On Independence and Separability between Points and Marks of Marked Point Processes

Tonglin Zhang

Purdue University

Supplementary Material

The online supplememntary material includes the definitions of the Brownian sheet, pinned Brownian sheet, and Brownian pillow, the defintions of (F_1, F_2)-functional Brownian pillow and (p_1, p_2)-standard Brownian pillow. It contains the proofs of Theorem 1, Theorem 2, Theorem 3, Corollary 1, and Theorem 4, as well as the associated lemmas.

S1 Definitions of Brownian Sheet, Pinned Brownian Sheet, and Brownian Pillow

A Brownian pillow is a mean zero Gaussian process on $[0,1]^p$ with a covariance function given by

$$\mathbb{E}[\mathbb{W}_p(\mathbf{t})\mathbb{W}_p(\mathbf{t}')] = \prod_{j=1}^{p}(t_j \wedge t'_j - t_j t'_j)$$

for any $\mathbf{t} = (t_1, \cdots, t_p)$ and $\mathbf{t}' = (t'_1, \cdots, t'_p)$ in $[0,1]^p$. Related concepts to $\mathbb{W}_p$ are the (standard) *Brownian sheet* (denoted by $\mathbb{B}_p$) and the (standard) *pinned Brownian sheet* (denoted by $\tilde{\mathbb{W}}_p$) (Yeh (1960)), where $\mathbb{B}_p$ is a mean zero Gaussian process on $\mathbb{R}_+^p = [0,\infty)^p$ and $\tilde{\mathbb{W}}_p$ is a mean zero Gaussian process on $[0,1]^p$. The covariance function of $\mathbb{B}_p$ is

$$\mathbb{E}[\mathbb{B}_p(\mathbf{t})\mathbb{B}_p(\mathbf{t}')] = \prod_{j=1}^{p} t_j \wedge t'_j, \mathbf{t}, \mathbf{t}' \in \mathbb{R}_+^p.$$

The covariance function of $\tilde{\mathbb{W}}_p$ is

$$\mathbb{E}[\tilde{\mathbb{W}}_p(\mathbf{t})\tilde{\mathbb{W}}_p(\mathbf{t}')] = \prod_{j=1}^{p} t_j \wedge t'_j - \prod_{j=1}^{p} t_j t'_j, \mathbf{t}, \mathbf{t}' \in [0,1]^p.$$

Both $\tilde{\mathbb{W}}_p$ and $\mathbb{W}_p$ can be derived using $\mathbb{B}_p$. For example if $p = 2$, there are

$$\tilde{\mathbb{W}}_2(\mathbf{t}) = \mathbb{B}_2(t_1, t_2) - t_1 t_2 \mathbb{B}_2(1, 1)$$

and

$$\mathbb{W}_2(\mathbf{t}) = \mathbb{B}_2(t_1, t_2) - t_1\mathbb{B}_2(1, t_2) - t_2\mathbb{B}_2(t_1, 1) + t_1 t_2 \mathbb{B}_2(1, 1)$$

for any $t_1, t_2 \in [0,1]$. Since the sample paths of $\mathbb{B}^p$ are continuous with probability one (Czörgö and Révécs (1981); Dalang (2003); Orey and Pruitt (1973); Walsh (1982)), the sample paths of $\tilde{\mathbb{W}}_p$ and $\mathbb{W}_p$ are also continuous with probability one.

S2 Definitions of (F_1, F_2)-Functional Brownian Pillow and (p_1, p_2)-Standard Brownian Pillow

A mean zero Gaussian process $\mathbb{B}_F$ is called an F-functional Brownian sheet on $\mathbb{R}^p$, where F is a CDF on $\mathbb{R}^p$, if its covariance function is

$$\mathbb{E}[\mathbb{B}_F(\mathbf{t})\mathbb{B}_F(\mathbf{t}')] = F(t_1 \wedge t'_1, \cdots, t_p \wedge t'_p),$$

where $\mathbf{t} = (t_1, \cdots, t_p) \in \mathbb{R}^p$ and $\mathbf{t}' = (t'_1, \cdots, t'_p) \in \mathbb{R}^p$.

A mean zero Gaussian process $\mathbb{W}_{F_1,F_2}$ is called an (F_1, F_2)-functional Brownian pillow on $\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, where F_1 is a CDF on $\mathbb{R}^{p_1}$ and F_2 is a CDF on $\mathbb{R}^{p_2}$, if its covariance function is

$$\begin{aligned}
&\mathbb{E}[\mathbb{W}_{F_1,F_2}(\mathbf{t}_1, \mathbf{t}_2)\mathbb{W}_{F_1,F_2}(\mathbf{t}'_1, \mathbf{t}'_2)] \\
=&[F_1(t_{11} \wedge t'_{11}, \cdots, t_{1p_1} \wedge t_{1p_1}) - F_1(t_1, \cdots, t_{1p_1})F_1(t'_1, \cdots, t'_{1p_1})] \\
&\times [F_2(t_{21} \wedge t'_{21}, \cdots, t_{2p_2} \wedge t'_{2p_2}) - F_2(t_{21}, \cdots, t_{2p_2})F_2(t'_{21}, \cdots, t'_{2p_2})],
\end{aligned}$$

where $\mathbf{t}_1 = (t_{11}, \cdots, t_{1p_1}) \in \mathbb{R}^{p_1}$, $\mathbf{t}_2 = (t_{21}, \cdots, t_{2p_2}) \in \mathbb{R}^{p_2}$, $\mathbf{t}'_1 = (t'_{11}, \cdots, t'_{1p_1}) \in \mathbb{R}^{p_1}$, and $\mathbf{t}'_2 = (t'_{21}, \cdots, t'_{2p_2}) \in \mathbb{R}^{p_2}$.

If F_1 and F_2 are marginal CDFs of F, then $\mathbb{W}_{F_1,F_2}$ can be defined using $\mathbb{B}_F$ with

$$\mathbb{W}_{F_1,F_2}(\mathbf{t}_1, \mathbf{t}_2) = \mathbb{B}_F(\mathbf{t}_1, \mathbf{t}_2) - F_1(\mathbf{t}_1)\mathbb{B}_F(\infty_{p_1}, \mathbf{t}_2) - F_2(\mathbf{t}_2)\mathbb{B}_F(\mathbf{t}_1, \infty_{p_2}) + F(\mathbf{t}_1, \mathbf{t}_2)\mathbb{B}_F(\infty_{p_1}, \infty_{p_2}),$$

where ∞_p is the p-dimensional vector with all elements equal to ∞. Since the sample path of $\mathbb{B}_F$ is continuous, the sample path of $\mathbb{W}_{F_1,F_2}$ is also continuous.

A mean zero Gaussian process $\mathbb{W}_{p_1,p_2}$ is called the (p_1, p_2)-*standard Brownian pillow* on $[0,1]^{p_1} \times [0,1]^{p_2}$ if F_1 and F_2 are the uniform distributions on $[0,1]^{p_1}$ and $[0,1]^{p_2}$ in $\mathbb{W}_{F_1,F_2}$, respectively. The (p_1, p_2)-standard Brownian pillow is a mean zero process with the covariance function given by

$$\begin{aligned}
&\mathbb{E}[\mathbb{W}_{p_1,p_2}(\mathbf{t}_1, \mathbf{t}_2)\mathbb{W}_{p_1,p_2}(\mathbf{t}'_1, \mathbf{t}'_2)] \\
=&\left[\prod_{i=1}^{p_1}(t_{1i} \wedge t'_{1i}) - \prod_{i=1}^{p_1}(t_{1i}t'_{1i})\right]\left[\prod_{i=1}^{p_2}(t_{2i} \wedge t'_{2i}) - \prod_{i=1}^{p_2}(t_{2i}t'_{2i})\right].
\end{aligned}$$

If $\mathbb{B}_p$ is the standard Brownian sheet on $\mathbb{R}^p_+$ with $p = p_1 + p_2$, then

$$\begin{aligned}
\mathbb{W}_{p_1,p_2}(\mathbf{t}_1, \mathbf{t}_2) =&\mathbb{B}_p(\mathbf{t}'_1, \mathbf{t}'_2) - \mathbb{B}_p(\mathbf{1}_{p_1}, \mathbf{t}_2)\prod_{i=1}^{p_1} t_{1i} \\
&- \mathbb{B}_p(\mathbf{t}_1, \mathbf{1}_{p_2})\prod_{i=1}^{p_2} t_{2i} + \mathbb{B}_p(\mathbf{1}_{p_1}, \mathbf{1}_{p_2})(\prod_{i=1}^{p_1} t_{1i})(\prod_{i=1}^{p_2} t_{2i}),
\end{aligned}$$

where $\mathbf{1}_p$ represents the vector with all elements 1. Therefore, the sample path of $\mathbb{W}_{p_1,p_2}$ is continuous.

S3 Proofs

Lemma 1. *Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathscr{F})$. Then $\mathbb{P}$ is uniquely determined by its restriction on $\mathcal{G} = \{C = \sum_{n=0}^{\infty} C^{(n)} : C^{(n)} \in \mathscr{X}^n\}$ or its restriction on $\mathcal{F} = \{C = \sum_{n=0}^{\infty} C^{(n)} : C^{(n)} = A^{(n)} \times B^{(n)}, A^{(n)} \in \mathscr{S}^n, B^{(n)} \in \mathscr{M}^n\}$.*

Proof: Since $\mathcal{F} \subseteq \mathcal{G}$ and $\sigma(\mathcal{S}^n \times \mathcal{M}^n) = \mathscr{X}^n$, $\sigma(\mathcal{G}) = \sigma(\mathcal{F})$. The theory of *Dynkin*'s π-λ theorem (Billingsley (1995)) states that if two probability measures agree on a π-system (a π-system is a collection of subsets which is closed under interaction) then they agree on the σ-field of the π-system. If $\mathcal{F}$ is a π-system, then $\mathbb{P}$ is uniquely determined by $\mathcal{F}$. Therefore, it is enough to show $\mathcal{F}$ is a π-system. The proof is straightforward. For any $C, \tilde{C} \in \mathcal{F}$ there exist $A^{(n)}, \tilde{A}^{(n)} \in \mathscr{S}^n$ and $B^{(n)}, \tilde{B}^{(n)} \in \mathscr{M}^n$ such that $C \cap \tilde{C} = \sum_{n=0}^{\infty}(C^{(n)} \cap \tilde{C}^{(n)}) = \sum_{n=0}^{\infty}[(A^{(n)} \times \tilde{A}^{(n)}) \cap (B^{(n)} \times \tilde{B}^{(n)})]$. Since $(A^{(n)} \times \tilde{A}^{(n)}) \cap (B^{(n)} \times \tilde{B}^{(n)}) \in \mathscr{S}^n \times \mathscr{M}^n$ for any given n, we have $C \cap \tilde{C} \in \mathcal{F}$ and hence $\mathcal{F}$ is a π-system. Then, the final conclusion is drawn as $\mathscr{F} = \sigma(\mathcal{F})$. □

Proof of Theorem 1: Clearly $\mathbb{P} \geq 0$ and $P(\Omega) = \sum_{n=0}^{\infty} P^{(n)}(\mathcal{X}^n) = 1$. If C_k is a disjoint sequence of sets in $\mathcal{F}$, then there exist $A_k^{(n)} \in \mathscr{S}^n$ and $B_k^{(n)} \in \mathscr{M}^n$ satisfying $(A_k^{(n)} \times B_k^{(n)}) \cap (A_{k'}^{(n)} \times B_{k'}^{(n)}) = \phi$ for any $k \neq k'$ such that $C_k = \sum_{n=0}^{\infty} A_k^{(n)} \times B_k^{(n)}$. Thus, $\mathbb{P}(\sum_{k=1}^{\infty} C_k) = \mathbb{P}(\sum_{k=1}^{\infty}\sum_{n=0}^{\infty} A_k^{(n)} \times B_k^{(n)}) = \sum_{n=0}^{\infty} P^{(n)}(\sum_{k=1}^{\infty} A_k^{(n)} \times B_k^{(n)}) = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} P^{(n)}(A_k^{(n)} \times B_k^{(n)}) = \sum_{k=1}^{\infty} \mathbb{P}(C_k)$. Therefore, $\mathbb{P}$ is σ-additive and hence a probability measure on $\mathscr{F}$. The uniqueness of $\mathbb{P}$ is directly implied by Lemma 1. □

Proof of Theorem 2: Consider the right side of Equation (8). There is

$$\sum_{n=r}^{\infty} \frac{n!}{(n-r)!} p_n f_{r,n}(\mathbf{x}_1, \cdots, \mathbf{x}_r) \leq H_r(\mathbf{x}_1, \cdots, \mathbf{x}_r)\mathbb{E}(N^r).$$

The left side is bounded for every $\mathbf{x}_1, \cdots, \mathbf{x}_r \in \mathcal{X}$, implying that $\lambda_r(\mathbf{x}_1, \cdots, \mathbf{x}_r)$ is well-defined for every $r \leq k$. □

Proof of Theorem 3: Since λ_r exists for every $r \leq k$, there is

$$\lambda_r(\mathbf{x}_1, \cdots, \mathbf{x}_r) = f_r(\mathbf{x}_1, \cdots, \mathbf{x}_r) \sum_{n=r}^{\infty} \frac{n!}{(n-r)!} p_n. \tag{S3.1}$$

For sufficienct, assume $\mathcal{N}$ is kth-order independent. For any $r \leq k$, there is $f_r(\mathbf{x}_1, \cdots, \mathbf{x}_r) = f_{r,m}(\mathbf{m}_1, \cdots, \mathbf{m}_r) f_{r,s}(\mathbf{s}_1, \cdots, \mathbf{x}_r)$, where $f_{r,m} = f_{r,m|s}$ does not depend on $\mathbf{s}^{(r)}$. Therefore,

$$\lambda_r(\mathbf{x}_1, \cdots, \mathbf{x}_r) = f_{r,m}(\mathbf{m}_1, \cdots, \mathbf{m}_r) f_{r,s}(\mathbf{s}_1, \cdots, \mathbf{s}_r) \sum_{n=r}^{\infty} \frac{n!}{(n-r)!} p_n,$$

implying that $\mathcal{N}$ is kth-order separable. For necessity, assume that $\mathcal{N}$ is kth-order separable. Using (S3.1) with any $r \leq k$,

$$\lambda_{r,s}(\mathbf{s}_1, \cdots, \mathbf{s}_r) = f_{r,s}(\mathbf{s}_1, \cdots, \mathbf{s}_r) \sum_{n=r}^{\infty} \frac{n!}{(n-r)!} p_n$$

and

$$\lambda_{r,m}(\mathbf{m}_1, \cdots, \mathbf{m}_r) = f_{r,m}(\mathbf{m}_1, \cdots, \mathbf{m}_r) \sum_{n=r}^{\infty} \frac{n!}{(n-r)!} p_n.$$

Then,

$$\frac{\lambda_r(\mathbf{x}_1,\cdots,\mathbf{x}_r)}{\lambda_{r,s}(\mathbf{s}_1,\cdots,\mathbf{s}_r)\lambda_{r,m}(\mathbf{m}_1,\cdots,\mathbf{m}_r)}=\frac{f_r(\mathbf{x}_1,\cdots,\mathbf{x}_r)\sum_{n=r}^{\infty}\frac{n!}{(n-r)!}p_n}{f_{r,s}(\mathbf{s}_1,\cdots,\mathbf{s}_r)f_{r,m}(\mathbf{m}_1,\cdots,\mathbf{m}_r)}.$$

Let c_r be the value of the left side of the above equation. Then, c_r does not depend on $\mathbf{x}_1,\cdots,\mathbf{x}_r$ for any $r\le k$, and

$$f_{r,m|s}(\mathbf{m}^{(r)}|\mathbf{s}^{(r)})=\frac{c_r}{\sum_{n=r}^{\infty}\frac{n!}{(n-r)!}}f_{r,m}(\mathbf{m}_1,\cdots,\mathbf{m}_r).$$

Therefore, $f_{r,m|s}(\mathbf{m}^{(r)}|\mathbf{s}^{(r)})$ does not depend on $\mathbf{s}^{(r)}$, and $\mathcal{N}$ is kth-order independent. □

Proof of Corollary 1: The conclusion can be directly implied from Theorem 3. □

Proof of Theorem 4: Let $A_{\mathbf{k},\eta}=\mathcal{S}_\eta\cap([0,1]^d+\mathbf{k})$, where $\mathbf{k}=(k_1,\cdots,k_d)\in\mathbb{Z}^d$. Write $N_\eta=N(\mathcal{S}_\eta\times\mathcal{M})$ and denote $\kappa_\eta=E(N_\eta)$. For any given $\mathbf{k}$, if η is sufficiently large, then $A_{\mathbf{k}}=A+\mathbf{k}$. Since Condition (A3) implies that Lemma 1 of Herrdnorf (1984) holds, using Conditions (A1) and (A2) there is $\kappa_\eta/n\xrightarrow{P}1$ as $\eta\to 1$. Still using Condition (A3), there is $\sup_{\mathbf{d}\in\mathbf{Z}^d}\|\mathcal{N}(A_{\mathbf{k},\eta})\|_\beta<\infty$. Therefore, Corollary 1 of Herrdnorf (1984) can be applied, which implies that there exists $\sigma>0$ such that $\sqrt{n}[\mathcal{N}_\eta(\mathbf{A}_\mathbf{s}\times\mathbf{B}_\mathbf{m})/n-F(\mathbf{x})]/\sigma$ weakly converges to $\mathbb{B}_F$, where $\mathbb{B}_F$ is a mean zero Gaussian process with the covariance function given by $\mathbb{E}[\mathbb{B}_F(\mathbf{x})\mathbb{B}_F(\mathbf{x}')]=F(\mathbf{x}\wedge\mathbf{x}')$, $\mathbf{x},\mathbf{x}'\in\mathbb{R}^{d+q}$. Note that conditioning on n, $\mathbf{m}_1,\cdots,\mathbf{m}_n$ are iid $f_m(\mathbf{m}|\mathbf{s})$. For any $B_0\in\mathscr{M}$, $\mathcal{N}_{B_0}(A\times B)=\mathcal{N}_\eta(A\times(B\cap B_0))$ is an MPP which also satisfies Conditions (A1)–(A5) for $\mathcal{N}_{B_0}$. For any partition $\{B_1,\cdots,B_I\}$ of $\mathcal{M}$, the number of events occurred in $A\times B_1,\cdots,A\times B_I$ are independent. Given N, $(\mathcal{N}_\eta(A\times B_1),\cdots,\mathcal{N}_\eta(A\times B_I))$ follows a multinomial distribution with total N and probability vector equal to $(\pi(B_1),\cdots,\pi(B_I))$, where $\pi(B)=\int_B f_m(\mathbf{u})d\mathbf{u}$. The rest of the proof is omitted since it is similar to the method used in the proof of Theorem 4 of Zhang (2014). □

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