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Complete List of Authors	Jilin Wu, Ruike Wu and Zhijie Xiao
Corresponding Authors	Jilin Wu
E-mails	rainforest1061@126.com
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Robust Tests for Changing Volatility

*Jilin Wu**, *Ruike Wu** *and Zhijie Xiao*†

*Xiamen University**, *Boston College*†

Abstract: This paper develops two modified CUSUM and QS tests to examine structural changes in volatility based on least absolute deviation (LAD) regression and consistent estimation of the long-run variance (LRV). We establish fairly mild conditions under which the new tests have standard null distributions and are consistent against any fixed alternatives that deviate from the null, including smooth changes, single or multiple breakpoints in volatility. In addition, the tests also have asymptotic unit powers against two classes of local alternatives approaching the null at different rates. Simulations are conducted to show better finite sample performance of the new tests relative to other popular tests especially in the presence of heavy-tailed innovations. Finally, two empirical applications to detection of structural changes in volatilities of U.S. dollar/Russian Ruble exchange rate and S&P 500 index highlight the usefulness of our tests in real datasets.

Key words and phrases: CUSUM test, Heavy-tailed innovation, Least absolute deviation, Nonparametric estimation, QS test, Structural change in volatility.

1. Introduction

Up to now, most popular macroeconomic and financial econometric models are constructed, whether explicitly or implicitly, under the assumption of global stationarity in unconditional volatility. While this assumption can help simplify inference and estimation procedures in time series analysis, it seems implausible over long periods of time since the underlying economic mechanisms are likely to be disturbed by various factors such as business cycles, institutional changes and technological progress. Numerous studies have demonstrated that structural changes in volatilities are widely existing in macroeconomic and financial data. For example, Sensier and Dijk (2002) report that about 80% of 214 USA macroeconomic time series displayed breaks in volatilities during the period 1959-1999. Kim and Nelson (1999), and Justiniano and Primiceri (2008) demonstrate that the volatilities of U.S. major macroeconomic variables, especially GDP, have declined since 1980s. Clark (2011) provides empirical evidence strongly suggesting that the volatilities of U.S. macroeconomic variables rise sharply during the severe recession of 2007–2009. Similarly, Andreou and Ghysels (2002) discover that the Asian and Russian financial crises have caused obvious structural breaks in volatility dynamics of international financial markets. Mikosch and Stărică (2004) and Liu and Maheu (2008) also find strong evidence of structural changes relating to shifts in the volatility of S&P 500 returns.

The presence of time-varying unconditional volatility could invalidate conventional statistical inference and hypothesis testing. Lamoureux and Lastrapes (1990), and Granger and Hyung (2004) prove that changes in unconditional volatilities can cause spurious persistence and long memory effects in volatility dynamics. Hamori and Tokihisa (1997), Kim et al. (2002) and Cavaliere (2005) show that ignoring the effect of a volatility shift could result in big over-sized distortion in unit root tests. Hansen (1995), Xu and Phillips (2008) and Linton and Xiao (2019) point out that time-varying unconditional variances can lead to inefficient estimation and unreliable inference in parametric and nonparametric models. Hammoudeh and Li (2008) and Groen et al. (2013) conclude that the macroeconomic predictors not allowing for structural breaks in volatilities can bring about very poor point and density forecasts. Vilasuso (2001) and Patilea and Raissi (2012) show that the presence of unconditional heteroskedasticity can damage Granger causality testing and lead to erroneous conclusions. Therefore, it is of great necessity and extreme importance to examine structural changes in volatility before formally carrying out econometric analysis.

A lot of tests have been developed to detect structural changes in volatilities in literature. One of the most widely used tests is based on the cumulative sum (CUSUM) of squared series, see Inclan and Tiao (1994), Andreou and Ghysels (2002), Cavaliere and Taylor (2008) and Xu (2008), among others. These tests differ in how they accommodate serial dependence in asset returns. Chu (1995) considers a supremum Lagrange multiplier (LM) test to check structural breaks in GARCH models. Berkes et al. (2004) construct a sequential likelihood ratio (GLR) test to detect whether the coefficients of a GARCH model keep constant over time. However, these tests often suffer from power loss because of not considering any explicit alternative information. In order to improve testing powers, Xu (2013) proposes modified CUSUM and LM tests that are built on robust estimation of the long-run variance (LRV) of squared series. Chen and Hong (2016) propose a nonparametric test for smooth structural changes in GARCH models by comparing the distance between the log likelihood of the null and that of the alternatives. Similarly, Wu and Xiao (2018) also develop a nonparametric test for structural changes in volatility, which is derived from an L_2-type test statistic and takes the form of a U-statistic. Although all of the aforementioned tests have achieved power gains due to exploiting the information of alternatives and can provide optimal performance under Gaussian conditions, they are derived by assuming the existence of at least the fourth moment. Under departure from the Gaussian models, particularly for the innovations with leptokurtic and heavy-tailed features, these testing methods can exhibit rather poor power performance. Because many applications, particularly for macroeconomic and financial data, have notoriously heavy-tailed behaviors without the fourth moment, even sometimes the second moment also does not exist, see Akgiray and Booth (1988), Loretan and Phillips (1994), Chen and Zhu (2015) and Wang et al. (2022) for more discussion. Hence, it is important to develop the tests for structural changes in volatility that are robust to departure from Gaussian conditions.

This paper proposes robust CUSUM and QS (quadratic sums) tests for structural changes in volatility that allow for various types of heavy-tailed innovations. The main contributions of this paper can be summarized as follows: Firstly, the test statistics of the interest are constructed in the framework of the LAD regression rather than the least squares (LS) regression. It is well known that the LAD is generally more robust than the LS whenever the errors have a heavy-tailed distribution. By employing the LAD testing procedure, we require no moment conditions on the innovations, and hence make the proposed tests more robust to heavy-tailed innovations. Secondly, in order to improve testing powers of the proposed tests, we suggest estimating the LRV based on the LAD nonparametric residuals.

We show that the tests constructed using such an LRV estimator not only retain correct size under the null, but also achieve decent power gains under alternatives. Thirdly, the new tests are consistent against various alternatives that deviate from the null hypothesis, and no prior information about the alternatives is required. Specifically, we do not need to know whether the structural changes in volatility are smooth or abrupt, and in the cases of abrupt breaks, we do not need to know the exact dates or the number of changepoints.

The plan of this study is organized as follows. Section 2 presents the basic CUSUM and QS tests that only make use of the LAD regression residuals under the null. In Section 3 we propose two modified CUSUM and QS tests that are built on LAD nonparametric estimation of the unknown volatility function, and then study their asymptotic distributions under the null. Next we also investigate their asymptotic power properties under a fixed alternative and two sequences of local alternatives. We report the results of Monte Carlo simulations to assess finite sample performance of our tests compared with other popular tests in Section 4. Section 5 applies the proposed tests to detect changing volatilities in two financial markets. Section 6 discusses how to extend our testing procedure to the case of multivariate volatility. The conclusion is given in Section 7. Further simulation results and mathematical proofs of the main results can be found in supplementary materials.

2. The basic tests

We consider the following model for a sequence u_t (e.g. log returns)

$$u_t = \sigma_t \varepsilon_t, t = 1, \cdots, T, \tag{2.1}$$

where $\sigma_t = \sigma(t/T)$ is a positively deterministic and bounded sequence on $[0, 1]$ functioning as a proxy for all factors that affect the long-run component of volatility; $\{\varepsilon_t\}$ is a strictly stationary mixing error process, and can be modeled by various kinds of stationary GARCH-type processes to capture short-run dynamics in conditional volatility. The approach to modeling the unconditional volatility by a deterministic function of scaled time index can be referred to, e.g., Cavaliere (2005), Xu and Phillips (2008), and Kristensen (2012). Model (2.1) also covers the popular multiplicative volatilities, see Engle and Rangel (2008), Hafner and Linton (2010), and Jiang et al. (2021) for more discussion.

The research question of this paper is to test whether there exist structural changes in σ_t over time t. Thus, the null hypotheses of interest is given by

$$\mathbb{H}_0 : \sigma_t = \sigma_0, \tag{2.2}$$

and the alternative hypothesis is $\mathbb{H}_A : \sigma_t \neq \sigma_0$. Under the null, σ_t is a constant over time; Under the alternatives, σ_t can change smoothly or abruptly. It is worth noting that the conventional tests are focusing on testing the null hypothesis of $\mathbb{H}_0 : \sigma_t^2 = \sigma_0^2$, see Xu (2008, 2013) and Wu and Xiao (2018), while we aim at testing the null hypothesis of $\mathbb{H}_0 : \sigma_t = \sigma_0$. The two nulls are essentially the same.

Let c represent the median of $|\varepsilon_t|$, and denote $g_t = c\sigma_t$ and $sgn\,(z) = I\,(z > 0) - I\,(z < 0)$, where $I\,(\cdot)$ is the indicator function, then the model (2.1) can be rewritten as

$$|u_t| = g_t + e_t, \tag{2.3}$$

where $e_t = \sigma_t\,(|\varepsilon_t| - c)$ satisfies $E\,(sgn\,(e_t)) = 0$. As a result, testing the null hypothesis of $\mathbb{H}_0 : \sigma_t = \sigma_0$ is equivalent to checking whether $g_t = g_0$ holds over time, where $g_0 = \sigma_0 c$ is a positive constant.

Under the null, the LAD estimator for g_0 is given by

$$\tilde{g} = \arg\min_{g \in R} \sum_{t=1}^{T} ||u_t| - g|\,, \tag{2.4}$$

and the LAD residual is $\tilde{e}_t = |u_t| - \tilde{g}$. To devise a statistical test for (2.2), we propose to look at the following empirical process

$$E_n = \frac{1}{\sqrt{T}} \sum_{t=1}^{n} sgn\,(\tilde{e}_t)\,, \tag{2.5}$$

where $n = 1, \ldots, T$. Under $\mathbb{H}_0$ and some regularity conditions, E_n will converge weakly to a demeaned Brownian bridge with the LRV $\omega^2 = \sum_{i=-\infty}^{\infty} \gamma(i)$, where $\gamma(i) = E[sgn(e_{t+i})\, sgn(e_t)]$ is the autocovariance functions of $sgn(e_t)$. In the presence of structural changes in σ_t, E_n will deviate away from mean zero. By using the different performance of E_n under the null and the alternatives, we can judge whether the structural changes occur in σ_t.

Since the limiting process of E_n has the variance ω^2, appropriate standardization is needed to remove this nuisance parameter. Let $\tilde{\omega}^2$ be an LRV estimator for ω^2, we may consider the following standardized empirical process:

$$SE_n = \frac{1}{\sqrt{T}\tilde{\omega}} \sum_{t=1}^{n} sgn(\tilde{e}_t). \tag{2.6}$$

Based on the Kolmogorov–Smirnoff measure and the Cramér-von Mises measure, we construct the following $CUSUM$ and QS test statistics

$$CSM_M = \max_{1 \le n \le T} |SE_n|, \ QS_M = \frac{1}{T} \sum_{n=1}^{T} (SE_n)^2. \tag{2.7}$$

Compared with those traditional $CUSUM$ and QS tests derived in the LS framework, the new tests CSM_M and QS_M as well as the modified tests CSM_M^* and QS_M^* proposed in the next section possess the merit of robustness because the fluctuation in $\{sgn(\tilde{e}_t)\}_{t=1}^{T}$ always stays between -1 and 1, and is not affected by outliers.

The above tests make use of an estimated LRV, which is proportional to the spectral density of $sgn\,(e_t)$ at zero frequency. This quantity is usually estimated by a nonparametric kernel method in the following form:

$$\tilde{\omega}^2 = \sum_{i=-T+1}^{T-1} l(i/q_T)\tilde{\gamma}(i), \tag{2.8}$$

where $\tilde{\gamma}(i) = T^{-1}\sum_{t=i+1}^{T} sgn\,(\tilde{e}_t)\, sgn\,(\tilde{e}_{t-i})$, for $i \geq 0$, and $\tilde{\gamma}(i) = \tilde{\gamma}(-i)$ for $i < 0$, $l(\cdot)$ is the kernel function, and q_T is the truncation parameter.

In order to study the asymptotic properties of CSM_M and QS_M under the null and under the alternatives, we introduce the following regularity conditions.

Assumption 1. *(i)*$\{sgn\,(|\varepsilon_t| - c)\}$ *is a strictly stationary and strong mixing process, where* c *is the median of* $|\varepsilon_t|$, *with the mixing coefficients* $\{\alpha(i)\}_{i=1}^{\infty}$ *satisfying* $\sum_{i=1}^{\infty}\alpha(i) < \infty$. *(ii) The LRV* $\xi^2 = \sum_{i=-\infty}^{\infty} E\,[sgn\,(|\varepsilon_{t+i}| - c)\, sgn\,(|\varepsilon_t| - c)]$ *is strictly positive.*

Assumption 2. (i) *The probability density function* $f_{|\varepsilon|}(\cdot)$ *of* $|\varepsilon_t|$ *satisfies* $0 < f_{|\varepsilon|}(x) < \infty$ *for any* $x \in R^+$, *and is continuous at* c, *where* c *is the median of* $|\varepsilon_t|$; *(ii) The conditional cumulative density function* $F_{|\varepsilon|,t-1}(x) = P(|\varepsilon_t| < x|\varepsilon_{t-1}, \varepsilon_{t-1}, \cdots)$ *has the first derivative* $f_{|\varepsilon|,t-1}(x)$ *almost surely with* $f_{|\varepsilon|,t-1}(c)$ *uniformly integrable and* $E(f^r_{|\varepsilon|,t-1}(c)) < \infty$ *for some* $r > 1$.

Assumption 3. *(i) The kernel function $l(\cdot)$ has bounded support on $[-1,1]$ and $|l(x)| \leq 1$ for all x on the real line, $l(x) = l(-x)$, $l(0) = 1$; $l(x)$ is continuous at zero and for almost all x. (ii) $q_T \to \infty$, $q_T/\sqrt{T} \to 0$ as $T \to \infty$.*

Assumption 1(i) ensures the functional weak convergence of partial sum process $sgn\left(|\varepsilon_t| - c\right)$ to Brownian motion with the LRV $\xi^2 = \sum_{i=-\infty}^{\infty} E\left[sgn\left(|\varepsilon_{t+i}| - c\right) sgn\left(|\varepsilon_t| - c\right)\right]$, where $0 \leq \xi^2 < \infty$, see Theorem 2.20 (ii) and Theorem 2.21(ii) of Fan and Yao (2003). To avoid the possibility of ξ^2 being zero, we further assume it to be strictly positive in Assumption 1(ii). The two LRVs ω^2 and ξ^2 are actually equal since $sgn\left(e_t\right)$ is the same as $sgn\left(|\varepsilon_t| - c\right)$. In addition, Assumption 1 does not impose any restriction on the moments of ε_t, and hence allows for various types of heavy-tailed innovations, including the case of infinite variance.

Assumption 2(i) is a general set-up for the LAD-type estimator; see, e.g., Peng and Yao (2003), Li and Li (2005), and Zhu and Ling (2015). Assumption 2(ii) defines the conditional cumulative distribution function and conditional density function for $|\varepsilon_t|$, which is also a technique requirement for the LAD regression under serial dependence, see Assumption 1(iii) of Herce (1996). Additionally, it is worthwhile to mention that although the new tests are constructed by employing the LAD method, we do not assume the data come from the distribution with zero (conditional or unconditional) median.

Assumption 3 ensures consistent estimation of ω^2 under the null, see Jansson (2002).

Theorem 1. *Suppose Assumptions 1-3 hold, then under $\mathbb{H}_0$, as $T \to \infty$ we have*

$$CSM_M \xrightarrow{d} \sup_{0<r\leq 1} \left|\underline{W}(r)\right|, QS_M \xrightarrow{d} \int_0^1 \underline{W}^2(r)dr,$$

where $\underline{W}(r) = W(r) - rW(1)$ is a Brownian bridge, and $W(\cdot)$ is the standard Brownian motion.

This theorem implies that the tests CSM_M and QS_M still converge to the classical asymptotic null distributions, and hence their critical values can be found, for example, in Andrews (1993). The proof of the asymptotic distributions here, however, is much more complicated than that in the LS framework, this is because the function $sgn(\cdot)$ is not everywhere differentiable and we proceed by treating the function $sgn(\cdot)$ as a generalized function with a smooth regular sequence $sgn_m(\cdot)$ defined on an appropriate set of test functions, see more in supplementary material.

The consistency property of the test rejecting $\mathbb{H}_0$ for large values of CSM_M and QS_M is stated in the following theorem.

Theorem 2. *Suppose that Assumptions 1-3 hold, then for any sequence of nonstochastic constants* $\left\{C_T = o\left(\sqrt{T/q_T}\right)\right\}$, *we have*

$$\Pr(CSM_M > C_T) \to 1, \Pr(QS_M > C_T^2) \to 1$$

under $\mathbb{H}_A$ *as* $T \to \infty$.

Theorem 2 shows that under the fixed alternative, CSM_M and QS_M diverge to positive infinity with the rates $\sqrt{T/q_T}$ and T/q_T respectively. This is because both the numerator E_n and the denominator $\tilde{\omega}$ in SE_n diverge to infinity under the fixed alternative as $T \to \infty$, but E_n diverges faster than $\tilde{\omega}$. In particular, $E_n = O_p\left(T^{1/2}\right)$ and $\tilde{\omega} = O_p\left(q_T^{1/2}\right)$ under $\mathbb{H}_A$. The factor q_T arises from poor estimation of the LRV ω^2 by using the null residuals $\{\tilde{e}_t\}_{t=1}^T$, which leads to $\tilde{\omega}^2 = O_p(q_T)$ under $\mathbb{H}_A$, and hence causes power loss.

3. The modified tests

In order to improve the testing power of CSM_M and QS_M under $\mathbb{H}_A$, we endeavor to consistently estimate the nuisance parameter ω^2 under both the null and the alternatives. In this section, we propose a modified inference procedure that is built on nonparametric estimation of the unknown volatility function, and can ameliorate the diagnostic strength of the tests.

Specifically, we first employ the nonparametric LAD method to estimate g_t and then obtain a robust LRV estimator $\hat{\omega}^2$ by using the nonparametric residual. Hall et al. (2002) and Ziegelmann (2008) employed the local linear LAD method to estimate the conditional mean and the conditional volatility respectively by assuming the regressors are stationary. Here we also consider the local linear LAD estimator for g_t, which is given by

$$\begin{pmatrix} \hat{g}_t \\ \hat{\varphi}_t \end{pmatrix} = \arg\min_{g,\varphi \in R^2} \sum_{s=1}^{T} \left| |u_s| - g - \left(\frac{s-t}{T}\right)\varphi \right| k\left(\frac{s-t}{Th}\right), \tag{3.1}$$

where $k(\cdot)$ is a kernel function defined on $[-1,1]$ and h is a bandwidth parameter satisfying $h \to 0$ and $Th \to \infty$ as $T \to \infty$. We denote the nonparametric LAD residual as $\hat{e}_t = |u_t| - \hat{g}_t$. Based on $\{sgn\,(\hat{e}_t)\}_{t=1}^{T}$, the LRV can be estimated by

$$\hat{\omega}^2 = \sum_{i=-T+1}^{T-1} l(i/q_T)\hat{\gamma}(i), \tag{3.2}$$

where $\hat{\gamma}(i) = T^{-1}\sum_{t=i+1}^{T} sgn\,(\hat{e}_t)\, sgn\,(\hat{e}_{t-i})$, for $i \geq 0$, and $\hat{\gamma}(i) = \hat{\gamma}(-i)$ for $i < 0$. By comparing (3.2) with (2.8), we note that $\hat{\omega}^2$ is constructed in the same way as $\tilde{\omega}^2$ only by replacing $\tilde{e}_t$ with $\hat{e}_t$ in (2.8).

To obtain asymptotic properties of $\hat{g}_t$ and $\hat{\omega}^2$, we need the following assumptions:

Assumption 4. *The positively deterministic and bounded volatility function $\sigma(\cdot)$ has continuous second derivatives on $[0,1]$.*

Assumption 5. *(i)The kernel* $k(u)$: $[-1,1] \mapsto \mathbb{R}^+$ *is a symmetric and bounded probability density function. (ii) The bandwidth* h *satisfies* $h \to 0$ *and* $1/Th \to 0$ *as* $T \to \infty$.

Assumption 6. *(i)Assumption 3(i) holds; (ii) The truncation parameter* q_T *satisfies* $q_T \to \infty$ *and* $q_T h^2 + q_T^2/Th \to 0$ *as* $T \to \infty$.

Assumption 4 imposes certain smoothness condition on the volatility function $\sigma(\cdot)$ and covers both the null and the alternatives. Although this condition rules out single or multiple breakpoints, our modified tests still have testing powers against abrupt breaks in $\sigma(\cdot)$ since broken volatility can be well approximated by certain smooth functions. Note that we can approximate the indicator function $I(t/T \geq \tau^*)$ for some given $\tau^* \in (0,1)$ by a smooth function $\psi(\cdot)$ such that $I(t/T \geq \tau^*) \approx \psi((t/T - \tau^*)/r_T)$, where $\psi(\cdot)$ behaves like a cumulative distribution function (CDF) and r_T ($r_T \to 0$) is a smoothing parameter (also see Case II of the local sharp alternative in Section 3). A popular example of $\psi(\cdot)$ is the CDF of the standard normal distribution. In the simulation part, we have considered two alternatives (i.e. DGPP.1 and DGPP.2) where abrupt breaks are allowed in $\sigma(\cdot)$. Our tests appear to work satisfactorily for them.

Assumption 5 is a standard assumption in the kernel regression literature. Under Assumption 5(i), we have $\int_{-1}^{1} k(u)du = 1$, $\int_{-1}^{1} uk(u)du = 0$

and $\int_{-1}^{1} u^2 k(u)du < \infty$.

Assumption 6(ii) imposes a more strict restriction on q_T than that in Assumption 3(ii).

Lemma 1. *Suppose that Assumptions 1-2, 4-5 hold, then for any* $\tau \in (0,1)$ *we have*

$$\sqrt{Th}\left(\hat{g}(\tau) - g(\tau) - \frac{1}{2}h^2 c\sigma''(\tau)\mu_2\right) \xrightarrow{d} N\left(0, \frac{\sigma^2(\tau)\xi^2\upsilon_2}{4f_{|\varepsilon|}^2(c)}\right), \tag{3.3}$$

where $\mu_2 = \int_{-1}^{1} u^2 k(u)du$, $\upsilon_2 = \int_{-1}^{1} k^2(u)du$ *, and* $\sigma''(\cdot)$ *is the second derivative of* $\sigma(\cdot)$.

We see from Lemma 1 that the asymptotic bias $\frac{1}{2}h^2 c\sigma''(\cdot)\mu_2$ of $\hat{g}(\cdot)$ is the same as that of the corresponding local linear LS estimator, whereas their asymptotic variances are different. In most cases, the local linear LAD estimator can be more efficient than the local linear LS one. Additionally, the bias disappears when the null holds since $\sigma''(\cdot) = 0$ under $\mathbb{H}_0$.

Lemma 2. *Suppose that Assumptions 1-2, 4-6 hold, then the corrected LRV estimator*

$$\hat{\omega}^2 \xrightarrow{p} \omega^2$$

under both the null and the alternatives.

Let $SE_n^* = \left(\sqrt{T}\hat{\omega}\right)^{-1} \sum_{t=1}^{n} sgn(\tilde{e}_t)$, then the modified CUSUM and

QS tests based on $\hat{\omega}^2$ are given by

$$CSM_M^* = \max_{1\leq n\leq T} |SE_n^*|\,,\ QS_M^* = \frac{1}{T}\sum_{n=1}^{T}(SE_n^*)^2 \tag{3.4}$$

By Lemma 2, the corrected LRV estimator $\hat{\omega}^2$ always converges to the true LRV ω^2 under both the null and the alternatives due to exploiting the nonparametric residuals. Thus the modified tests CSM_M^* and QS_M^* are expected to have correct sizes and enjoy higher testing powers. The following two theorems show that this intuition is correct.

Theorem 3. *Suppose Assumptions 1-2, 4-6 hold, then under $\mathbb{H}_0$, as $T \to \infty$ we have*

$$CSM_M^* \xrightarrow{d} \sup_{0<r\leq 1} |\underline{W}(r)|\,, QS_M^* \xrightarrow{d} \int_0^1 \underline{W}^2(r)dr,$$

where $\underline{W}(r) = W(r) - rW(1)$ is a Brownian bridge, and $W(\cdot)$ is the standard Brownian motion.

Theorem 4. *Suppose that Assumptions 1-2, 4-6 hold. Then under $\mathbb{H}_A$, for any nonstochastic sequence $C_T = o(T^{1/2})$, as $T \to \infty$ we have*

$$\Pr(CSM_M^* > C_T) \to 1, \Pr(QS_M^* > C_T^2) \to 1.$$

The two modified tests CSM_M^* and QS_M^* still converge to the null asymptotic distributions as before when the null holds, but achieve higher testing powers under alternatives since they diverge to positive infinity with the rates $T^{1/2}$ and T, faster than $\sqrt{T/\,q_T}$ and $T/\,q_T$, the rates of the two basic tests CSM_M and QS_M. This is because $\hat{\omega}^2$ always converges to the true LRV ω^2 under both the null and alternatives, which helps remove the estimated bias.

In order to study the local powers of CSM_M^* and QS_M^*, the next theorem establishes their asymptotic distributions under two sequences of local alternatives that converge to the null at different rates; namely, $\mathbb{H}_{LA}^1$ and $\mathbb{H}_{LA}^2$ given by (3.5) and (3.6) below.

Case I: Local smooth alternatives:

$$\mathbb{H}_{LA}^1 : \sigma_t = \sigma_0 + b_T \pi\left(t/T\right), \tag{3.5}$$

where $b_T \to 0$ and $\pi(\cdot)$ is a twice continuously differentiable function. The term $b_T\pi\left(t/T\right)$ characterizes the degree of departure of the smoothly-changing volatility σ_t from the null σ_0 at time t. Specifically, $\pi\left(\cdot\right)$ denotes the direction of departure, while b_T is the speed at which the departure of σ_t from σ_0 at each time point t vanishes to 0 as $T \to \infty$.

Case II: Local sharp alternatives at some point τ^*:

$$\mathbb{H}_{LA}^2 : \sigma_t = \sigma_0 + d_T \psi\left(\frac{t/T - \tau^*}{r_T}\right), \tag{3.6}$$

where τ^* is a given point in $(0,1)$, $\psi\left(\cdot\right)$ is a twice continuously differentiable function that is unknown and satisfies $\sup_{z\in R}|\psi(z)| < C$ and $\sup_{z\in R}|\psi''(z)| < C$ with $\psi''(\cdot)$ the second derivative of $\psi(\cdot)$, $d_T = d(T) \to 0$ and $r_T = r(T) \to 0$ as $T \to \infty$. This type of alternatives is also considered by Chen and Hong (2012) and Chen and Huang (2018) in different contexts and can be regarded as a type of high frequency local alternatives. Under $\mathbb{H}_{LA}^2$, the volatility function σ_t becomes a non-smooth spike at location τ^* as $T \to \infty$, due to the shrinking width parameter r_T. Here, r_T controls the sharpness of the deviation from the null around τ^*, while d_T is the speed at which the departure of σ_t from σ_0 at each t vanishes to zero.

Theorem 5. *Suppose that Assumptions 1-2, 4-6 hold. (i) Under $\mathbb{H}_{LA}^1$ with $b_T = T^{-1/2}$, as $T \to \infty$ we have*

$$CSM_M^* \xrightarrow{d} \sup_{0<r\leq 1}\left|\underline{W}(r) + \zeta_1(r)\right|, QS_M^* \xrightarrow{d} \int_0^1 \left[\underline{W}(r) + \zeta_1(r)\right]^2 dr,$$

where $\zeta_1(r) = 2cf_{|\varepsilon|}(c)\left(\int_0^r \pi(s)\,ds - r\int_0^1 \pi(s)\,ds\right)/\sigma_0\omega$.

(ii) Under $\mathbb{H}_{LA}^2$ with $d_T \to 0$, $r_T \to 0$, and $d_T r_T = T^{-1/2}$, as $T \to \infty$ we have

$$CSM_M^* \xrightarrow{d} \sup_{0<r\leq 1}\left|\underline{W}(r) + \zeta_2(r)\right|, QS_M^* \xrightarrow{d} \int_0^1 \left[\underline{W}(r) + \zeta_2(r)\right]^2 dr,$$

where $\zeta_2(r) = 2cf_{|\varepsilon|}(c)\left(\int_{-\infty}^{(r-\tau^*)/r_T} \psi(s)\,ds - r\int_{-\infty}^{\infty} \psi(s)\,ds\right)/\sigma_0\omega$.

Clearly, when $\pi(\cdot) = 0$ in Case I or $\psi(\cdot) = 0$ in Case II, the results in Theorem 5 degenerate to Theorem 4. The "non-centrality parameters" $\zeta_1(\cdot)$

and $\zeta_2(\cdot)$ represent the shifts in charge of asymptotic local powers against $\mathbb{H}^1_{LA}$ and $\mathbb{H}^2_{LA}$. Theorem 5(i) means that CSM^*_M and QS^*_M have non-trivial asymptotic powers against $\mathbb{H}^1_{LA}$ that diverge from the null hypothesis at a parametric rate of $T^{-1/2}$. On the other hand, under $\mathbb{H}^2_{LA}$, as long as $\psi(\cdot)$ is a non-zero function, the non-centrality parameter $\zeta_2(\cdot)$ is non-trivial. Then, by choosing suitable sequences of d_T and r_T, Theorem 5(ii) implies that our tests also have asymptotic powers in detecting various kinds of structural breaks in volatilities such as single or multiple breaks since the local non-smooth and sharp spikes under $\mathbb{H}^2_{LA}$ are similar to jumps or temporal structural breaks.

4. Monte Carlo simulations

Section 4.1 discusses how to choose the bandwidth h for the nonparametric estimation, and Section 4.2 then conducts a set of simulations to assess the finite sample performance of the new proposed tests.

4.1 Bandwidth selection

The choice of the bandwidth parameter h plays an important role in nonparametric estimation. For independent observations, the leave-one-out cross-validation (CV) is an attractive data-driven method for choosing the bandwidth, although it also suffers from sample noise, see Härdle et al. (1988) for detailed discussion. However, if the observations are serially dependent, then the leave-one-out CV is known to be severely affected by the dependence and will not produce good bandwidths. In order to eliminate the effect caused by serial dependence, Chu and Marron (1991) suggested using leave-$(2p+1)$-out CV to choose the bandwidth h. Define a "leave-$(2p+1)$-out" estimator as

$$\begin{pmatrix} \hat{g}_{cv,t} \\ \hat{\varphi}_{cv,t} \end{pmatrix} = \arg\min_{g,\varphi \in R^2} \sum_{s:|s-t|>p_T}^{T} \left| |u_s| - g - \left(\frac{s-t}{T}\right)\varphi \right| k\left(\frac{s-t}{Th}\right), \quad (4.1)$$

where $\hat{g}_{cv,t}$ is the same as $\hat{g}_t$ except that the summation is taken for $|s-t| > p_T$, where p_T satisfies $p_T \to \infty$ and $p_T/Th \to 0$ as $T \to \infty$.

In addition, the asymptotic mean integrated squared error (AMISE) of $\hat{g}(\cdot)$ in (3.3) is given by

$$AMISE = \frac{\upsilon_2 \xi^2 \int_0^1 \sigma^2(r)\, dr}{4Th f_{|\varepsilon|}^2(c)} + \frac{h^4 \mu_2^2 c^2 \int_0^1 (\sigma''(r))^2\, dr}{4}. \quad (4.2)$$

By minimizing the AMISE, we obtain the optimal bandwidth

$$h_{opt} = \delta \xi^{2/5} T^{-1/5}, \quad (4.3)$$

where ξ^2 is the LRV of $\{sgn(|\varepsilon_t| - c)\}_{t=1}^{T}$, and

$$\delta = \left(\frac{\upsilon_2 \int_0^1 \sigma^2(r)\, dr}{4c^2 f_{|\varepsilon|}^2(c)\, \mu_2^2 \int_0^1 (\sigma''(r))^2\, dr} \right)^{1/5}.$$

The formula(4.3) means that the optimal bandwidth h_{opt} should be proportional to $T^{-1/5}$. Consequently, a data-driven choice of h is given by

$$h_{cv} = \underset{c_1 T^{-1/5} \leq h \leq c_2 T^{-1/5}}{\arg\min} \sum_{t=1}^{T} ||u_t| - \hat{g}_{cv,t}| \tag{4.4}$$

where c_1 and c_2 are two prespecified constants.

4.2 Simulation results

In this subsection we study finite sample performance of the modified tests CSM_M^* and QS_M^*, and compare them with the basic tests CSM_M and QS_M, the standard tests LM (LM_B) and CUSUM (CSM_B), Xu's (2013) modified tests LM (LM_X) and CUSUM (CSM_X).For the tests LM_B, CSM_B,LM_X and CSM_X, we completely follow the testing procedures of Xu (2013). For nonparametric estimation of g_t in (3.1), we use the Epanechnikov kernel and let the truncation parameter $p_T = [T^{1/3}]$ in (4.1). Additionally, we set $c_1 = 0.5$ and $c_2 = 4$ in (4.4), which allows h to be chosen from a wide range. For the LRV estimation in the tests CSM_M, QS_M, CSM_M^* and QS_M^* , we choose the Bartlett kernel and specify the truncation parameter as $q_T = \left[T^{1/3}\right]$. Additionally, just as suggested by one of the referees it seems more reasonable and efficient to directly test the null $\mathbb{H}_0 : \sigma_t = \sigma_0$ in mean of the transformed data $\{|u_t|\}$, which can also help relax moment restriction on innovations. Hence, we also include two mean-based CUSUM(CSM_A)

and QS(QS_A) tests as competitors, and they are constructed in the same way as our modified tests except that the calculation are based on the least squares.

First, to investigate the size performance of our tests under $\mathbb{H}_0$, we generate 1000 data sets of random sample $\{u_t\}_{t=1}^T$ for each T = 250, 500 and 750, and consider the following data generating processes (DGPs)

$$u_t = \sigma_t \varepsilon_t,\ \varepsilon_t = \phi_t \eta_t,$$

$$\phi_t^2 = \mu_0 + \alpha_0 \varepsilon_{t-1}^2 + \beta_0 \phi_{t-1}^2,$$

where the GARCH(1,1) parameters are set as $\mu_0 = \alpha_0 = 0.1$ and $\beta_0 = 0.6$, and $\{\eta_t\}$ is an independently identically distributed (i.i.d.) random variable. We consider both normal and non-normal disturbances in our experiment, and specify four types of errors for $\{\eta_t\}$: (1) The standard normal, $N(0,1)$; (2) The student-t-distribution with 2 degrees of freedom, $t(2)$; (3) The skewed student-t-distribution with 3 degrees of freedom and the skewness parameter being -0.8, $st(3,-0.8)$; (4) The centered chi-squared distribution with 1 degrees of freedom, $x^2(1)$. We specify $\sigma_0 = 1$ for the null, and thus u_t is the stationary GARCH(1,1) process under the null.

Table 1 reports the rejection rates of all tests under the null at the 5% significance level, using asymptotic critical values. We find that the Xu's tests LM_X and CSM_X as well as the test QS_A suffer from some over-

Table 1: Empirical rejection probabilities of the tests under the null.

Error	T	LM_B	CSM_B	LM_X	CSM_X	CSM_A	QS_A	CSM_M	QS_M	CSM_M^*	QS_M^*
$N(0,1)$	250	0.061	0.049	0.112	0.116	0.062	0.072	0.037	0.050	0.060	0.071
	500	0.056	0.061	0.094	0.085	0.067	0.079	0.039	0.047	0.051	0.055
	750	0.056	0.053	0.084	0.085	0.052	0.055	0.048	0.053	0.053	0.058
$t(2)$	250	0.044	0.018	0.064	0.038	0.035	0.062	0.037	0.040	0.055	0.057
	500	0.027	0.017	0.042	0.019	0.037	0.056	0.047	0.048	0.062	0.063
	750	0.039	0.015	0.048	0.024	0.039	0.067	0.053	0.062	0.055	0.062
$st(3,-0.8)$	250	0.041	0.014	0.072	0.037	0.041	0.066	0.062	0.066	0.078	0.078
	500	0.047	0.023	0.065	0.037	0.049	0.067	0.044	0.048	0.059	0.060
	750	0.050	0.029	0.071	0.047	0.063	0.084	0.048	0.052	0.058	0.056
$x^2(1)$	250	0.066	0.026	0.105	0.069	0.068	0.086	0.059	0.062	0.063	0.065
	500	0.053	0.034	0.073	0.053	0.061	0.080	0.057	0.063	0.059	0.063
	750	0.044	0.035	0.063	0.055	0.074	0.079	0.053	0.060	0.055	0.062

sized distortion under the distributions $N(0,1)$ and $x^2(1)$. The tests CSM_B and CSM_X display some under-sized distortion under the distributions $t(2)$ and $st(3,-0.8)$. For the tests LM_B and CSM_A as well the four LAD-based tests CSM_M, QS_M, CSM_M^* and QS_M^*, they all have acceptable sizes in most cases although they also exhibit a little size distortion, and increasing sample size T seems to help improve the size performance more or less. Comparatively speaking, the sizes of the LS-based tests LM_B, CSM_A and QS_A, especially CSM_B, LM_X and CSM_X, are a little more sensitive to the types of the error distributions, and the other four LAD-based tests are more robust and have more stable size performance.

To investigate the empirical powers of the proposed tests under $\mathbb{H}_A$, we still employ the above DGPs except that we consider the following specifications for σ_t:

DGPP.1—Single structural break:

$$\sigma_t = \begin{cases} \sigma_0 & t/T \leq 0.5 \\ \sigma_0 + \sigma_1 & t/T > 0.5 \end{cases},$$

DGPP.2—Two structural breaks:

$$\sigma_t = \begin{cases} \sigma_0 & t/T < 0.3 \\ \sigma_0 + \sigma_1 & 0.3 \leq t/T \leq 0.7 \\ \sigma_0 & t/T > 0.7 \end{cases},$$

DGPP.3— Quadratic smooth structural change:

$$\sigma_t = \sigma_0 + \sigma_1 \left(t/T\right)^2,$$

DGPP.4—Non-monotonic smooth structural change:

$$\sigma_t = \sigma_0 + \sigma_1 G\left(t/T\right), G\left(t/T\right) = 1 - \exp\left[-15\left(t/T - 0.5\right)^2\right].$$

DGPP.5—Oscillating smooth structural change:

$$\sigma_t = \sigma_0 + 0.5\sigma_1(sin(2\pi t/T) + 1).$$

DGPP.1 corresponds to one abrupt change of volatility from σ_0 to $\sigma_0+\sigma_1$ at the middle time $[0.5T]$. In DGPP.2, the volatility jumps from the level of σ_0 to $\sigma_0+\sigma_1$ at time $[0.3T]$ and stays for a period before jumping back at time $[0.7T]$. DGPP.3 is a quadratic form in structural change. In DGPP.4, the volatility changes smoothly by exhibiting a symmetric 'U' shape. Lastly, the volatility in DGPP.5 oscillates between σ_0 and $\sigma_0 + \sigma_1$. Without loss of generality, we specify $\sigma_0 = \sigma_1 = 1$.

Table 2: Empirical rejection probabilities of the tests under DGPP.1.

Error	T	LM_B	CSM_B	LM_X	CSM_X	CSM_A	QS_A	CSM_M	QS_M	CSM_M^*	QS_M^*
$N(0,1)$	250	0.992	0.998	1.000	1.000	1.000	0.999	0.971	0.969	0.979	0.975
	500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	750	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$t(2)$	250	0.177	0.176	0.260	0.255	0.699	0.747	0.925	0.930	0.942	0.947
	500	0.236	0.258	0.302	0.314	0.848	0.873	0.998	0.998	0.998	1.000
	750	0.319	0.348	0.377	0.394	1.000	1.000	1.000	1.000	1.000	1.000
$st(3,-0.8)$	250	0.294	0.333	0.411	0.433	0.885	0.907	0.995	0.992	0.999	0.995
	500	0.454	0.481	0.528	0.546	0.968	1.000	1.000	1.000	1.000	1.000
	750	0.535	0.556	0.583	0.607	1.000	1.000	1.000	1.000	1.000	1.000
$x^2(1)$	250	0.348	0.422	0.521	0.577	0.934	0.942	1.000	1.000	1.000	1.000
	500	0.645	0.712	0.733	0.769	0.995	0.997	1.000	1.000	1.000	1.000
	750	0.830	0.861	0.868	0.899	0.999	0.999	1.000	1.000	1.000	1.000

Tables 2-6 report the testing powers for DGPP.1-5, which are still calculated at the 5% significance level. When the error follows the normal distribution, we find that the six LS-based tests clearly outperform the other four LAD-based tests in almost all cases. The tests LM_X and CSM_X are the best ones; the tests CSM_A and QS_A also perform very well and their powers are very close to those of LM_X and CSM_X in most cases. However, when we turn to the non-normal errors, the four LAD-based tests CSM_M, QS_M, CSM_M^* and QS_M^* come to exhibit obvious advantages over

Table 3: Empirical rejection probabilities of the tests under DGPP.2.

Error	T	LM_B	CSM_B	LM_X	CSM_X	CSM_A	QS_A	CSM_M	QS_M	CSM_M^*	QS_M^*
$N(0,1)$	250	0.111	0.139	0.757	0.777	0.622	0.718	0.329	0.273	0.505	0.458
	500	0.852	0.872	0.990	0.991	0.992	0.997	0.797	0.776	0.887	0.857
	750	0.982	0.984	0.998	0.998	1.000	1.000	0.963	0.952	0.986	0.972
$t(2)$	250	0.003	0.002	0.027	0.033	0.110	0.132	0.272	0.211	0.422	0.355
	500	0.010	0.008	0.028	0.029	0.309	0.385	0.702	0.663	0.790	0.767
	750	0.013	0.014	0.033	0.039	0.527	0.570	0.915	0.886	0.965	0.930
$st(3,-0.8)$	250	0.008	0.008	0.046	0.047	0.233	0.286	0.450	0.407	0.639	0.606
	500	0.019	0.023	0.058	0.082	0.620	0.684	0.911	0.913	0.955	0.950
	750	0.050	0.063	0.140	0.156	0.802	0.847	0.997	0.996	1.000	0.998
$x^2(1)$	250	0.004	0.008	0.036	0.056	0.239	0.322	0.678	0.689	0.839	0.814
	500	0.022	0.036	0.114	0.145	0.687	0.775	0.995	0.996	0.997	0.998
	750	0.071	0.098	0.268	0.301	0.926	0.939	1.000	1.000	1.000	1.000

Table 4: Empirical rejection probabilities of the tests under DGPP.3.

Error	T	LM_B	CSM_B	LM_X	CSM_X	CSM_A	QS_A	CSM_M	QS_M	CSM_M^*	QS_M^*
$N(0,1)$	250	0.924	0.916	0.957	0.958	0.923	0.955	0.638	0.740	0.722	0.776
	500	0.995	0.995	0.997	0.997	0.995	0.998	0.923	0.954	0.934	0.959
	750	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$t(2)$	250	0.143	0.103	0.204	0.163	0.404	0.510	0.535	0.646	0.609	0.680
	500	0.192	0.155	0.243	0.196	0.621	0.718	0.855	0.906	0.874	0.915
	750	0.241	0.191	0.277	0.234	0.736	0.808	0.947	0.972	0.963	0.977
$st(3,-0.8)$	250	0.242	0.217	0.331	0.294	0.605	0.720	0.794	0.873	0.844	0.886
	500	0.353	0.290	0.420	0.362	0.832	0.891	0.986	0.992	0.987	0.993
	750	0.430	0.397	0.486	0.455	0.911	0.939	0.999	1.000	0.999	1.000
$x^2(1)$	250	0.289	0.223	0.391	0.345	0.656	0.755	0.983	0.995	0.990	0.996
	500	0.477	0.440	0.556	0.529	0.883	0.923	0.999	1.000	0.999	1.000
	750	0.650	0.625	0.705	0.690	0.958	0.980	1.000	1.000	1.000	1.000

Table 5: Empirical rejection probabilities of the tests under DGPP.4.

Error	T	LM_B	CSM_B	LM_X	CSM_X	CSM_A	QS_A	CSM_M	QS_M	CSM_M^*	QS_M^*
$N(0,1)$	250	0.555	0.363	0.810	0.704	0.521	0.576	0.173	0.160	0.324	0.307
	500	0.837	0.659	0.946	0.915	0.813	0.890	0.458	0.511	0.613	0.641
	750	0.957	0.882	0.991	0.977	0.962	0.993	0.691	0.814	0.832	0.893
$t(2)$	250	0.133	0.066	0.202	0.115	0.189	0.189	0.153	0.120	0.251	0.214
	500	0.121	0.055	0.174	0.089	0.286	0.270	0.375	0.393	0.482	0.478
	750	0.168	0.076	0.208	0.119	0.389	0.412	0.574	0.657	0.702	0.744
$st(3,-0.8)$	250	0.171	0.073	0.260	0.165	0.291	0.282	0.276	0.262	0.467	0.458
	500	0.236	0.111	0.304	0.188	0.449	0.490	0.663	0.800	0.808	0.874
	750	0.267	0.153	0.343	0.218	0.619	0.662	0.880	0.975	0.951	0.992
$x^2(1)$	250	0.225	0.108	0.341	0.227	0.314	0.302	0.515	0.725	0.744	0.838
	500	0.312	0.174	0.426	0.274	0.509	0.526	0.958	0.999	0.986	0.998
	750	0.413	0.243	0.525	0.385	0.717	0.777	0.999	1.000	1.000	1.000

Table 6: Empirical rejection probabilities of the tests under DGPP.5.

Error	T	LM_B	CSM_B	LM_X	CSM_X	CSM_A	QS_A	CSM_M	QS_M	CSM_M^*	QS_M^*
$N(0,1)$	250	0.857	0.896	0.938	0.962	0.943	0.937	0.713	0.734	0.771	0.769
	500	0.992	0.995	0.997	0.998	0.997	0.997	0.950	0.949	0.963	0.958
	750	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.996	0.999	0.998
$t(2)$	250	0.088	0.088	0.146	0.139	0.375	0.429	0.601	0.598	0.645	0.643
	500	0.112	0.130	0.161	0.157	0.575	0.619	0.901	0.894	0.912	0.906
	750	0.152	0.165	0.200	0.202	0.766	0.781	0.980	0.979	0.985	0.982
$st(3,-0.8)$	250	0.139	0.147	0.226	0.250	0.620	0.654	0.851	0.864	0.890	0.893
	500	0.234	0.273	0.307	0.337	0.826	0.859	0.990	0.989	0.991	0.990
	750	0.295	0.354	0.378	0.412	0.923	0.915	1.000	1.000	1.000	1.000
$x^2(1)$	250	0.176	0.196	0.291	0.318	0.675	0.717	0.995	0.995	0.997	0.997
	500	0.347	0.410	0.453	0.511	0.900	0.902	1.000	1.000	1.000	1.000
	750	0.496	0.566	0.588	0.649	0.964	0.969	1.000	1.000	1.000	1.000

the six LS-based tests. Specifically, the two modified tests CSM_M^* and QS_M^* always enjoy higher testing powers than the corresponding basic tests CSM_M and QS_M all the time, this is because the modified ones are constructed by using information of the alternatives, so no doubt they are expected to have better power performance in finite samples. Additionally, although the tests CSM_A and QS_A are inferior to the four LAD-based tests when the errors are non-normally distributed, they are the best among the six LS-based ones. After all, they only require the existence of the second moment, and hence are also robust to heavy-tailed innovations.

To confirm that the proposed tests exhibit monotonic powers when the deviation from the null is increased, we assume the empirical powers of all tests to be a function of σ_1. Here we keep all specifications unchanged as before except that we let $T = 250$ (since the results for $T = 500, 750$ are similar). The resulting power functions of DGPP.1-5 with four types of error distributions are plotted in Figure 1. The rows denote the different specifications for error distributions, and the columns represent the different alternatives for σ_t. From Figure 1, it appears that the power functions of the ten tests except LM_B and CSM_B all enjoy monotonic powers for the different distributions and the different alternatives although they exhibit different climbing rates. Comparatively speaking, when the errors are nor-

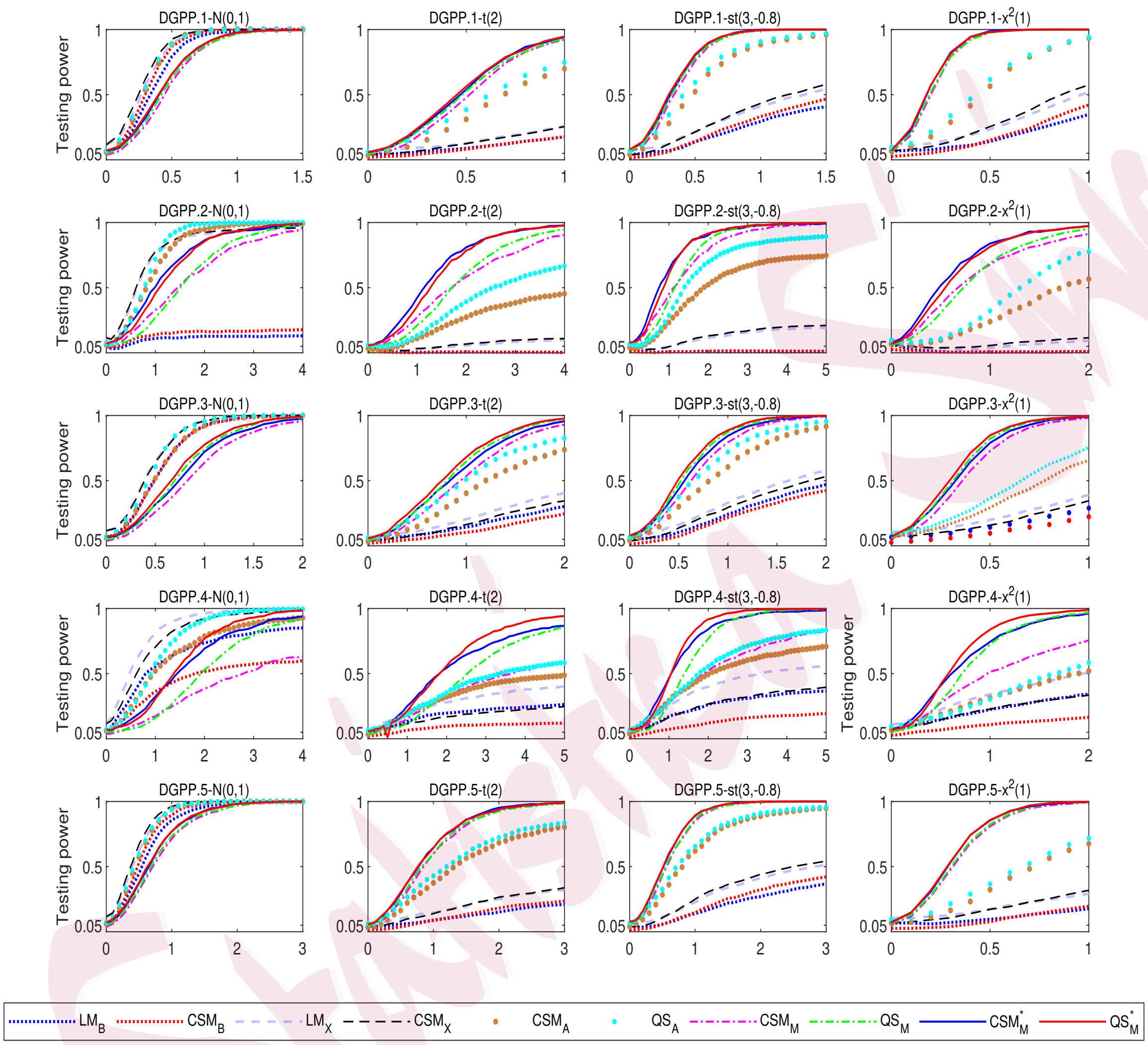

Figure 1: Empirical power curves of all tests under DGPP.1-5 with different distributions.

maly distributed the tests CSM_M^* and QS_M^* perform well although they are still inferior to Xu's tests LM_X and CSM_X as well as the tests CSM_A and QS_A. When the errors are non-normally distributed, CSM_M^* and QS_M^* have much faster climbing-up rates than the other tests. In addition, we note that the tests CSM_M and QS_M also display excellent performance under DGPP.1-5.

In the supplemental material we also examine residual-based tests for changing volatility when u_t is not directly observable and is estimated by the regression residuals. Suppose that the time series follows an $AR(1)$ model: $y_t = 0.1 + 0.5y_{t-1} + u_t$, where u_t is still generated by the model (2.1). The whole Monte Carlo experiment designs for u_t are the same as before. To test for changing volatility, we first estimate the $AR(1)$ model by the OLS regression and obtain the estimated residuals $\hat{u}_t$. Then the residual-based tests are constructed by employing $\hat{u}_t$ to replace u_t. We find the effects of estimating u_t are almost negligible, and the residual-based tests have very similar finite sample performance under both $\mathbb{H}_0$ and $\mathbb{H}_A$ as if u_t were observed.

Additionally, we also study the time-varying coefficient $GARCH(1,1)$ model for u_t:

$$u_t = f_t\eta_t,\ f_t^2 = \mu_t + \alpha_t\varepsilon_{t-1}^2 + \beta_t f_{t-1}^2,$$

where $\{\eta_t\}$ is specified the same as before, allowing for four types of errors, and the coefficients μ_t, α_t and β_t are time dependent under alternatives. Although this time-varying coefficient GARCH model does not directly fit in our testing framework, our simulation results show that our proposed tests also have testing powers against this sort of alternatives. For simplicity, we only consider one break occurring at the middle time in the following two DGPs:

DGPP.6: Structural break in the ARCH coefficient α_t:

$$\mu_t = 0.1, \alpha_t = \begin{cases} 0.1 & t/T \leq 0.5 \\ 0.3 & t/T > 0.5 \end{cases}, \beta_t = 0.6,$$

DGPP.7: Structural break in the GARCH coefficient β_t:

$$\mu_t = 0.1, \alpha_t = 0.1, \beta_t = \begin{cases} 0.4 & t/T \leq 0.5 \\ 0.8 & t/T > 0.5 \end{cases},$$

The testing powers for DGPP.6-7 are reported in Tables 7- 8. We find that all tests have powers to detect the structural break in the GARCH coefficients α_t and β_t. Under the normal distribution, the four LAD-based tests are still a little inferior to the six LS-based tests in most cases. On the contrary, under the non-normal distributions, the former four tests obviously outperform the latter six ones, and the proposed tests CSM_M^* and QS_M^* always enjoy highest testing powers in all tests considered. These testing results are also consistent with those findings in Tables 2-6.

Table 7: Empirical rejection probabilities of the tests under DGPP.6.

Error	T	LM_B	CSM_B	LM_X	CSM_X	CSM_A	QS_A	CSM_M	QS_M	CSM_M^*	QS_M^*
$N(0,1)$	250	0.559	0.601	0.715	0.756	0.493	0.573	0.485	0.489	0.523	0.512
	500	0.809	0.855	0.880	0.912	0.766	0.809	0.746	0.753	0.766	0.757
	750	0.904	0.921	0.940	0.957	0.914	0.927	0.884	0.872	0.891	0.876
$t(2)$	250	0.058	0.031	0.105	0.057	0.088	0.112	0.078	0.093	0.098	0.105
	500	0.045	0.021	0.072	0.042	0.095	0.144	0.126	0.135	0.139	0.146
	750	0.047	0.031	0.076	0.040	0.110	0.148	0.158	0.167	0.173	0.174
$st(3,-0.8)$	250	0.098	0.062	0.144	0.115	0.123	0.173	0.213	0.240	0.245	0.267
	500	0.113	0.071	0.153	0.125	0.149	0.216	0.339	0.359	0.357	0.366
	750	0.120	0.079	0.157	0.134	0.194	0.262	0.432	0.426	0.469	0.444
$x^2(1)$	250	0.133	0.090	0.216	0.193	0.280	0.301	0.490	0.498	0.508	0.523
	500	0.177	0.142	0.232	0.214	0.403	0.439	0.725	0.722	0.734	0.726
	750	0.228	0.210	0.302	0.278	0.546	0.578	0.816	0.815	0.823	0.818

Table 8: Empirical rejection probabilities of the tests under DGPP.7.

Error	T	LM_B	CSM_B	LM_X	CSM_X	CSM_A	QS_A	CSM_M	QS_M	CSM_M^*	QS_M^*
$N(0,1)$	250	0.996	0.999	1.000	1.000	0.999	0.997	0.980	0.981	0.985	0.982
	500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	750	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$t(2)$	250	0.163	0.159	0.247	0.250	0.672	0.744	0.854	0.870	0.888	0.891
	500	0.249	0.249	0.324	0.305	0.853	0.874	0.996	0.994	0.996	0.994
	750	0.304	0.326	0.377	0.384	0.921	0.930	1.000	1.000	1.000	1.000
$st(3,-0.8)$	250	0.343	0.362	0.467	0.477	0.822	0.868	0.991	0.989	0.993	0.992
	500	0.505	0.517	0.590	0.605	0.940	0.947	1.000	1.000	1.000	1.000
	750	0.599	0.622	0.678	0.672	0.976	0.986	1.000	1.000	1.000	1.000
$x^2(1)$	250	0.472	0.525	0.633	0.664	0.967	0.972	1.000	1.000	1.000	1.000
	500	0.720	0.768	0.810	0.838	0.999	0.997	1.000	1.000	1.000	1.000
	750	0.862	0.880	0.901	0.921	0.999	0.998	1.000	1.000	1.000	1.000

To sum up, our simulation results suggest that the two modified tests CSM_M^* and QS_M^* exhibit reasonable size accuracy for both normal and non-normal errors. When the data are characterized by heavy-tailed distributions, they also have all-around better powers in detecting various types of changing volatilities than other popular tests.

5. Real data applications

In this section we compare the modified tests CSM_M^* and QS_M^* with six other tests LM_B,CSM_B, LM_X, CSM_X, CSM_A, QS_A, CSM_M and QS_M in detecting whether there exist structural changes in the volatilities of daily U.S. dollar/Russian Ruble exchange rate during Russia-Ukraine war and of daily S&P 500 index during COVID-19. The whole testing procedures of building these tests completely follow the simulation section.

5.1 U.S. dollar/Russian Ruble exchange rate

In this subsection, we consider the log return ($\times 100\%$) of U.S. dollar/Russian Ruble exchange rate over the period September 1, 2021 to August 31, 2022 with a total of 255 observations $\{r_t\}_{t=1}^{255}$. Figure 2(a) displays the time plot of the absolute values of $\{r_t\}_{t=1}^{255}$ as the proxies for volatilities and Figure 2(b) is the plot of the absolute values of AR(1) residuals $\{e_t\}_{t=1}^{254}$

based on the autoregressive model with lag order one. Visual inspection of Figure 2(a)-(b) reveals that the exchange rate return experienced a highly volatile period, and exhibited completely different dynamic characteristics since the middle time of the samples, which coincided with Russian invasion of Ukraine that started on February 24, 2022. So, any reasonable tests are expected to find such abnormal dynamics in volatility due to outbreak of the war. By simple calculation we have the skewness 2.571 and the kurtosis 22.53 for $\{r_t\}_{t=1}^{255}$, and the skewness 2.004 and the kurtosis 21.73 for $\{e_t\}_{t=1}^{254}$. Figure 3(a)–(b) also plot the kernel densities of $\{r_t\}_{t=1}^{255}$ and $\{e_t\}_{t=1}^{254}$ with their corresponding normal densities. By comparison, we know that the exchange rate sequences obviously deviate away from the normal distribution. Hence it is more suitable to use the LAD-based tests in this case.

	LM_B	CSM_B	LM_X	CSM_X	CSM_A	QS_A	CSM_M	QS_M	CSM_M^*	QS_M^*
Return($T = 255$)	5.549	1.177	6.995	1.321*	1.885***	0.813***	2.042***	1.265***	2.228***	1.640***
Residual($T = 254$)	5.299	1.149	6.625	1.284*	1.822***	0.747***	1.973***	1.107***	2.240***	1.427***

Table 9: The testing results for U.S. dollar/Russian Ruble exchange rate based on the original series and AR(1) residuals with * and *** representing 10% and 1% significance levels respectively.

Table 9 reports the testing results for the null hypothesis. As a by-product of our testing procedure, the LAD nonparametric estimate of g_t

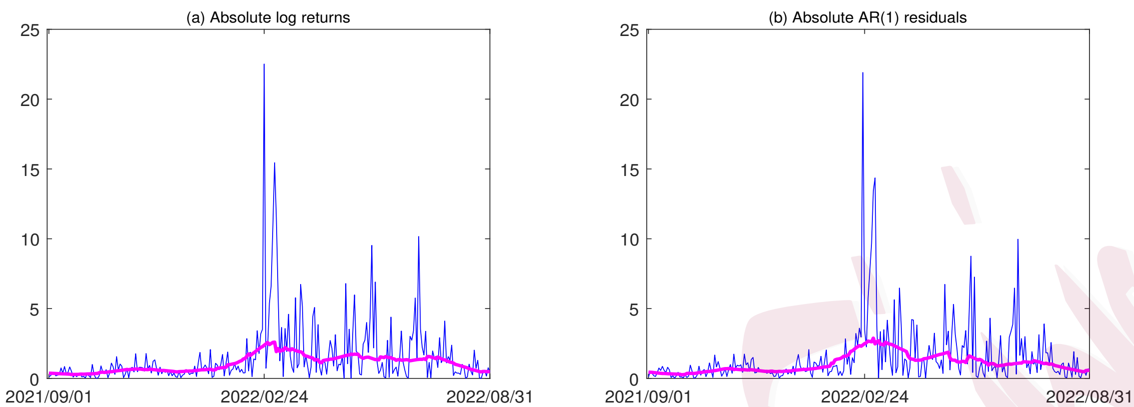

Figure 2: Absolute log returns, absolute AR(1) residuals, and nonparametric volatility estimate (thick solid lines) for U.S. dollar/Russian Ruble exchange rate.

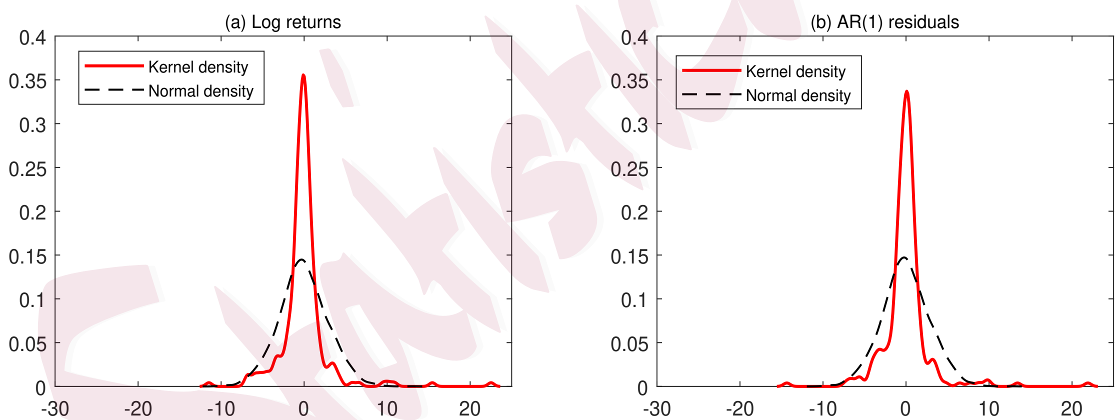

Figure 3: The kernel densities of log returns and AR(1) residuals and the normal densities with the corresponding same means and variances.

that is proportional to the deterministic volatility σ_t is also plotted as a thick solid line in Figure 2(a)-(b). From Table 9, we find that the four LAD-based tests reject the null at 1% significance level by using either the log returns or the AR(1) residuals. Of course, the two modified tests CSM_M^* and QS_M^* have bigger values than the corresponding tests CSM_M and QS_M, respectively. The test CSM_X of Xu (2013) is only significant at 10% level, and his test LM_X as well as the conventional tests LM_B and CSM_B can not reject the null even at 10% significance level. We also find that the tests CSM_A and QS_A suggested by one of the referees also work well, and reject the null at 1% level, but their statistical values are smaller than those of CSM_M and QS_M, let alone the values of CSM_M^* and QS_M^* .This result manifests the robustness of the proposed tests for heavy-tailed financial data. As a consequence, we draw the conclusion that the volatility of the U.S. dollar/Russian Ruble exchange rate return experienced structural changes during September, 2021 to August, 2022.

5.2 S&P 500 index

The second example is to apply our proposed tests to detect structural changes in volatility of daily S&P 500 index during the first year of the outbreak of COVID-19. The data cover the period from November 1, 2019 to October 29, 2020 with 252 daily observations $\{R_t\}_{t=1}^{252}$, which is calculated by taking logdifference ($\times 100\%$) of the daily closing price index. Similarly, Figure 4(a)-(b) give the time plots of absolute returns $\{R_t\}_{t=1}^{252}$ and the absolute AR(1) residuals $\{e_t\}_{t=1}^{251}$. It is obvious that the two series suffer big volatilities after March, 2020. Similarly, we also calculate the skewness and the kurtosis, with -0.860 and 11.83 for $\{R_t\}_{t=1}^{252}$ and -1.336 and 11.41 for $\{e_t\}_{t=1}^{251}$. Figure 5(a)-(b) then plot the kernel densities of $\{R_t\}_{t=1}^{252}$ and $\{e_t\}_{t=1}^{251}$ as well as their corresponding normal densities. All evidence demonstrates that the S&P 500 index follows a non-gaussian distribution.

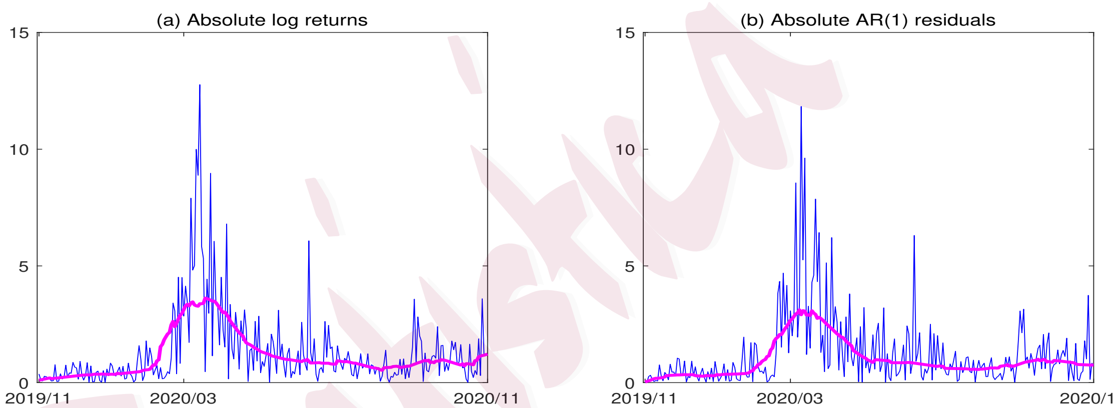

Figure 4: Absolute log returns, absolute AR(1) residuals, and nonparametric volatility estimate (thick solid lines) for S&P 500 index

Now we formally examine whether there exist structural changes in volatility of daily stock returns during the period of COVID-19. Table 10

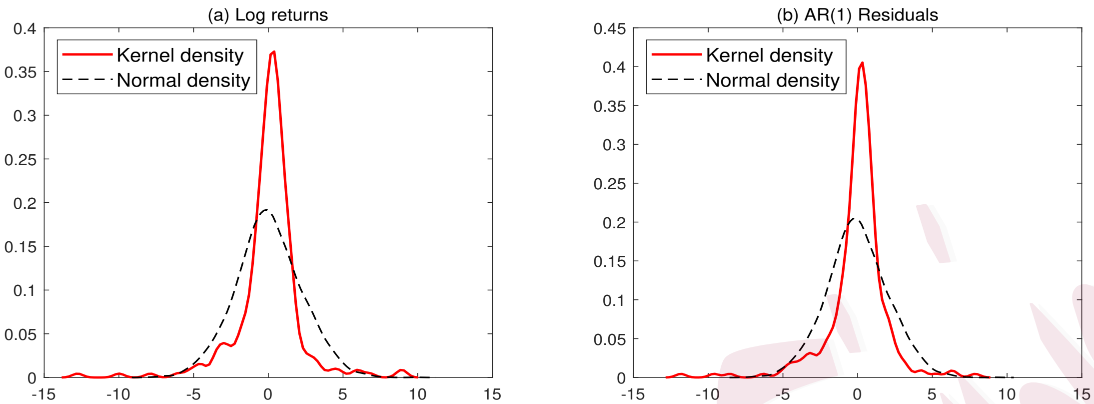

Figure 5: The kernel densities of log returns and AR(1) residuals and the normal densities with the corresponding same means and variances.

shows that the four LAD-based tests LM_B, CSM_B, LM_X and CSM_X are all statistically insignificant even at 10% significance level whether the original series or AR residuals are used. The two tests CSM_A and QS_A reject the null only at 10% significance level. In contrast, the four LAD-based tests CSM_M, QS_M, CSM_M^* and QS_M^* all reject the constant volatility at 1% significance level.

The above two examples demonstrate that the presence of heavy tails in empirical data can invalidate the traditional CUSUM and QS tests as well as the modified tests of Xu(2013). Although the two tests suggested by one of the referees also work well and have some powers to detect structural changes in volatilities, our proposed LAD-based tests , especially the

	LM_B	CSM_B	LM_X	CSM_X	CSM_A	QS_A	CSM_M	QS_M	CSM_M^*	QS_M^*
Return($T = 252$)	3.321	0.903	4.755	1.080	1.234*	0.385*	2.000***	0.796***	2.287***	0.996***
Residual($T = 251$)	3.635	0.950	5.612	1.180	1.319*	0.410*	1.877***	0.613***	2.091***	0.765***

Table 10: The testing results for S&P 500 index based on the original series and AR(1) residuals with * and *** representing 10% and 1% significance levels respectively.

two modified ones, are more robust and more powerful under non-normal distributions.

6. Extension to multivariate volatility

As pointed out by one of the referees, it is important to consider testing for structural changes in multivariate volatility. This section extends univariate volatility models to multivariate ones. For simplifying exposition, some notations are introduced first. For an $m \times 1$ vector $\alpha = (\alpha_1, \ldots, \alpha_m)'$,$\|\alpha\|_1 = \sum_{i=1}^{m} |a_i|$. For a $d \times d$ matrix A, $A^{(i,j)}$ denotes its $(i, j)\, th$ element, $vech(A)$ is a vector obtained by stacking all columns of the lower triangular part of A, and $\boldsymbol{sgn}\,(A)$ denotes calculating every element $A^{(i,j)}$ in A with the function $sgn\,(\cdot)$.

Let $u_t = (u_{1t}, \ldots, u_{dt})'$, $t = 1, \ldots, T$, be a d-dimensional vector of random variables. The multivariate volatility model is given by

$$u_t = \Sigma_t^{1/2}\varepsilon_t, t = 1, \cdots, T, \tag{6.1}$$

where $\Sigma_t = \Sigma(t/T)$ is assumed to be an unknown $d \times d$ positive definite and deterministic covariance matrix of the scaled time t/T, and $\varepsilon_t = (\varepsilon_{1t}, \cdots, \varepsilon_{dt})'$ is a $d \times 1$ strictly stationary and strong mixing error process. Here we are interested in testing the null hypothesis

$$\mathbb{H}_0 : \Sigma_t = \Sigma_0, \tag{6.2}$$

and the alternative is $\mathbb{H}_A : \Sigma_t \neq \Sigma_0$. Let C represent the median of $\varepsilon_t\varepsilon_t'$ and denote $G_t = \Sigma_t^{1/2} C \Sigma_t^{1/2}$, then the model (6.1) can be rewritten as

$$\boldsymbol{u}_t\boldsymbol{u}_t' = G_t + e_t, \tag{6.3}$$

where $e_t = \Sigma_t^{1/2}(\varepsilon_t\varepsilon_t' - C)\Sigma_t^{1/2}$ satisfies $E(\boldsymbol{sgn}(e_t)) = \mathbf{0}$. As a result, testing the null hypothesis of $\mathbb{H}_0 : \Sigma_t = \Sigma_0$ is equivalent to checking whether $G_t = G_0$ holds over time, where $G_0 = \Sigma_0^{1/2} C \Sigma_0^{1/2}$ is a positive definite constant matrix.

Under the null, the LAD estimator for G_0 is given by

$$\tilde{G} = \arg\min_{G \in R^{d\times d}} \sum_{t=1}^{T} \left\| vech\left(\boldsymbol{u}_t\boldsymbol{u}_t' - G\right)\right\|_1, \tag{6.4}$$

and the LAD residual is $\tilde{e}_t = \boldsymbol{u}_t\boldsymbol{u}_t' - \tilde{G}$. Define the following empirical process

$$\tilde{S}_n = \frac{1}{\sqrt{T}} \sum_{t=1}^{n} vech \left[\boldsymbol{sgn} \left(\tilde{e}_t \right) \right], \tag{6.5}$$

where $n = 1, \ldots, T$. Under $\mathbb{H}_0$, the fluctuation in $\{vech \left[\boldsymbol{sgn} \left(\tilde{e}_t \right) \right]\}_{t=1}^{T}$ is stable around zero vector, and $\tilde{S}_n$, under some regularity conditions, converges weakly to a multivariate Brownian bridge with the LRV $\Omega = \sum_{i=-\infty}^{\infty} \Upsilon(i)$ with $\Upsilon(i) = E\left[vech \left[\boldsymbol{sgn} \left(\boldsymbol{e}_t \right) \right] \left(vech \left[\boldsymbol{sgn} \left(\boldsymbol{e}_{t-i} \right) \right] \right)' \right]$. In the presence of structural changes in Σ_t, $\tilde{S}_n$ will deviate away from mean zero vector. Since the limiting process of $\tilde{S}_n$ has the LRV Ω, appropriate standardization is needed to remove it. Let $\tilde{\Omega}$ be an LRV estimator for Ω, we may consider the multivariate versions of the CUSUM and QS test statistics as follows:

$$MS_M = \max_{1 \le n \le T} \tilde{S}_n' \tilde{\Omega}^{-1} \tilde{S}_n, MQ_M = \frac{1}{T} \sum_{n=1}^{T} \tilde{S}_n' \tilde{\Omega}^{-1} \tilde{S}_n, \tag{6.6}$$

where $\tilde{\Omega} = \sum_{i=-T+1}^{T-1} l(i/q_T) \tilde{\Upsilon}(i)$, $\tilde{\Upsilon}(i) = T^{-1} \sum_{t=i+1}^{T} vech \left[\boldsymbol{sgn} \left(\tilde{e}_t \right) \right] \left(vech \left[\boldsymbol{sgn} \left(\tilde{e}_{t-i} \right) \right] \right)'$ for $i \ge 0$, and $\tilde{\Upsilon}(i) = \tilde{\Upsilon}'(-i)$ for $i < 0$, $l(\cdot)$ is the kernel function, and q_T is the truncation parameter.

Similarly, the LRV estimator $\tilde{\Omega}$ is not consistent for Ω under alternatives, and may lead to power loss in our testing procedure. In order to improve testing powers, we can construct the modified multivariate CUSUM and QS tests that are built on nonparametric estimation of G_t. The local linear LAD estimator for G_t , denoted as $\hat{G}_t$, is obtained by minimizing the following target function

$$\sum_{s=1}^{T} \left\| vech\left(\boldsymbol{u}_s \boldsymbol{u}_s' - G - \left(\frac{s-t}{T}\right) G^* \right) \right\|_1 k\left(\frac{s-t}{Th}\right), \tag{6.7}$$

and the LAD nonparametric residual, say $\hat{e}_t = \boldsymbol{u}_t \boldsymbol{u}_t' - \hat{G}_t$, is obtained as well. Then the two modified test statistics are given by

$$MS_M^* = \max_{1 \le n \le T} \tilde{S}_n' \hat{\Omega}^{-1} \tilde{S}_n, MQ_M^* = \frac{1}{T} \sum_{n=1}^{T} \tilde{S}_n' \hat{\Omega}^{-1} \tilde{S}_n, \tag{6.8}$$

where $\hat{\Omega}$ is estimated in the same way as $\tilde{\Omega}$ except that we replace the null residual $\tilde{e}_t$ by the nonparametric residual $\hat{e}_t$.

Under certain assumptions, we can derive the asymptotic distributions of MS_M, MQ_M, MS_M^* and MQ_M^* under the null and under the alternatives, whose proofs are beyond the scope of the present paper, and are left for future researches.

7. Conclusion

This paper proposes two modified CUSUM and LM tests to examine structural changes in volatility. The two tests are constructed in the framework of LAD regression so that they are robust to various kinds of heavy-tailed innovations and outliers. In addition, we also utilize information of the alternatives to estimate the long-run variance so that the two modified tests enjoy higher testing powers compared with the tests that are built only using information of the nulls. The asymptotic results demonstrate that the new tests weakly converge to the supremum of the absolute value of a Brownian bridge and the integral of a squared Brownian bridge respectively under the null, and are consistent against any fixed alternatives that deviate from the null. They also have nontrivial asymptotic powers against two sequences of local alternatives. Monte Carlo simulations indicate that our proposed tests have both acceptable sizes and all-around good powers in finite samples, obviously outperforming the tests built on the LS regression if the errors are heavy-tailed and skewed. So the proposed tests can be used as effective supplements to existing tests when the data are characterized by non-Gaussian distributions. Two empirical examples concerning detection of the structural changes in volatilities of U.S. dollar/Russian Ruble exchange rate and S&P 500 index also illustrate the usefulness of our testing methods in real datasets. One possible extension is to generalize our proposed tests to the context of multivariate volatility, which is left for future research.

Supplementary Material

The online Supplementary Material contains two Appendices S1 and S2, where Appendix S1 provides the technique details of Theorem 1-5 and

Lemma 1-2, and Appendix S2 gives some additional simulation results when u_t is not directly observable.

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Department of Finance, School of Economics, Gregory and Paula Chow Institute for Studies in Economics, and Wang Yanan Institute for Studies in Economics (WISE), Xiamen University

E-mail: rainforest1061@126.com

Department of Finance, School of Economics, Xiamen University

E-mail: ruikewu@foxmail.com

Department of Economics, Boston College

E-mail: xiaoz@bc.edu