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TESTING FOR HIGH-DIMENSIONAL WHITE NOISE

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Abstract: Testing multi-dimensional white noise has been an important subject of statistical inference in time series. Such test in the high-dimensional case becomes an open problem waiting to be further investigated, especially when the dimension of a time series is comparable to or even greater than the sample size. To detect an arbitrary form of departure from high-dimensional white noise, a few tests have been developed. Some of these tests are based on max-type statistics, while others are based on sum-type ones. Despite the progress, an urgent issue awaits to be resolved: none of these tests is robust to the sparsity of the serial correlation structure. Motivated by this, we propose a Fisher's combination test by combining the max-type and the sum-type statistics, taking advantage of the established asymptotic independence between them. This combination test can achieve robustness to the sparsity of the serial correlation structure, and combine the advantages of the two types of tests. We thoroughly study the theoretical properties of the proposed combination test, and demonstrate its advantages over some existing tests through extensive numerical results and an empirical analysis.

Key words and phrases: asymptotic independence, Fisher's combination test, high-dimensional white noise, hypothesis test, robustness.

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1. Introduction

Testing for white noise or serial correlation is an important problem in statistical modeling and inference, especially in diagnostic checking for linear regression and linear time series modeling. In recent years, researchers are increasingly interested in modeling high-dimensional time series data, which are becoming one of the most common data types, and frequently appear in many applications, including meteorology, genomics, chemometrics, biological and environmental research, finance and econometrics, etc. This brings further challenge to diagnostic checking, as we need to perform test for high-dimensional white noise, where the dimension of time series is comparable to or even greater than the sample size, i.e. the observed length of the time series.

For univariate time series, many widely used white noise tests have been proposed in the literature (Li, 2004). Some of these tests have been extended for testing multivariate time series (Hosking, 1980; Li and Mcleod, 1981), which are, however, only suitable for the case that the dimension of the time series is small compared to the sample size. Specifically, for univariate time series, the celebrated Box-Pierce portmanteau test and its variations are considered to be among the most popular omnibus tests for detecting non-specific forms of deviation from white noise. These tests are particularly convenient in practical applications, due to the fact that they are asymptotically distribution-free and are

χ^2-distributed under the null hypothesis (Li, 2004; Lütkepohl, 2005). However, it is widely known that when extended to the multivariate cases, these tests suffer slow convergence to their asymptotic null distributions (Li et al., 2019).

Recently, multivariate white noise tests have undergone rapid development. Some new omnibus tests, such as the tests proposed by Chang et al. (2017), Li et al. (2019) and Tsay (2020), can even deal with high-dimensional time series, where the dimension of the time series is comparable to or even greater than the sample size. Specifically, Chang et al. (2017) proposed a max-type test for high-dimensional white noise, using the maximum absolute auto-correlations and cross-correlations of the component series. Based on an approximation by the L_∞-norm of a normal random vector, the critical value of the max-type test can be evaluated by bootstrapping from a multivariate normal distribution. Subsequently, Tsay (2020) proposed a rank-based max-type test for high-dimensional white noise by using Spearman's rank correlation, and established the limiting null distribution based on the theory of extreme values. On the other hand, Li et al. (2019) proposed a sum-type test for high-dimensional white noise, using sum of squared singular values of several lagged sample autocovariance matrices. Using the random matrix theory, the asymptotic normality for the test statistic under the null is established under the Marcenko-Pastur asymptotic regime.

In general, the max-type test performs well in the case of sparse correlations, i.e. there is a small amount of large absolute auto- or cross-correlations at any nonzero lag. In contrast, the sum-type test performs well in the case of non-sparse correlations, which encapsulates the serial correlations within and across all component series. These two types of tests have their own applicability, but neither of them can perform well in both cases. In other words, neither of these two types of tests is applicable in the case of sparse serial correlations. This motivates us to establish a new test, which can combine the advantages of both types and is therefore applicable to sparse and non-sparse serial correlations. To this end, we first reconsider both the max-type test and the sum-type test, and establish their asymptotic independence. Taking advantage of the newly established independence, we propose to combine them to construct a combination test. The general idea of constructing combination tests after establishing asymptotic independence of max-type and sum-type statistics has appeared in the literature of independence tests and covariance matrix tests for high-dimensional random vectors, for example, in Li and Xue (2015) and Yu et al. (2024).

To combine the asymptotically independent tests, we employ the framework of combining the p-values of independent tests (Littell and Folks, 1971). In many earlier literatures, the problem of combining independent tests of hypotheses has been widely considered, such as in Pearson (1938), Fisher (1950), Wilk and Shapiro (1968) and Naik (1969). Among these methods, the well known Fisher's combination test proposed in Fisher (1950) is usually regarded as one of the best choices, whose advantages were discussed in Littell and Folks (1971). It should be noted that in addition to combining p-values of independent tests, there are other ways for combining independent tests. For example, if all test statistics asymptotically follow Gaussian distributions, then a linear combination of the statistics can be used to construct a combined test statistic. However, in situation where the test statistics have different types of asymptotic distributions, such as normal distribution and Gumble distribution, it is usually difficult to directly combine these statistics, hence combining p-values becomes more practical.

In this paper, to test high-dimensional white noise, we propose a Fisher's combination test by combining the p-values of the max-type and sum-type tests, which is suitable to detect sparse and non-sparse serial correlations. Employing tools in extreme value theory and martingale's central limit theorem, we establish the limiting null distributions of the max-type and sum-type statistics, respectively. Further, we establish the asymptotic independence between the two statistics under the null hypothesis, which enables us to use Fisher's framework of combining independent tests. We demonstrate the advantages of the proposed Fisher's combination test over its competitors through extensive numerical results. In the empirical application, we demonstrate the robust performance of the proposed Fisher's combination test.

The main contributions of this paper are listed as follows.

1. We established the limiting null distribution of the max-type statistic for testing high-dimensional white noise, proved that this max-type test is rate-optimal and investigated its local power function in special cases.

2. We proposed a new sum-type test for testing high-dimensional white noise, where the relationship between the sample size and the dimension is not constrained. This improves the existing sum-type test, which is restricted to the Marčenko-Pastur regime, i.e. the ratio of the sample size to the dimension is required to go to a constant.

3. We proved the asymptotic independence between the above max-type and sum-type test statistics under both Gaussian and non-Gaussian distributions. The establish of the independence is the most important contribution of this paper. The proof of the non-Gaussian case, where we eliminated the Gaussianity requirement of the error distribution, is especially novel and can potentially be used in other context, such as in the high-dimensional cross-sectional independence test (Feng et al., 2022), the high-dimensional location test (Xu et al., 2016) and the high-dimensional covariance matrix test (Li and Xue, 2015; Yu et al., 2024).

4. Based on the newly established asymptotic independence, we constructed

the Fisher's combination test that is suitable to detect both sparse or non-sparse serial correlations.

5. Under a specific local alternative hypothesis, we established the asymptotic independence between the max-type and sum-type test statistics. Based on this, we obtained a lower bound of the power function of the proposed combination test.

The rest of this paper is organized as follows. In Section 2, we describe the problem of testing for high-dimensional white noise, reconsider the max-type and sum-type tests, establish their asymptotic independence and then construct the Fisher's combination test. In Section 3, we present extensive numerical results of the proposed test in comparison with some of its competitors, followed by an empirical application in Section 4. Then, we conclude the paper with some discussions in Section 5, and relegate the technical proofs to Supplementary Material.

2. Methodology

2.1 Notations and the testing problem

Consider a p-dimensional weakly stationary time series $\{\boldsymbol{\varepsilon}_t\}_{t=1}^n$ with mean zero, where $\boldsymbol{\varepsilon}_t = (\varepsilon_{t1}, \cdots, \varepsilon_{tp})^\top$, $t \in \{1, \cdots, n\}$, are identically distributed random vectors. Let $\boldsymbol{\Sigma}(k) = \{\sigma_{ij}(k)\}_{1\leqslant i,j\leqslant p} \doteq \mathrm{cov}(\boldsymbol{\varepsilon}_{t+k}, \boldsymbol{\varepsilon}_t)$ denote the autocovariance of $\boldsymbol{\varepsilon}_t$ at lag k, and let $\boldsymbol{\Gamma}(k) = \{\rho_{ij}(k)\}_{1\leqslant i,j\leqslant p} \doteq \mathrm{diag}\{\boldsymbol{\Sigma}(0)\}^{-1/2}\boldsymbol{\Sigma}(k)\mathrm{diag}\{\boldsymbol{\Sigma}(0)\}^{-1/2}$ denote the autocorrelation of $\boldsymbol{\varepsilon}_t$ at lag k, where for any matrix $\mathbf{M}$, $\mathrm{diag}(\mathbf{M})$ denotes the diagonal matrix consisting of the diagonal elements of $\mathbf{M}$ only. Let $\boldsymbol{\Sigma} = \{\sigma_{ij}\}_{1\leqslant i,j\leqslant p} = \boldsymbol{\Sigma}(0)$ and $\boldsymbol{\Gamma} = \{\rho_{ij}\}_{1\leqslant i,j\leqslant p} = \boldsymbol{\Gamma}(0)$. Let $\sigma_i^2 = \sigma_{ii}$, for each $i \in \{1, \cdots, p\}$.

With the observations $\{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n\}$, let

$$\hat{\boldsymbol{\Gamma}}(k) = \{\hat{\rho}_{ij}(k)\}_{1\leqslant i,j\leqslant p} \doteq \mathrm{diag}\{\hat{\boldsymbol{\Sigma}}(0)\}^{-1/2}\hat{\boldsymbol{\Sigma}}(k)\,\mathrm{diag}\{\hat{\boldsymbol{\Sigma}}(0)\}^{-1/2}$$

denote the sample autocorrelation matrix at lag k, where

$$\hat{\boldsymbol{\Sigma}}(k) = \{\hat{\sigma}_{ij}(k)\}_{1\leqslant i,j\leqslant p} \doteq \frac{1}{n}\sum_{t=1}^{n-k} \boldsymbol{\varepsilon}_{t+k}\boldsymbol{\varepsilon}_t^{\mathrm{T}}$$

denotes the sample autocovariance matrix at lag $k \geq 0$ and $\hat{\boldsymbol{\Sigma}}(k) = \hat{\boldsymbol{\Sigma}}(-k)^{\top}$ for $k < 0$.

We consider the following testing problem:

$$H_0 : \{\boldsymbol{\varepsilon}_t\} \text{ is white noise } \ v.s. \ H_1 : \{\boldsymbol{\varepsilon}_t\} \text{ is not white noise,} \qquad (2.1)$$

where the dimension of time series p is comparable to or even greater than the sample size n. Here, we call an identically distributed time series $\{\varepsilon_t\}$ white noise if ε_t's are independent and identically distributed following the definition of white noise in Tsay (2005).

2.2 The max-type test

Before proposing the Fisher's combination test for testing high-dimensional white noise, we need to re-examine the max-type and sum-type tests, which will be proved to be asymptotically independent and combined to construct the combination test.

Since $\boldsymbol{\Gamma}(k) = 0$ for any $k \geqslant 1$ under H_0, the max-type test statistic T_{MAX} is defined as

$$T_{\text{MAX}} \doteq \max_{1 \leqslant k \leqslant K} T_{n,k}, \tag{2.2}$$

which was first proposed by Chang et al. (2017), where

$$T_{n,k} \doteq \max_{1 \leqslant i,j \leqslant p} n^{1/2} \left|\hat{\rho}_{ij}(k)\right|$$

and $K \geqslant 1$ is an integer. For this max-type test statistic, Chang et al. (2017) evaluated the critical value by bootstrapping from a multivariate normal distribution, which is a widely recognized practice in the case of sparse correlations.

To establish the Fisher's combination test in this paper, we first derive the limiting null distribution of the max-type test statistic, which will be presented in the following Theorem 1. Specifically, Theorem 1 states that $T^2_{\text{MAX}} - 2\log(Kp^2) + \log\log(Kp^2)$ has an asymptotic extreme-value distribution when both n and p go to infinity. Hence, a level-α test with $\alpha \in (0,1)$ will be performed by rejecting H_0 when $T^2_{\text{MAX}} - 2\log(Kp^2) + \log\log(Kp^2)$ is larger than q_α, i.e. the $1-\alpha$ quantile of the distribution with the cumulative distribution function (CDF) $G(y) \doteq \exp\left\{-\pi^{-1/2}\exp(-y/2)\right\}$.

In deriving Theorem 1, we impose the following three conditions.

(C1) ε_{ti}'s have one of the following two types of tails: (i) sub-gaussian-type tails, i.e. there exist some constant $\eta > 0$ and $M > 0$, such that $\mathbb{E}e^{\eta\varepsilon_{ti}^2/\sigma_i^2} \leq M$ for all $i \in \{1, \cdots, p\}$ and $t \in \{1, \cdots, n\}$, where p satisfies $\log p = o(n^{1/5})$; (ii) polynomial-type tails, i.e. for some γ_0 and $c_1 > 0$, $p \leq c_1 n^{\gamma_0}$ and for some $\epsilon > 0$ and $M > 0$, $\mathbb{E}|\varepsilon_{ti}/\sigma_i|^{4\gamma_0+4+\epsilon} \leq M$ for all $i \in \{1, \cdots, p\}$ and $t \in \{1, \cdots, n\}$.

(C2) There exists a positive constant C such that

$$C^{-1} \leq \min_{1\leq i\leq p}\sigma_i^2 \leq \max_{1\leq i\leq p}\sigma_i^2 \leq C.$$

(C3) There exists $\varrho \in (0,1)$ s.t. $|\rho_{ij}| \leq \varrho$ for all $1 \leq i < j \leq p$ with $p \geq 2$. $|C_p|/p^2 \to 0$ as $p \to \infty$ if (C1)-(i) holds; and $|C_p|/n^{\epsilon/8} \to 0$ if (C1)-

(ii) holds. Here $C_p \doteq \{(i,j) : |B_{p,(i,j)}| \geq p^{\kappa_p}\}$ and $B_{p,(i,j)} \doteq \{(s,l) : |\rho_{ij}\rho_{sl}| \geq \delta_p\}$ for $1 \leq i,j \leq p$ with $\delta_p, \kappa_p > 0$, $\delta_p = o(1/\log p)$ and $\kappa_p = o(1)$ as $p \to \infty$.

Remark 1. Condition (C1) requires that the tail of the distributions of ε_{ti}'s is sub-gaussian-type or polynomial-type, which is the same as Condition (C2) or (C2*) used in Cai et al. (2013). It is a more general moment condition than the normal distribution assumption. Condition (C2) requires that all the variances of ε_{ti}'s are bounded. Condition (C3) requires that the number of variable pairs with strong correlation cannot be too large. Below, we provide some cases where Condition (C3) holds. First, consider the case where $\mathbf{\Gamma}$ is a banded matrix, i.e. $\rho_{ij} = 0$ if $|i-j| > \zeta$. In this case, $|C_p| \leq 2\zeta p$, because $|B_{p,(i,j)}| \equiv 0$ when $|i-j| > \zeta$. Let $\delta_p = o(1/\log p)$ and $\kappa_p = o(1)$. If (C1)-(i) and $\zeta = o(p)$ hold, then Condition (C3) holds, because $|C_p| = o(p^2)$. If (C1)-(2) and $\zeta p = o(n^{\epsilon/8})$ hold, Condition (C3) also holds, because $|C_p| = o(n^{\epsilon/8})$. Next, consider the case where $\mathbf{\Gamma}$ has an AR(1) structure, i.e. $\rho_{ij} = \rho^{|i-j|}$ for each $i,j \in \{1,\cdots,p\}$. Let $\delta_p = (\log p)^{-2}$ and $\kappa_p = o(1)$. Hence, $|C_p| \leq -2p\log\log p/\log\rho$, because $|\rho_{ij}\rho_{sl}| \geq \delta_p$ is equivalent to $|i-j| + |s-l| \leq -2\log\log p/\log\rho$. If (C1)-(i) holds, or if (C1)-(2) and $p\log\log p = o(n^{\epsilon/8})$ hold, Condition (C3) holds.

Theorem 1. *Assume Conditions (C1)-(C3) to hold. Then, under H_0, for any*

$y \in \mathbb{R}$, we have

$$\mathrm{P}\left\{T^2_{MAX} - 2\log(Kp^2) + \log\log(Kp^2) \leqslant y\right\} \to G(y)$$

as $n, p \to \infty$, where $G(y) = \exp\left\{-\pi^{-1/2}\exp(-y/2)\right\}$.

We recall that all technical proofs are relegated to the supplementary.

Let $\mathcal{U}(c)$ be a set of matrices indexed by a constant c, which is given by

$$\left[\left\{\boldsymbol{\Gamma}(1), \cdots, \boldsymbol{\Gamma}(K)\right\} \in \mathbb{R}^{p\times Kp} : \max_{1\le k\le K, 1\leqslant i<j\leqslant p} |\rho_{ij}(k)| \geqslant c(\log p/n)^{1/2}\right].$$

Consider the following sparse alternative

$$H^R_{\mathrm{a}}(c) \doteq \left[F(\boldsymbol{\varepsilon}_1, \cdots, \boldsymbol{\varepsilon}_n) : \left\{\mathrm{cor}_F(\boldsymbol{\varepsilon}_{t+1}, \boldsymbol{\varepsilon}_t), \cdots, \mathrm{cor}_F(\boldsymbol{\varepsilon}_{t+K}, \boldsymbol{\varepsilon}_t)\right\} \in \mathcal{U}(c)\right], \tag{2.3}$$

where $F(\boldsymbol{\varepsilon}_1, \cdots, \boldsymbol{\varepsilon}_n)$ denotes the joint distribution of $\{\boldsymbol{\varepsilon}_1, \cdots, \boldsymbol{\varepsilon}_n\}$ and cor_F denotes the autocorrelation matrix under the joint distribution F. Let $\mathcal{T}_\alpha$ denote the set of all measurable size-α tests.

The following theorem characterizes the conditions under which the power of the proposed max-type test $\mathbb{I}\{T^2_{\mathrm{MAX}} - 2\log(Kp^2) + \log\log(Kp^2) \geq q_\alpha\}$ tends to 1 as $n \to \infty$, under the alternative $H^R_{\mathrm{a}}(b_0)$ for some constant b_0.

Theorem 2. *Assume that $\{\boldsymbol{\varepsilon}_t\}$ is a strictly stationary time series and the long-run variance $\gamma_i^L \doteq \lim_{n\to\infty} \mathrm{var}\left(n^{-1/2}\sum_{t=1}^n \varepsilon_{it}^2\right)$ is bounded for all $1 \le i \le p$, i.e. $\gamma_i^L \in (c_L, c_U)$ for some positive constant c_L, c_U. Then, we have*

$$\inf_{F(\boldsymbol{\varepsilon}_1,\cdots,\boldsymbol{\varepsilon}_n)\in H_{\mathrm{a}}^R(b_0)} \mathrm{P}\left[\mathbb{I}\{T_{\mathrm{MAX}}^2 - 2\log(Kp^2) + \log\log(Kp^2) \ge q_\alpha\} = 1\right] = 1 - o(1),$$

for all $b_0 > 3$, where the infimum is taken over the joint distribution family $H_{\mathrm{a}}^R(b_0)$ of $\{\boldsymbol{\varepsilon}_1,\cdots,\boldsymbol{\varepsilon}_n\}$ defined in (2.3).

Theorem 2 indicates that the above max-type test can detect alternatives of order $(\log p/n)^{1/2}$. In Theorem 3, we further show that this test is rate-optimal, i.e. the rate of the signal gap, $(\log p/n)^{1/2}$, cannot be further relaxed.

Theorem 3. *Suppose $c_0 < 1$ is a positive constant, and let β be a positive constant satisfying $\alpha + \beta < 1$. If $\log p/n = o(1)$, we have*

$$\inf_{T_\alpha\in\mathcal{T}_\alpha}\ \sup_{F(\boldsymbol{\varepsilon}_1,\cdots,\boldsymbol{\varepsilon}_n)\in H_{\mathrm{a}}^R(c_0)} \mathrm{P}\left(T_\alpha = 0\right) \geqslant 1 - \alpha - \beta$$

as $n, p \to \infty$, where the supremum is taken over the joint distribution family $H_{\mathrm{a}}^R(c_0)$ of $\{\boldsymbol{\varepsilon}_1,\cdots,\boldsymbol{\varepsilon}_n\}$ defined in (2.3).

Theorem 3 indicates that any measurable size-α test cannot differentiate between the null hypothesis H_0 and the sparse alternative when

$$\max_{1\leq k\leq K, 1\leqslant i<j\leqslant p} |\rho(k)_{ij}| < c_0(\log p/n)^{1/2}$$

for some constant $c_0 < 1$.

Remark 2. Note that if the condition $c_0 < 1$ in Theorem 3 is not satisfied, varying power results may arise depending on the specific alternative setting, and a universal power conclusion cannot be definitively drawn. This statement is substantiated by the following example. Let $\varepsilon_{t1} = z_{t1} + \rho z_{t-1,1}$, where $\rho = O(\sqrt{\log p/n})$ and $z_{t1} \overset{i.i.d}{\sim} \mathcal{N}(0,1)$ for all $t \in \{1, \cdots, n\}$. Let $\varepsilon_{ti} \overset{i.i.d}{\sim} \mathcal{N}(0,1)$, for all $t \in \{1, \cdots, n\}$ and $i \in \{2, \cdots, p\}$. Hence, $\{\varepsilon_{t1}\}_{t=1}^n$ are independent of $\{\varepsilon_{ti}\}_{t=1}^n$ for each $i \in \{2, \cdots, p\}$. As presented in Section S1 of Supplementary Material, when $\rho = c_0\sqrt{\log p/n}$, we have: (1) $\beta_{\text{MAX}}(\rho) \in (0, \alpha)$, if $0 < c_0 < 2$; (2) $\beta_{\text{MAX}}(\rho) = 1$, if $2 < c_0 < 3$. On the other hand, when $\sqrt{n}\rho = \sqrt{4\log p + c_1\sqrt{\log p}}$, we have $\beta_{\text{MAX}}(\rho) \in (\Phi(c_1/4), \alpha + \Phi(c_1/4))$.

Remark 3. As mentioned in Chang et al. (2017), the max-type tests based on asymptotic Gumble distributions usually have conservative size performance. The proposed MAX test also has such limitation. To solve this problem, some resampling methods can be employed, such as the bootstrap procedure used in the max-type test proposed by Chang et al. (2017). The resampling methods can also relax the conditions imposed on ε_t's. Unfortunately, such methods generally require much heavier computational cost, especially in high-dimensional situations. Hence, whether to use a test based on asymptotic distribution or based on resampling heavily depends on the computation capacity.

2.3 The sum-type test

We reconsider the sum-type test, with the test statistic defined as

$$T_{\mathrm{SUM}} \doteq \frac{1}{n(n-1)} \sum_{l=1}^{K} \sum\sum_{t \neq s} \boldsymbol{\varepsilon}_t^\top \boldsymbol{\varepsilon}_s \boldsymbol{\varepsilon}_{t+l}^\top \boldsymbol{\varepsilon}_{s+l}. \tag{2.4}$$

It can be seen from the following Theorem 4 and Proposition 1 that under H_0, $T_{\mathrm{SUM}}/\hat{\sigma}_S$ has an asymptotically standard normal distribution when both n and p go to infinity, where $\hat{\sigma}_S^2 \doteq \frac{2K}{n(n-1)} \widehat{\mathrm{tr}(\boldsymbol{\Sigma}^2)}^2$ and $\widehat{\mathrm{tr}(\boldsymbol{\Sigma}^2)} \doteq \frac{1}{n(n-1)} \sum\sum_{t \neq s} (\boldsymbol{\varepsilon}_t^\top \boldsymbol{\varepsilon}_s)^2$. Hence, a level-$\alpha$ test will be performed by rejecting H_0 when $T_{\mathrm{SUM}}/\hat{\sigma}_S$ is larger than z_α, i.e. the $1-\alpha$ quantile of the standard normal distribution.

Note that the test statistic in (2.4) is similar to the sum-type test statistic proposed by Li et al. (2019). The differences between them are twofold: first, the test statistic in (2.4) removes the diagonal elements $\boldsymbol{\varepsilon}_t^\top \boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_{t+l}^\top \boldsymbol{\varepsilon}_{t+l}$ from the summation to lesson the requirement on $\boldsymbol{\Sigma}$; second, we use the martingale central limit theorem to establish the limiting null distribution of the test statistic, while

Li et al. (2019) used the random matrix theory.

In deriving the asymptotic properties of T_{SUM}, we impose the following two conditions.

(C4) Let $\boldsymbol{\varepsilon}_t = \boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_t$ under H_0, where $\{\boldsymbol{z}_t\}$ with $\boldsymbol{z}_t = (z_{t1}, \cdots, z_{tp})^\top$ is a sequence of p-dimensional independent random vectors with independent components z_{ti}'s, satisfying $\mathbb{E}z_{ti} = 0$, $\mathbb{E}z_{ti}^2 = 1$ and $\mathbb{E}z_{ti}^4 < \infty$.

(C5) $\mathrm{tr}(\boldsymbol{\Sigma}^4) = o\{\mathrm{tr}^2(\boldsymbol{\Sigma}^2)\}$.

Remark 4. Condition (C5) is mild and holds automatically if all the eigenvalues of $\boldsymbol{\Sigma}$ are bounded, i.e. Condition (C5) is weaker than the condition of bounded eigenvalues of $\boldsymbol{\Sigma}$ imposed in Li et al. (2019), which indeed lessons the requirement on $\boldsymbol{\Sigma}$. Note that Condition (C5) is also commonly adopted in the literature of testing high-dimensional covariance matrices, such as in Chen et al. (2010).

Theorem 4. *Suppose Conditions (C4)-(C5) hold. Then, under H_0, we have $T_{\mathrm{SUM}}/\sigma_S \xrightarrow{d} \mathcal{N}(0,1)$, where $\sigma_S^2 \doteq \frac{2K}{n(n-1)}\mathrm{tr}^2(\boldsymbol{\Sigma}^2)$.*

Following the result in Proposition 1 below, we use the above $\widehat{\mathrm{tr}(\boldsymbol{\Sigma}^2)}$ to estimate $\mathrm{tr}(\boldsymbol{\Sigma}^2)$.

Proposition 1. *If $\boldsymbol{\varepsilon}_t = \boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_t$ and $\mathrm{tr}(\boldsymbol{\Sigma}^4) = o\{\mathrm{tr}^2(\boldsymbol{\Sigma}^2)\}$, then under H_0, $\widehat{\mathrm{tr}(\boldsymbol{\Sigma}^2)}/\mathrm{tr}(\boldsymbol{\Sigma}^2) \xrightarrow{p} 1$.*

Further, we present the asymptotic power function of the sum-type test $\mathbb{I}(T_{\mathrm{SUM}}/\hat{\sigma}_S \geq z_\alpha)$, when an alternative hypothesis H_1 is specified. Here, we assume that under H_1, the observations $\{\boldsymbol{\varepsilon}_1, \cdots, \boldsymbol{\varepsilon}_n\}$ follow a p-dimensional first-order vector moving average process, abbreviated as VMA(1), of the form

$$H_1 : \boldsymbol{\varepsilon}_t = \boldsymbol{A}_0 \boldsymbol{z}_t + \boldsymbol{A}_1 \boldsymbol{z}_{t-1}, \tag{2.5}$$

where $\boldsymbol{A}_0, \boldsymbol{A}_1 \in \mathbb{R}^{p\times p}$ are the coefficient matrices. We consider the asymptotic distribution of T_{SUM} in the case of $K = 1$ in the following Theorem 5.

Note that the VMA process represents a relevant framework, widely discussed and employed by the recent time series literature (Brockwell and Davis, 2009; Poloni and Sbrana, 2019). The VMA process as well as its univariate version, the MA process, are useful in modeling financial time series. For example, the bid-ask bounce in stock trading may introduce a MA structure in a return series, while bivariate series of monthly log returns in percentages of the IBM stock and the S&P 500 index are modeled with VMA (Tsay, 2005).

Theorem 5. *Under H_1 in (2.5) with $K = 1$, $(T_{\mathrm{SUM}} - \mu_S)/\sigma_{S1} \xrightarrow{d} \mathcal{N}(0, 1)$, where*

$$\mu_S \doteq \mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_0 \tilde{\boldsymbol{\Sigma}}_1) + \frac{2}{n}\mathrm{tr}^2(\tilde{\boldsymbol{\Sigma}}_{01}), \quad \tilde{\boldsymbol{\Sigma}}_0 \doteq \mathbf{A}_0^\top \mathbf{A}_0, \quad \tilde{\boldsymbol{\Sigma}}_1 \doteq \mathbf{A}_1^\top \mathbf{A}_1, \quad \tilde{\boldsymbol{\Sigma}}_{01} \doteq \mathbf{A}_0^\top \mathbf{A}_1,$$

$$\sigma_{S1}^2 \doteq \frac{2}{n^2}\mathrm{tr}^2(\tilde{\boldsymbol{\Sigma}}_0^2 + \tilde{\boldsymbol{\Sigma}}_1^2) + \frac{6}{n^2}\mathrm{tr}^2(\tilde{\boldsymbol{\Sigma}}_0 \tilde{\boldsymbol{\Sigma}}_1)$$

$$
\begin{aligned}
&+ \frac{4}{n}\left[2\,\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_0\tilde{\boldsymbol{\Sigma}}_1)^2 + (\nu_4 - 3)\,\mathrm{tr}\left\{D^2(\tilde{\boldsymbol{\Sigma}}_0\tilde{\boldsymbol{\Sigma}}_1)\right\}\right] \\
&+ \frac{8}{n^2}\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_{01}\tilde{\boldsymbol{\Sigma}}_{01}^\top)\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_0^2 + \tilde{\boldsymbol{\Sigma}}_1^2) + \frac{16}{n^2}\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_{01}\tilde{\boldsymbol{\Sigma}}_1)\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_{01}\tilde{\boldsymbol{\Sigma}}_0) \\
&+ \frac{16}{n^2}\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_0 + \tilde{\boldsymbol{\Sigma}}_1)\left\{\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_{01}^\top\tilde{\boldsymbol{\Sigma}}_{01}\tilde{\boldsymbol{\Sigma}}_0) + \mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_{01}\tilde{\boldsymbol{\Sigma}}_{01}^\top\tilde{\boldsymbol{\Sigma}}_1)\right\} \\
&+ \frac{16}{n^2}\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_{01})\left\{\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_0^2\tilde{\boldsymbol{\Sigma}}_{01}^\top) + \mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_1^2\tilde{\boldsymbol{\Sigma}}_{01}) + 2\,\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_1\tilde{\boldsymbol{\Sigma}}_{01}\tilde{\boldsymbol{\Sigma}}_0)\right\} \\
&+ \frac{4}{n}\,\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_{01}^\top\tilde{\boldsymbol{\Sigma}}_{01}\tilde{\boldsymbol{\Sigma}}_0^2 + \tilde{\boldsymbol{\Sigma}}_{01}\tilde{\boldsymbol{\Sigma}}_{01}^\top\tilde{\boldsymbol{\Sigma}}_1^2 + 2\tilde{\boldsymbol{\Sigma}}_{01}^\top\tilde{\boldsymbol{\Sigma}}_1\tilde{\boldsymbol{\Sigma}}_{01}\tilde{\boldsymbol{\Sigma}}_0) \\
&+ \frac{4}{n}\,\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_{01}\tilde{\boldsymbol{\Sigma}}_{01}^\top\tilde{\boldsymbol{\Sigma}}_{01}^\top\tilde{\boldsymbol{\Sigma}}_{01}) + \frac{12}{n^2}\,\mathrm{tr}^2(\tilde{\boldsymbol{\Sigma}}_{01}\tilde{\boldsymbol{\Sigma}}_{01}^\top) \\
&+ \frac{16}{n^2}\,\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_{01})\,\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_{01}\tilde{\boldsymbol{\Sigma}}_{01}^\top\tilde{\boldsymbol{\Sigma}}_{01}^\top) \\
&+ \frac{4}{n^2}\,\mathrm{tr}^2(\tilde{\boldsymbol{\Sigma}}_0\tilde{\boldsymbol{\Sigma}}_{01}) + \frac{4}{n^2}\,\mathrm{tr}^2(\tilde{\boldsymbol{\Sigma}}_1\tilde{\boldsymbol{\Sigma}}_{01}) + r_n,
\end{aligned}
$$

and the remainder $r_n = o(\sigma_{S1}^2)$. Here, for each square matrix A, $D(A)$ denotes the diagonal matrix consisting of the main diagonal elements of A.

Similar to Proposition 1, $\widehat{\mathrm{tr}(\boldsymbol{\Sigma}^2)}/\xi_0 \xrightarrow{p} 1$ under H_1 in (2.5), where $\xi_0 \doteq \mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_0^2 + \tilde{\boldsymbol{\Sigma}}_1^2) + 2\mathrm{tr}(\tilde{\boldsymbol{\Sigma}}_{01}^\top\tilde{\boldsymbol{\Sigma}}_{01})$. Hence, the asymptotic power of the proposed sum-type test $\mathbb{I}(T_{\mathrm{SUM}}/\hat{\sigma}_S \geq z_\alpha)$ under H_1 in (2.5) is approximately equal to

$$
\beta_{\mathrm{SUM}} \doteq \Phi\left(\frac{\mu_S}{\sigma_{S1}} - z_\alpha\frac{\sqrt{2}n^{-1}\xi_0}{\sigma_{S1}}\right). \tag{2.6}
$$

2.4 Independence and Fisher's combination test

We now further establish the asymptotic independence between T_{MAX} and T_{SUM}. This will allow us to to combine the two tests via, for example, the Fisher's combination test, to provide a new test that can potentially benefit from both existing tests. We first establish the independence under Gaussian errors in Theorems 6 and under non-Gaussian errors in Theorem 7. The proof of Theorem 7 is innovative and generates a new set of tools for similar problems. Specifically, let $p_{\mathrm{MAX}} \doteq 1 - G\{T_{\mathrm{MAX}}^2 - 2\log(Kp^2) + \log\log(Kp^2)\}$ and $p_{\mathrm{SUM}} \doteq 1 - \Phi\left(T_{\mathrm{SUM}}/\widehat{\sigma}_S\right)$ denote the p-values with respect to the test statistics T_{MAX} and T_{SUM} respectively. Based on p_{MAX} and p_{SUM}, the proposed Fisher's combination test rejects H_0 at the significance level α, if

$$T_{\mathrm{FC}} \doteq -2\log p_{\mathrm{MAX}} - 2\log p_{\mathrm{SUM}}$$

is larger than c_α, i.e. the $1-\alpha$ quantile of the chi-squared distribution with 4 degrees of freedom (Fisher, 1950; Littell and Folks, 1971).

In deriving the asymptotic independence between T_{MAX} and T_{SUM}, we need to impose an additional condition as follows. Let $\lambda_{\min}(\mathbf{\Sigma})$ and $\lambda_{\max}(\mathbf{\Sigma})$ denote the minimum and maximum eigenvalues of $\mathbf{\Sigma}$, respectively.

(C6) $\mathrm{tr}^{-1}(\mathbf{\Sigma}^2)(\log p)^\gamma \max\{\lambda_{\max}(\mathbf{\Sigma})M_p, M_p^2, M_p^{3/2}\lambda_{\max}^{1/2}(\mathbf{\Sigma})\} \to 0$ for some positive constant $\gamma > 1$, where $M_p \doteq \max_{1\le i\le p}\sum_{j\neq i}^{p}\sigma_{ij}^2$.

Remark 5. Condition (C6) requires that the covariance between each pair of variables is not too large, which holds in many common situations. For example, it automatically holds when all the variables are independent, i.e. $M_p = 0$. It also holds if all the eigenvalues of $\mathbf{\Sigma}$ are bounded and M_p is also bounded. In addition, if $\mathbf{\Sigma}$ is a banded covariance matrices, i.e. $\sigma_{ij} = 0$ if $|i-j| > k$ for some fixed integer k, and all nonzero σ_{ij}'s are bounded by c, then Condition (C6) holds when $\lambda_{\max}(\mathbf{\Sigma})(\log p)^{\gamma}\mathrm{tr}^{-1}(\mathbf{\Sigma}^2) \to 0$.

Note that Condition (C6) is significantly different from Assumption 1 (ii) of Yu et al. (2024) for testing high-dimensional covariance matrix. For example, when $M_p = 0$, we do not need to impose conditions on $\lambda_i(\mathbf{\Sigma})$'s. In fact, these two type of conditions do not subset each other.

Theorem 6. *Suppose $\boldsymbol{\varepsilon}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ for $t = 1, \cdots, n$ and Conditions (C2), (C3), (C5) and (C6) hold. Then, under H_0, we have*

$$\mathrm{P}\left\{T_{\mathrm{MAX}}^2 - 2\log(Kp^2) + \log\log(Kp^2) \le x, T_{\mathrm{SUM}}/\widehat{\sigma}_S \le y\right\} \to G(x)\cdot\Phi(y), \tag{2.7}$$

as $n, p \to \infty$, i.e. T_{MAX} and T_{SUM} are asymptotically independent.

Further, we relax the assumption of Gaussian distribution of $\boldsymbol{\varepsilon}_t$ in Theorem 6 to non-Gaussian distributions, with sub-gaussian-type or polynomial-type tails. To establish the theoretical result under non-Gaussian distributions, Condition (C1) is modified as follows.

(C1′) ε_{ti}'s have one of the following two types of tails: (i) sub-gaussian-type tails, i.e. there exist some constant $\eta > 0$ and $M > 0$, such that $\mathbb{E}e^{\eta\varepsilon_{ti}^2/\sigma_i^2} \leq M$ for all $i \in \{1, \cdots, p\}$ and $t \in \{1, \cdots, n\}$, where p satisfies $\log p = o(n^{1/6})$; (ii) polynomial-type tails, i.e. for some γ_0 and $c_1 > 0$, $p \leq c_1 n^{\gamma_0}$ and for some $\epsilon > 0$ and $M > 0$, $\mathbb{E}|\varepsilon_{ti}/\sigma_i|^{6\gamma_0+6+\epsilon} \leq M$ for all $i \in \{1, \cdots, p\}$ and $t \in \{1, \cdots, n\}$.

Theorem 7. *Assume Conditions (C1′) and (C2)-(C6) hold. Then, under H_0, we have*

$$\mathrm{P}\left\{T_{\mathrm{MAX}}^2 - 2\log(Kp^2) + \log\log(Kp^2) \leq x, T_{\mathrm{SUM}}/\widehat{\sigma}_S \leq y\right\} \to G(x)\cdot\Phi(y), \tag{2.8}$$

as $n, p \to \infty$, i.e. T_{MAX} and T_{SUM} are asymptotically independent.

Remark 6. Note that relaxing the assumption of Gaussian distribution in establishing the asymptotically independence between the max-type and the sum-type statistics is an important contribution of this paper, since all the existing literatures on establishing such asymptotic independence, including Li and Xue

(2015), Xu et al. (2016), Feng et al. (2022) and Yu et al. (2024), are limited by the assumption of Gaussian distribution. In this paper, we have developed a novel theoretical tool so that we can weaken the Gaussian distribution to non-Gaussian distributions with sub-gaussian-type or polynomial-type tails. Its theoretical framework is enlightening, which can be generalized to analogous studies.

Based on Theorem 6 or Theorem 7, we immediately have the following result for T_{FC}.

Corollary 1. *Assume the same conditions as in Theorem 6 or Theorem 7, then we have* $T_{FC} \xrightarrow{d} \chi_4^2$ *as* $n, p \to \infty$.

Under the alternative hypothesis (2.3), we have $p_{\text{MAX}} \to 0$ under the sparse alternatives due to Theorem 2. On the other hand, under the dense alternative hypothesis (2.5), we have $p_{\text{SUM}} \to 0$ if $\mu_S/\sigma_{S1} \to \infty$ due to Theorem 5.

According to the definition of T_{FC}, if $p_{\text{MAX}} \to 0$ or $p_{\text{SUM}} \to 0$, we have $T_{\text{FC}} \to \infty$, hence we reject the null hypothesis.

Remark 7. Note that Conditions (C2), (C3), (C5) and (C6) are all about $\mathbf{\Sigma}$, which hold automatically if all the eigenvalues of $\mathbf{\Sigma}$ are bounded. This indicates that these conditions are compatible and the intersection of these conditions is routinely considered, which means that the scope of application of the proposed

Fisher's combination test is relatively broad.

Next, we show that T_{SUM} is still asymptotically independent of T_{MAX} under a specific alternative hypothesis. Based on this result, we obtain a low bound of the power function of T_{FC}.

Theorem 8. *Assume Conditions (C1′) and (C2)-(C5) hold. Assume that all eigenvalues of* $\boldsymbol{\Sigma} = \text{cov}(\boldsymbol{\varepsilon}_t)$ *are bounded. Let* $K = 1$*. Then, under the alternative hypothesis (2.5) with*

$$\mathbf{A}_0 = \begin{pmatrix} \mathbf{A}_{011} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{022} \end{pmatrix}, \ \mathbf{A}_1 = \begin{pmatrix} \mathbf{A}_{111} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

$\mathbf{A}_{011} \in \mathbb{R}^{d\times d}$, $\mathbf{A}_{022} \in \mathbb{R}^{(p-d)\times(p-d)}$, $\mathbf{A}_{111} \in \mathbb{R}^{d\times d}$ *and* $d = o(p)$*, we have*

$$\begin{aligned} &\text{P}\left\{T_{\text{MAX}}^2 - 2\log(Kp^2) + \log\log(Kp^2) \le x, T_{\text{SUM}}/\widehat{\sigma}_S \le y\right\} \\ &\to \text{P}\left\{T_{\text{MAX}}^2 - 2\log(Kp^2) + \log\log(Kp^2) \le x\right\}\text{P}\left\{T_{\text{SUM}}/\widehat{\sigma}_S \le y\right\}, \quad (2.9) \end{aligned}$$

as $n, p \to \infty$*, i.e.* T_{MAX} *and* T_{SUM} *are still asymptotically independent.*

Define a minimal p-value test statistic $T_{\min} = \min(p_{\text{SUM}}, p_{\text{MAX}})$. According to Theorem 7, the minimal p-value test based on $T_{\min}$ rejects the null hypothesis if $p_{\text{SUM}} \le 1 - \sqrt{1-\alpha} \approx \alpha/2$ or $p_{\text{MAX}} \le 1 - \sqrt{1-\alpha} \approx \alpha/2$.

To explain the power gain of the minimal p-value test, let $\beta_{T_{\text{SUM}},\alpha}$ and $\beta_{T_{\text{MAX}},\alpha}$ denote the power functions of the sum-type test based on T_{SUM} and max-type test based on T_{MAX} at significant level α, respectively. Let $\beta_{T_{\min}}$ denote the power of the minimal p-value test based on $T_{\min}$ at significant level α. Then, $\beta_{T_{\min}} \geq \max\{\beta_{T_{\text{SUM}},\alpha/2}, \beta_{T_{\text{MAX}},\alpha/2}\}$ for all alternative hypothesis. In addition, according to the inclusion-exclusion principle together with Theorem 8, we have

$$\begin{aligned}\beta_{T_{\min}} &\geq \beta_{T_{\text{SUM}},\alpha/2} + \beta_{T_{\text{MAX}},\alpha/2} - \beta_{T_{\text{SUM}},\alpha/2}\beta_{T_{\text{MAX}},\alpha/2} \\ &\geq \max\{\beta_{T_{\text{SUM}},\alpha/2}, \beta_{T_{\text{MAX}},\alpha/2}\}\end{aligned}$$

under the specific serial correlation structure considered in Theorem 8. Further, if the condition $\beta_{T_{\text{MAX}},\alpha/2} \geq \frac{\beta_{T_{\text{SUM}},\alpha}-\beta_{T_{\text{SUM}},\alpha/2}}{1-\beta_{T_{\text{SUM}},\alpha/2}}$ holds, then $\beta_{T_{\min}} \geq \beta_{T_{\text{SUM}},\alpha}$. Similarly, $\beta_{T_{\min}} \geq \beta_{T_{\text{MAX}},\alpha}$ if $\beta_{T_{\text{SUM}},\alpha/2} \geq \frac{\beta_{T_{\text{MAX}},\alpha}-\beta_{T_{\text{MAX}},\alpha/2}}{1-\beta_{T_{\text{MAX}},\alpha/2}}$. These imply that under the condition where the tests based on T_{MAX} and T_{SUM} have certain power values, the test based on $T_{\min}$ is more powerful than both of them. As mentioned in Littell and Folks (1971), the power of a Fisher's combination test often has very similar power performance to a minimal p-value test. Hence, under the above condition, the proposed Fisher's combination test based on T_{FC} may also be more powerful than both the test based on T_{MAX} and that based on T_{SUM}.

Finally, in the following remark, we make some discussions on how the dimension p affects the results of the proposed tests.

Remark 8. We start the discussion from a theoretical perspective. Recall that Condition (C1) is used in deriving the asymptotic null distribution of the max-type statistic T_{MAX} in (2.2). It requires that the dimension p relates to the sample size n at: (1) the exponential rate, such as $p = ce^{n^{\upsilon}}$ for some constants c and υ, in the case where ε_{ti}'s have sub-gaussian-type tails; (2) the polynomial rate, such as $p = cn^{\upsilon}$ for some constants c and υ, in the case when ε_{ti}'s have polynomial-type tails. This indicates that under different tail types, the order requirements for the dimension of the max-type test are significantly different in theory. In contrast, when deriving the asymptotic null distribution of the sum-type statistic T_{SUM} in (2.4), after imposing Condition (C5) on $\boldsymbol{\Sigma}$, there is no need to impose additional order requirements for p. In some other literature, such as Li et al. (2019), when establishing the asymptotic distribution of a sum-type test statistic, it is necessary to assume that (p, n) satisfies the Marčenko-Pastur regime, i.e. $p/n \to c > 0$. Due to the construction of T_{FC}, establishing its asymptotic distribution requires more stringent conditions. Indeed, for the pair (p, n), a condition slightly more stringent than that for T_{MAX} is imposed.

Next, from the results of the simulation study in the following section, different testing methods show significant differences when the sample size is given but the dimension increases. For example, suggested by Tables 1 and 2, the empirical size of the sum-type test proposed by Li et al. (2019) significantly decreases as the dimension increases, while in contrast, our sum-type test has a very stable size performance.

3. Numerical results

We now present some numerical results to demonstrate the performance of the max-type test, sum-type test and Fisher's combined probability test, abbreviated as MAX, SUM and FC respectively, as well as their comparison with the sum-type test proposed by Li et al. (2019), abbreviated as LY. Note that the max-type test is based on the asymptotic results established in this paper, not the resampling method in Chang et al. (2017).

3.1 Size performance

For the cases under H_0, we let $\boldsymbol{\varepsilon}_t = \mathbf{A}\boldsymbol{z}_t$ with $\boldsymbol{z}_t = (z_{t1}, \cdots, z_{tp})^\top$ and $\mathbf{A} = \{a_{ij}\}_{1\le i,j\le p}$. We consider the following two distributions of $\boldsymbol{z}_t$: (i) $\boldsymbol{z}_t \overset{i.i.d}{\sim} \mathcal{N}(\mathbf{0}, \mathbf{I}_p)$; (ii) $z_{ti} \overset{i.i.d}{\sim} Ga(4, 0.5) - 2$, and the following three settings of $\mathbf{A}$:

(I) $\mathbf{A} = \boldsymbol{\Sigma}^{1/2}$, $\boldsymbol{\Sigma} = \{\sigma_{ij}\}_{1\leqslant i,j\leqslant p}$, $\sigma_{ii} = 1$, $i = 1, \cdots, p$, $\sigma_{ij} = 0.5(i-j)^{-2}$ with $i \neq j$;

(II) $\mathbf{A} = \boldsymbol{\Sigma}^{1/2}$, $\boldsymbol{\Sigma} = \{\sigma_{ij}\}_{1\leqslant i,j\leqslant p}$, $\sigma_{ij} = 0.5^{|i-j|}$;

(III) $a_{ij} \overset{i.i.d}{\sim} U(-1,1)$.

Tables 1 and 2 summarize the empirical size performance of MAX, LY, SUM and FC under settings (I)-(III) with the distributions (i) and (ii), respectively. The simulation results presented here, along with all subsequent results, are derived from the mean of 1,000 repetitions. The corresponding standard deviation is indicated in parentheses following each mean. Please note that all empirical size values have been scaled up by a factor of 100. Results in both tables suggest that in terms of size performance, SUM is the best and MAX is the most conservative, while FC and LY are in between. In addition, we find that FC has much better size performance than LY in situations where p/n is relatively large or $K > 1$.

3.2 Power comparison

In this subsection, we compare the empirical power performance of the above four tests. For the cases under the alternative hypothesis, we only consider the above distribution (i) to avoid redundancy and consider the following three new settings of ε_t:

(IV) VAR(1) model: $\varepsilon_t = \mathbf{A}\varepsilon_{t-1} + z_t$;

(V) VMA(1) model: $\varepsilon_t = z_t + \mathbf{A}z_{t-1}$;

Table 1: Size performance in the case of $\varepsilon_t = \mathbf{A}z_t$ with distribution (i).

		$K=1$				$K=2$				$K=3$			
n	p	MAX	LY	SUM	FC	MAX	LY	SUM	FC	MAX	LY	SUM	FC
		Setting (I)											
100	30	1.0(0.3)	4.9(0.7)	5.9(0.7)	4.4(0.6)	0.9(0.3)	2.7(0.5)	4.8(0.7)	2.9(0.5)	1.0(0.3)	1.8(0.4)	5.3(0.7)	4.0(0.6)
100	60	1.0(0.3)	2.8(0.5)	5.0(0.7)	3.1(0.5)	1.2(0.3)	2.0(0.4)	5.5(0.7)	3.1(0.5)	0.6(0.2)	1.0(0.3)	4.5(0.7)	2.1(0.5)
100	90	0.9(0.3)	1.5(0.4)	4.2(0.6)	2.7(0.5)	0.3(0.2)	0.6(0.2)	5.7(0.7)	2.4(0.5)	0.6(0.2)	0.0(0.0)	4.8(0.7)	2.3(0.5)
100	120	0.9(0.3)	1.0(0.3)	4.0(0.6)	2.3(0.5)	0.6(0.2)	0.5(0.2)	5.2(0.7)	2.1(0.5)	0.3(0.2)	0.3(0.2)	5.6(0.7)	3.1(0.5)
200	30	2.0(0.4)	4.6(0.7)	5.3(0.7)	3.9(0.6)	1.8(0.4)	3.8(0.6)	5.3(0.7)	4.5(0.7)	1.0(0.3)	4.4(0.6)	5.7(0.7)	4.3(0.6)
200	60	1.2(0.3)	3.2(0.6)	4.0(0.6)	2.8(0.5)	1.2(0.3)	3.0(0.5)	6.0(0.8)	3.7(0.6)	1.1(0.3)	1.9(0.4)	5.4(0.7)	3.3(0.6)
200	90	0.8(0.3)	3.4(0.6)	5.0(0.7)	3.7(0.6)	1.2(0.3)	2.4(0.5)	5.3(0.7)	2.9(0.5)	1.2(0.3)	0.7(0.3)	4.0(0.6)	2.2(0.5)
200	120	1.3(0.4)	3.3(0.6)	5.9(0.7)	3.6(0.6)	1.3(0.4)	1.5(0.4)	4.9(0.7)	2.4(0.5)	1.1(0.3)	1.0(0.3)	5.4(0.7)	3.2(0.6)
		Setting (II)											
100	30	1.5(0.4)	3.0(0.5)	4.1(0.6)	3.1(0.5)	1.1(0.3)	3.5(0.6)	5.7(0.7)	4.4(0.6)	0.8(0.3)	2.0(0.4)	4.9(0.7)	2.9(0.5)
100	60	1.6(0.4)	2.3(0.5)	4.4(0.6)	3.0(0.5)	1.0(0.3)	1.2(0.3)	5.1(0.7)	3.0(0.5)	1.6(0.4)	1.0(0.3)	5.2(0.7)	3.2(0.6)
100	90	1.2(0.3)	2.0(0.4)	5.4(0.7)	3.4(0.6)	0.6(0.2)	0.7(0.3)	3.7(0.6)	2.3(0.5)	0.3(0.2)	0.3(0.2)	5.4(0.7)	2.2(0.5)
100	120	0.8(0.3)	1.9(0.4)	5.8(0.7)	2.6(0.5)	0.9(0.3)	0.2(0.1)	6.0(0.8)	2.2(0.5)	0.2(0.1)	0.1(0.1)	5.4(0.7)	2.3(0.5)
200	30	2.1(0.5)	4.7(0.7)	5.8(0.7)	3.7(0.6)	1.6(0.4)	4.9(0.7)	6.3(0.8)	4.6(0.7)	1.3(0.4)	4.2(0.6)	6.4(0.8)	5.1(0.7)
200	60	1.5(0.4)	4.0(0.6)	5.5(0.7)	2.9(0.5)	1.5(0.4)	2.9(0.5)	5.2(0.7)	3.9(0.6)	1.8(0.4)	2.3(0.5)	6.0(0.8)	4.0(0.6)
200	90	2.2(0.5)	3.9(0.6)	5.0(0.7)	3.3(0.6)	1.3(0.4)	1.8(0.4)	5.1(0.7)	3.7(0.6)	1.2(0.3)	1.3(0.4)	5.4(0.7)	3.2(0.6)
200	120	1.0(0.3)	3.3(0.6)	5.3(0.7)	3.4(0.6)	1.7(0.4)	1.8(0.4)	6.5(0.8)	4.0(0.6)	1.5(0.4)	0.8(0.3)	4.8(0.7)	3.1(0.5)
		Setting (III)											
100	30	1.4(0.4)	3.8(0.6)	5.0(0.7)	4.3(0.6)	0.9(0.3)	3.7(0.6)	5.2(0.7)	3.8(0.6)	0.5(0.2)	2.2(0.5)	5.1(0.7)	3.2(0.6)
100	60	0.3(0.2)	3.0(0.5)	4.9(0.7)	2.1(0.5)	1.0(0.3)	1.7(0.4)	5.0(0.7)	2.5(0.5)	0.9(0.3)	1.1(0.3)	4.9(0.7)	2.5(0.5)
100	90	0.8(0.3)	2.7(0.5)	4.5(0.7)	3.1(0.5)	1.1(0.3)	1.1(0.3)	5.6(0.7)	4.0(0.6)	0.6(0.2)	0.7(0.3)	4.6(0.7)	2.1(0.5)
100	120	0.8(0.3)	1.7(0.4)	4.2(0.6)	2.8(0.5)	0.2(0.1)	0.8(0.3)	5.2(0.7)	2.2(0.5)	0.4(0.2)	0.1(0.1)	3.5(0.6)	2.2(0.5)
200	30	1.8(0.4)	4.5(0.7)	5.3(0.7)	4.3(0.6)	1.4(0.4)	4.5(0.7)	5.1(0.7)	4.3(0.6)	1.0(0.3)	3.0(0.5)	4.8(0.7)	3.5(0.6)
200	60	1.6(0.4)	4.1(0.6)	5.2(0.7)	3.3(0.6)	1.6(0.4)	3.1(0.5)	5.6(0.7)	3.9(0.6)	1.3(0.4)	1.7(0.4)	4.7(0.7)	2.7(0.5)
200	90	1.6(0.4)	2.8(0.5)	4.0(0.6)	2.3(0.5)	1.7(0.4)	2.3(0.5)	4.9(0.7)	4.3(0.6)	1.7(0.4)	0.9(0.3)	5.0(0.7)	3.1(0.5)
200	120	1.5(0.4)	3.5(0.6)	5.0(0.7)	3.7(0.6)	1.1(0.3)	1.3(0.4)	4.3(0.6)	2.8(0.5)	1.1(0.3)	0.8(0.3)	5.2(0.7)	3.0(0.5)

Table 2: Size performance in the case of $\boldsymbol{\varepsilon}_t = \mathbf{A}\boldsymbol{z}_t$ with distribution (ii).

		$K=1$				$K=2$				$K=3$			
n	p	MAX	LY	SUM	FC	MAX	LY	SUM	FC	MAX	LY	SUM	FC
		Setting (I)											
100	30	2.3(0.5)	4.7(0.7)	5.6(0.7)	5.2(0.7)	1.6(0.4)	3.1(0.5)	4.5(0.7)	4.4(0.6)	1.5(0.4)	1.2(0.3)	4.6(0.7)	3.7(0.6)
100	60	2.2(0.5)	3.0(0.5)	4.8(0.7)	3.9(0.6)	2.2(0.5)	1.0(0.3)	4.7(0.7)	3.4(0.6)	2.4(0.5)	1.0(0.3)	5.6(0.7)	5.3(0.7)
100	90	1.6(0.4)	1.9(0.4)	5.0(0.7)	4.3(0.6)	1.9(0.4)	1.3(0.4)	4.9(0.7)	4.9(0.7)	2.1(0.5)	0.0(0.0)	5.1(0.7)	4.6(0.7)
100	120	2.0(0.4)	1.9(0.4)	4.4(0.6)	4.2(0.6)	1.4(0.4)	0.2(0.1)	4.6(0.7)	3.0(0.5)	1.7(0.4)	0.0(0.0)	5.2(0.7)	3.8(0.6)
200	30	2.6(0.5)	5.7(0.7)	5.9(0.7)	5.8(0.7)	2.3(0.5)	4.8(0.7)	6.7(0.8)	6.3(0.8)	3.0(0.5)	2.0(0.4)	4.4(0.6)	3.8(0.6)
200	60	2.5(0.5)	4.1(0.6)	5.7(0.7)	5.7(0.7)	2.7(0.5)	2.9(0.5)	5.2(0.7)	4.8(0.7)	2.8(0.5)	2.1(0.5)	4.7(0.7)	5.5(0.7)
200	90	2.5(0.5)	3.0(0.5)	5.1(0.7)	4.8(0.7)	2.9(0.5)	1.9(0.4)	5.0(0.7)	4.1(0.6)	2.6(0.5)	1.1(0.3)	6.1(0.8)	5.5(0.7)
200	120	2.7(0.5)	3.0(0.5)	5.2(0.7)	4.8(0.7)	3.7(0.6)	1.0(0.3)	4.3(0.6)	4.7(0.7)	3.0(0.5)	0.4(0.2)	3.8(0.6)	3.8(0.6)
		Setting (II)											
100	30	1.6(0.4)	4.2(0.6)	5.2(0.7)	4.9(0.7)	2.4(0.5)	2.8(0.5)	4.8(0.7)	4.8(0.7)	2.7(0.5)	2.4(0.5)	5.9(0.7)	5.1(0.7)
100	60	2.4(0.5)	2.7(0.5)	4.5(0.7)	4.0(0.6)	2.2(0.5)	1.2(0.3)	4.9(0.7)	4.2(0.6)	2.3(0.5)	1.1(0.3)	5.4(0.7)	4.1(0.6)
100	90	2.0(0.4)	2.1(0.5)	5.4(0.7)	3.8(0.6)	1.6(0.4)	1.0(0.3)	3.9(0.6)	3.5(0.6)	1.7(0.4)	0.3(0.2)	5.2(0.7)	4.4(0.6)
100	120	1.7(0.4)	1.3(0.4)	3.9(0.6)	2.9(0.5)	1.3(0.4)	0.6(0.2)	4.1(0.6)	2.8(0.5)	1.9(0.4)	0.0(0.0)	5.4(0.7)	3.4(0.6)
200	30	2.6(0.5)	4.3(0.6)	4.7(0.7)	4.9(0.7)	2.8(0.5)	4.5(0.7)	6.1(0.8)	5.5(0.7)	2.4(0.5)	3.4(0.6)	5.4(0.7)	5.4(0.7)
200	60	3.1(0.5)	3.9(0.6)	5.5(0.7)	6.8(0.8)	3.0(0.5)	2.3(0.5)	5.1(0.7)	5.3(0.7)	2.3(0.5)	1.6(0.4)	5.0(0.7)	5.4(0.7)
200	90	3.3(0.6)	3.5(0.6)	5.3(0.7)	5.6(0.7)	2.1(0.5)	2.4(0.5)	5.0(0.7)	3.8(0.6)	2.9(0.5)	0.9(0.3)	5.0(0.7)	5.0(0.7)
200	120	3.1(0.5)	3.4(0.6)	5.8(0.7)	5.1(0.7)	2.2(0.5)	2.2(0.5)	4.9(0.7)	4.7(0.7)	3.1(0.5)	0.3(0.2)	5.8(0.7)	5.2(0.7)
		Setting (III)											
100	30	1.0(0.3)	4.9(0.7)	6.1(0.8)	4.8(0.7)	0.3(0.2)	3.2(0.6)	5.3(0.7)	3.5(0.6)	1.1(0.3)	2.1(0.5)	4.4(0.6)	3.2(0.6)
100	60	1.2(0.3)	2.3(0.5)	4.2(0.6)	2.9(0.5)	1.3(0.4)	2.2(0.5)	5.3(0.7)	3.8(0.6)	0.8(0.3)	1.6(0.4)	6.3(0.8)	3.7(0.6)
100	90	0.7(0.3)	2.9(0.5)	5.3(0.7)	2.8(0.5)	0.8(0.3)	0.8(0.3)	5.1(0.7)	2.4(0.5)	0.3(0.2)	0.2(0.1)	5.5(0.7)	3.1(0.5)
100	120	0.5(0.2)	1.7(0.4)	5.2(0.7)	2.5(0.5)	0.5(0.2)	0.6(0.2)	4.0(0.6)	1.6(0.4)	0.9(0.3)	0.2(0.1)	5.1(0.7)	2.3(0.5)
200	30	1.6(0.4)	4.5(0.7)	5.7(0.7)	4.4(0.6)	1.9(0.4)	4.5(0.7)	5.9(0.7)	4.5(0.7)	1.9(0.4)	4.4(0.6)	6.0(0.8)	5.1(0.7)
200	60	1.3(0.4)	4.2(0.6)	5.5(0.7)	3.8(0.6)	0.7(0.3)	2.3(0.5)	4.4(0.6)	3.0(0.5)	1.5(0.4)	1.9(0.4)	5.0(0.7)	2.9(0.5)
200	90	1.5(0.4)	3.3(0.6)	4.6(0.7)	3.7(0.6)	1.3(0.4)	2.7(0.5)	7.0(0.8)	4.0(0.6)	1.6(0.4)	1.5(0.4)	4.9(0.7)	3.1(0.5)
200	120	0.9(0.3)	2.4(0.5)	4.8(0.7)	2.9(0.5)	1.8(0.4)	2.7(0.5)	5.0(0.7)	4.2(0.6)	0.8(0.3)	1.3(0.4)	5.3(0.7)	2.8(0.5)

(VI) VARMA(1) model: $\boldsymbol{\varepsilon}_t = 0.5\mathbf{A}\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{z}_t + 0.5\mathbf{A}\boldsymbol{z}_{t-1}$.

Here "VAR(1)", "VMA(1)" and "VARMA(1)" are the abbreviations of 1-order vector autoregressive process, vector moving average process and vector autoregressive moving average process, respectively. Let $\mathbf{A} = \{a_{ij}\}_{1\le i,j\le p}$. For the alternative hypothesis, we let the first $a_{ij} \neq 0$ for $1 \le i,j \le m$ and $a_{ij} = 0$ otherwise. Note that m controls the signal strength and sparsity of $\mathbf{A}$. For the VAR(1) model, if $m = 1$, $a_{ij} \sim U(0.4, 0.8)$; if $2 \le m \le 10$, $a_{ij} \sim U(-1.4/m, 1.4/m)$. For the VMA(1) model, if $m = 1$, $a_{ij} \sim U(0.4, 0.9)$; if $2 \le m \le 10$, $a_{ij} \sim U(-1.8/m, 1.8/m)$. For the VARMA(1) model, if $m = 1$, $a_{ij} \sim U(0.4, 0.8)$; if $2 \le m \le 10$, $a_{ij} \sim U(-1.6/m, 1.6/m)$. Specifically, as m decreases, both the signal strength and sparsity of $\mathbf{A}$ increase. Let $n = 200$, $p \in \{60, 90\}$ and $K \in \{1, 2, 3\}$.

Figures 1 and 2 present the empirical power curves of MAX, LY, SUM and FC under settings (IV)-(VI) and distribution (i) for $(n, p) = (200, 60)$ and $(200, 90)$, respectively. In each panel of these figures, the abscissa m varies between 1 and 10, corresponding to the power performance of the involved tests with different signal strength and sparsity of $\mathbf{A}$. Results in both figures suggest that in terms of empirical power performance, FC is better than its competitors in most cases regardless $\mathbf{A}$ is sparse or non-sparse, which has robust performance due to the combination of the advantages of both MAX and SUM. Although FC

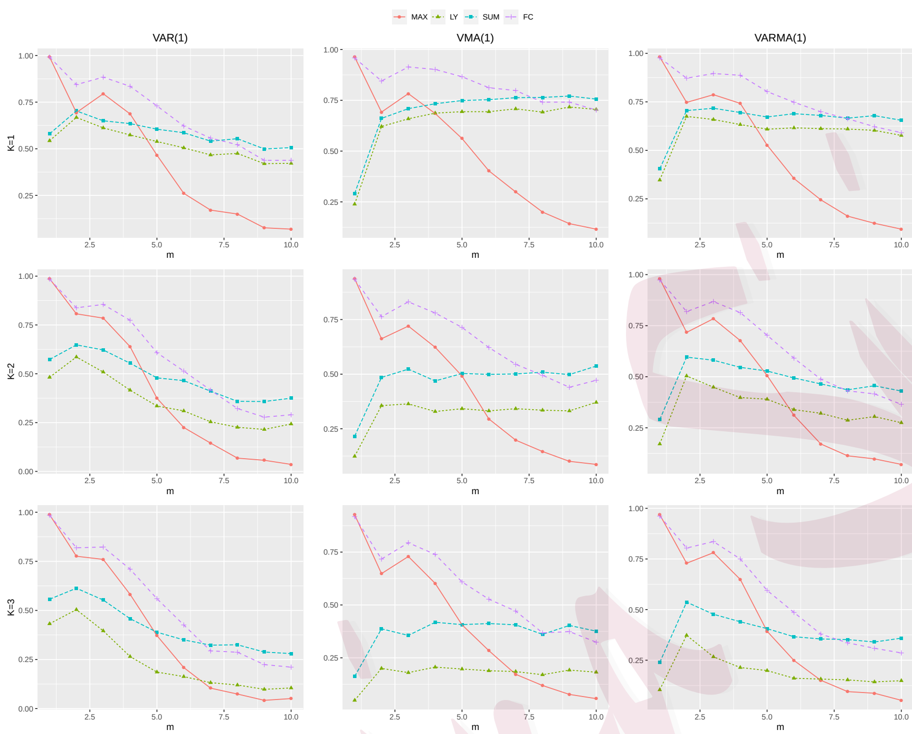

Figure 1: Power curves of the involved tests with $m = 1, 2, \cdots, 10$ and $(n, p) = (200, 60)$.

does not outperform its competitors in all cases, it is indeed applicable to both sparse and non-sparse cases of $\mathbf{A}$. In contrast, MAX generally fails when $\mathbf{A}$ is sufficiently dense, while SUM and LY generally fail when $\mathbf{A}$ is very sparse.

4. Application

In this section, we are interested in testing whether the identically distributed error series $\{\boldsymbol{\varepsilon}_t\}$ under the Fama-French three-factor model (Fama and French, 1993) is white noise, where $\boldsymbol{\varepsilon}_t = (\varepsilon_{t1}, \cdots, \varepsilon_{tp})^\top$ and p is the number of secu-

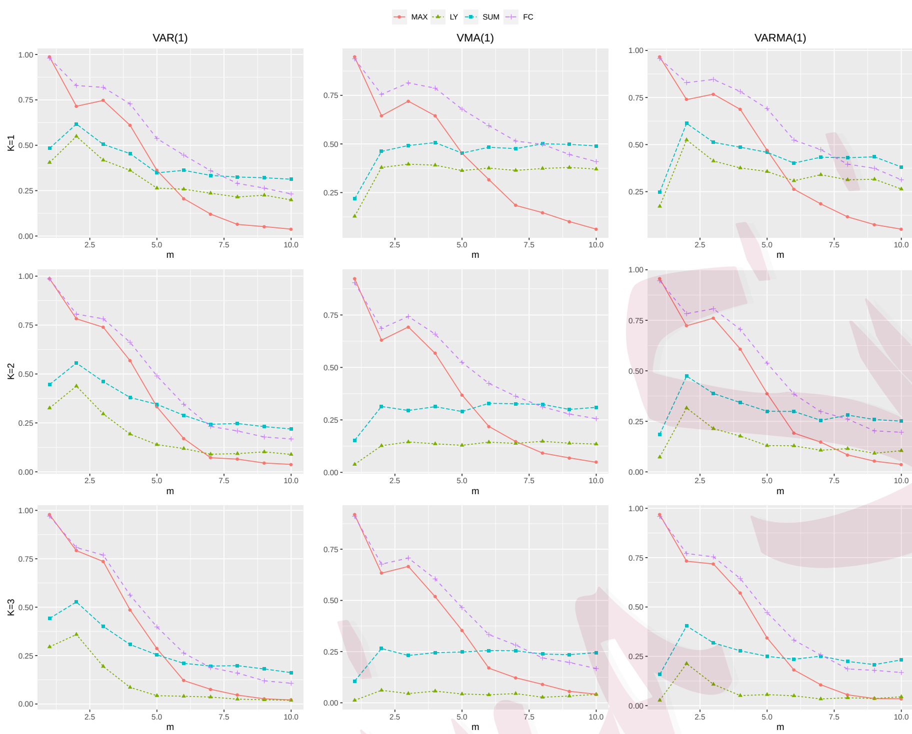

Figure 2: Power curves of the involved tests with $m = 1, 2, \cdots, 10$ and $(n, p) = (200, 90)$.

rities. The Fama-French three-factor model is one of the most popular factor pricing models in finance, which has the explicit form

$$Y_{ti} = r_{ti} - r_{ft} = \alpha_i + \beta_{i1}(r_{mt} - r_{ft}) + \beta_{i2}SMB_t + \beta_{i3}HML_t + \varepsilon_{ti}$$

for $t \in \{1, \cdots, n\}$ and $i \in \{1, \cdots, p\}$, where r_{ti} is the return of the i-th security at time t, r_{ft} is the risk free rate at time t, $Y_{ti} = r_{ti} - r_{ft}$ is the excess return of the i-th security at time t and r_{mt} is the market return at time t.

We collected the return data of the securities in the S&P 500 index and considered two forms of data compilation. First, we compiled the monthly returns on all the securities that constitute the S&P 500 index each month over the period from January 2005 to November 2018. Because the securities that make up the index change over time, we only consider $p = 374$ securities that were included in the S&P 500 index during the entire period. A total of $T = 165$ consecutive observations were obtained. The time series data on the safe rate of return, and the market factors are obtained from Ken French's data library web page. The one-month US treasury bill rate is chosen as the risk-free rate (r_{ft}). The value-weighted return on all NYSE, AMEX, and NASDAQ stocks from CRSP is used as a proxy for the market return (r_{mt}). The average return on the three small portfolios minus the average return on the three big portfolios (SMB_t), and the average return on two value portfolios minus the average return on two growth portfolios (HML_t) are calculated based on the stocks listed on the NYSE, AMEX and NASDAQ.

Second, we compiled the weekly returns on all the securities that constitute the S&P 500 index over the period from January 2005 to November 2018. The weekly data were calculated using the security prices on Fridays. Similar to the monthly data, we only considered a total of $p = 381$ stocks that were included in the S&P 500 index during the entire period. We formed a total of $T = 716$

weekly return rates for each stock during this period after excluding the Fridays that happened to be holidays.

Under these two forms of data compilation, we test the hypotheses in (2.1) using the proposed Fisher's combination test as well as its competitors, respectively. Specifically, we let $\hat{\varepsilon}_{ti} \doteq Y_{ti} - \hat{\alpha}_i - \hat{\beta}_{i1}(r_{mt} - r_{ft}) - \hat{\beta}_{i2} SMB_t - \hat{\beta}_{i3} HML_t$, where $\hat{\alpha}_i$, $\hat{\beta}_{i1}$, $\hat{\beta}_{i2}$ and $\hat{\beta}_{i3}$ are the ordinary least squares (OLS) estimators of α_i, β_{i1}, β_{i2} and β_{i3}, respectively. To demonstrate the usefulness of the proposed test, we treat the residual $\hat{\boldsymbol{\varepsilon}}_t = (\hat{\varepsilon}_{t1}, \cdots, \hat{\varepsilon}_{tp})^\top$ as the observation of $\boldsymbol{\varepsilon}_t$, instead of considering the testing problem within the Fama-French three-factor model. We assume that $\{\boldsymbol{\varepsilon}_t\}$ follows a VMA(1) process in (2.5). This assumption is partially supported by the observation that for most $j \in \{1, \cdots, p\}$, as shown in the ACF plots in Figure 3, $\{\hat{\varepsilon}_{tj}\}_{t=1}^T$ appears to have been generated by a MA(1) process.

We use the sliding window method for the subsequent application. Given a fixed length n, for each $\tau \in \{1, \cdots, T-n\}$, we implement each of the involved tests on the data compiled from the period from τ to $\tau+n-1$, where $\{\tau, \cdots, \tau+n-1\}$ is the sliding window of length n. Then, we record the rate of rejecting the null hypothesis in these $T-n$ testing results corresponding to the $T-n$ sliding windows.

Due to the great complexity and diversity of the financial market, the Fama-

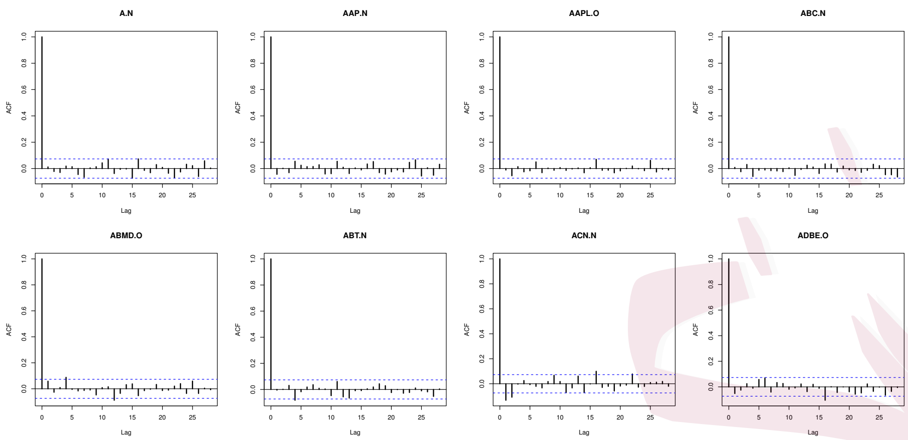

Figure 3: AFC plots of the residuals of some stocks in the S&P 500 index.

French three-factor model is only an approximation and the three included factors may often fail to accurately describe the generating mechanism of the excess returns of a large number of securities. Nevertheless, it has played an important role in pricing analysis of securities. This certainly motivates the investigation on whether a certain factor pricing model is sufficient and whether more advanced factor pricing models with more explanatory factors are needed. It is not irrational to suspect that the Fama-French three-factor model is not sufficient hence the null hypothesis may not be true, especially in the high-dimensional situations. To this end, of course a testing method with more tendency of rejection may be considered to perform better, as long as the test can control the effect size.

Table 3 summarizes the rejecting rates for each $n \in \{40, 50, 60, 70\}$ and

Table 3: Rejecting rates for the weekly and monthly data respectively.

		$K=2$				$K=3$			
n	p	MAX	LY	SUM	FC	MAX	LY	SUM	FC
		Weekly data							
40	381	0.74	0.19	0.83	0.93	0.72	0.24	0.63	0.86
50	381	0.86	0.26	0.79	0.98	0.83	0.32	0.63	0.94
60	381	0.88	0.34	0.73	0.98	0.87	0.39	0.57	0.97
70	381	0.89	0.46	0.58	0.98	0.90	0.41	0.64	0.99
		Monthly data							
40	374	0.00	0.45	0.64	0.36	0.00	0.45	0.86	0.67
50	374	0.00	0.55	0.68	0.55	0.01	0.62	0.90	0.75
60	374	0.00	0.50	0.70	0.49	0.01	0.59	0.95	0.84
70	374	0.21	0.56	0.69	0.53	0.18	0.69	0.99	0.93

each form of data compilation, where the prescribed integer $K \in \{2, 3\}$ is used to establish the proposed test statistics. It suggests that for the weekly data, FC, MAX and SUM are more inclined to reject the null hypothesis than LY, where FC is the most powerful and MAX is the second. This may be due to the stronger dependence of securities on time series of weekly data, compared with the monthly data, which may lead to some correlation matrices with larger signal strength. In such a circumstance, both the max-type and sum-type of tests can perform well, and the Fisher's combined probability test FC outperform them as a combination of them.

Compared with weekly time series, the time dependence of monthly time series is much weaker, which leads to some correlation matrices with much weaker signal strength. This may be the reason why MAX fails to deal with the monthly data, while SUM, FC and LY have good performance. In particular, SUM outperform all the remaining methods in such circumstance.

Overall, SUM, LY and MAX can only have good performance in their respective suitable situations, while FC can have robust performance in both situations.

5. Conclusion

Driven by the task of testing high-dimensional white noise, we adopt the strategy of combining independent tests of hypotheses. To this end, we first rigorously establish the asymptotic properties of an existing max-type test under both null and alternative. We then propose a new sum-type test and establish its corresponding theoretical properties. These results are established under conditions weaker than the existing literature. We then proceed to establish the independence between the max-type and sum-type statistics, and employ the well known Fisher's combination test to form a combined test. We rigorously establish the theoretical properties of the test in terms of both level and power. Through extensive numerical experiments, we demonstrate that the proposed test has clear advantages in power comparison, due to its robustness to sparsity of the serial correlation structure. Furthermore, via an empirical application, we demonstrate the robust performance of the proposed test in testing white noise of the return data of the S&P 500 securities under the Fama-French three-factor model.

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Supplementary Materials

Supplementary Material presents the technical details of Remark 2, some additional simulation results and the technical proofs.

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