\right)^2$$

$$-\left\{\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}I\big(X_i\geq Y_j\big)I\big\{Y_j\geq S_{G,n}^{-1}(P_0)\big\}\right\}^2\Big]+o_p(1)$$

$$=\frac{(m+n-1)^2 m}{(m+n)(m-1)^2}\Big[\frac{1}{m}\sum_{h=1}^{m}I(X_h\geq S_{G,n}^{-1}(P_0))\left(\frac{1}{n}\sum_{j=1}^{n}\{1-I\big(X_h\leq Y_j\big)\}I\big\{Y_j\geq S_{G,n}^{-1}(P_0)\big\}\right)^2$$

$$-\left\{\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}I\big(X_i\geq Y_j\big)I\big\{Y_j\geq S_{G,n}^{-1}(P_0)\big\}\right\}^2\Big]+o_p(1)$$

$$=\int_{+\infty}^{S_G^{-1}(P_0)}\{P_0-S_G(t)\}^2dS_F(t)-\left\{\int_{+\infty}^{S_G^{-1}(P_0)}S_F(t)dS_G(t)\right\}^2+o_p(1).$$

For part V,

$$
\begin{aligned}
&\frac{(m+n-1)^2}{m+n}\sum_{h=m+1}^{m+n}\left\{\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n-1,h-m}(X_i),P_0\}-\frac{1}{m}\sum_{i=1}^{m}\min\{S_{G,n}(X_i),P_0\}\right\}^2\\
=&\frac{(m+n-1)^2}{m^2(m+n)}\sum_{h=m+1}^{m+n}\Big[\sum_{i=1}^{m}I\big\{S_{G,n-1,h-m}(X_i)\le P_0\big\}I\big\{S_{G,n}(X_i)\le P_0\big\}\Big[S_{G,n-1,h-m}(X_i)-S_{G,n}(X_i)\Big]\\
&+\sum_{i=1}^{m}\Big(S_{G,n-1,h-m}(X_i)-P_0\Big)I\big\{S_{G,n-1,h-m}(X_i)<P_0\big\}I\big\{S_{G,n}(X_i)>P_0\big\}\\
&+\sum_{i=1}^{m}\big(P_0-S_{G,n}(X_i)\big)I\big\{S_{G,n}(X_i)<P_0<S_{G,n-1,h-m}(X_i)\big\}\Big]^2\\
=&\frac{(m+n-1)^2}{m^2(m+n)}\sum_{h=m+1}^{m+n}\left\{\sum_{i=1}^{m}I\big\{S_{G,n-1,h-m}(X_i)\le P_0\big\}I\big\{S_{G,n}(X_i)\le P_0\big\}\Big[S_{G,n-1,h-m}(X_i)-S_{G,n}(X_i)\Big]\right\}\\
&+o_p(1)\\
=&\frac{(m+n-1)^2}{m^2(m+n)}\sum_{h=1}^{n}\left\{\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}\Big[S_{G,n-1,h-m}(X_i)-S_{G,n}(X_i)\Big]\right\}^2+o_p(1)\\
=&\frac{(m+n-1)^2}{m^2(m+n)}\sum_{h=1}^{n}\left\{\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}\Big[\frac{1}{n-1}\sum_{j\neq h}^{n}I\{Y_h>X_i\}-\frac{1}{n}\sum_{j=1}^{n}I\{Y_h>X_i\}\Big]\right\}^2+o_p(1)\\
=&\frac{(m+n-1)^2}{m^2(m+n)}\sum_{h=1}^{n}\left\{\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}\Big[\frac{1}{n(n-1)}\sum_{j=1}^{n}I\{Y_h>X_i\}-\frac{1}{n-1}I\{Y_h>X_i\}\Big]\right\}^2+o_p(1)\\
=&\frac{(m+n-1)^2}{m^2(m+n)(n-1)^2}\sum_{h=1}^{n}\left\{\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}S_{G,n}(X_i)-\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}I\{Y_h>X_i\}\right\}^2\\
&+o_p(1)\\
=&\frac{(m+n-1)^2}{m^2(m+n)(n-1)^2}\Big[n\left\{\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}S_{G,n}(X_i)\right\}^2\\
&+\sum_{h=1}^{n}\left\{\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}I\{Y_h>X_i\}\right\}^2\Big]\\
&\underbrace{-\frac{2(m+n-1)^2}{m^2(m+n)(n-1)^2}\left[\sum_{h=1}^{n}\left(\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}S_{G,n}(X_i)\right)\left(\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}I\{Y_h>X_i\}\right)\right]}_{\mathrm{VI}}\\
&+o_p(1)\\
=&\frac{(m+n-1)^2}{m^2(m+n)(n-1)^2}\Big[\sum_{h=1}^{n}\left\{\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}I\{Y_h>X_i\}\right\}^2\\
&-n\left\{\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}S_{G,n}(X_i)\right\}^2\Big]+o_p(1),
\end{aligned}
$$

because the term VI can be represented as

$$
\begin{aligned}
VI =&\frac{2(m+n-1)^2}{m^2(m+n)(n-1)^2}\Big[\sum_{h=1}^{n}\left(\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}S_{G,n}(X_i)\right)\\
&\left(\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}I\{Y_h>X_i\}\right)\Big]\\
=&\frac{2(m+n-1)^2}{m^2(m+n)(n-1)^2}\Big[n\left(\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}S_{G,n}(X_i)\right)\\
&\left(\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}\frac{1}{n}\sum_{h=1}^{n}I\{Y_h>X_i\}\right)\Big]\\
=&\frac{2(m+n-1)^2}{m^2(m+n)(n-1)^2}\left[n\left(\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}S_{G,n}(X_i)\right)^2\right].
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\frac{1}{n}\sum_{h=1}^{n}\left\{\frac{1}{m}\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le P_0\big\}I\{Y_h>X_i\}\right\}^2\\
=&\frac{1}{n}\sum_{h=1}^{n}\left\{\frac{1}{m}\sum_{i=1}^{m}I\big\{S_{G,n}(X_i)\le S_{G,n}(S_{G,n}^{-1}(P_0))\big\}I\{Y_h>X_i\}\right\}^2+o_p(1)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{n}\sum_{h=1}^{n}\left\{\frac{1}{m}\sum_{i=1}^{m}I\big\{X_i\ge S_{G,n}^{-1}(P_0)\big\}I\{Y_h>X_i\}\right\}^2 I\big\{Y_h\ge S_{G,n}^{-1}(P_0)\big\}+o_p(1)\\
=&\frac{1}{n}\sum_{h=1}^{n}\left\{\frac{1}{m}\sum_{i=1}^{m}I\big\{X_i\ge S_{G,n}^{-1}(P_0)\big\}(1-I\{Y_h<X_i\})\right\}^2 I\big\{Y_h\ge S_{G,n}^{-1}(P_0)\big\}+o_p(1)\\
=&\frac{1}{n}\sum_{h=1}^{n}\left\{\frac{1}{m}\sum_{i=1}^{m}I\big\{X_i\ge S_{G,n}^{-1}(P_0)\big\}-\frac{1}{m}\sum_{i=1}^{m}I\big\{X_i\ge S_{G,n}^{-1}(P_0)\big\}I\{Y_h<X_i\}\right\}^2 I\big\{Y_h\ge S_{G,n}^{-1}(P_0)\big\}\\
&+o_p(1)\\
=&\frac{1}{n}\sum_{h=1}^{n}\left\{\frac{1}{m}\sum_{i=1}^{m}I\big\{X_i\ge S_{G,n}^{-1}(P_0)\big\}-\frac{1}{m}\sum_{i=1}^{m}I\{Y_h<X_i\}\right\}^2 I\big\{Y_h\ge S_{G,n}^{-1}(P_0)\big\}+o_p(1)\\
=&\int_{+\infty}^{S_G^{-1}(P_0)}[S_F(t)-S_F\{S_G^{-1}(P_0)\}]^2dS_G(t)
\end{aligned}
$$

and

$$\left\{\frac{1}{m}\sum_{i=1}^{m} I\{S_{G,n}(X_i) \le P_0\} S_{G,n}(X_i)\right\}^2$$

$$= \left\{\frac{1}{m}\sum_{i=1}^{m} I\{S_{G,n}(X_i) \le S_{G,n}(S_{G,n}^{-1}(P_0))\} \frac{1}{n}\sum_{h=1}^{n} I\{Y_h > X_i\}\right\}^2 + o_p(1)$$

$$= \left\{\frac{1}{n}\sum_{h=1}^{n}\left[\frac{1}{m}\sum_{i=1}^{m} I\{X_i \ge S_{G,n}^{-1}(P_0)\} I\{Y_h > X_i\}\right] I\{Y_h \ge S_{G,n}^{-1}(P_0)\}\right\}^2 + o_p(1)$$

$$= \left\{\frac{1}{n}\sum_{h=1}^{n}\left[\frac{1}{m}\sum_{i=1}^{m} I\{X_i \ge S_{G,n}^{-1}(P_0)\}(1 - I\{Y_h < X_i\})\right] I\{Y_h \ge S_{G,n}^{-1}(P_0)\}\right\}^2 + o_p(1)$$

$$= \left\{\frac{1}{n}\sum_{h=1}^{n}\left[\frac{1}{m}\sum_{i=1}^{m} I\{X_i \ge S_{G,n}^{-1}(P_0)\} - \frac{1}{m}\sum_{i=1}^{m} I\{X_i \ge S_{G,n}^{-1}(P_0)\} I\{Y_h < X_i\}\right] I\{Y_h \ge S_{G,n}^{-1}(P_0)\}\right\}^2$$
$$+ o_p(1)$$

$$= \left\{\frac{1}{n}\sum_{h=1}^{n}\left[\frac{1}{m}\sum_{i=1}^{m} I\{X_i \ge S_{G,n}^{-1}(P_0)\} - \frac{1}{m}\sum_{i=1}^{m} I\{Y_h < X_i\}\right] I\{Y_h \ge S_{G,n}^{-1}(P_0)\}\right\}^2 + o_p(1)$$

$$= \left(\int_{+\infty}^{S_G^{-1}(P_0)} [S_F(t) - S_F\{S_G^{-1}(P_0)\}] dS_G(t)\right)^2.$$

Hence,

$$\frac{1}{m+n}\sum_{h=1}^{m+n}(V_h(P_0) - pAUC(P_0))^2 \xrightarrow{p} \frac{\sigma_3^2}{\lambda} + \frac{\sigma_4^2}{1-\lambda},$$

$$S^2_{\widetilde{pAUC}}(P_0) = \frac{\sigma_3^2(P_0)}{\lambda} + \frac{\sigma_4^2(P_0)}{1-\lambda} + o_p(1).$$

□

Proof of Theorem 2 Based on Lemmas 2-3 and Slusky's theorem, Theorem 2 is obtained. □

Proof of Theorem 3 Combining Lemmas 2-3 and similar arguments in Owen (1990), we can prove that the jackknife empirical log-likelihood ratio $l_{pAUC}\{P_0, pAUC(P_0)\}$ converges to χ_1^2 in distribution. □

Proofs of Corollaries 2-4 The proofs are similar to ones of Theorems 1-3 and Lemmas 2-3. Thus, we escape them. □

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