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## Robust optimization and inference on manifolds

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*Abstract:* We propose a robust and scalable procedure for general optimization and inference problems on manifolds leveraging the classical idea of ‘median-of-means’ estimation. This is motivated by ubiquitous examples and applications in modern data science in which a statistical learning problem can be cast as an optimization problem over manifolds. Being able to incorporate the underlying geometry for inference while addressing the need for robustness and scalability presents great challenges. We address these challenges by first proving a key lemma that characterizes some crucial properties of geometric medians on manifolds. In turn, this allows us to prove robustness and tighter concentration of our proposed final estimator in a subsequent theorem. This estimator aggregates a collection of subset estimators by taking their geometric median over the manifold. We illustrate bounds on this estimator via calculations in explicit examples. The robustness and scalability of the procedure is illustrated in numerical examples on both simulated and real data sets.

*Key words and phrases:* Geometric median on manifolds; Median-of-means; Optimization on manifolds; Robust inference; Robust principal geodesic analysis (RPGA); Scalability

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## 1. Introduction

There is a rapidly growing collection of learning problems and applications in data science that can be formalized as optimization problems over non-Euclidean spaces, such as non-linear Riemannian manifolds. Advancement in technology and computing leads to the increasing prevalence of complex data that are in non-Euclidean forms, such as positive definite matrices (diffusion matrices) in diffusion tensor imaging (Alexander et al., 2007), shape objects in medical vision (Kendall, 1984), network data objects (Kolaczyk et al., 2020), subspaces or orthonormal frames and so on (Lin et al., 2017). Proper statistical inference from such data involves optimization over the underlying manifold to which the data are constrained. For example, there is a vibrant line of research based on estimation of Fréchet means (Fréchet, 1948), which are minimizers of Fréchet functions on manifolds (Bhattacharya and Bhattacharya, 2012; Bhattacharya and Lin, 2017). In this case, both the data and parameters of interest are on manifolds. In addition, it is common to represent lower-dimensional structure in high-dimensional data as a manifold. Learning such a manifold is a non-trivial optimization problem. In each of the above problems, developing algorithms that are robust to data contamination and heavy tails and that scale efficiently to large datasets is crucial.

With this motivation, our main aim is to propose a robust and scalable

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procedure for general optimization on manifolds. We generalize the powerful ‘median-of-means’ estimator (Nemirovski and Yudin, 1983), to manifolds by establishing some key properties of the geometric median on manifolds with which we can prove tighter concentration bounds of our proposed estimator. The key idea is to obtain optimizers from subset data which are aggregated to form a final estimator. Our estimator can be shown to be robust to outliers and contaminations of arbitrary nature and has provable robustness. Scalability of the algorithm is automatically gained via the divide-and-conquer nature of combining subset-based estimators.

There is a related literature outside of the non-Euclidean manifold setting. For example, Minsker (2015) applies the ‘median-of-means’ procedure for robust estimation in Banach spaces. In (Minsker et al., 2017) and Minsker et al. (2014), a robust Bayesian estimator is proposed as the geometric median of subset posteriors measures. Characterizing properties of the geometric median on manifolds requires a substantially different approach, which deals with the underlying geometry. We prove a key lemma characterizing the robustness property of geometric medians on manifolds, which allows us to show our estimator has tighter concentration bounds than subset estimators. This is done for both the *extrinsic geometric median* and the *intrinsic geometric median* with the former employing an embedding of manifolds into some higher-dimensional Euclidean space and the later adopting a Rieman-

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nian structure. We illustrate the bounds with explicit calculations in both the extrinsic and intrinsic cases. Our procedure is demonstrated in a class of manifolds through both simulated and real data examples. Manifolds considered include the sphere, positive definite matrices and the planar shape spaces, all of which are commonly applicable in real data analyses.

The paper is organized as follows: in section 2, we introduce the general procedure and prove a key property of the geometric median on manifolds. Section 3 is devoted to robust estimation and optimization on manifolds. In particular, we prove the concentration property of our final estimator in estimating the population parameter of interest and provide examples of calculations of the bounds. In section 4, we consider a simulation study and data analysis illustrating both the robustness and scalability of our procedure. The paper ends with a discussion.

## 2. Geometric median and robust estimation on manifolds

Let  $Q$  be a probability distribution on some space  $\mathcal{X}$  and  $\mathcal{M}$  be a manifold. We consider the problem of estimating the *population parameter*

$$\mu = \arg \min_{p \in \mathcal{M}} L^*(p), \quad (2.1)$$

where  $L^*(p)$  is defined as

$$L^*(p) = \int_{\mathcal{X}} L(p, x) Q(dx)$$

for some loss function  $L$ .

Let  $\mathbf{x} = \{x_1, \dots, x_n\}$  where  $x_1, \dots, x_n$  are sampled from  $Q$ . The parameter  $\mu$  is often estimated by the *empirical risk estimator*

$$\hat{\mu}_n = \arg \min_{p \in \mathcal{M}} L_n(p, \mathbf{x}) = \arg \min_{p \in \mathcal{M}} \frac{1}{n} \sum_{i=1}^n L(p, x_i). \quad (2.2)$$

**Remark 1.** An important example is the *Fréchet mean* in which the risk function is

$$L^*(p) = \int \rho^2(p, x) Q(dx),$$

with  $Q$  supported on a manifold  $\mathcal{X} = \mathcal{M}$  and  $\rho$  a metric defined on  $\mathcal{M}$ . There is significant literature on nonparametric statistical inference on manifolds in which estimation of the Fréchet mean is addressed (see Bhattacharya and Bhattacharya (2012); Bhattacharya and Lin (2017)). Similarly, in a regression problem with manifold-valued output, the underlying problem can be cast as an optimization problem on manifolds (Lin et al., 2017). In many other applications, we do not have  $\mathcal{X} = \mathcal{M}$  with  $\mathcal{X}$  a higher-dimensional ambient space and optimization done over a lower-dimensional manifold such as the Grassmannian (Lohit and Turaga, 2017; Saparbayeva et al., 2018), which has

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abundant applications in manifold learning and low-rank estimation matrix problems (Dai et al., 2012; Boumal and Absil, 2015).

Real data sets often contain outliers that can be errors, extreme observations or contamination of various sorts which occur when sampling from heavy tailed or mixture distributions. Thus, there is interest in robust estimation of population parameters by estimators which are stable and not unduly effected by the presence of outliers.

In this paper, we consider the classic and intuitive estimator formed by taking the geometric median of a collection of subset estimators or optimizers. Before formally introducing our procedure in the next section, we introduce the notion of the *geometric median on a manifold* and prove an important lemma about its properties.

For a metric space  $(\mathcal{M}, \rho)$  the *geometric median*,  $p^*$ , of points  $p_1, \dots, p_m \in \mathcal{M}$  minimizes the sum of distances to the points, i.e.,

$$p^* = \text{med}(p_1, \dots, p_m) = \arg \min_{p \in \mathcal{M}} \frac{1}{m} \sum_{k=1}^m \rho(p, p_k) \quad (2.3)$$

assuming that  $p^*$  exists and is unique. When  $\mathcal{M}$  is a manifold, there are different ways to metrize the space. Let  $J : \mathcal{M} \rightarrow \mathbb{R}^D$  be an embedding of a manifold  $\mathcal{M}$  into some higher-dimensional Euclidean space  $\mathbb{R}^D$ . We denote the image of  $\mathcal{M}$  after the embedding  $J$  as  $\tilde{\mathcal{M}}$ . That is,  $\tilde{\mathcal{M}} = J(\mathcal{M})$ . Note that  $\tilde{\mathcal{M}}$  is a submanifold in  $\mathbb{R}^D$ . One can define an *extrinsic distance* on  $M$

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induced from the embedding  $J$  using the Euclidean distance on  $\tilde{\mathcal{M}}$ . That is,

$$\rho(p, q) = \|J(p) - J(q)\|,$$

where  $\|\cdot\|$  is the Euclidean norm on  $\mathbb{R}^D$ . Alternatively, one can take  $\rho$  to be the *intrinsic distance* as the geodesic distance arising from a Riemannian structure on  $\mathcal{M}$ .

With the choice of  $\rho$  as the extrinsic or intrinsic distance in (2.3), we have corresponding definitions of the *extrinsic geometric median* and the *intrinsic geometric median*, respectively. Some properties of the intrinsic geometric median are studied in Fletcher et al. (2008) by, for example, characterizing the uniqueness conditions of the intrinsic sample median along with a Weiszfeld algorithm for finding the median. Our theoretical results below on robustness are of a fundamentally different nature, allowing us to construct an estimator that is not only robust but also has tighter bounds around the true parameter of interest.

We prove the following lemma, which says if  $\omega \in \mathcal{M}$  is at least a constant,  $C_\alpha$ , times  $\epsilon$  distance away from the geometric median  $p^* = \text{med}(p_1, \dots, p_m)$ , then  $\omega$  is at least  $\epsilon$  distance away from at least an  $\alpha$  fraction of the points  $p_1, \dots, p_m$ . This result is illustrated in Figure 1. A similar result was proved in Minsker (2015) in the case of Banach spaces. The proof of the following, a general lemma for manifolds, requires additional machinery.

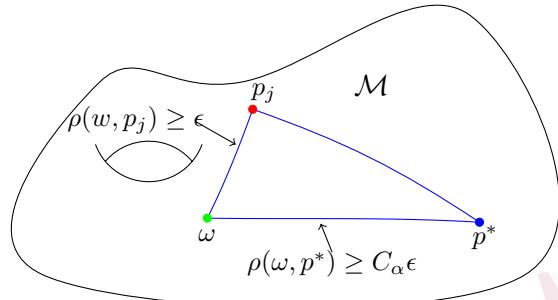


Figure 1: Geometric Illustration of Lemma 1 on Manifold  $\mathcal{M}$

**Lemma 1.** Let  $p_1, \dots, p_m \in \mathcal{M}$ ,  $p^* = \text{med}(p_1, \dots, p_m)$  as in (2.3). Then (a) and (b) below hold.

(a) Let  $\rho$  be the extrinsic distance for some embedding  $J : \mathcal{M} \rightarrow \tilde{\mathcal{M}} \subset \mathbb{R}^D$ .

Let  $\omega \in \mathcal{M}$ ,  $\psi$  be angle between  $J(\omega) - J(p^*)$  and the tangent space

$T_{J(p^*)}\tilde{\mathcal{M}}$  and let

$$C_\alpha = \frac{1 - \alpha}{\sqrt{1 - 2\alpha} \cos \psi - \alpha \sin \psi}$$

where  $\alpha \in (0, \cot \psi \tan \frac{\psi}{2})$ . If  $\rho(\omega, p^*) \geq C_\alpha \epsilon$ , then there exists an  $\alpha$  portion of elements of  $p_1, \dots, p_m$  which are at least  $\epsilon$  distance away from  $\omega$ . That is, there exists an index set  $T \subset \{1, \dots, m\}$  with  $|T| \geq \alpha m$ , and  $\rho(p_j, \omega) \geq \epsilon$  for any  $j \in T$ .

(b) Let  $\rho$  be an intrinsic distance on  $M$  with respect to some Riemannian structure. Let  $\omega \in \mathcal{M}$ , the log map  $\log_{p^*}$ , namely, the inverse exponential map  $\log_{p^*} = \exp_{p^*}^{-1}$ , be  $K$ -Lipschitz continuous from  $B(\omega, \epsilon)$  to  $T_{p^*}\mathcal{M}$ ,

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where the distance on  $T_{p^*}\mathcal{M}$  is the Euclidean distance, and let

$$C_\alpha = K(1 - \alpha)\sqrt{\frac{1}{1 - 2\alpha}}$$

where  $\alpha \in (0, 1/2)$ . If  $\rho(\omega, p^*) \geq C_\alpha\epsilon$ , then there exists an  $\alpha$  portion of elements of  $p_1, \dots, p_m$  which are at least  $\epsilon$  distance away from  $\omega$ .

The detailed proof of Lemma 1 can be found in the appendix. The key ideas are to consider the directional derivative of the objective function at the geometric median and the ‘angle’ between the curve connecting  $p^*$  and  $w$  and ones that connect  $p^*$  and  $p_j$ . Assuming the Lemma doesn’t hold leads to the directional derivative at the median being negative - a contradiction with the shown fact that it is positive. At the same time, the directional derivatives, curves and angles depend on whether we are using the extrinsic or intrinsic distance. For example, for the extrinsic case, the directional derivative of the objective function is defined along the curve that connects  $J(p^*)$  and  $J(w)$  on the image of the manifold  $\tilde{\mathcal{M}}$ .

There are many known Riemannian manifolds with  $K$ -Lipschitz continuous log maps as required in part (b) of the above lemma. Below we provide a few examples including the sphere, the planar shape space and the space of positive definite matrices, which are commonly encountered manifolds in the statistics and medical imaging literature.

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**Proposition 2.1.** *which has expression*

$$\log_p(q) = \frac{\arccos(p^T q)}{\sqrt{1 - (p^T q)^2}} (q - (p^T q)p),$$

*is 2-Lipschitz continuous from  $B(p, \pi/2)$  to  $T_p S^d$  for all  $p \in S^d$ .*

The following proposition shows that the log map in similarity-shape spaces Kendall (1984) also satisfies the  $K$ -Lipschitz condition.

**Proposition 2.2.** *The similarity or planar shape space is given as*

$$\Sigma_2^k = S^{2k-3}/S^1. \quad (2.4)$$

*The inverse exponential map,  $\log_p$ , which has expression*

$$\log_p(q) = \frac{\arccos(p^T q)}{\sqrt{1 - (p^T q)^2}} (q - (p^T q)p),$$

*on  $\Sigma_2^k$  is 2-Lipschitz continuous from  $B(p, \pi/4)$  to  $T_p \Sigma_2^k$  for all  $p \in \Sigma_2^k$ .*

**Proposition 2.3.** *The manifold of positive definite  $n$  by  $n$  matrices  $\text{PD}(n)$  has a 1-Lipchitz continuous inverse exponential map at any  $p \in \text{PD}(n)$ . For given metric we have the following exponential and logarithm mappings*

$$\exp_p A = p^{1/2} \exp(p^{-1/2} A p^{-1/2}) p^{1/2},$$

$$\log_p q = p^{1/2} \log(p^{-1/2} q p^{-1/2}) p^{1/2},$$

where

$$\begin{aligned}\exp X &= I + \frac{Y}{1!} + \frac{Y^2}{2!} + \dots + \frac{Y^n}{n!} + \dots, \\ \log x &= (x - I) - \frac{(x - I)^2}{2} + \dots + (-1)^{n-1} \frac{(x - I)^n}{n} + \dots,\end{aligned}$$

for any  $A, X \in \text{Sym}(n)$  and any  $p, q, x \in \text{PD}(n)$ .

### 3. Robust optimization on manifolds: concentration properties

In this section, we introduce our proposed estimator, which aggregates a collection of subset optimizers of the empirical risk function. We first divide the data set  $x_1, \dots, x_n$  into  $m$  subsets  $U_1, \dots, U_m$  each of roughly size  $\lfloor n/m \rfloor$ . Let  $\mu_1, \dots, \mu_m$  be the optimizers of the empirical risk function from each subset,  $U_1, \dots, U_m$ , respectively. That is,

$$\mu_j = \arg \min_{p \in \mathcal{M}} L_{|U_j|}(p, U_j) \text{ for } j = 1, \dots, m \quad (3.5)$$

as in (2.2). Our estimator  $\mu^*$  is the *geometric median of the subset optimizers*, that is,

$$\mu^* = \arg \min_{p \in M} \sum_{j=1}^m \rho(p, \mu_j). \quad (3.6)$$

We will show that  $\mu^*$  has desired robustness properties in estimating the population parameter  $\mu$ .

In Minsker (2015), it is proven that the geometric median of a collection of weakly concentrated estimators admits a tighter deviation bound in a Hilbert

space. With the help of the Lemma 1, we generalise this result to manifolds in the following theorem.

**Theorem 1.** Let  $\mu_1, \dots, \mu_m$  be a collection of independent estimators of the parameter  $\mu$ , and let geometric median  $\mu^* = \text{med}(\mu_1, \dots, \mu_m)$ .

- (a) Let  $\rho$  be the extrinsic distance on  $\mathcal{M}$  for some embedding  $J : \mathcal{M} \rightarrow \tilde{\mathcal{M}} \subset \mathbb{R}^D$ . Assume for any  $\omega \in \mathcal{M}$  the angle between  $J(\omega) - J(\mu^*)$  and the tangent space  $T_{J(\mu^*)}\tilde{\mathcal{M}}$  is no bigger than  $\bar{\psi}$ . For any  $\alpha \in (0, \cot \bar{\psi} \tan \frac{\bar{\psi}}{2})$  set

$$\bar{C}_\alpha = \frac{1 - \alpha}{\sqrt{1 - 2\alpha} \cos \bar{\psi} - \alpha \sin \bar{\psi}}.$$

- (b) Let  $\rho$  be an intrinsic distance on  $\mathcal{M}$  with respect to some Riemannian structure. Assume  $\log_{\mu^*}$  is  $K$ -Lipschitz continuous from  $B(\mu^*, \epsilon)$  to  $T_{\mu^*}\mathcal{M}$ . For any  $\alpha \in (0, \frac{1}{2})$  set

$$\bar{C}_\alpha = K(1 - \alpha) \sqrt{\frac{1}{1 - 2\alpha}}.$$

Under (a) or (b), if

$$P(\rho(\mu_j, \mu) > \epsilon) \leq \eta \text{ for } i = 1, \dots, n \quad (3.7)$$

where  $\eta < \alpha$  then

$$P(\rho(\mu^*, \mu) > \bar{C}_\alpha \epsilon) \leq \exp(-m\phi(\alpha, \eta)), \quad (3.8)$$

where

$$\phi(\alpha, \eta) = (1 - \alpha) \log \frac{1 - \alpha}{1 - \eta} + \alpha \log \frac{\alpha}{\eta}.$$

**Remark 2.** One important aspect in constructing the estimator  $\mu^*$  is the choice of the number of subsets  $m$ . By (3.8), a larger number of subset estimators,  $m$ , yields more robustness and a tighter concentration around the true parameter. At the same time, there must be enough data in each subset to ensure that each subset estimator behaves well and  $\eta$  in (3.7) is sufficiently small. For a given confidence level  $\epsilon$ , one can determine the number of subsets to achieve  $\eta$  in (3.7) and the desired bound on the concentration or confidence level in (3.8).

In the following, we provide examples, in both the intrinsic and extrinsic cases, of finding an  $\eta$  in (3.7) which allows the computation of the bound in (3.8).

**Example 1.** Consider the embedding  $J : \mathcal{M} \rightarrow \mathbb{R}^D$ . We have the induced measure  $\tilde{Q}$  on the image where  $\tilde{Q} = Q \circ J^{-1}$ . Let  $x_1, \dots, x_n$  be an i.i.d. sample from a distribution  $Q$ , such that we have the extrinsic mean  $\mu$  for the random variable  $x_1$

$$\mu = J^{-1} \left( \mathcal{P} \left( \int_{\mathbb{R}^D} u \tilde{Q}(du) \right) \right).$$

Divide the sample  $x_1, \dots, x_n$  into  $m$  disjoint groups  $U_1, \dots, U_m$  of size  $[n/m]$

each, and define

$$\tilde{\mu}_j = \frac{1}{|U_j|} \sum_{i \in U_j} J(x_i) \quad j = 1, \dots, m,$$

$$\mu_j \in J^{-1}(\mathcal{P}(\tilde{\mu}_j)).$$

One can easily conclude that

$$\begin{aligned} \rho(\mu, \mu_j) &= \|J(\mu) - J(\mu_j)\| \\ &= \|J(\mu) - \tilde{\mu}_j + \tilde{\mu}_j - J(\mu_j)\| \\ &\leq \|J(\mu) - \tilde{\mu}_j\| + \|\tilde{\mu}_j - J(\mu_j)\| \\ &\leq 2\|J(\mu) - \tilde{\mu}_j\|. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}\rho^2(\mu, \mu_j) &\leq 4\mathbb{E}\|J(\mu) - \tilde{\mu}_j\|^2 \\ &= \frac{4}{|U_j|^2} \sum_{i \in U_j} \mathbb{E}\|J(\mu) - J(x_i)\|^2 \\ &\leq \frac{4}{|U_j|^2} \sum_{i \in U_j} \mathbb{E}\rho^2(\mu, x_i) \\ &= \frac{4}{|U_j|} \mathbb{E}\rho^2(\mu, x_1) \leq 4 \left[ \frac{m}{n} \right] \mathbb{E}\rho^2(\mu, x_1). \end{aligned}$$

So by Chebyshev's inequality

$$P(\rho(\mu_j, \mu) \geq \epsilon) = P(\rho^2(\mu_j, \mu) \geq \epsilon^2) \leq \frac{1}{\epsilon^2} \mathbb{E}\rho^2(\mu_j, \mu) \leq \frac{4}{\epsilon^2} \left[ \frac{m}{n} \right] \mathbb{E}\rho^2(\mu, x_1). \tag{3.9}$$

Finally, we have the collection of independent estimators  $\mu_1, \dots, \mu_m$ , such that

$$P(\rho(\mu_j, \mu) > \epsilon) \leq \eta,$$

where  $\eta = \frac{4}{\epsilon^2} \left[ \frac{m}{n} \right] \mathbb{E}\rho^2(\mu, x_1)$ . So by theorem 1 for any  $\alpha \in (0, \cot \bar{\psi} \tan \frac{\bar{\psi}}{2})$

$$P(\rho(\mu^*, \mu) > \bar{C}_\alpha \epsilon) \leq \exp(-m\phi(\alpha, \eta)),$$

where

$$\mu^* = \text{med}(\mu_1, \dots, \mu_m),$$

$$\bar{C}_\alpha = \frac{1 - \alpha}{\sqrt{1 - 2\alpha} \cos \bar{\psi} - \alpha \sin \bar{\psi}},$$

$$\phi(\alpha, \eta) = (1 - \alpha) \log \frac{1 - \alpha}{1 - \eta} + \alpha \log \frac{\alpha}{\eta}.$$

**Example 2.** Let  $x_1, \dots, x_n$  be an i.i.d. sample from a distribution  $Q$ , such that we have the Fréchet mean  $\mu$  for the random variable  $x_1$ .

Divide the sample  $x_1, \dots, x_n$  into  $m$  disjoint groups  $U_1, \dots, U_m$  each of size  $[n/m]$ , and define

$$\mu_j = \arg \min_{y \in \mathcal{M}} \frac{1}{|U_j|} \sum_{i \in U_j} d_g^2(y, x_i), \quad j = 1, \dots, m.$$

Considering the  $j$ th subsample corresponding to  $U_j$  on the tangent space at  $\mu_j$ ,

$$\log_{\mu_j} \mu_j = \frac{1}{|U_j|} \sum_{x_i \in U_j} \log_{\mu_j} x_i = 0.$$

Thus on the tangent space  $T_{\mu_j} \mathcal{M}$ , we can obtain the equality

$$d_g^2(\mu, \mu_j) = \|\log_{\mu_j} \mu\|^2 = \frac{1}{|U_j|^2} \left\| \sum_{x_i \in U_j} (\log_{\mu_j} x_i - \log_{\mu_j} \mu) \right\|^2.$$

Thus,

$$\begin{aligned} \mathbb{E} d_g^2(\mu, \mu_j) &= \frac{1}{|U_j|^2} \sum_{x_i \in U_j} \mathbb{E} \|\log_{\mu_j} x_i - \log_{\mu_j} \mu\|^2 \\ &\leq \frac{K^2}{|U_j|^2} \sum_{i \in U_j} \mathbb{E} d_g^2(\mu, x_i) = \frac{K^2}{|U_j|} \mathbb{E} d_g^2(\mu, x_1) \leq K^2 \left[ \frac{m}{n} \right] \mathbb{E} d_g^2(\mu, x_1). \end{aligned}$$

So by Chebyshev's inequality,

$$P(d_g(\mu_j, \mu) \geq \epsilon) = P(d_g^2(\mu_j, \mu) \geq \epsilon^2) \leq \frac{1}{\epsilon^2} \mathbb{E} d_g^2(\mu_j, \mu) \leq \frac{K^2}{\epsilon^2} \left[ \frac{m}{n} \right] \mathbb{E} d_g^2(\mu, x_1). \quad (3.10)$$

Finally, we have the collection of independent estimators  $\mu_1, \dots, \mu_m$ , such that

$$P(d_g(\mu_j, \mu) > \epsilon) \leq \eta,$$

where  $\eta = K^2 \left[ \frac{m}{n} \right] \mathbb{E} d_g^2(\mu, x_1)$ . So by theorem 1 for any  $\alpha \in (0, \frac{1}{2})$

$$P(\rho(\mu^*, \mu) > \bar{C}_\alpha \epsilon) \leq \exp(-m\phi(\alpha, \eta)),$$

where

$$\mu^* = \text{med}(\mu_1, \dots, \mu_m),$$

$$\bar{C}_\alpha = K(1 - \alpha) \sqrt{\frac{1}{1 - 2\alpha}},$$

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$$\phi(\alpha, \eta) = (1 - \alpha) \log \frac{1 - \alpha}{1 - \eta} + \alpha \log \frac{\alpha}{\eta}.$$

**Remark 3.** Our proposed median-of-means estimator, in addition to being robust, is *scalable* over large data sets. Dividing the data into  $m = 2, \dots, \lfloor n/2 \rfloor$  subsets, for example, avoids expensive gradient descent steps computed over the entire data set in finding an overall sample median ( $m = n$ ) or an overall sample mean ( $m = 1$ ). Parallel processing, as well, can be applied in computing subset estimates simultaneously.

#### 4. Simulations and Applications

In this section, through extensive numerical examples, we show robustness and improved concentration about the population parameter of the geometric median of subset estimators in agreement with theorem 1. We first consider some simulated examples in estimating population means in  $S^d$  and  $PD(3)$ . We then formulate a robust procedure for estimating explanatory directions for dimension reduction in  $PD(3)$  and do a simulation study using this procedure. Finally, we apply the median-of-means method in the shape space to a hand shape data set as in Fletcher et al. (2008).

Numerical results from both simulated and real data analysis in this section agree with the robustness and concentration properties of the estimator. We see in these results that

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1. In simulations 1, 2, 3, and 4, and with various numbers of outliers, the average distance of the median-of-means is always an improvement over the average distances of the subset means.
  2. The average distance of the median-of-means is almost always an improvement over the overall mean in the presence of outliers.
  3. In the case of  $PD(3)$ , in Simulation 4, the average distance of the median-of-means for  $m = 5, 10, 15$  often gives an improvement over the overall median ( $m = 60$ ) in the presence of outliers. Number of groups  $m = 15$  seems to provide the best concentration overall. That the effect in more pronounced seems to agree with the log map in  $PD(3)$  being 1-Lipschitz as in proposition 2.3 and with the bound given in theorem 1 with  $K = 1$ .

In simulation 5, the median-of-means estimator is applied in estimating both the center of operations and explanatory directions for dimension reduction. The robustness property is shown as explanatory submanifolds maintain their fit to data in terms of intrinsic sum-of-squared residuals in the presence of outliers better than the ordinary PGA procedure. All code and data used in this section can be found here <https://github.com/DrewLazar/RobustManifold>.

## 4.1 Simulation Study on $S^d$

### 4.1 Simulation Study on $S^d$

In this subsection, we provide examples with data simulated from the von Mises-Fisher distribution on the sphere. We consider the estimation of both intrinsic and extrinsic means in the presence of various numbers of outliers. As shown by the numerical comparisons below, the estimator obtained from the robust estimation procedure shows improved concentration over subset-based estimators and often is closer to the true parameter of interest compared to the overall sample mean and overall sample median. The algorithms used for computing various summary statistics related to our estimators in  $S^d$  including intrinsic mean, extrinsic mean, intrinsic median and extrinsic median are provided in the Supplementary Material.

#### 4.1.1 Simulations in $S^d$

**Simulation 1.** Estimating Intrinsic Mean in  $S^2$ : Using Jung (2010) we sample  $n = 60$  data points from the von Mises-Fisher distribution on  $S^2$ . We take the concentration parameter  $\kappa = 30$ , which guarantees with probability  $\approx 1$  that the sample is within a hemisphere and thus the intrinsic mean and median uniquely exist.

We include  $k = 0, 5, 10$ , and 15 outliers outside a symmetric 95% confidence region about the mean with the confidence region. We then apply the

#### 4.1 Simulation Study on $S^d$

median-of-means technique of section 3 for  $m = 1, 5, 15, 30$  and 60 groups.

Over 1000 runs, we compute

1. the average intrinsic distance  $\overline{\rho(\mu^*, \mu)}$  from the true mean  $\mu$  to the geometric median of subsets estimator  $\mu^*$ .
2. the average intrinsic distance  $\overline{\overline{\rho(\mu_i, \mu)}}$  from  $\mu$  to the average of the subset means  $\mu_i$ ,  $i = 1, \dots, m$ .

$k$	$\overline{\rho(\hat{\mu}, \mu)}$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\overline{\rho(\mu_i, \mu)}}$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\overline{\rho(\mu_i, \mu)}}$
0	0.0597	0.0583	0.0947	0.0514	0.1496
5	0.0647	0.0615	0.1159	0.0531	0.1652
10	0.1194	0.1116	0.1414	0.1018	0.2113
15	0.1819	0.1731	0.1973	0.1631	0.2419
	sample mean (m=1)		m=5		m=15

$k$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\overline{\rho(\mu_i, \mu)}}$	$\overline{\rho(\hat{m}, \mu)}$	$\overline{\overline{\rho(\mu_i, \mu)}}$
0	0.0455	0.2118	0.0424	0.2829
5	0.0453	0.2350	0.0447	0.2959
10	0.0776	0.2501	0.0614	0.3259
15	0.1383	0.2954	0.0925	0.3738
	m=30		sample median (m=60)	

Table 1: Results from Simulation 1 showing performance for various estimators of the mean under a von Mises-Fisher distribution in  $S^2$ , with  $k$  the number of outliers and  $\rho$  intrinsic distance.

Note that when  $m = 1$ ,  $\mu_i$  and  $\mu^*$  are both the sample Fréchet mean of the whole data set, which we denote as  $\hat{\mu}$ . Also, when  $m = 60$ ,  $\mu^*$  is the sample median and  $\mu_i = p_i$  for  $i = 1, \dots, 60$ . The same situation holds in simulations 2, 3 and 4.

#### 4.1 Simulation Study on $S^d$

In Figure 2 we have a sample of  $n = 60$  from the von Mises-Fisher Distribution including 5 added outliers. We take  $m = 5$  subsets and we see the improved concentration about the population mean of the geometric median of the 5 subset means.

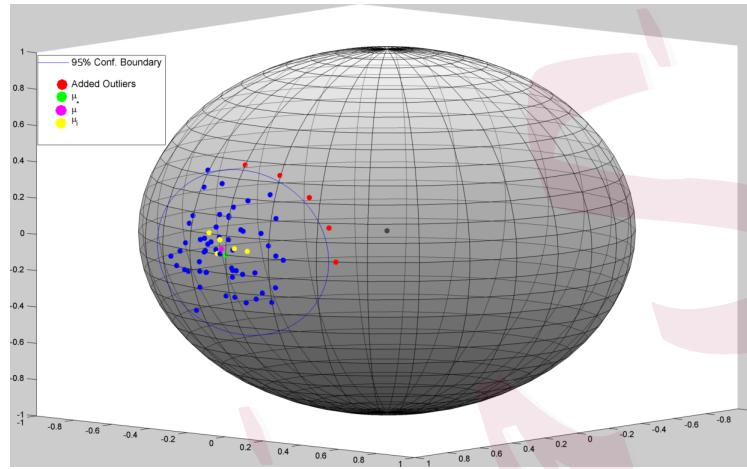


Figure 2: Von-Mises Fisher,  $\kappa = 30$ , 5 added outliers

**Simulation 2.** Approximation of the Intrinsic Mean in  $S^7$ : We repeat the first part of the experiment in Simulation 1 in  $S^7$  except with  $n = 200, \kappa = 20, k = 0, 10, 20, 40$  outliers and  $m = 1, 10, 50, 100, 200$  groups.

**Simulation 3.** Approximation of the Extrinsic Mean in  $S^2$ : We repeat the experiment in Simulation 1, but with  $\rho$  as the extrinsic distance and with each average taken over 1200 runs.

The results in Tables 1-3, showing the performance of the various estimators in Simulations 1-3 respectively, demonstrates that the median-of-means

#### 4.1 Simulation Study on $S^d$

$k$	$\overline{\rho(\hat{\mu}, \mu)}$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$
0	0.0396	0.0399	0.1186	0.0384	0.2570
10	0.0565	0.0541	0.1258	0.0514	0.2669
20	0.0897	0.0900	0.1462	0.0834	0.2827
40	0.1656	0.1678	0.2082	0.1596	0.3376
	Sample mean (m=1)		m=10		m=50

$k$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$	$\overline{\rho(\hat{m}, \mu)}$	$\overline{\rho(\mu_i, \mu)}$
0	0.0398	0.3590	0.0387	0.4896
10	0.0469	0.3676	0.0457	0.4978
20	0.0760	0.3896	0.0682	0.5301
40	0.1513	0.5176	0.1305	0.5987
	m=100		sample median (m=200)	

Table 2: Results from Simulation 2 showing performance for various estimators of the mean under a von Mises-Fisher distribution in  $S^7$ , with  $k$  the number of outliers and  $\rho$  intrinsic distance.

$k$	$\overline{\rho(\hat{\mu}, \mu)}$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$
0	0.0272	0.0330	0.0676	0.0312	0.1179
5	0.0621	0.0634	0.0943	0.0541	0.1512
10	0.1231	0.1190	0.1456	0.1083	0.1952
15	0.1771	0.1688	0.1956	0.1632	0.2337
	Sample mean (m=1)		m=5		m=15

$k$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$	$\overline{\rho(\hat{m}, \mu)}$	$\overline{\rho(\mu_i, \mu)}$
0	0.0305	0.1681	0.0312	0.2312
5	0.0453	0.2034	0.0411	0.2745
10	0.0847	0.2479	0.0612	0.3241
15	0.1453	0.2971	0.0837	0.3728
	m=30		sample median (m=60)	

Table 3: Results from Simulation 3 showing performance for various estimators of the mean under a von Mises-Fisher distribution in  $S^7$ , with  $k$  the number of outliers and  $\rho$  intrinsic distance.

## 4.2 Simulation study on $PD(3)$

estimator almost always improves over the average of subset means and overall Fréchet sample mean estimators in the presence of outliers.

### 4.2 Simulation study on $PD(3)$

In this subsection, we consider simulated data from a generalized log-normal distribution on the space of  $3 \times 3$  positive definite matrices,  $PD(3)$ . As in subsection 4.1, we consider the estimation of intrinsic means in the presence of various numbers of outliers. There are multiple applications in which it is of interest to estimate the mean of a sample of positive definite matrices. This includes principal geodesic analysis (PGA), as in Fletcher and Joshi (2007), where optimization to find explanatory directions is done in the tangent space at the sample mean. Using our median-of-means procedure, we formulate a robust PCA procedure (RPGA). We first describe algorithms used for computing various summary statistics related to our estimators in  $PD(3)$ .

#### 4.2.1 Computation of Sample Statistics on $PD(3)$

To compute the sample intrinsic mean in the following simulation, we use the damped gradient descent algorithm as in Fletcher and Joshi (2007). As shown in Karcher (1977), as  $PD(3)$  is of non-negative curvature, the intrinsic mean is guaranteed to exist and to be unique. To compute the sample intrinsic median, we use the generalization of Weiszfeld's algorithm given in Fletcher

## 4.2 Simulation study on $PD(3)$

et al. (2008) where the sample intrinsic median is shown to exist and to be unique. Computations of projection to subspaces and of principal geodesic directions are done using MATLAB minimization routines and user-supplied gradients as formulated in Sommer et al. (2010) with the derivative of the matrix exponential map provided by (Najfeld and Havel, 1995, Theorem 4.5).

### 4.2.2 Robust Principal Geodesic Analysis (RPGA)

Principal Geodesic Analysis (PGA) as in Lazar and Lin (2017) is a two-step procedure which involves 1) computing a center of the data and 2) successively finding orthogonal tangent vectors at that center so that their exponentiated span best fits the data according to intrinsic sum-of-squared residuals.

We propose a Robust PGA procedure (RPGA) which 1) uses the median-of-means estimate as the center of the data and 2) finds orthogonal directions in the tangent space using the robust median-of-means Principal Component Analysis (PCA) procedure given in Minsker (2015). Specifically, in RPGA

1. Divide the data into  $m$  subsets  $U_1, \dots, U_m$  and for each compute an intrinsic mean  $\mu_j$  as in (3.5) and then compute  $\mu^* = \text{med}(\mu_1, \dots, \mu_m)$  as in (3.6).
2. Compute  $V_i = \text{vec}(\text{Log}_{\mu^*}(U_i))$  where  $\text{Log}_{\mu^*}(U_i)$  is the image of  $U_i$  under the Riemannian log map. As in Minsker (2015), compute sample

## 4.2 Simulation study on $PD(3)$

covariance matrices  $\Sigma_i$  for each  $V_i$  and then compute

$$\hat{\Sigma} = \text{med}(\Sigma_1, \dots, \Sigma_n)$$

where the median is taken with respect to Frobenius norm  $\|A\|_F = \text{trace}(A^\top A)$ . We take the eigenvectors of  $\hat{\Sigma}$ ,  $\{w_1, \dots, w_6\}$ , arranged in order by largest to smallest eigenvalue. Then our robust principal geodesic directions in the tangent space at  $\mu^*$  are  $\{v_1, \dots, v_6\}$  where  $v_i$  is the vector corresponding to  $w_i$  by the vec operator. To form explanatory subspaces we then exponentiate the span of  $\{v_1, \dots, v_k\}$  at  $\mu^*$  for  $k = 1, \dots, 6$ .

This procedure is robust as it ensures both the located center of the data and the located explanatory directions are not unduly affected by the presence of outliers.

### 4.2.3 Simulations in $PD(3)$

**Simulation 4.** Estimating the Intrinsic Mean in  $PD(3)$ : We sample  $n = 60$  data points from a log-normal distribution where if the random variable  $X$  has this distribution then  $\text{vec}(\text{Log}_I(X)) \sim \mathcal{N}(\mathbf{0}, \kappa \mathbf{I})$  with  $\kappa$  a scaling parameter. We repeat the experiment of Simulation 1 of section 4.1.1 with each average taken over 1200 runs.

The results are shown in Table 4. Again, in this example, the median-

## 4.2 Simulation study on $PD(3)$

$k$	$\overline{\rho(\hat{\mu}, \mu)}$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$
0	0.2630	0.2781	0.5909	0.2753	1.0408
5	0.2640	0.2512	0.5776	0.2683	1.0745
10	0.3568	0.3179	2.7485	0.2986	1.3158
15	0.5292	0.3001	1.0433	0.3437	1.4246
	Sample mean ( $m=1$ )		$m=5$		$m=15$

$k$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$	$\overline{\rho(\hat{m}, \mu)}$	$\overline{\rho(\mu_i, \mu)}$
0	0.2750	1.5230	0.2728	2.3449
5	0.2724	1.5930	0.2675	2.4139
10	0.3306	1.7607	0.3482	2.5002
15	0.4183	1.8107	0.5265	2.5617
	$m=30$		Sample median ( $m=60$ )	

Table 4: Results for Simulation 4 with data simulated from a log-normal distribution in  $PD(3)$ ,  $k$  the number of outliers, and  $\rho$  the intrinsic distance.

of-mean estimator always improves over the average of the means and almost always over the overall sample Fréchet mean. The average distance from the truth of the median-of-means for  $m = 5, 10, 15$  is an improvement over the overall median ( $m = 60$ ) in the presence of outliers. The number of groups  $m = 15$  seems to provide the best concentration overall.

**Simulation 5.** Estimating Explanatory Directions in  $PD(3)$  with RPGA: We sample from a log-normal distribution, where if the random variable  $X$  has this distribution then  $\text{vec}(\text{Log}_I(X)) \sim \mathcal{N}(\mathbf{0}, \kappa \Sigma)$  with  $\kappa$  a scaling parameter.  $\Sigma$  is diagonal with diagonal entries which vary from 1 to 20 to ensure that population PGA directions exist.

Over 200 runs, we add 0, 5, 10, 15 outliers outside a 95% confidence region

### 4.3 Hand Shape Data in $\Sigma_2^K$

in  $n = 60$  data points and compute PGA and RPGA explanatory directions.

We then find the intrinsic mean sum of squared residuals (mSSRs) of the data without outliers relative to the estimated explanatory submanifolds. Table 5 gives the average of the mSSRs over 200 runs for submanifolds of 1, 2, and 3 dimensions for PGA and for RPGA computed with 5, 10 and 15 groups.

We see that without outliers, the PGA procedure, which sequentially optimizes a fit to the data at the intrinsic mean, produces the lowest average mSSR, regardless of the number of groups for RPGA. However, as outliers are added, the mSSR for PGA increases to a greater extent than RPGA. Note that RPGA with  $m = 1$  groups is the linear approximation of the PGA procedure given in Fletcher and Joshi (2007).

### 4.3 Hand Shape Data in $\Sigma_2^K$

We consider the hand shape data set in Cootes et al. (1995) of 18 hands with each hand in planar shape space  $\Sigma_2^{72}$ . A planar shape  $\Sigma_2^K$  consists of objects with  $K$  landmarks in  $\mathbb{R}^2$  modulo the Euclidean motions including rotation, scaling and translation (Bhattacharya and Bhattacharya, 2012; Kendall, 1984). As in Fletcher et al. (2008), we use ellipses as outliers with each one as

$$\{(a \cos(k\pi/36), b \sin(k\pi/36); k = 0, \dots, 71\}$$

### 4.3 Hand Shape Data in $\Sigma_2^K$

k	PGA	RPGA	RPGA	RPGA
0	0.4206	0.4265	0.4259	0.4320
5	0.4529	0.4465	0.4314	0.4342
10	0.4541	0.4438	0.4508	0.4374
15	0.4540	0.4445	0.4492	0.4442
20	0.4527	0.4473	0.4507	0.4496
$m$ groups		m=5	m=10	m=15

k	PGA	RPGA	RPGA	RPGA
0	0.2629	0.2686	0.2691	0.2751
5	0.2924	0.2870	0.2803	0.2795
10	0.2963	0.2838	0.2925	0.2791
15	0.2994	0.2835	0.2758	0.2850
20	0.3041	0.2841	0.2889	0.2775
$m$ groups		m=5	m=10	m=15

k	PGA	RPGA	RPGA	RPGA
0	0.1472	0.1497	0.1533	0.1608
5	0.1919	0.1801	0.1600	0.1588
10	0.2242	0.2102	0.1940	0.1743
15	0.2208	0.2149	0.2134	0.2079
20	0.2305	0.2259	0.2169	0.2206
$m$ groups		m=5	m=10	m=15

Table 5: Average mSSRs to explanatory submanifolds computed with  $k$  outliers to data without outliers in  $PD(3)$

where  $a, b$  are sampled from the uniform distribution on  $[0.5, 1]$ . With  $k = 3$  added outliers, we divide the data of size  $n = 21$  into  $m = 7$  random subsets, each of size 3. We then compute and observe the geometric median and the sample mean.

#### 4.3.1 Computation of Sample Statistics on $\Sigma_2^K$

We identify  $\Sigma_2^{72}$  with  $S^{69}/S^1$  as in (2.4), and compute intrinsic sample means and medians using direct modifications of the algorithms in section S2.

In Figure 3 (a) we show  $n = 21$  hands with 3 outliers. In (b) we show 7 randomly assigned subsets indicated with seven different colors, and in (c) we show the subset means of each group. In (d) we see less influence of the outliers in the geometric median, as it retains the shape of a hand similar to the original 18 hands.

## 5. Discussion

We propose a robust and scalable procedure for general optimization problems on manifolds. Scalability is of particular importance in dealing with the difficult computational issues that arise in estimating sample statistics for manifold data or extracting low-dimensional manifold in high-dimensional data. Along these lines, parallel computation can be implemented trivially from the subsampling procedure.

It is shown through lemma 1, which provides an important property of geometric medians on manifolds, and the following theorem 1, that the resulting estimator yields provable robustness and tighter concentration bounds about the true parameter of interest. Numerical results from both simulated and

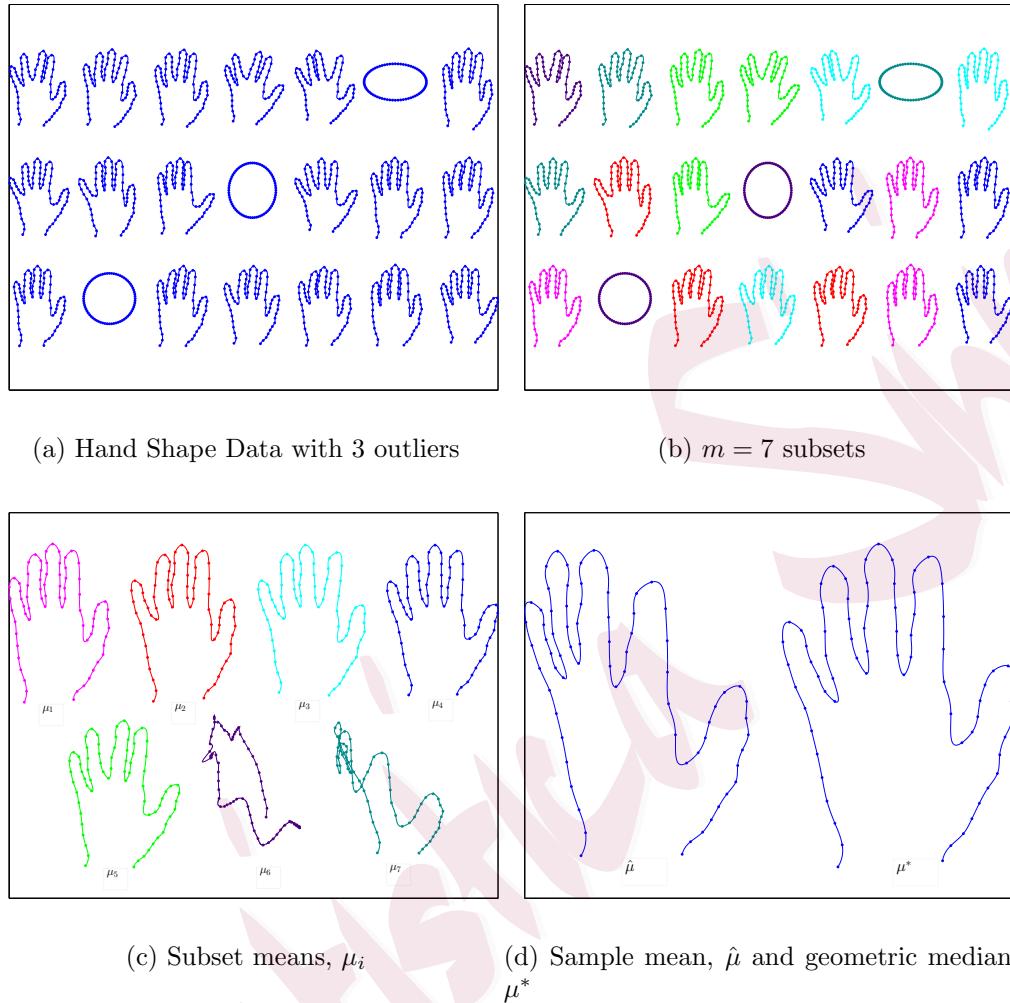


Figure 3: Median-of-Means on Hand Shape Data

real data analysis in Section 4 agree with the robustness and concentration properties of the estimator.

Future research might include considering the optimal numbers and sizes of subgroups for estimation as discussed in remark 2. In theorem 1, for a given  $\epsilon$ , more groups provide a larger  $m$  but also a larger  $\eta$  in the bound

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provided by 3.8. This is reflected in Examples 1 and 2 in (3.9) and (3.10). In finding the optimal  $m$ , the number of outliers and amount of contamination in the data also must be factored in. In addition, challenging computational considerations for large data sets on manifolds and advantages in partitioning data needs to be considered. Also, the second step of the RPGA procedure in 4.2.2 might be done instead by partitioning the data in the manifold rather than their Riemannian logs in the tangent space at  $\mu^*$ . Computation of RPGA, as formulated in 4.2.2, only requires the computation of the median-of-means  $\mu^*$ , and then the linear operation of computing sample covariance matrices of the Riemannian logs of data in the tangent space at  $\mu^*$ . Robust estimation on manifolds in other contexts such as manifold regression (Aswani et al., 2011) might also be considered. As in the case of estimation of the mean, additional machinery and complications arise in the more general context of a manifold.

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### Appendix: Proof of Lemma 1

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*Proof.* (a) Let  $L(J(p)) = \sum_{j=1}^m \rho(p, p_j) = \sum_{j=1}^m \|J(p) - J(p_j)\|$  for  $J(p) \in \tilde{\mathcal{M}}$ . Let  $\gamma(t)$  be a curve from  $J(p^*)$  to  $J(\omega)$  on  $\tilde{\mathcal{M}}$ , where  $\gamma(0) = J(p^*)$ ,  $\gamma(1) = J(\omega)$ , and  $\gamma'(0) = v$ . The directional derivative of  $L$  at  $J(p^*)$  evaluated at  $v$  is given by

$$dL_{J(p^*)}(v) = \lim_{t \rightarrow 0^+} \frac{L(\gamma(t)) - L(\gamma(0))}{t} = \lim_{t \rightarrow 0^+} \frac{L(\gamma(t)) - L(J(p^*))}{t} \geq 0 \quad (5.11)$$

with the above inequality holding as  $J(p^*)$  minimizes  $L$  for  $p \in \mathcal{M}$ . Let

$$\gamma(t) = \mathcal{P}_{\tilde{\mathcal{M}}}\left(J(p^*) + t(J(\omega) - J(p^*))\right),$$

where  $\mathcal{P}$  is the projection of  $\mathbb{R}^D$  onto  $\tilde{\mathcal{M}}$ , that is,  $\mathcal{P}(x) = \arg \min_{y \in \tilde{\mathcal{M}}} \rho(y, x)$ . We assume the projection map  $\mathcal{P}$  is differentiable at  $t = 0$ . Denote  $\mathcal{J}$  as the Jacobian matrix of the projection map  $\mathcal{P}$  at  $J(p^*)$ . Then one has

$$v = \gamma'(0) = \mathcal{J}(J(\omega) - J(p^*)),$$

which will be needed in determining the constant  $C_\alpha$ . One can see that

$$L(\gamma(t)) - L(J(p^*)) = \sum_{j=1}^m (\|\gamma(t) - J(p_j)\| - \|\gamma(0) - J(p_j)\|).$$

Let

$$A_j = \frac{\|\gamma(t) - J(p_j)\| - \|\gamma(0) - J(p_j)\|}{t} \text{ for } j = 1, \dots, m.$$

Then

$$A_j = \frac{\|\gamma(t) - J(p_j)\|^2 - \|\gamma(0) - J(p_j)\|^2}{t(\|\gamma(t) - J(p_j)\| + \|\gamma(0) - J(p_j)\|)} \text{ for } j = 1, \dots, m.$$

One has

$$\lim_{t \rightarrow 0^+} (\|\gamma(t) - J(p_j)\| + \|\gamma(0) - J(p_j)\|) = 2\|\gamma(0) - J(p_j)\|. \quad (5.12)$$

Also,

$$\|\gamma(t) - J(p_j)\|^2 = \langle \gamma(t) - J(p_j), \gamma(t) - J(p_j) \rangle = \langle \gamma(t), \gamma(t) \rangle - 2\langle \gamma(t), J(p_j) \rangle + \langle J(p_j), J(p_j) \rangle,$$

and

$$\|\gamma(0) - J(p_j)\|^2 = \langle \gamma(0), \gamma(0) \rangle - 2\langle \gamma(0), J(p_j) \rangle + \langle J(p_j), J(p_j) \rangle.$$

Then

$$\begin{aligned} & \|\gamma(t) - J(p_j)\|^2 - \|\gamma(0) - J(p_j)\|^2 \\ &= \langle \gamma(t), \gamma(t) \rangle - \langle \gamma(0), \gamma(0) \rangle - 2\langle \gamma(t) - \gamma(0), J(p_j) \rangle \\ &= (\langle \gamma(t), \gamma(t) \rangle - \langle \gamma(0), \gamma(t) \rangle) + (\langle \gamma(0), \gamma(t) \rangle - \langle \gamma(0), \gamma(0) \rangle) - 2\langle \gamma(t) - \gamma(0), J(p_j) \rangle \\ &= \langle \gamma(t) - \gamma(0), \gamma(t) \rangle + \langle \gamma(0), \gamma(t) - \gamma(0) \rangle - 2\langle \gamma(t) - \gamma(0), J(p_j) \rangle \\ &= \langle \gamma(t) - \gamma(0), \gamma(t) + \gamma(0) - 2J(p_j) \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{\|\gamma(t) - J(p_j)\|^2 - \|\gamma(0) - J(p_j)\|^2}{t} &= \lim_{t \rightarrow 0^+} \left\langle \frac{\gamma(t) - \gamma(0)}{t}, \gamma(t) + \gamma(0) - 2J(p_j) \right\rangle \\ &= \langle \gamma'(0), \gamma(0) + \gamma(0) - 2J(p_j) \rangle \\ &= 2\langle \gamma'(0), \gamma(0) - J(p_j) \rangle = 2\langle \gamma'(0), J(p^*) - J(p_j) \rangle. \end{aligned}$$

Thus, by (5.12) and the above equation, if  $J(p_j) \neq J(p^*)$ , one has

$$\lim_{t \rightarrow 0^+} A_j = \frac{\langle \gamma'(0), J(p^*) - J(p_j) \rangle}{\|J(p^*) - J(p_j)\|}.$$

Otherwise, if  $J(p_j) = J(p^*)$ , then

$$\lim_{t \rightarrow 0^+} A_j = \lim_{t \rightarrow 0^+} \frac{\|\gamma(t) - J(p_j)\|}{t} = \|\gamma'(0)\|.$$

Therefore,

$$dL_{J(p^*)}(v) = \sum_{j=1}^m \lim_{t \rightarrow 0^+} A_j = \sum_{j:p_j \neq p^*} \frac{\langle \gamma'(0), J(p^*) - J(p_j) \rangle}{\|J(p^*) - J(p_j)\|} + \|\gamma'(0)\| \sum_{j=1}^m I(p_j = p^*),$$

where  $I(\cdot)$  is the indicator function. The above implies

$$\frac{dL_{p^*}(v)}{\|\gamma'(0)\|} = \sum_{j=1}^m \lim_{t \rightarrow 0^+} \frac{A_j}{\|\gamma'(0)\|} = \sum_{j:p_j \neq p^*} \frac{\langle \gamma'(0), J(p^*) - J(p_j) \rangle}{\|\gamma'(0)\| \|J(p^*) - J(p_j)\|} + \sum_{j=1}^m I(p_j = p^*). \quad (5.13)$$

The Jacobian matrix of the projection map  $\mathcal{P}$  at  $J(p^*)$ ,  $\mathcal{J}$ , is the orthogonal projection of  $T_{J(p^*)}\mathbb{R}^D \equiv \mathbb{R}^D$  to  $T_{J(p^*)}\tilde{\mathcal{M}}$ . That is, for  $a \in T_{J(p^*)}\mathbb{R}^D$ ,  $\mathcal{J}(a) = a_1$ , where  $a = a_1 + a_2$  is the unique orthogonal decomposition of  $a$  with  $a_1 \in T_{J(p^*)}\tilde{\mathcal{M}}$ . Now assume that there does *not* exist an  $\alpha$  portion of elements of  $p_1, \dots, p_m$  which are at least  $\epsilon$  distance away from  $\omega$ , that is, without loss of generality,

$$\|J(p_j) - J(\omega)\| \leq \epsilon \text{ for } j = 1, \dots, \lfloor(1-\alpha)m\rfloor + 1.$$

Let us denote by  $\angle(J(\omega) - J(p^*), J(p_j) - J(p^*))$  the angle between the vectors  $J(\omega) - J(p^*)$  and  $J(p_j) - J(p^*)$ . Then for  $j = 1, \dots, \lfloor(1-\alpha)m\rfloor + 1$ ,

$$\sin(\angle(J(\omega) - J(p^*), J(p_j) - J(p^*))) < \frac{1}{C_\alpha}$$

and so

$$\cos \left( \angle(J(\omega) - J(p^*), J(p_j) - J(p^*)) \right) > \sqrt{1 - \frac{1}{C_\alpha^2}}.$$

Notice that

$$\begin{aligned} & \angle(J(J(\omega)) - J(p^*), J(\omega) - J(p^*)) + \angle(J(\omega) - J(p^*), J(p_j) - J(p^*)) \\ &= \psi + \angle(J(\omega) - J(p^*), J(p_j) - J(p^*)) \geq \angle(J(J(\omega)) - J(p^*), J(p_j) - J(p^*)). \end{aligned}$$

Therefore,

$$\begin{aligned} \cos \left( \angle(J(J(\omega)) - J(p^*), J(p_j) - J(p^*)) \right) &\geq \cos \left( \psi + \angle(J(\omega) - J(p^*), J(p_j) - J(p^*)) \right) \\ &> \sqrt{1 - \frac{1}{C_\alpha^2}} \cos \psi - \frac{1}{C_\alpha} \sin \psi. \end{aligned}$$

We have

$$\frac{\langle \gamma'(0), p_j - J(p^*) \rangle}{\|\gamma'(0)\| \|p_j - J(p^*)\|} = \cos \left( \angle(J(J(\omega)) - J(p^*), J(p_j) - J(p^*)) \right) > \sqrt{1 - \frac{1}{C_\alpha^2}} \cos \psi - \frac{1}{C_\alpha} \sin \psi.$$

Then for any  $\alpha \in (0, \cot \psi \tan \frac{\psi}{2})$  from (5.13)

$$\frac{dL_{J(p^*)}(v)}{\|\gamma'(0)\|} < -(1-\alpha)m \left( \sqrt{1 - \frac{1}{C_\alpha^2}} \cos \psi - \frac{1}{C_\alpha} \sin \psi \right) + \alpha m \leq 0,$$

when

$$C_\alpha \geq \frac{1-\alpha}{\sqrt{1-2\alpha} \cos \psi - \alpha \sin \psi}$$

which is a contradiction with (5.11).

(b) The intrinsic median requires a different proof. Let  $L(p) = \sum_{j=1}^m \rho(p, p_j)$  where  $\rho$  is the

intrinsic distance; we use the Riemannian exponential map  $\exp_{p^*} : T_{p^*}\mathcal{M} \rightarrow \mathcal{M}$ . Let  $v = \log_{p^*} \omega \in T_{p^*}\mathcal{M}$  and consider the geodesic curve  $\gamma(t) = \exp_{p^*}(tv)$ . Then

$$dL_{p^*}(v) = \lim_{t \rightarrow 0} \frac{L(\gamma(t)) - L(\gamma(0))}{t} = \lim_{t \rightarrow 0} \frac{L(\gamma(t)) - L(p^*)}{t} \geq 0. \quad (5.14)$$

Denote

$$A = \lim_{t \rightarrow 0+} \sum_{j=1}^m \left( \frac{\sqrt{\langle \gamma_{js}(s, t), \gamma_{js}(s, t) \rangle} - \sqrt{\langle \gamma_{js}(s, 0), \gamma_{js}(s, 0) \rangle}}{t} \right),$$

where  $\gamma_j(s, t) = \exp_{\gamma(t)}(s \log_{\gamma(t)} p_j) = \exp_{\gamma(t)}(sv_j(t))$  is the geodesic curve connecting  $\gamma(t)$  with  $p_j$ ,

then  $\gamma_{js}(s, t) = \frac{\partial \gamma_j(s, t)}{\partial s}$ . Set

$$A_j = \frac{\sqrt{\langle \gamma_{js}(s, t), \gamma_{js}(s, t) \rangle} - \sqrt{\langle \gamma_{js}(s, 0), \gamma_{js}(s, 0) \rangle}}{t}, \text{ for } j = 1, \dots, m.$$

Then

$$A_j = \frac{1}{t} \frac{\langle \gamma_{js}(s, t), \gamma_{js}(s, t) \rangle - \langle \gamma_{js}(s, 0), \gamma_{js}(s, 0) \rangle}{\sqrt{\langle \gamma_{js}(s, t), \gamma_{js}(s, t) \rangle} + \sqrt{\langle \gamma_{js}(s, 0), \gamma_{js}(s, 0) \rangle}}, \text{ for } j = 1, \dots, m.$$

We see that

$$\lim_{t \rightarrow 0+} \left( \sqrt{\langle \gamma_{js}(s, t), \gamma_{js}(s, t) \rangle} + \sqrt{\langle \gamma_{js}(s, 0), \gamma_{js}(s, 0) \rangle} \right) = 2\sqrt{\langle \gamma_{js}(s, 0), \gamma_{js}(s, 0) \rangle}.$$

On the other hand,

$$\begin{aligned} \lim_{t \rightarrow 0+} \frac{\langle \gamma_{js}(s, t), \gamma_{js}(s, t) \rangle - \langle \gamma_{js}(s, 0), \gamma_{js}(s, 0) \rangle}{t} &= 2 \left\langle \frac{D}{dt} \gamma_{js}(s, 0), \gamma_{js}(s, 0) \right\rangle \\ &= 2 \left\langle \frac{D}{ds} \gamma_{jt}(s, 0), \gamma_{js}(s, 0) \right\rangle = 2 \frac{d}{ds} \langle \gamma_{jt}(s, 0), \gamma_{js}(s, 0) \rangle. \end{aligned}$$

Thus if  $p_j \neq p^*$ , one has

$$\lim_{t \rightarrow 0^+} A_j = \frac{\frac{d}{ds} \langle \gamma_{jt}(s, 0), \gamma_{js}(s, 0) \rangle}{\sqrt{\langle \gamma_{js}(s, 0), \gamma_{js}(s, 0) \rangle}}.$$

Otherwise, if  $p_j = p^*$ , then

$$\lim_{t \rightarrow 0^+} A_j = \lim_{t \rightarrow 0^+} \frac{\sqrt{\langle -t\gamma'((1-s)t), -t\gamma'((1-s)t) \rangle}}{t} = \lim_{t \rightarrow 0^+} \frac{t\|v\|}{t} = \|v\|.$$

Therefore,

$$\begin{aligned} dL_{p^*}(v) &= \sum_{j=1}^m \int_0^1 \lim_{t \rightarrow 0^+} A_j ds \\ &= \sum_{j:p_j \neq p^*} \int_0^1 \frac{\frac{d}{ds} \langle \gamma_{jt}(s, 0), \gamma_{js}(s, 0) \rangle}{\sqrt{\langle \gamma_{js}(s, 0), \gamma_{js}(s, 0) \rangle}} ds + \|v\| \sum_{j=1}^m I(p_j = p^*) \\ &= \sum_{j:p_j \neq p^*} \frac{\langle \gamma_{jt}(1, 0), \gamma_{js}(1, 0) \rangle}{\|v_j\|} + \|v\| \sum_{j=1}^m I(p_j = p^*) \\ &= \sum_{j:p_j \neq p^*} \frac{\langle (d \exp_{p^*})_{v_j} (1 \cdot v'_j(0)), (d \exp_{p^*})_{v_j} v_j \rangle}{\|v_j\|} + \|v\| \sum_{j=1}^m I(p_j = p^*) \\ &= \sum_{j:p_j \neq p^*} \frac{\langle v'_j(0), v_j \rangle}{\|v_j\|} + \|v\| \sum_{j=1}^m I(p_j = p^*) \\ &= - \sum_{j:p_j \neq p^*} \frac{\langle v, v_j \rangle}{\|v_j\|} + \|v\| \sum_{j=1}^m I(p_j = p^*), \end{aligned}$$

where  $I(\cdot)$  is the indicator function. Then one has,

$$\frac{dL_{p^*}(v)}{\|v\|} = - \sum_{j:p_j \neq p^*} \frac{\langle v, v_j \rangle}{\|v\| \|v_j\|} + \sum_{j=1}^m I(p_j = p^*) = - \sum_{j:p_j \neq p^*} \cos(\widehat{v}, \widehat{v}_j) + \sum_{j=1}^m I(p_j = p^*).$$

From the condition that  $\log_{p^*}$  is  $K$ -Lipschitz continuous from  $B(\omega, r)$  to  $T_{p^*}\mathcal{M}$ ,

$$\|v_j - v\| \leq K d_g(\exp_{p^*} v_j, \exp_{p^*} v).$$

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Then this yields

$$\frac{dL_{p^*}(v)}{\|v\|} < -(1-\alpha)m\sqrt{1 - \frac{K^2}{C_\alpha^2}} + \alpha m \leq 0,$$

whenever  $C_\alpha \geq K(1-\alpha)\sqrt{\frac{1}{1-2\alpha}}$ , which leads to a contradiction with (5.14).  $\square$

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