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How Sensitive are Tail-related Risk Measures in a Contamination Neighbourhood?

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Abstract: **Contents of the Abstract. Estimation or mis-specification errors in the portfolio return distribution can have a considerable impact on risk measures. This paper investigates the sensitivity of tail-related risk measures including the Value-at-Risk, expected shortfall and the expectile-quantile transformation level in an epsilon-contamination neighbourhood. The findings give the different approximations according to the tail heaviness of the contamination models and its contamination levels. Illustrating examples and an empirical study on the Royalton CRIX capturing and displaying the market movements are given. The codes used to obtain the results in this paper are available via QuantLet/SRMC.**

Key words and phrases: Sensitivity, expected shortfall, expectile, risk measures, Royalton CRIX.

1. Introduction

Risk measures are used for both financial institutions' internal risk management and external regulation (e.g., in the Basel Accord for risk-based requirements for regulatory capital [Chernobai et al., 2008]. Both academics and practitioners are devoted to developing appropriate risk measures with good properties, including robustness, elicitability, and backtesting [He et al., 2022, Gneiting, 2011]. Note that risk measures are defined as functionals of the unknown porfolio loss distributions, and the particular difficulty in measuring risk is that the tail part of a loss distribution bears substantial model uncertainty. On the other hand, estimation or mis-specification errors in the portfolio loss distribution can have a considerable impact on risk measures, and it is important to examine the robustness of risk measures to these errors [Bernard et al., 2022, Cont et al., 2010]. In this work, we focus on the three most common risk measures, Value-at-Risk, expectile and expected shortfall, and examine how the tail of these risk measures varies in the contamination field in terms of Huber [1964].

Value-at-Risk (VaR) measures the maximum loss which is not exceeded with a given high probability. Given a portfolio return $X \sim F$ and its quantile function at probability level $\alpha \in (0, 1)$ as

$$q_\alpha = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\},$$

we define parallelly the VaR as

$$\mathrm{VaR}_\alpha = -q_\alpha, \quad \alpha \in (0, 1/2),$$

i.e., the maximum loss incurred by the portfolio not exceeded with a given high probability $100(1-\alpha)\%$. Although VaR has become some sort of standard measure of financial market risk, it has been criticized for reporting only a tail probability, and thus neglecting effects like the amount of loss beyond the quantile. Additionally, VaR does not take diversification and risk aggregation effects into account. Expected shortfall (ES), a natural coherent alternative to VaR, overcomes these weaknesses, is becoming increasingly used in financial risk management, Artzner et al. [1999], Delbaen [2002]. Specifically, ES is defined as the conditional expectation of the portfolio given that its return is smaller than a certain value at some probability level, i.e.,

$$ES_\alpha = -\,\mathsf{E}\left[X|X < q_\alpha\right] = -\frac{1}{\alpha}\int_{-\infty}^{q_\alpha} x \, \mathrm{d}F(x) = -\frac{1}{\alpha}\int_0^\alpha q_\beta \, \mathrm{d}\beta \tag{1.1}$$

provided that the underlying distribution function F is continuous. Cont et al. [2010] pointed out that ES appears to lack robustness with respect to small changes in the underlying cdf. The recent contribution by Mihoci et al. [2021] provides evidence on expected shortfall robustness through its link with expectile, which is given as the minimizer of the expected value of an asymptotic piece-wise quadratic loss [Newey and Powell, 1987]:

$$e_\alpha = \arg\min_\theta \mathsf{E}\left[|\alpha - \mathbf{I}\{X - \theta < 0\}||X - \theta|^2\right],$$

where $\mathbf{I}\{\cdot\}$ stands for the indicator function. It is well-known that the expectile is the only coherent risk measure possessing elicitability, a desirable property for model selection, generalized regression, forecast ranking and comparative backtesting, Nolde et al. [2017], Xu et al. [2018]. Further, the expectile is the so-called index of prudentiality in financial set-up, i.e., the amount of money added to a position with a pre-specified, sufficiently high gain-loss ratio, Bellini and Di Bernardino [2017], Daouia et al. [2017].

To find the expectile-quantile transformation level is practically useful for the regulators to set a proper level of quantile with the extreme loss taken into account, Kuan et al. [2009], Borke and Härdle [2018]. Specifically, using Jones [1994] it is not hard to verify that the expectile is obtainable through a one-to-one mapping with VaR. In other words, if $e_{w_\alpha} = q_\alpha$ for some α given, then the corresponding level w_α is such that [Yao and Tong, 1996]

$$w_\alpha = \frac{LPM_\alpha - q_\alpha \alpha}{2(LPM_\alpha - q_\alpha \alpha) + q_\alpha - \mathsf{E}[X]}. \tag{1.2}$$

Here $LPM_\alpha = \int_{-\infty}^{q_\alpha} x \, \mathrm{d}F(x)$ stands for the lower partial moment at α quantile. As a consequence, we get an alternative expression of ES as follows [Taylor, 2008, Mihoci et al., 2021].

$$ES_\alpha = -\left(q_\alpha + \frac{q_\alpha - \mathsf{E}[X]}{1 - 2w_\alpha} \frac{w_\alpha}{\alpha} \right). \tag{1.3}$$

The aforementioned risk measures are widely useful in risk management and receive continuous attention in financial practice in the determination of insurance premium and economic capital etc. However, financial asset returns and fundamental factor exposure data often contain outliers, observations that are inconsistent with the majority of the data. One might then be interested in the contamination case as follows.

$$F_\epsilon(x) = (1-\epsilon)F(x) + \epsilon H(x), \quad x \in \mathbb{R},\ \epsilon \in [0,1], \tag{1.4}$$

where ϵ reflects the amount of uncertainty in F, and H represents plausible deviations from F. Note that (1.4) is a flexible mixture model if one takes H as another mixture model.

Huber [1964] initially employed (1.4) for the robust estimation of a location parameter, Zhu and Fukushima [2009] considered generally mixture models concerning the worst-case ES of robust portfolio management. In the spirit of (1.4), Ghosh [2017], Vandewalle et al. [2007] established robust estimations of extreme value index. We remark that model (1.4) is different from the model mis-specification studied by Blanchet and Murthy [2020], Engelke and Ivanovs [2017], Escobar-Bach et al. [2017] concerning the worst VaR and extreme dependence. We refer to Nataliya et al. [2010] for optimal robust estimations of risk measures under max-bias and max mean square error.

The aim of this paper is to study how the common risk measures vary as a function of the contamination neighbourhood. Basically, we investigate the sensitivity of VaR, ES and the expectile-quantile transformation level for small α, indicating a potential severe loss (or an extreme negative return) caused by a portfolio investment. In contrast with the easier robustness against outliers with moderate small α, risk measures are more sensitive to potential high risks corresponding to extreme small level α. We notice that, although very large losses are observed rarely, they still have a tremendous effect. As a consequence, usually only few observations will have an overwhelming impact on the computed regulatory capital. In realistic modelling, it is hard to tell whether these events are singular outliers or contribute valuable evidence for future losses. This question of relevance for future losses gets even more severe in the common and Basel-II-recommended practice of data pooling used to overcome the scarcity of very large loss data. This motivates our asymptotic study on the sensitivity of risk measures with heavier contamination distribution (see Eq.(1.5) specified below).

Our methodology is from extreme value theory, a powerful tool in financial risk management. Therefore, a common assumption is thus that both F and H belong to the max-domain attraction, i.e., the linear normalization of sample maxima possesses a non-degenerate limit distribution Resnick [2007]. This enables the possible extrapolation from moderate tail inference to extreme tail analysis in a semi-parametric framework ensuring the practical applications of our study since little and even no information on the exact distribution involved is known in reality [Ferreira et al., 2003]. We refer to Castillo et al. [1989], Embrechts et al. [2013], Gomes and Guillou [2015] for the identification of the distribution via probability plots, or the tail conditions on the max-domain attractions. In addition, we consider that H has a heavier left tail than that of F (recall portfolio return $X \sim F$ in our context). Namely,

$$\lim_{x \to -\infty} \frac{F(x)}{H(x)} = 0. \tag{1.5}$$

We refer to McNeil et al. [2015] for the monograph of heavy tail analysis in finance and insurance fields.

The contributions of this paper are as follows: a) Sensitivities of common risk measures including VaR, ES and expectile-quantile transformation level are systematically studied and compared; b) Effects on VaR, ES and expectile-quantile transformation level of an infinitesimal contamination to a known F, are investigated by use of influence functions; c) Efficiency of the theoretical results is illustrated by several typical examples and numerical study; d) As an application of the sensitivity of VaR and ES, empirical study involved in the Royalton CRIX index, the first scientifically-backed proxy to the crypto-market well studied by Härdle and Trimborn [2015], Härdle et al. [2020], Trimborn et al. [2020], is given. We expect our research would be beneficial to both financial practitioners and theoretical experts focusing on risk management and extreme value statistics.

The paper is outlined as follows. Section 2 is devoted to establishing the sensitivity of tail-related risk measures in the framework of robustness analysis of Huber [1964]. Numerical studies on several illustrating examples are given in Section 3, followed by an empirical study concerning the CRIX in Section 4. We give an overall discussion and conclusion of the sensitivity of risk measures in

Section 5. Proofs are postponed to Section 6.

2. Main Results

Throughout the paper, we keep VaR_α for the Value-at-Risk of F and write $\mathrm{VaR}_\alpha(\epsilon)$ for the VaR of the contamination model F_ϵ. The same understanding applies for $q_\alpha, e_\alpha, ES_\alpha, w_\alpha$ and LPM_α involved in (1.2) and (1.3). Further, we write

$$h_1 \sim h_2 \quad \text{or} \quad h_1 = h_2 \{1 + o(1)\}$$

if two functions $h_i(\cdot), i = 1, 2$ are such that h_1/h_2 goes to 1 as the argument takes limits. Similarly, we write $f \simeq g$ if f is asymptotically equal to g.

Our first result below concerns the approximations of VaR_α, ES_α and w_α when the underlying cdf is contaminated by a heavier tail loss H with a fixed level $\epsilon \in (0, 1]$, in terms of (1.4).

Theorem 2.1. Consider the contamination model (1.4) with $\epsilon \in (0, 1]$ given. Suppose that F and H satisfying (1.5) are continuous with infinite left endpoint and finite means. We have as $\alpha \to 0$

$$\mathrm{VaR}_\alpha(\epsilon) \sim \mathrm{VaR}_{\alpha'}(1), \quad ES_\alpha(\epsilon) \sim -\frac{\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}H(x)}{\alpha'} \quad \text{with} \quad \alpha' \stackrel{\text{def}}{=} \alpha/\epsilon.$$

Further, if the α quantile equals the $w_\alpha(\epsilon)$ expectile, then

$$w_\alpha(\epsilon) \sim \epsilon \left(\frac{\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}H(x)}{q_\alpha(\epsilon)} - \alpha' \right).$$

Remark 1. a) We see that once the reference model F is contaminated by a heavier tail distribution H, the tail event involved will be completely determined by the contamination model H with a scaled probability level α/ϵ. This should be taken as a caveat for the practitioners when they take the underlying cdf for a lighter left-tailed cdf F.

b) If

$$\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}H(x) \sim \int_{-\infty}^{q_{\alpha'}(1)} x\,\mathrm{d}H(x) \stackrel{\text{def}}{=} LPM_{\alpha'}(1), \tag{2.1}$$

then

$$ES_\alpha(\epsilon) \sim ES_{\alpha'}(1), \quad w_\alpha(\epsilon) \sim \epsilon w_{\alpha'}(1)$$

implying that the ratio w_α/α satisfies that

$$\frac{w_\alpha(\epsilon)}{\alpha} \sim \frac{w_{\alpha'}(1)}{\alpha'}.$$

c) Since $LPM_\alpha \to 0$ as $\alpha \to 0$, the ratio w_α/α satisfies that

$$\frac{w_\alpha}{\alpha} \sim \frac{LPM_\alpha - q_\alpha\alpha}{q_\alpha\alpha} = \mathsf{E}\left[(X/q_\alpha - 1)|\{X < q_\alpha\}\right]$$

depicting the relative distance of the underlying loss X from the α quantile at the left tail. Therefore, the heavier the underlying loss is, the bigger the ratio w_α/α becomes for sufficiently small α.

Note that (2.1) holds for instance when H exhibits considerably heavier tail than typically selected distributions in practice such that $H(x) = |x|^{-\tau}\ell(x)$, $\tau > 1$

with $\ell(\cdot)$ a slowly varying function, that is, $\ell(tx) \sim \ell(t)$, $x > 0$ as $t \to -\infty$, see Example 3.3 below.

According to the latest revisions of the Basel Accords, the risk level α should be determined by the risk measure without changing too much its resulting value and the corresponding capital requirements, Bellini and Di Bernardino [2017]. To this end, one may shrink the contamination neighbourhood to reduce the influence of the outliers. In other words, we consider $\epsilon = \epsilon_\alpha \to 0$ as $\alpha \to 0$. This is similar to the common approach in asymptotic robust statistics, shrinking neighbourhood so as to protect against deviations which can not be detected. More specifically, we expect that the potential risk measures based on the contamination model F_ϵ deviate little from the one according to the reference model F. A natural idea is to restrict the distribution difference of $\epsilon H(x)$ (in comparison of $F(x)$) for small x. This motivates our investigation stated in Theorem 2.2 below. We will keep the same notation aforementioned when considering ϵ to vary in α.

Theorem 2.2. Under the same assumptions of Theorem 2.1, suppose further that $\epsilon = \epsilon_\alpha \to 0$ as $\alpha \to 0$.

a). If H is much heavier than F and/or $\epsilon = \epsilon_\alpha \to 0$ very slowly, such that $F(q_\alpha(\epsilon)) = o(1)\epsilon H(q_\alpha(\epsilon))$, i.e., $F(q_\alpha(\epsilon))/[\epsilon H(q_\alpha(\epsilon))] \to 0$ as $\alpha \to 0$, then we have $\lim_{\alpha\to 0} \alpha/\epsilon = 0$, and with $\alpha' \stackrel{\text{def}}{=} \alpha/\epsilon$

$$\mathrm{VaR}_\alpha(\epsilon) \sim \mathrm{VaR}_{\alpha'}(1), \quad ES_\alpha(\epsilon) \sim -\frac{\int_{-\infty}^{q_\alpha(\epsilon)} x \, \mathrm{d}H(x)}{\alpha'}, \quad w_\alpha(\epsilon) \sim \epsilon \left(\frac{\int_{-\infty}^{q_\alpha(\epsilon)} x \, \mathrm{d}H(x)}{q_\alpha(\epsilon)} - \alpha' \right).$$

b). If H is heavier than F and/or $\epsilon = \epsilon_\alpha \to 0$ with certain convergence, such that $F(q_\alpha(\epsilon)) \sim c'\epsilon H(q_\alpha(\epsilon))$ with some $c' > 0$, then

$$\begin{aligned}
\mathrm{VaR}_\alpha(\epsilon) &\sim \mathrm{VaR}_{\alpha/((c'+1)\epsilon)}(1) \sim \mathrm{VaR}_{\alpha c'/(c'+1)}, \\
ES_\alpha(\epsilon) &\sim -\frac{(1+1/c')\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}F(x)}{\alpha} \sim -\frac{(1+c')\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}H(x)}{\alpha'}, \\
w_\alpha(\epsilon) &\sim \frac{(1+1/c')\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}F(x)}{q_\alpha(\epsilon)} - \alpha \sim \frac{(1+c')\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}H(x)}{q_\alpha(\epsilon)} - \alpha.
\end{aligned}$$

c). If H is slightly heavier than F and/or $\epsilon = \epsilon_\alpha \to 0$ very quickly such that $\epsilon H(q_\alpha(\epsilon)) = o(1)F(q_\alpha(\epsilon))$, then

$$\mathrm{VaR}_\alpha(\epsilon) \sim \mathrm{VaR}_\alpha, \quad ES_\alpha(\epsilon) \sim -\frac{\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}F(x)}{\alpha}, \quad w_\alpha(\epsilon) \sim \frac{\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}F(x)}{q_\alpha(\epsilon)} - \alpha.$$

Remark 2. a) Theorem 2.2 covers all three cases by considering the interplay role in the determination of the risk measures between $F(x)$ and $\epsilon H(x)$ with $x = q_\alpha(\epsilon) \to -\infty$ as $\alpha \to 0$.

b) Case a) of Theorem 2.2 indicates the same claim as for ϵ given, while Case c) implies that the tail-related risk measures are robust with very slight contamination level ϵ_α for instance $\epsilon_\alpha = c\alpha^\tau$ for some $c > 0$ and all $\tau \geq 1$.

c) A typical example of contamination model $(1-\epsilon)\Phi(x\sqrt{1-\epsilon}) + \epsilon\Phi((x-\mu)\sqrt{\epsilon})$ with μ a constant, and $\Phi(\cdot)$ the standard normal cdf, discussed in Kuan et al. [2009], gives different sensitivity with respect to the contamination level $\epsilon = \epsilon_\alpha$.

The influence function approach, known also as the "infinitesimal approach", is generally employed to give qualitative robustness measure, for instance Fermanian and Scaillet [2005] investigated robust risk portfolios under netting agreements when the level of contamination in the data gradually decreases to zero. Recall that the influence function of some risk measure ϱ is defined as follows.

$$IF(\varrho; F, H) = \lim_{\epsilon \to 0} \frac{\varrho(\epsilon) - \varrho(0)}{\epsilon} = \frac{\partial \varrho(\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}$$

with $\varrho(\epsilon)$ standing for the risk measure ϱ of the contamination model $F_\epsilon(x) = (1-\epsilon)F(x) + \epsilon H(x)$.

Below, we study the influence function (IF) of VaR and ES evaluating its approximate bias if the corresponding risk measures are based loosely on the presupposed ideal model F.

Theorem 2.3. Assume that F has positive continuous differential at the $100\alpha\%$ quantile, and H is continuous at q_α. We have

$$IF(\mathrm{VaR}_\alpha; F, H) = \frac{H(q_\alpha) - \alpha}{F'(q_\alpha)}, \qquad IF(ES_\alpha; F, H) = \frac{\int_{-\infty}^{q_\alpha} (H(x) - F(x))\,\mathrm{d}x}{\alpha}.$$

Remark 3. a) Note that if H has heavier left tail than that of F, i.e., (1.5) holds, then in view of Theorem 2.3, both influence functions are positive for small α. Further, it holds for sufficiently small ϵ that

$$\mathrm{VaR}_\alpha \simeq \mathrm{VaR}_\alpha(\epsilon) - \epsilon IF(\mathrm{VaR}_\alpha; F, H), \qquad ES_\alpha \simeq ES_\alpha(\epsilon) - \epsilon IF(ES_\alpha; F, H).$$

We conclude that both VaR and ES based on the reference model F have a slightly negative bias in comparison to those strictly based on the contamination model.

b) The influence function leads to some robustness measure such as gross error sensitivity of the estimation of the tail-related risk measure through the following worst-case scenario

$$\sup_{H\in\mathcal{H}} |IF(\varrho; F, H)| \quad \text{or} \quad \sup_{H\in\mathcal{H}} |IF(\varrho; F, H)|^2,$$

where $\mathcal{H}$ is the class of contamination models H's. This might evoke the min-max global robustness analysis in risk management [Brazauskas, 2003, Nataliya et al., 2010].

3. Numerical studies

In this section, We illustrate our theoretical results through Monte-carlo simulation of three examples introduced below. We take the standard normal cdf as the presupposed ideal model F, and the contamination distribution H as normal, Laplace and power-like distributions in Examples 3.1-3.3, subsequently.

Example 3.1. Let $F_\epsilon(x) = (1-\epsilon)\Phi(x) + \epsilon\Phi(x/\sigma)$ with $\sigma > 1$, the scale parameter of the contamination model $H(x) = \Phi(x/\sigma)$(the same as below). Clearly, condition (1.5) holds, and the larger σ is, the heavier tail H possesses. Recall $q_\alpha(\epsilon)$ denotes the α quantile of F_ϵ. We have by Theorem 2.1, with $\alpha' = \alpha/\epsilon \to 0$

$$\mathrm{VaR}_\alpha(\epsilon) \sim \mathrm{VaR}_{\alpha'}(1) = \sigma\Phi^{-1}(1-\alpha'), \qquad \int_{-\infty}^{u} x \,\mathrm{d}\Phi(x/\sigma) = -\sigma\varphi(u/\sigma), \quad u \to -\infty$$

with $\varphi(\cdot)$ standing for the probability density function (pdf) of a standard normal random variable. Further,

$$ES_\alpha(\epsilon) \;=\; \frac{(1-\epsilon)\varphi(q_\alpha(\epsilon)) + \epsilon\sigma\varphi\left(q_\alpha(\epsilon)/\sigma\right)}{\alpha} \sim \frac{\sigma\varphi\left(q_\alpha(\epsilon)/\sigma\right)}{\alpha'}.$$

Further, a straightforward calculation yields by setting $u = q_\alpha(\epsilon)$:

$$\begin{aligned} w_\alpha(\epsilon) &= \frac{(1-\epsilon)\varphi(u) + \epsilon\sigma\varphi(u/\sigma) + u\alpha}{2\left((1-\epsilon)\varphi(u) + \epsilon\sigma\varphi(u/\sigma) + u\alpha\right) - u} \\ &\sim -\frac{(1-\epsilon)\varphi(u) + \epsilon\sigma\varphi(u/\sigma)}{u} - \alpha \\ &\sim \epsilon\left(-\frac{\sigma\varphi(u/\sigma)}{u} - \alpha'\right). \end{aligned}$$

Next, we consider the case with contamination level ϵ varying in α, denoted thus by ϵ_α. It follows by the Mills' ratio $\Phi(x) \sim \varphi(x)/|x|$, $x \to -\infty$ that, for $\epsilon = \sqrt{\alpha}$ and $1 < \sigma^2 \le 2$

$$ES_\alpha(\epsilon) = \frac{(1-\epsilon)\varphi(u) + \epsilon\sigma\varphi(u/\sigma)}{\alpha} \sim \frac{\varphi(u)}{\alpha},$$

where $u = q_\alpha(\epsilon) \sim \Phi^{-1}(\alpha)$.

Similarly, we have for $\epsilon = \sqrt{\alpha}$ and $\sigma^2 > 2$

$$ES_\alpha(\epsilon) = -\frac{(1-\epsilon)\varphi(u) + \epsilon\sigma\varphi(u/\sigma)}{\alpha} \sim -\frac{\sigma\varphi(u/\sigma)}{\alpha'},$$

where $u = q_\alpha(\epsilon) \sim \sigma\Phi^{-1}(\alpha/\epsilon)$.

We conclude the sensitivity of VaR, ES and w_α via the contamination level ϵ and the heaviness parameter σ, coinciding the claims established in Theorems 2.1 and 2.2.

Example 3.2. Let $F_\epsilon(x) = (1-\epsilon)\Phi(x) + \epsilon L(\sqrt{2}x/\sigma)$, $\sigma > 0$ with $L(\cdot)$ the standard Laplace distribution (double-sided exponential distribution), i.e., the density function $l(\cdot)$ is given by

$$l(x) = \frac{1}{2}\exp\left\{-|x|\right\}, \quad x \in \mathbb{R}.$$

It follows that (1.5) holds, and for $\alpha < 0.5, u < 0$

$$L^{-1}(\alpha) = \log(2\alpha), \quad \int_{-\infty}^{u} x \, \mathrm{d}L(x) = \frac{1}{2}(u-1)\mathrm{e}^{u} \overset{\text{def}}{=} LP(u).$$

Therefore, by Theorem 2.1, we have with $\alpha' = \alpha/\epsilon$

$$\mathrm{VaR}_{\alpha}(\epsilon) \sim \mathrm{VaR}_{\alpha'}(1) = \frac{\sigma}{\sqrt{2}} L^{-1}(1-\alpha') = -\frac{\sigma}{\sqrt{2}} \log(2\alpha'), \quad \alpha \to 0$$

and

$$\begin{aligned} ES_{\alpha}(\epsilon) &= \frac{(1-\epsilon)\varphi(q_{\alpha}(\epsilon)) - \epsilon\sigma/\sqrt{2}LP\left(\sqrt{2}q_{\alpha}(\epsilon)/\sigma\right)}{\alpha} \\ &\sim \frac{\sigma}{2\sqrt{2}\alpha'}\left(1 - \frac{\sqrt{2}}{\sigma} q_{\alpha}(\epsilon)\right) \exp\left\{\frac{\sqrt{2}}{\sigma} q_{\alpha}(\epsilon)\right\}. \end{aligned}$$

Further, we have by setting $u = q_{\alpha}(\epsilon)$

$$\begin{aligned} w_{\alpha}(\epsilon) &= \frac{-(1-\epsilon)\varphi(u) + \epsilon\sigma/\sqrt{2}LP(\sqrt{2}u/\sigma) - u\alpha}{2\left(-(1-\epsilon)\varphi(u) + \epsilon\sigma/\sqrt{2}LP(\sqrt{2}u/\sigma) - u\alpha\right) + u} \\ &\sim \frac{-(1-\epsilon)\varphi(u) + \epsilon\sigma/\sqrt{2}LP(\sqrt{2}u/\sigma)}{u} - \alpha \\ &\sim \epsilon\left(\frac{LP(\sqrt{2}u/\sigma)}{\sqrt{2}u/\sigma} - \alpha'\right). \end{aligned}$$

Next, we consider the case $\epsilon = \epsilon_{\alpha}$. It follows by Theorem 2.2 together with elementary calculations that, for $\epsilon = \sqrt{\alpha}$,

$$\begin{aligned} ES_{\alpha}(\epsilon) &= \frac{(1-\epsilon)\varphi(u) - \epsilon\sigma/\sqrt{2}LP\left(\sqrt{2}u/\sigma\right)}{\alpha} \\ &\sim \frac{\sigma}{2\sqrt{2}\alpha'}\left(1 - \frac{\sqrt{2}}{\sigma}u\right) \exp\left\{\frac{\sqrt{2}}{\sigma}u\right\}, \end{aligned}$$

where $u = q_{\alpha}(\epsilon) \sim q_{\alpha'}(1) = (\sigma/\sqrt{2})\log(2\sqrt{\alpha})$.

If we take $\epsilon = \alpha$, we have $u = q_\alpha(\epsilon) \sim q_\alpha = \Phi^{-1}(\alpha)$ and

$$ES_\alpha(\epsilon) = \frac{(1-\epsilon)\varphi(u) - \epsilon(\sigma/\sqrt{2})LP\left(\sqrt{2}u/\sigma\right)}{\alpha} \sim \frac{\varphi(u)}{\alpha}.$$

Comparing the Laplace with normal contamination models, we see that the risk measures are more sensitive for heavier contamination models with even infrequent contamination data. Financial practitioners should therefore take care of the extreme value risk due to asset-specific events and market-wide events.

Example 3.3. Let $F_\epsilon(x) = (1-\epsilon)\Phi(x) + \epsilon H(x/\sigma)$, $\sigma > 0$, with H a symmetric distribution such that

$$H(x) = \frac{1}{2}\left(1 - \left(1 - \frac{4}{4+x^2}\right)^{0.5}\right), \quad x < 0.$$

Clearly, Remark 1 a) holds with $H(x) \sim |x|^{-2}/2$ as $x \to -\infty$, i.e., H decays slowly like a power function with index -2. Hence, H is the so-called power-like distribution with scale parameter $\sigma > 0$ in the context. Further, with $\alpha' = \alpha/\epsilon$, we have by Theorem 2.1

$$\mathrm{VaR}_\alpha(\epsilon) \sim \mathrm{VaR}_{\alpha'}(1) = \frac{1-2\alpha'}{\sqrt{\alpha'(1-\alpha')}}, \quad \alpha \to 0, \tag{3.1}$$

$$\int_{-\infty}^{u} x\,\mathrm{d}H(x) = \int_{-\infty}^{u} \frac{2x}{(4+x^2)^{3/2}}\,\mathrm{d}x = -\frac{2}{\sqrt{4+u^2}} \stackrel{\text{def}}{=} LP(u).$$

We see that $|LP(\cdot)|$ is a regular varying function at $-\infty$ with index -1. Therefore,

$$\begin{aligned} ES_\alpha(\epsilon) &= \frac{(1-\epsilon)\varphi(q_\alpha(\epsilon)) - \epsilon\sigma LP(q_\alpha(\epsilon)/\sigma)}{\alpha} \\ &\sim -\frac{\sigma LP(q_\alpha(\epsilon)/\sigma)}{\alpha'} \sim -\frac{\sigma LP(q_{\alpha'}(1)/\sigma)}{\alpha'} = ES_{\alpha'}(1). \end{aligned}$$

Further, we have by setting $u = q_\alpha(\epsilon)$

$$
\begin{aligned}
w_\alpha(\epsilon) &= \frac{-(1-\epsilon)\varphi(u) + \epsilon\sigma LP(u/\sigma) - u\alpha}{2\left(-(1-\epsilon)\varphi(u) + \epsilon\sigma LP(u/\sigma) - u\alpha\right) + u} \\
&\sim \frac{-(1-\epsilon)\varphi(u) + \epsilon\sigma LP(u/\sigma)}{u} - \alpha \\
&\sim \epsilon\left(\frac{\sigma LP(u/\sigma)}{u} - \alpha'\right) \sim \epsilon w_{\alpha'}(1).
\end{aligned}
$$

Next, applying Theorem 2.2 we consider two cases of $\epsilon = \epsilon_\alpha$ so that the tail-related risk measures determined by F and H, respectively.

- For $\epsilon = \sqrt{\alpha}$ tending to zero slowly, we have with $u = q_\alpha(\epsilon) \sim q_{\sqrt{\alpha}}(1)$ given by (3.1)

$$
ES_\alpha(\epsilon) = \frac{(1-\epsilon)\varphi(u) - \epsilon\sigma LP(u/\sigma)}{\alpha} \sim -\frac{\sigma LP(u/\sigma)}{\sqrt{\alpha}} \sim ES_{\alpha'}(1).
$$

- For $\epsilon = \alpha$, we have

$$
ES_\alpha(\epsilon) \sim \frac{\varphi(u)}{\alpha}, \quad u = \mathrm{VaR}_\alpha(\epsilon) \sim \mathrm{VaR}_\alpha = \Phi^{-1}(1-\alpha).
$$

We remark that the power-like contamination distribution is a typical example that the probability level ratio $w_\alpha/\alpha = 1$. For a realistic degree of tail heaviness, the ratio w_α/α is less than 1, and increases with the degree of tail heaviness [Mihoci et al., 2021].

In the following, we investigate the behavior of the theoretical results indicated by Theorems 2.1, 2.2 and 2.3 with the three examples given above.

In Fig. 1, we fix the contamination level ϵ to be 0.5 and show performance of our asymptotical results indicated by Theorem 2.1. Here the contamination models are normal, Laplace and power-like distributions with the same scale parameter $\sigma = 1.5$. Here, the approximations of the VaR, ES and the expectile-quantile transformation level ratio (indicated by the dotted line) are given by the heavier contamination distribution of normal, Laplace and power-like risk at level $\alpha' = \alpha/\epsilon$, i.e., $\mathrm{VaR}_{\alpha'}(1), ES_{\alpha'}(1)$ and $w_{\alpha'}(1)/\alpha'$. We see that, as the risk level $\alpha \to 0$, the more accurate approximations to the true values based strictly on the underlying model are obtained.

In Fig. 2, we fix small risk level $\alpha = 0.5\%$ and investigate how the approximations vary with the contamination level ϵ. Clearly, the VaR, ES and the probability level ratio w_α/α become smaller and smaller as ϵ becomes larger, in other words, the smaller ratio level indicates the heavier left tails of the contamination model. Further, the approximations perform better for larger ϵ. Finally, we conclude that the level ratio of expectile vs. quantile is not monotonic for moderate ϵ, and the normal-power-like contamination model has ratio level around 1.

Fig. 3 and 4 show the approximations of VaR, ES and expectile-quantile transformation level ratio with varying contamination level $\epsilon = \alpha^\tau, \tau = 0.05, 0.1$ based on Theorem 2.2, respectively. In Fig. 3, we conduct the numerical approximations based on the reference normal model with normal and power-like contamination models. For both cases, we take the same scale parameter $\sigma = 1.2$. Generally, the efficiency of approximations supports the claim in c) of Theorem 2.2. The approximations of expectile-quantile transformation level ration seem slower than that for Value-at-Risk and expectile, which is common since its approximation depends on the estimated Value-at-Risk for the mixed models (see Eq.(1.2) and Theorem 2.2). Meanwhile, the approximations based on the heavier contamination model (normal and Laplace) with the same scale parameter $\sigma = 0.8$ and lighter contamination changes according to $\epsilon = \alpha^{0.1}$ were illustrated in Fig 4. We see that the efficiency of approximations for ES is more obvious than that for VaR.

Finally, in view of Theorem 2.3, we estimate the risk measure ϱ by

$$\widetilde{\varrho}(\epsilon) \stackrel{\text{def}}{=} \varrho + \epsilon IF(\varrho; F, H) \tag{3.2}$$

for small ϵ, provided that both F and H are asymptotically known. Define the relative error (RE) of the estimations as follows.

$$\mathrm{RE}(\varrho) = \frac{\widetilde{\varrho}(\epsilon) - \varrho(\epsilon)}{\varrho(\epsilon)}. \tag{3.3}$$

In Table 1, we fix $\alpha = 10\%$ and see that, the smaller the contamination ϵ is, the less RE is for all three contamination cases. Further, the RE of ES is not greater than that of VaR in absolute value when the contamination level $\epsilon > 1.10\%$. Therefore, we conclude that Theorem 2.3 gives nice estimations of VaR as well as ES for moderate α.

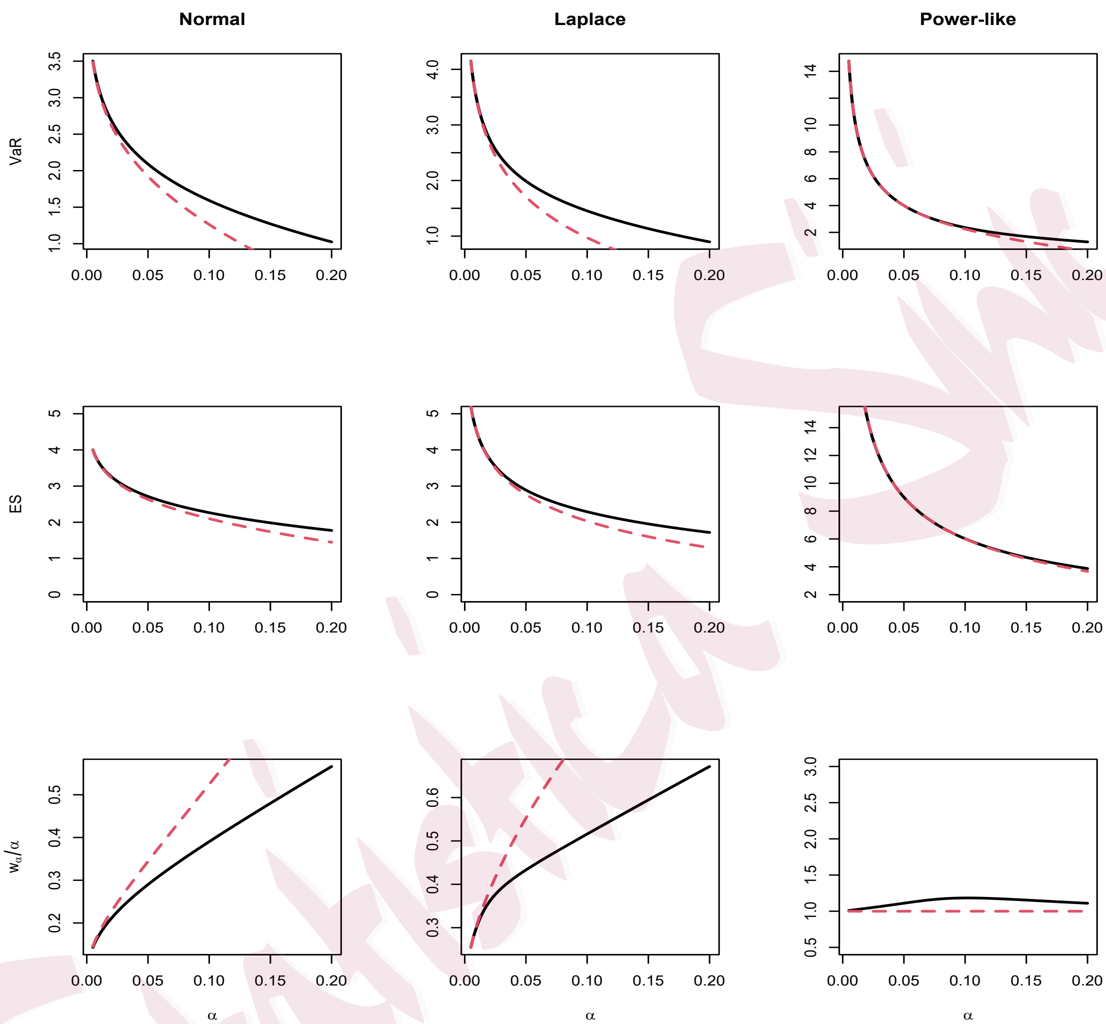

Figure 1: Comparisons of the **true values** (solid line) based strictly on the contamination model F_ϵ and **approximations** (dotted line) based loosely on the contamination distribution H at level $\alpha' = \alpha/\epsilon$ via Value-at-Risk, Expected-shortfall and the expectile-quantile transformation level ratio w_α/α. Here $\epsilon = 0.5$ and H is the normal, Laplace and power-like distribution with the same scale parameter $\sigma = 1.5$.

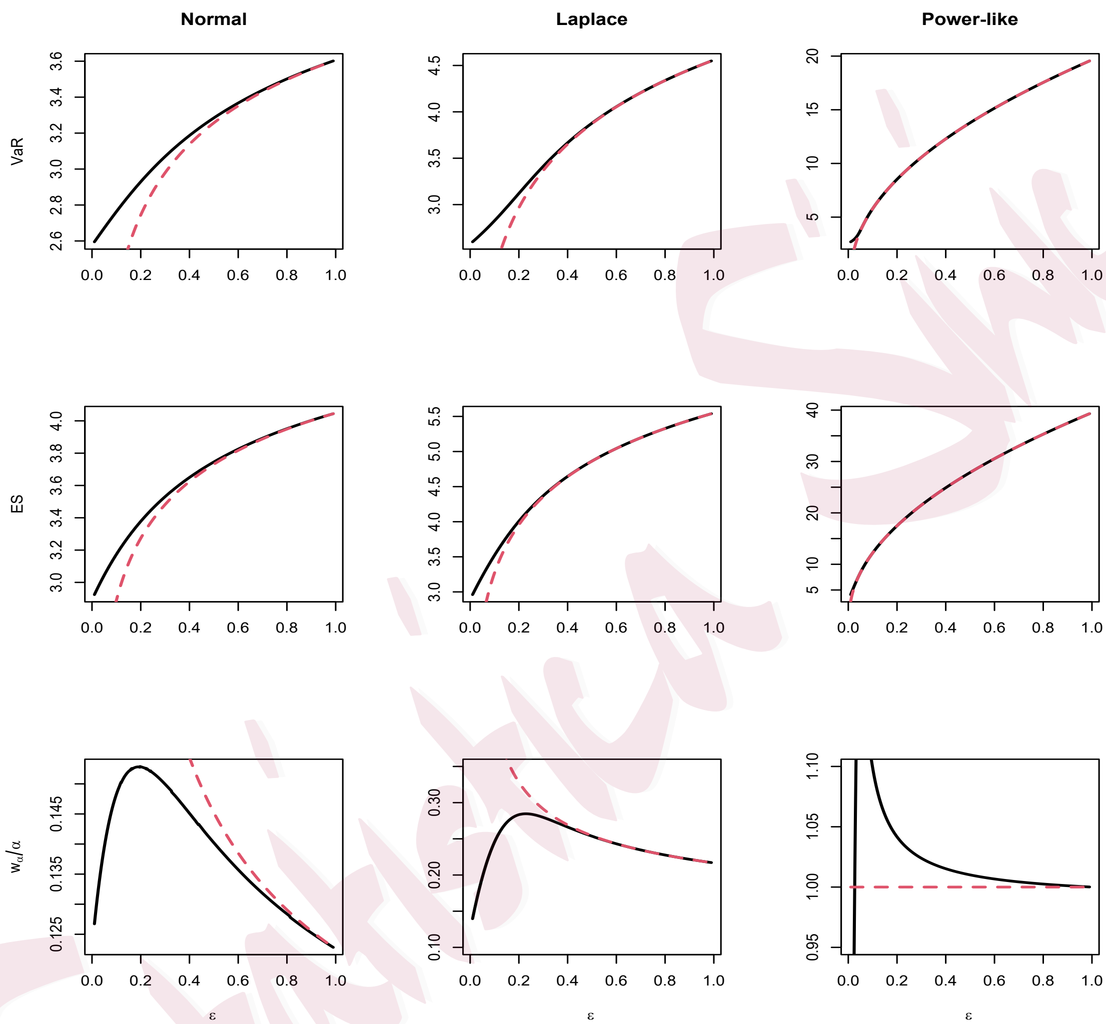

Figure 2: Comparisons of the **true values** (solid line) and **approximations** (dotted line) based on the contamination model H at level α/ϵ via Value-at-Risk, Expected-shortfall and the expectile-quantile transformation level ratio w_α/α. Here $\alpha = 0.5\%$, and ϵ varies in $(0.01, 1)$, and H is normal, Laplace and power-like distribution with the same scale parameter $\sigma = 1.4$.

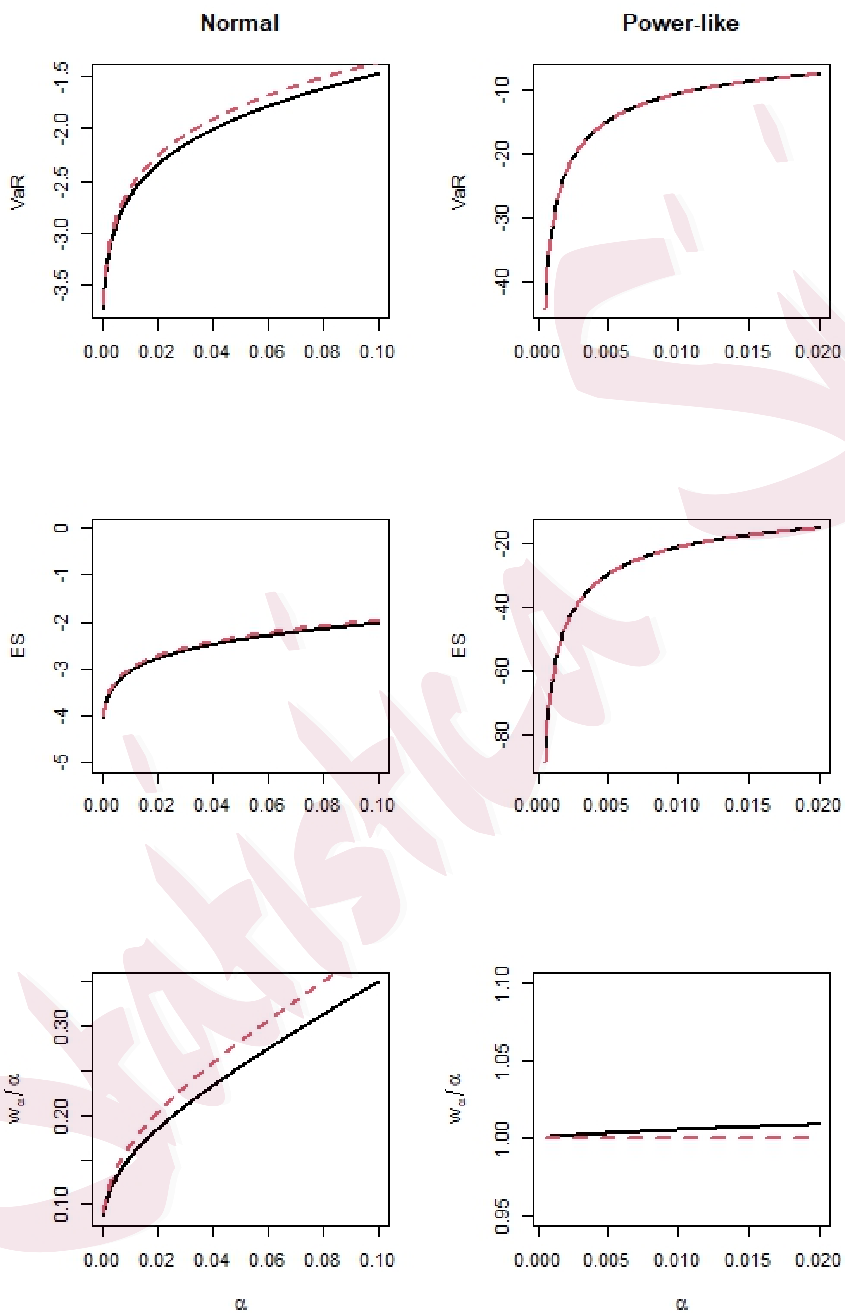

Figure 3: Comparisons of the **true values** (solid line) and **approximations** (red dotted line) based on the pre-supposed Normal model. Here we take H as normal and power-like distribution with same scale parameter $\sigma = 1.2$ and $\epsilon = \alpha^\tau$ with $\tau = 0.05$.

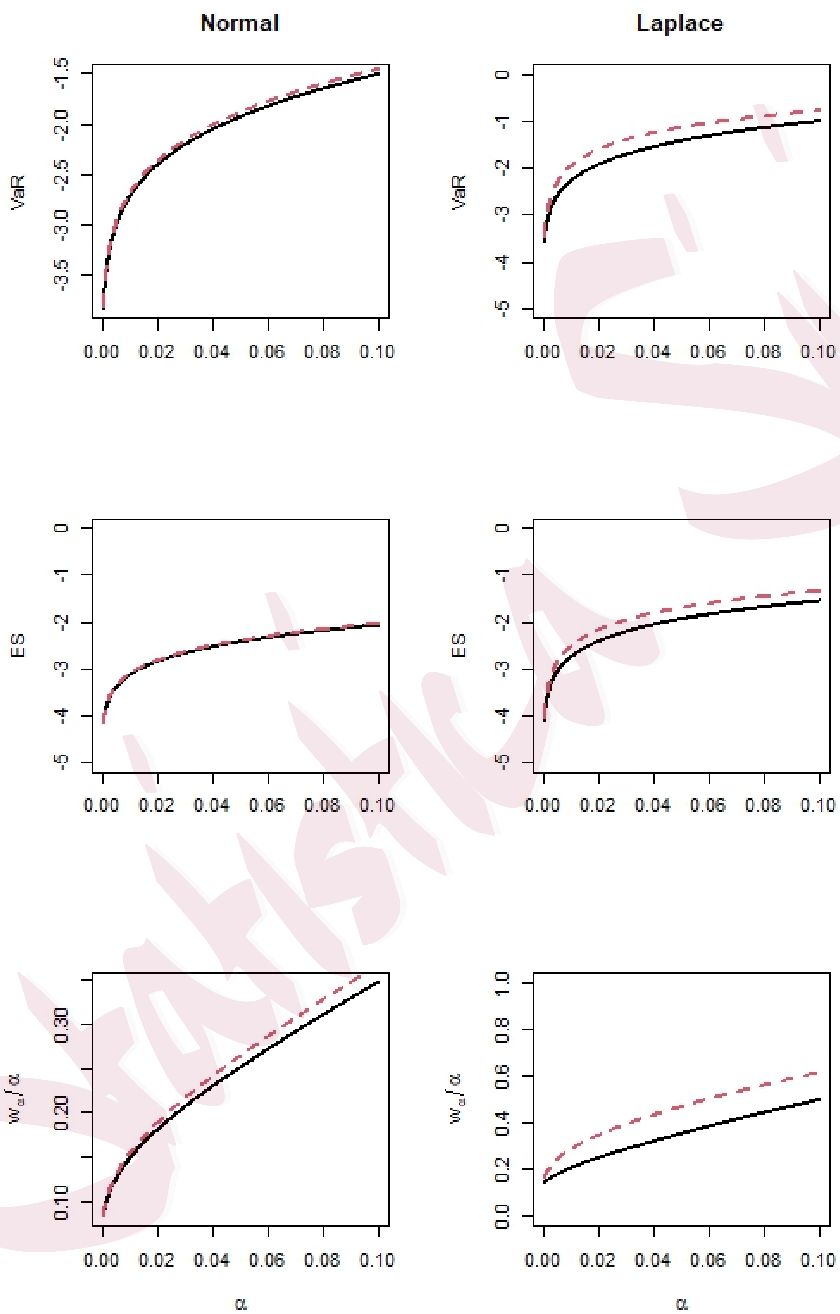

Figure 4: Comparisons of the **true values** (solid line) and **approximations** (red dotted line) based on the contamined normal (left) and Laplace (right). Here we take H as normal, and Laplace distribution with the same scale parameter $\sigma = 0.8$ and $\epsilon = \alpha^\tau$ with $\tau = 0.1$.

Table 1: Relative error (RE) of VaR and ES at $\alpha = 0.10$ with varying ϵ for contamination model H being normal, Laplace and power-like with scale parameter $\sigma = 2, 1.2, 1$ based on (3.3). Here, VaR_α and ES_α are estimated by Eq.(3.2), and the unit of $\epsilon, \widehat{\mathrm{VaR}}_\alpha, \widehat{ES}_\alpha$ is % while for $\mathrm{RE}(\widehat{\mathrm{VaR}}_\alpha), \mathrm{RE}(\widehat{ES}_\alpha)$ is ‰.

Normal										
ϵ	0.10	1.10	2.10	3.10	4.10	5.10	6.10	7.10	8.10	9.10
$\widehat{\mathrm{VaR}}_\alpha$	128.25	129.17	130.11	131.06	132.02	132.99	133.97	134.97	135.99	137.01
$\widehat{ES}_\alpha$	175.76	178.43	181.09	183.73	186.36	188.99	191.58	194.16	196.72	199.27
$\mathrm{RE}(\widehat{\mathrm{VaR}}_\alpha)$	−0.01	−0.07	−0.22	−0.47	−0.81	−1.19	−1.71	−2.31	−3.01	−3.79
$\mathrm{RE}(\widehat{ES}_\alpha)$	0.02	0.08	0.21	0.42	0.71	1.00	1.44	1.95	2.53	3.18
Laplace										
ϵ	0.10	1.10	2.10	3.10	4.10	5.10	6.10	7.10	8.10	9.10
$\widehat{\mathrm{VaR}}_\alpha$	128.16	128.22	128.28	128.34	128.40	128.46	128.53	128.59	128.65	128.71
$\widehat{ES}_\alpha$	175.54	176.00	176.47	176.93	177.39	177.86	178.32	178.78	179.25	179.71
$\mathrm{RE}(\widehat{\mathrm{VaR}}_\alpha)$	−0.01	−0.02	−0.02	−0.03	−0.04	−0.05	−0.07	−0.09	−0.11	−0.14
$\mathrm{RE}(\widehat{ES}_\alpha)$	0.02	0.02	0.03	0.03	0.03	0.03	0.04	0.04	0.04	0.05
Power-like										
ϵ	0.10	1.10	2.10	3.10	4.10	5.10	6.10	7.10	8.10	9.10
$\widehat{\mathrm{VaR}}_\alpha$	128.23	128.98	129.74	130.50	131.28	132.07	132.86	133.67	134.48	135.31
$\widehat{ES}_\alpha$	175.99	180.98	185.96	190.93	195.89	200.84	205.78	210.72	215.64	220.55
$\mathrm{RE}(\widehat{\mathrm{VaR}}_\alpha)$	−0.01	−0.06	−0.18	−0.37	−0.63	−0.96	−1.37	−1.80	−2.36	−2.98
$\mathrm{RE}(\widehat{ES}_\alpha)$	0.02	0.06	0.14	0.27	0.45	0.67	0.93	1.16	1.50	1.87

We notice that our theoretical findings are well illustrated by the above numerical analysis. The question of relevance is that the potential contamination distribution as well as its contamination level are completely unknown in reality. To this point, we recommend a splicing model with a classical distribution for the bulk of the data and the tail distribution based on extreme value theory [Reynkens et al., 2017]. We refer to Mohamed and Odile [1999] for the likelihood ratio test whether a known density contaminated by another density of the same parametric family and among others. On the other hand, statistician considered the alternative robust statistical methods which thus avoid the issue of identifying the contamination distribution and contamination level in reality [Zhu and Fukushima, 2009, Blanchet and Murthy, 2020]. The next section will show us the empirical study on Royalton CRIX with numerical estimation of the contamination level and subjective selection of the contamination distribution as well [Trimborn and Härdle, 2018, Kim et al., 2021].

4. Empirical study on CRIX

The Royalton CRIX, a market index (benchmark), is designed by Trimborn and Härdle [2018]. It enables each interested party to study the performance of the crypto market as a whole or single cryptos, and therefore attracts increasing attention of risk managers and regulators [Chen et al., 2018, Härdle et al., 2020, Trimborn et al., 2020, Petukhina et al., 2021, Kim et al., 2021]. Consequently, we focus on its tail feature and give the estimations of the tail-related risk measures.

As shown below, this is achieved by using the normal-Laplace contamination model and the approximations given in Section 2. Here, we use the daily CRIX dataset during 20140731-20180101, which is available on royalton-crix.com.

Firstly, we see the leptokurtic (fat-tailed) distributional feature of the financial dataset CRIX from the normal Q-Q plot in Fig. 5. Further, according to Chen et al. [2018], we standardize first the daily log returns of CRIX by GARCH(1,1) model explaining the heteroskedasticity of the CRIX, see also recent discussions by Venter and Maré [2020] and among others.

Next, in Fig. 6 we employ the empirical mean excess function from extreme value theory to visualize the heaviness of the standardized daily log returns of CRIX (X):

$$\widehat{m}_X(t) = \frac{\sum_{i=1}^{n} (X_i - t)\,\mathbf{I}\{X_i > t\}}{\sum_{i=1}^{n} \mathbf{I}\{X_i > t\}}, \quad t \text{ large},$$

where $X_1, \ldots, X_n$ are the observations of X. We see that both of upper and lower tails of CRIX decay exponentially since the graph $(t, \widehat{m}_X(t))$ becomes linear with slope zero for large threshold t (cf. Fig. 6 (upper)). Further, we see from Fig. 6 (bottom) that, the horizontal change of the log mean excess graph $(\log t, \log \widehat{m}_X(t))$ for large t, indicates the Laplace tail-decaying of the dataset [Dierckx et al., 2009]. We remark that the left tail feature of X is given by the right tail of $-X$. To illustrate the normality feature of standardized CRIX with possible heavier tail due to contamination model, we choose here the normal-Laplace contamination model with parameter $\epsilon, \boldsymbol{\mu} = (\mu_1, \mu_2), \boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ to fit the log returns of CRIX:

$$F_\epsilon(x) = (1-\epsilon)\Phi\left(\frac{x-\mu_1}{\sigma_1}\right) + \epsilon L\left(\sqrt{2}\frac{x-\mu_2}{\sigma_2}\right), \quad x \in \mathbb{R}. \tag{4.1}$$

We refer to Nasir et al. [2022] for the applications of Gaussian mixture and normal-Laplace mixture (one of heterogeneous mixtures) for return series. The maximum likelihood estimation of the parameters involved in (4.1) are listed in Table 2 utilizing the expectation-maximization (EM) algorithm [Dempster et al., 1977]. The EM algorithm, a general method to deal with the iterative computation of maximum likelihood estimation, works with initial parameter estimates and then iterates through two steps (i) the Expectation step and (ii) the Maximization step. The former step assumes fixed parameter estimates and computes the expected values of the latent variables in the model, and the maximization step updates the previous parameter estimates that maximize the likelihood function.

Further, for the CRIX during 20160401–20180101 the contamination level $\widehat{\epsilon} = 0.731$ is slightly bigger than those for the sub-period of 20140701–20160331 and the whole period of 20140401–20180101. We conclude the tail heaviness of CRIX time series might probably have an increasing tendency.

Finally, estimations of VaR and ES at level $\alpha = 0.5\%, 1\%, 5\%$ are compared also in Table 2 via three methods including the historical simulation, written by $\widehat{\mathrm{VaR}}^*_\alpha, \widehat{ES}^*_\alpha$; Laplace approximations at level $\alpha' = \alpha/\widehat{\epsilon}$ by use of Theorem 2.1, denoted by $\widehat{\mathrm{VaR}}_{\alpha'}(1), \widehat{ES}_{\alpha'}(1)$; and approximations based directly on the normal-Laplace mixture model, written by $\widehat{\mathrm{VaR}}_\alpha(\widehat{\epsilon}), \widehat{ES}_\alpha(\widehat{\epsilon})$. Note that historical simulation (HS) is a non-parametric method which assumes that past returns are anticipated to be the next period return, and thus a good guide for forecasting future returns. HS represents the easiest way of calculating VaR for many portfolios, and is more pliable and less sensitive to the odd outlier than the parametric method, which thus is extensively accepted by trading communities and management mostly [Andersen et al., 2009]. For all the estimations of ES, we keep the historical simulations of VaR, as in Mihoci et al. [2021].

We conclude that the estimations of $\widehat{\mathrm{VaR}}^*_\alpha, \widehat{\mathrm{VaR}}_{\alpha'}(1), \widehat{\mathrm{VaR}}_\alpha(\hat{\epsilon})$ and $\widehat{ES}_\alpha(\hat{\epsilon})$ for the sub-period of 20160401–20180101 are slightly larger (in absolute value) than those for the sub-period of 20140701–20160331 and the whole period of 20140401–20180101, and the same holds for the estimates of $\widehat{ES}^*_\alpha$ at $\alpha = 1\%$ and $\alpha = 5\%$. Further, the Laplace approximations of the VaR are rather close to the complete contamination model, illustrating again the efficiency of our theoretical approximation in Theorem 2.1.

Finally, given the scarcity of data in the tail, the estimates of VaR and ES based on historical simulations are not recommended for small α. In this case, we may employ the so-called extrapolation method in extreme value theory to get the out-of-sample estimates [Ferreira et al., 2003]. In our viewpoint, given the computation complexity of the mixture model with some model uncertainty in the tail, one may choose the estimates based on the heavier contamination model for extreme tail inference. In addition, our findings of the non-normality and certain heavy tails are in consistent with that by Chen et al. [2020].

Table 2: Estimated parameters of the normal-Laplace contamination model for the log return CRIX during 20140731–20180101 and the two sub-periods. Here $\widehat{\mathrm{VaR}}^*_\alpha$ and $\widehat{ES}^*_\alpha$ stand respectively for historical simulations of VaR and ES, and $\widehat{\mathrm{VaR}}_{\alpha'}(1), \widehat{\mathrm{VaR}}_\alpha(\widehat{\epsilon})$ and $\widehat{ES}_{\alpha'}(1), \widehat{ES}_\alpha(\widehat{\epsilon})$ are those estimations based on Laplace and the original mixture model with estimated parameters involved and $\alpha' = \alpha/\epsilon$.

period	parameter	α	$\widehat{\mathrm{VaR}}^*_\alpha$	$\widehat{\mathrm{VaR}}_{\alpha'}(1)$	$\widehat{\mathrm{VaR}}_\alpha(\epsilon)$	$\widehat{ES}^*_\alpha$	$\widehat{ES}_{\alpha'}(1)$	$\widehat{ES}_\alpha(\widehat{\epsilon})$
	$\widehat{\epsilon} = 0.622$	0.5%	0.136	0.127	0.127	0.183	0.125	0.000
2014.07-2018.01	$\widehat{\boldsymbol{\mu}} = (0.002, 0.004)$	1%	0.105	0.105	0.105	0.152	0.138	0.000
	$\widehat{\boldsymbol{\sigma}} = (0.010, 0.043)$	5%	0.054	0.055	0.054	0.091	0.085	0.000
	$\widehat{\epsilon} = 0.480$	0.5	0.118	0.125	0.125	0.171	0.184	0.000
2014.07-2016.03	$\widehat{\boldsymbol{\mu}} = (0.001, -0.002)$	1%	0.104	0.103	0.103	0.143	0.130	0.000
	$\widehat{\boldsymbol{\sigma}} = (0.014, 0.045)$	5%	0.046	0.052	0.052	0.086	0.094	0.000
	$\widehat{\epsilon} = 0.731$	0.5%	0.137	0.128	0.128	0.179	0.128	0.000
2016.04-2018.01	$\widehat{\boldsymbol{\mu}} = (0.002, 0.008)$	1%	0.108	0.106	0.106	0.155	0.130	0.000
	$\widehat{\boldsymbol{\sigma}} = (0.006, 0.045)$	5%	0.059	0.055	0.055	0.095	0.080	0.000

5. Discussions and Conclusion

Model mis-specifications and contamination data are very common in the growing fields of financial markets and insurance industry. Robustness against outliers are a desirable property of estimators based on the bulk data without being distorted by outliers. We argue that certain robustness of the VaR & ES remains under a contamination neighbourhood with lighter tailed contamination distribution

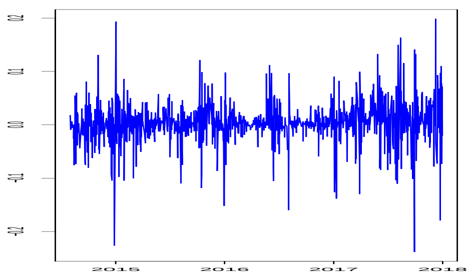

Figure 5: Time series of daily log returns of CRIX standardized by GARCH(1,1) during 2014.07.31 – 2018.01.01 (left) and normal Q-Q plot (right).

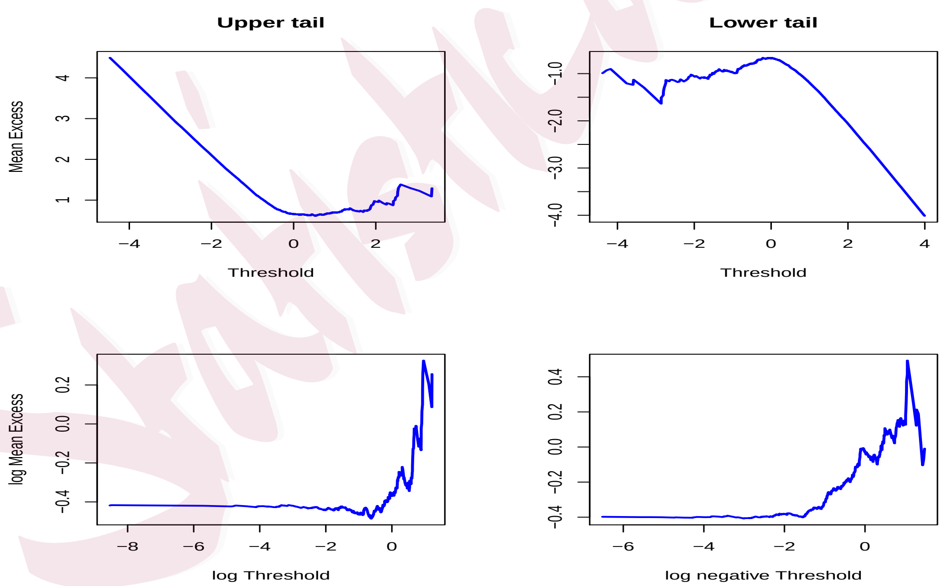

Figure 6: Mean Excess plot and log Mean Excess plot for log returns of CRIX standardized by GARCH(1,1) during 2014.07.31 – 2018.01.01.

or heavier-tailed contamination provided a suitable shrinked contamination level. This can lead to an understanding of the robust risk management role played by these extreme events. On the other hand, practitioners may implement outlier rejection approach and/or stress tests qualifying the effects of potentially hazardous events on VaR and other risk measures, according to the subjective judgement of the risk experts. Besides, it is worthy to emphasize the difference between outliers and extremes, and one may employ the splicing model to model both the bulk of data and extreme events [Reynkens et al., 2017]. We expect that the reference model with stable inference may improve the tail analysis through certain sharing effects in statistical applications. As an alternative topic on the sensitivity of portfolio strategy, one may study the sensitivity of the portfolio weight to changes in the portfolio loss distribution since in a long run, the portfolio loss distribution may violate to some extent. This will be considered in a forthcoming project [Caccioli et al., 2018, Nataliya et al., 2010].

6. Proofs

Proof of Theorem 2.1 Note by Eq.(1.1) that $ES_\alpha = -LPM_\alpha/\alpha$ with LPM_α the lower partial moment at $100\alpha\%$ quantile. Hence, in view of (1.3), it suffices to find $q_\alpha(\epsilon)$ and $LPM_\alpha(\epsilon)$. Since

$$(1-\epsilon)F\left(q_\alpha(\epsilon)\right) + \epsilon H\left(q_\alpha(\epsilon)\right) = \alpha, \tag{6.1}$$

we have by Eq.(1.5)

$$H\left(q_\alpha(\epsilon)\right) \sim \alpha/\epsilon.$$

Therefore, it follows further from Lemma 1.2.9 in de Haan and Ferreira [2006] that, for given $\epsilon \in (0, 1]$,

$$\mathrm{VaR}_\alpha(\epsilon) \sim \mathrm{VaR}_{\alpha/\epsilon}(1), \tag{6.2}$$

that is, the VaR for the contamination model equals approximately the VaR of H (the heavier df) at probability level α/ϵ.

Next, we return to $LPM_\alpha(\epsilon)$. It follows by integral by parts that

$$\frac{\int_{-\infty}^{a} x\,\mathrm{d}F(x)}{\int_{-\infty}^{a} x\,\mathrm{d}H(x)} = \frac{aF(a) - \int_{-\infty}^{a} F(x)\,\mathrm{d}x}{aH(a) - \int_{-\infty}^{a} H(x)\,\mathrm{d}x} \to 0, \quad a \to -\infty$$

since $F(a)/H(a) \to 0$ and $\int_{-\infty}^{a} F(x)\,\mathrm{d}x / \int_{-\infty}^{a} H(x)\,\mathrm{d}x \to 0$ by a straightforward application of L'Hôpital' rule and Eq.(1.5). Therefore, taking $a = q_\alpha(\epsilon)$, we have

$$\begin{aligned} LPM_\alpha(\epsilon) &= (1-\epsilon)\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}F(x) + \epsilon\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}H(x) \\ &\sim \epsilon\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}H(x), \end{aligned} \tag{6.3}$$

which goes to 0 by the fact that $\mathsf{E}[Y] < \infty$ with $Y \sim H$ and $q_\alpha(\epsilon) \to -\infty$.

Meanwhile, it follows by Eq.(1.2) that

$$w_\alpha = \frac{LPM_\alpha/q_\alpha - \alpha}{1 + 2(LPM_\alpha/q_\alpha - \alpha) - \mathsf{E}[X]/q_\alpha} \sim \frac{LPM_\alpha}{q_\alpha} - \alpha$$

since $w_\alpha \to 0$ as $\alpha \to 0$. Thus, we have (recall $\alpha' = \alpha/\epsilon$)

$$w_\alpha(\epsilon) \sim \frac{LPM_\alpha(\epsilon)}{q_\alpha(\epsilon)} - \alpha \sim \epsilon\left(\frac{\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}H(x)}{q_{\alpha'}(1)} - \alpha'\right) \tag{6.4}$$

Consequently, the claim follows by (6.2)-(6.4).

Proof of Theorem 2.2 Clearly, we have $\lim_{\alpha\to 0} F(q_\alpha(\epsilon)) = 0$ since $\lim_{\alpha\to 0}\max(\alpha, \epsilon_\alpha) = 0$ and $q_\alpha(\epsilon)$ satisfies (6.1). Hence, $\lim_{\alpha\to 0} q_\alpha(\epsilon) = -\infty$.

a) If $F(q_\alpha(\epsilon)) = o(1)\epsilon H(q_\alpha(\epsilon))$, then

$$\epsilon H(q_\alpha(\epsilon)) \sim (1-\epsilon)F(q_\alpha(\epsilon)) + \epsilon H(q_\alpha(\epsilon)) = \alpha.$$

Recalling that H is in the max-domain attraction, it follows by Lemma 1.2.9 by de Haan and Ferreira [2006] that

$$\mathrm{VaR}_\alpha(\epsilon) \sim \mathrm{VaR}_{\alpha/\epsilon}(1).$$

b) If $F(q_\alpha(\epsilon)) \sim c'\epsilon H(q_\alpha(\epsilon))$ with some fixed $c' > 0$, then

$$\mathrm{VaR}_\alpha(\epsilon) \sim \mathrm{VaR}_{\alpha/((c'+1)\epsilon)}(1) \sim \mathrm{VaR}_{\alpha c'/(c'+1)}.$$

c) If $\epsilon H(q_\alpha(\epsilon)) = o(1)F(q_\alpha(\epsilon))$, then

$$F(q_\alpha(\epsilon)) \sim (1-\epsilon)F(q_\alpha(\epsilon)) + \epsilon H(q_\alpha(\epsilon)) = \alpha$$

implying that $\mathrm{VaR}_\alpha(\epsilon) \sim \mathrm{VaR}_\alpha$.

Proof of Theorem 2.3 First, for given $\alpha \in (0,1)$, denote by θ the influence function of the VaR at quantile level $100\alpha\%$ with the underlying df in the ϵ-neighbourhood $\mathcal{F}_\epsilon = \{F_\epsilon | F_\epsilon(x) = (1-\epsilon)F(x) + \epsilon H(x)\}$. We have (recall $\mathrm{VaR}_\alpha = -q_\alpha$)

$$q_\alpha(\epsilon) \simeq q_\alpha - \epsilon\theta.$$

Therefore, we have by Taylor's expansion of F at q_α that

$$\lim_{\epsilon\to 0}\{(1-\epsilon)[\alpha-\epsilon\theta F'(q_\alpha)]+\epsilon H(q_\alpha)\}=\alpha.$$

This implies that $H(q_\alpha)-\theta F'(q_\alpha)=\alpha$. The first claim is obtained.

Similarly, we have for the ES

$$\begin{aligned}
IF(ES_\alpha;F,H) &= -\lim_{\epsilon\to 0}\frac{\int_{-\infty}^{q_\alpha(\epsilon)} x\,\mathrm{d}F_\epsilon(x)-\int_{-\infty}^{q_\alpha} x\,\mathrm{d}F(x)}{\alpha\epsilon}\\
&= -\frac{\int_{-\infty}^{q_\alpha} x\,\mathrm{d}(H(x)-F(x))+\lim_{\epsilon\to 0}\epsilon^{-1}\int_{q_\alpha}^{q_\alpha(\epsilon)} x\,\mathrm{d}F_\epsilon(x)}{\alpha}\\
&= -\frac{\int_{-\infty}^{q_\alpha} x\,\mathrm{d}(H(x)-F(x))+q_\alpha(\alpha-H(q_\alpha))}{\alpha}\\
&= \int_{-\infty}^{q_\alpha}(H(x)-F(x))\,\mathrm{d}x,
\end{aligned}$$

where the last second step follows by $q_\alpha(\epsilon)-q_\alpha\simeq\theta\epsilon$ and the continuity of F' at q_α, and the last step follows by integral by parts. □

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