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Double Happiness: Enhancing the Coupled Gains of L-lag Coupling via Control Variates

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Abstract: The recently proposed L-lag coupling for unbiased MCMC (Biswas et al., 2019; Jacob et al., 2020) calls for a joint celebration by MCMC practitioners and theoreticians. For practitioners, it circumvents the thorny issue of deciding the burn-in period or when to terminate an MCMC sampling process, and opens the door for safe parallel implementation. For theoreticians, it provides a powerful tool to establish elegant and easily estimable bounds on the exact error of MCMC approximation at any finite number of iterates. A serendipitous observation about the bias correcting term leads us to introduce naturally available control variates into the L-lag coupling estimators. In turn, this extension enhances the coupled gains of L-lag coupling, because it results in more efficient unbiased estimators as well as a better bound on the total variation error of MCMC iterations, albeit the gains diminish with the numerical value of L. Specifically, the new bound is theoretically guaranteed to never exceed the one given previously. We also argue that L-lag coupling represents a long-sought-after coupling for the future, breaking a logjam of the coupling-from-the-past type of perfect sampling, by reducing the generally un-achievable requirement of being *perfect* to being

1. IF BEING PERFECT IS IMPOSSIBLE, LET'S TRY BEING UNBIASED

unbiased, a worthwhile trade-off for ease of implementation in most practical situations. The theoretical analysis is supported by numerical experiments that show tighter bounds and a gain in efficiency when control variates are introduced.

Key words and phrases: Coupling from the Past, Maximum coupling, Median absolute deviation, Parallel implementation, Total variation distance, Unbiased MCMC.

1. If Being Perfect Is Impossible, Let's Try Being Unbiased

1.1 Perfect Coupling – Too Much To Hope For?

We thank Pierre Jacob and his team for a series of tantalizing articles (e.g., Jacob et al., 2020, 2019; Heng and Jacob, 2019; Biswas et al., 2019) that revitalize the excitement we experienced (e.g., Murdoch and Meng, 2001; Meng, 2000; Craiu and Meng, 2011; Stein and Meng, 2013) when working on the Coupling From The Past (CFTP; Propp and Wilson, 1996, 1998) and, more generally, perfect sampling. This time, we are dealing more with a form of Coupling For The Future (CFTF), literally and figuratively, as we shall discuss. The clever “cross-time coupling” idea of Glynn and Rhee (2014), on which CFTF is based upon, allows us to move away from the CFTP framework, which engulfed the MCMC (Markov chain Monte Carlo) community just around the turn of the century because of its promise to provide perfect/exact MCMC samplers (see, for instance, the annotated bibliography of Wilson,

1. IF BEING PERFECT IS IMPOSSIBLE, LET'S TRY BEING UNBIASED

1998). However, the research progress on perfect or exact samplers has slowed down significantly in the new millennium because they are very challenging, if not impossible, to develop for many routine Bayesian computational problems (see, for example, Murdoch and Meng, 2001).

In its most basic form, a CFTP-type perfect sampler couples a Markov chain $\{X_t, t \geq 0\}$ with itself but from different starting points, and runs two or more chains until they coalesce at a time τ . This apparent convergence does not guarantee in general that X_τ is from the desired stationary distribution $\pi(x)$. By shifting the entire chain to “negative time” (i.e., the past), $\{X_t, t \leq 0\}$, Propp and Wilson (1996) have shown that if we follow this coalescent chain until it reaches the present time, that is, $t = 0$, then the resulting X_0 will be exactly from $\pi(x)$. Perhaps the most intuitive way to understand this scheme is to realize that running a chain from its infinite past ($t = -\infty$) to present ($t = 0$) is mathematically equivalent to running the chain from the present ($t = 0$) to the infinite future ($t = +\infty$). The CFTP is a clever way to realize this seemingly impossible task, relying on the fact that if the coalescence occurs regardless how we have chosen the starting point, then the chain has “forgotten” its origin and hence settled in the perfect asymptotic land.

But there is no free lunch. Being perfect is never easy, especially in the mathematical sense. No error of any kind is allowed, and this requirement has manifested in two ways that greatly limit the practicality of perfect sampling. Firstly, constructing

1. IF BEING PERFECT IS IMPOSSIBLE, LET'S TRY BEING UNBIASED

a perfect sampler, especially for distributions with continuous and unbounded state spaces—which are ubiquitous in routine statistical applications—is a very challenging task in general, despite its great success for problems with some special structures such as certain monotonic properties (see Berthelsen and Møller, 2002; Corcoran and Tweedie, 2002; Huber, 2004, 2002; Ensor and Glynn, 2000; Huber, 2004; Murdoch and Takahara, 2006, for example). Secondly, even if a perfect sampler is devised, it can be excruciatingly slow because it refuses to deliver an output until it can guarantee its perfection, and one must devise problem-specific strategies for speed-up (e.g., Thönnies, 1999; Dobrow and Fill, 2003; Møller, 1999; Dobrow and Fill, 2003; Corcoran and Schneider, 2005).

1.2 Unbiased Coupling – A New Hope?

A relaxation of the exact sampling paradigm with important practical consequences has been proposed by Glynn and Rhee (2014) and Glynn (2016), who put forth strategies for exact estimation of integrals via Markov chain Monte Carlo. The contrast between exact sampling and exact estimation is a very large conceptual leap that allows us to bypass most of the difficulties of perfect sampling while maintaining some of its important benefits. Building on the work of Glynn and co-authors, the L-lag coupling of Biswas et al. (2019) and Jacob et al. (2020) aims to deliver unbiased estimators of $E[h(X_\pi)]$ for any (integrable) h , where X_π denote a random variable

1. IF BEING PERFECT IS IMPOSSIBLE, LET'S TRY BEING UNBIASED

defined by $\pi(X)$. An astute reader may immediately question if this is really a weaker requirement because the fact that $E[h(X_\pi)] = E[h(X_{\tilde{\pi}})]$ for all (integrable) h , would immediately imply $\pi(X) = \tilde{\pi}(X)$ (almost surely). However, this is where the innovation of L-lag coupling lies, because it does not couple a chain with itself from two or more starting points [e.g., two extreme states as with monotone coupling; see Propp and Wilson (1996)], but rather two chains which have the same transition probability, start from the same starting point, or more generally the same initial distribution π_0 , but are time-shifted by an integer lag, $L > 0$.

To illustrate, consider the case of $L = 1$, which was the focus of Jacob et al. (2020), where two chains $\mathcal{X} = \{X_t, t \geq 0\}$ and $\mathcal{Y} = \{Y_t, t \geq 0\}$ are coupled in such a way that both of them have the same transition kernel (and hence the same target stationary distribution), and that there exists with probability one a finite stopping time τ such that $X_t = Y_{t-1}$ for all $t \geq \tau$. This construction allows them to show that the following estimator based on both \mathcal{X} and \mathcal{Y} ,

$$H_k(\mathcal{X}, \mathcal{Y}) = h(X_k) + \sum_{j=k+1}^{\tau-1} [h(X_j) - h(Y_{j-1})] \quad (1.1)$$

is an unbiased estimator for $E[h(X_\pi)]$ for any $k \geq 0$ (under mild conditions). Heuristically, this is because the sum in (1.1) is the same as $\sum_{j=k+1}^{\infty} [h(X_j) - h(Y_{j-1})]$ since any term with $j \geq \tau$ must be zero by the coupling scheme. Furthermore, for the purpose of calculating expectations, we can replace $h(Y_{j-1})$ by $h(X_{j-1})$ for any j because X_{j-1} and Y_{j-1} have identical distributions by construction. However,

1. IF BEING PERFECT IS IMPOSSIBLE, LET'S TRY BEING UNBIASED

$h(X_k) + \sum_{j=k+1}^{\infty} [h(X_j) - h(X_{j-1})]$ is nothing but $\lim_{t \rightarrow \infty} h(X_t)$, which has the same distribution as $h(X_\pi)$.

The cleverness of constructing an estimator based on *both* \mathcal{X}, \mathcal{Y} to ensure $E[H_k(\mathcal{X}, \mathcal{Y})] = E[h(X_\pi)]$ for any h bypasses the requirement that X_τ itself must be perfect. The series of illustrative and practical examples in Jacob et al. (2020) as well as Jacob et al. (2019), Heng and Jacob (2019) and Biswas et al. (2019) provide good evidence of the practicality of this approach. The use of parallel computation for estimating $I = E_\pi[h(X)]$ supports using $E[E_\pi[h(x)|\mathcal{U}_j]]$ where the inner expectation is the estimate obtained from the j th parallel process, \mathcal{U}_j , and the outer mean averages over all processes. However, if each inner mean is a biased estimator for I , then the accumulation of errors can be seriously misleading. This has been documented in the Monte Carlo literature extensively, for instance, in Glynn and Heidelberger (1991) and Nelson (2016). Hence, unbiased MCMC designs allow one to take full advantage of parallel computation strategies without having to worry about the accumulation of bias as the number of parallel processes increases.

1.3 Using Control Variates – Even Higher Hope?

The expression (1.1) also opens a path to explore further improvements, and that is the starting point of our exploration. In Craiu and Meng (2020), we noticed that

1. IF BEING PERFECT IS IMPOSSIBLE, LET'S TRY BEING UNBIASED

(1.1) can be expressed equivalently as

$$H_k(\mathcal{X}, \mathcal{Y}) = h(X_{(\tau-1) \vee k}) + \sum_{j=k}^{\tau-2} [h(X_j) - h(Y_j)], \quad (1.2)$$

where $A \vee B = \max\{A, B\}$. Expression (1.1) renders the insight underlying Jacob et al. (2020), which is that $H_k(\mathcal{X}, \mathcal{Y})$ achieves the desired unbiasedness by providing a *time-forward bias correction* to $h(X_k)$'s, whenever $\tau > k + 1$; and hence Coupling for the Future. (No correction is needed when $\tau \leq k + 1$.) The dual expression (1.2) indicates that $H_k(\mathcal{X}, \mathcal{Y})$ can also be viewed as a *time-backward bias correction* to $h(X_{\tau-1})$ for its imperfection, because $k < \tau - 1$.

Most intriguingly, each correcting term $\Delta_j \equiv h(X_j) - h(Y_j)$ in (1.2) has mean zero by the construction of $\{\mathcal{X}, \mathcal{Y}\}$. However, the sum $\sum_{j=k}^{\tau-2} [h(X_j) - h(Y_j)]$ does not have mean zero because τ is random and it depends critically on $\{\mathcal{X}, \mathcal{Y}\}$. Indeed, if this sum had mean 0, then $X_{(\tau-1) \vee k}$ would have been a perfect draw from $\pi(X)$ because then $E[h(X_{(\tau-1) \vee k})] = E[h(X_\pi)]$ for any (integrable) h , which would imply that $X_{(\tau-1) \vee k} \sim \pi$.

However, the fact that $E(\Delta_j) = 0$ suggests that we can use any linear combination of Δ_j 's as a *control variate* for $H_k(\mathcal{X}, \mathcal{Y})$. Using control variates to reduce estimation errors is a well known technique in the literature of improving MCMC samplers and estimators via efficiency swindles, such as antithetic and control variates, Rao-Blackwellization, and so on, some of which we have explored in the past (e.g., Van Dyk and Meng, 2001; Craiu and Meng, 2001, 2005; Craiu and Lemieux,

2. THEORETICAL GAINS FROM INCORPORATING CONTROL VARIATES

2007; Yu and Meng, 2011). For example, for any finite constant $\eta > k + 1$, the estimator

$$H_k^*(\mathcal{X}, \mathcal{Y}; \eta) = H_k(\mathcal{X}, \mathcal{Y}) - \sum_{j=k}^{\eta-2} \Delta_j = h(X_{(\tau-1) \vee k}) + \sum_{j=k}^{\tau-2} \Delta_j - \sum_{j=k}^{\eta-2} \Delta_j, \quad (1.3)$$

shares the mean of $H_k(\mathcal{X}, \mathcal{Y})$, but can have a smaller variance with a judicious choice of η . Intuitively, this reduction of variance is possible because of the potential partial cancellation (on average) of the Δ_j terms in the last two summations in (1.3).

Indeed, Section 2 below investigates a more general class of control variates, and derives the optimal choice by establishing the minimal upper bound within the class on the total variation distance between the target π and π_k , the distribution of X_k . This leads to both an improved theoretical bound over the one in Biswas et al. (2019), as reported in Section 2, as well as a more efficient estimator than (1.1) via parallel implementation. Section 3 describes estimation methods and algorithms, and Section 4 provides examples and illustrations of both kinds of gains. Section 5 completes our first tour of CFTF with a discussion of future work.

2. Theoretical Gains From Incorporating Control Variates

2.1 L-lag Coupling: An Elegant and Powerful Method

The scheme of L-lag coupling extends the coupling of $\{X_k, Y_{k-1}\}$ to the more general form of coupling of $\{X_k, Y_{k-L}\}$ for some fixed $L \geq 1$, as detailed in Biswas et al.

2. THEORETICAL GAINS FROM INCORPORATING CONTROL VARIATES

(2019). The significance of this extension can be best understood by expressing the L -lag coupling idea in its mathematically equivalent form of seeking τ_L such that $X_{k+L} = Y_k$ for all $k \geq \tau_L$, and letting $L \rightarrow \infty$ while keeping k fixed. Heuristically, it is then clear that the larger the L , the closer is the distribution of Y_{τ_L} to the target because X_{τ_L+L} should converge to $X_\infty \sim \pi$ as $L \rightarrow \infty$, and \mathcal{X} and \mathcal{Y} share the same target π .

Indeed, by extending (1.1) to a general L , Biswas et al. (2019) have shown that (under mild regularity conditions) the total variation distance between π_k , the distribution of X_k , and π is bounded by a very simple function of τ_L and (k, L) :

$$d_{\text{TV}}(\pi_k, \pi) \leq \mathbb{E}[J_{k,L}], \quad \text{where } J_{k,L} = \max \left\{ 0, \left\lceil \frac{\tau_L - L - k}{L} \right\rceil \right\}, \quad (2.1)$$

and $\lceil a \rceil$ denotes the smallest integer that is equal to or exceeds a . We can clearly see the impact of increasing L or k , since larger values of either of them make it more likely that $\tau_L - L - k < 0$ and hence $J_{k,L} = 0$. Perhaps a crystal clear demonstration of this fact is when τ_L follows a geometric distribution with success probability p and state space $\{L + i, i \geq 0\}$ (since $\tau_L \geq L$ by definition), or equivalently $\delta = \tau - (L - 1) \sim \text{Geo}(p)$. Then, letting $q = 1 - p$, we have (see Biswas et al., 2019)

$$d_{\text{TV}}(\pi_k, \pi) \leq \mathbb{E}[J_{k,L}] = \frac{q^{k+1}}{1 - q^L}. \quad (2.2)$$

We see that the bound is a decreasing function of both k and L , though it decreases much faster with k , which controls the rate of convergence, than with L , which con-

2. THEORETICAL GAINS FROM INCORPORATING CONTROL VARIATES

trols only the (constant) scaling factor. We also observe that the bound can be trivial since it can be larger than 1 for small k and/or L , whereas d_{TV} cannot, suggesting there is room for improvement. Nevertheless, (2.1) is a remarkable bound because it encodes all the intricacies relevant for the convergence speed of \mathcal{X} , including the choice of X_0 , into a uni-variate (truncated) coupling time $J_{k,L}$. In the case of (2.2), the bound also immediately establishes the geometric ergodicity of \mathcal{X} , and provides a rather practical way to assess the bound by estimating p or more generally by assessing $J_{k,L}$ directly, say from a parallel implementation (see Section 3).

It is perhaps even more remarkable to see that the left hand side of (2.1) is a property of the marginal chain \mathcal{X} (and, equivalently, of the \mathcal{Y} chain), but its right hand side depends on the construction of the joint chain $\{\mathcal{X}, \mathcal{Y}\}$. This suggests that we can seek improvement by better coupling. Furthermore, as we establish below, even without changing the coupling scheme we can still obtain better bounds by using more efficient estimators than (1.1).

For a general L , the forward-correction expression in (1.1) becomes (Biswas et al., 2019),

$$H_{k,L}(\mathcal{X}, \mathcal{Y}) = h(X_k) + \sum_{j=1}^{J_{k,L}} [h(X_{k+jL}) - h(Y_{k+(j-1)L})], \quad (2.3)$$

and it is easy to verify that the backward-correction expression (1.2) takes the form

$$H_{k,L}(\mathcal{X}, \mathcal{Y}) = h(X_{k+LJ_{k,L}}) + \sum_{j=0}^{J_{k,L}-1} [h(X_{k+jL}) - h(Y_{k+jL})]. \quad (2.4)$$

2. THEORETICAL GAINS FROM INCORPORATING CONTROL VARIATES

Remark 1: The (random) subscript in X_{k+JL} cannot be reduced to $(\tau - L) \vee k$ when $L > 1$, the most obvious extension of the index $(\tau - 1) \vee k$ in (1.2). This is because $k + J_{k,L}L \geq (\tau - L) \vee k$, but the inequality can be strict when $\tau > k + L$. For example, if $\tau = L + k + M$, where M is a positive integer less than L (which does not exist when $L = 1$), $k + J_{k,L}L = k + L$, but $(\tau - L) \vee k = k + M$.

Remark 2: Whereas (2.3) and (2.4) are equivalent as equalities, they can and will lead to different inequalities depending on how we bound their respective right-hand sides. This is both a bonus and a trap, as we shall discuss below.

2.2 Deriving the Optimal Bound Over Choices of Control Variates

For notational simplicity, we drop the variables k, L from the notation of $J_{k,L}$ and we let $\Delta_{k,j} = h(X_{k+jL}) - h(Y_{k+jL})$. Then we know $\Delta_{k,j}$ has mean zero for any $\{k, j\}$ and L . This means that for any random sequence $\vec{\eta} \equiv \{\eta_j, j \geq 1\}$ such that: (A) it is independent of $\{\mathcal{X}, \mathcal{Y}\}$, and (B) $\sum_{j=1} \mathbb{E}_{\vec{\eta}} |\eta_j| < \infty$, we can use $C_{\vec{\eta}} = \sum_{j \geq 1} \eta_j \Delta_{k,j}$ as a control variate for $H_{k,L} \equiv H_{k,L}(\mathcal{X}, \mathcal{Y})$, because $\mathbb{E}[C_{\vec{\eta}}] = 0$. That is,

$$\tilde{H}_{k,L}^{(\vec{\eta})}(\mathcal{X}, \mathcal{Y}) = H_{k,L}(\mathcal{X}, \mathcal{Y}) - \sum_{j \geq 1} \eta_j \Delta_{k,j} \quad (2.5)$$

will also be an unbiased estimator of $\mathbb{E}[h(X_{\tau})]$ with a smaller variance than (2.4).

The question is: what will be a good choice of $\vec{\eta}$?

Instead of seeking $\vec{\eta}$ to minimize $\text{Var} \left[\tilde{H}_{k,L}^{(\vec{\eta})} \right]$, which is not an easy task and will also likely produce an h -dependent solution, we follow the argument used by Biswas

2. THEORETICAL GAINS FROM INCORPORATING CONTROL VARIATES

et al. (2019). They obtained their bound given in (2.1) to seek $\vec{\eta}$ that minimizes a class of bounds of $d_{\text{TV}}(\pi_t, \pi)$ over the choice of $\vec{\eta}$ that satisfies (A) and (B). This will lead to a sharper bound than (2.1), a special case corresponding to the special case of $\vec{\eta} = 0$, which in general is not an optimal choice, as shown below.

We proceed by using the same argument as in Biswas et al. (2019) for proving (2.1), but using (2.5) instead of (2.3). However, when applying (2.5), we must retain the expression of $H_{k,L}(\mathcal{X}, \mathcal{Y})$ as given by (2.3). (Interested readers are invited to try out and see what would happen if they use (2.4).) Specifically, the unbiasedness of (2.5) implies that, for any $k \geq 1$,

$$\begin{aligned} \mathbb{E}[h(X_\pi) - h(X_k)] &= \mathbb{E} \left\{ \sum_{j=1}^J [h(X_{k+jL}) - h(Y_{k+(j-1)L})] - \sum_{j \geq 1} \eta_j \Delta_{k,j} \right\} \\ &= \mathbb{E} \left\{ \sum_{j \geq 1} [h(X_{k+jL}) - h(Y_{k+(j-1)L})] 1_{\{j \leq J\}} - \sum_{j \geq 1} \eta_j [h(X_{k+jL}) - h(Y_{k+jL})] \right\} \quad (2.6) \\ &= \mathbb{E} \left\{ \sum_{j \geq 1} h(X_{k+jL}) [1_{\{j \leq J\}} - \eta_j] + \sum_{j \geq 1} h(Y_{k+jL}) [\eta_j - 1_{\{j+1 \leq J\}}] - h(Y_k) 1_{\{0 < J\}} \right\}. \end{aligned}$$

The interchanges of sum and expectation in the (infinite) sums hold under assumption (B) and the additional assumption that the h function is bounded. For computing the total variation distance, let $h \in \mathcal{H} = \{h : \sup_x |h(x)| \leq 1/2\}$, as in Biswas et al. (2019). Consequently, (2.6) implies

$$\begin{aligned} d_{\text{TV}}(\pi_k, \pi) &\leq \frac{1}{2} \left\{ \sum_{j \geq 1} \mathbb{E} |1_{\{j \leq J\}} - \eta_j| + \sum_{j \geq 1} \mathbb{E} |\eta_j - 1_{\{j \leq J-1\}}| + \Pr(0 < J) \right\} \\ &= \sum_{j \geq 1} \mathbb{E} |1_{\{j \leq J\}} - \eta_j| + 0.5 \Pr(J > 0), \quad (2.7) \end{aligned}$$

2. THEORETICAL GAINS FROM INCORPORATING CONTROL VARIATES

where $\tilde{J} = J - \xi$ and $\xi \sim \text{Bernoulli}(0.5)$ is independent of J . Note that the support for \tilde{J} is $\{-1, 0, 1, \dots\}$. Set

$$S_j = \Pr(\tilde{J} \geq j) = \Pr(J > j) + 0.5\Pr(J = j), \quad \text{for any } j \geq 0. \quad (2.8)$$

Recall that for any given random variable V , $\min_{U \perp V} E|V - U| = E|V - m_V|$, where m_V is a median of V and the notation $\min_{U \perp V}$ means to minimize over all U 's that are independent of V . Hence, in order to minimize (2.7) over $\vec{\eta}$, we should set η_j to be the median of the Bernoulli random variable $1_{\{j \leq \tilde{J}\}}$, i.e., $\eta_j = 1_{\{S_j > 0.5\}}$.

Let $m_{\tilde{J}}$ be the smallest integer median of \tilde{J} . Then, for any $j > m_{\tilde{J}}$, $S_j = 1 - \Pr(\tilde{J} < j) \leq 1 - \Pr(\tilde{J} \leq m_{\tilde{J}}) \leq 1/2$ because $\Pr(\tilde{J} \leq m_{\tilde{J}}) \geq 1/2$ by the definition of $m_{\tilde{J}}$, implying $\eta_j = 0$. Therefore we know the maximal number of non-zero η_j 's cannot exceed $m_{\tilde{J}}$. However, other than the ideal case with $\Pr(J = 0) = 1$, $m_{\tilde{J}}$ can be zero, but not -1 because $\Pr(\tilde{J} = -1) = 0.5\Pr(J = 0) < 0.5$. This automatically implies that condition (B) is trivially satisfied. For this choice of $\vec{\eta}$, (2.7) yields our new bound for $d_{\text{TV}}(\pi_k, \pi)$:

$$B_{k,L} = \sum_{j \geq 1} \min\{S_j, 1 - S_j\} + 0.5\Pr(J > 0) \quad (2.9)$$

$$= \sum_{j \geq 1} \min\{\Pr(J \geq j), \Pr(J \leq j)\}. \quad (2.10)$$

In deriving the last equality we use the fact that $S_j + 0.5\Pr(J = j) = \Pr(J \geq j)$ and $1 - S_j + 0.5\Pr(J = j) = \Pr(J \leq j)$, and $\Pr(J > 0) = \sum_{j \geq 1} \Pr(J = j)$.

2. THEORETICAL GAINS FROM INCORPORATING CONTROL VARIATES

2.3 Understand and Compare the Bounds

It is immediate from expression (2.10) that our new bound cannot exceed the bound of Biswas et al. (2019) as given in (2.1) because (2.10) obviously cannot exceed $\sum_{j \geq 1} \Pr(J \geq j)$, which is $E[J]$. The next result reveals alternative forms for the new bound, providing additional insights, including the optimality of the choice $\eta_j = 1_{\{j \leq m_j\}}$.

Theorem 1. *Under the same regularity conditions as in Biswas et al. (2019), we have*

$$B_{k,L} = E |\tilde{J}_{k,L} - m_{\tilde{J}_{k,L}}| + \Pr(J_{k,L} > 0) - 0.5 \quad (2.11)$$

$$= E |J_{k,L} - m_{J_{k,L}}| + \Pr(J_{k,L} > 0) - S_{k,L} \quad (2.12)$$

$$= 0.5 \sum_{j \geq 1} [1 - |\Pr(\tau > k + (j+1)L) + \Pr(\tau > k + jL) - 1|] \\ + 0.5 \Pr(\tau > k + L), \quad (2.13)$$

where $S_{k,L} = \max\{\Pr(J_{k,L} > m_{J_{k,L}}), \Pr(J_{k,L} < m_{J_{k,L}})\} \leq 0.5$, $m_{\tilde{J}_{k,L}}$ and $m_{J_{k,L}}$ are respectively the smallest integer median of $\tilde{J}_{k,L}$ and $J_{k,L}$.

Proof. To reduce the notation overload, we drop the k, L for J, \tilde{J}, m_J and $m_{\tilde{J}}$. We have already established that the optimal $\vec{\eta}$ must be of the form $\eta_j = 1_{\{j \leq m\}}$ for some $m \geq 0$. Note here the use of $m = 0$ permits $\vec{\eta} = 0$ since $j \geq 1$, and it is also consistent with setting $\eta_0 = 1$. We can minimize the right hand side of (2.7) with

2. THEORETICAL GAINS FROM INCORPORATING CONTROL VARIATES

respect to such a class, that is, with respect to the choice of m . But it is easy to see that

$$\begin{aligned}
 \sum_{j \geq 1} \mathbb{E}|1_{\{j \leq \tilde{J}\}} - \eta_j| &= \sum_{j \geq 0} \mathbb{E}|1_{\{j \leq J\}} - 1_{\{j \leq m\}}| - \mathbb{E}[1 - 1_{\{0 \leq \tilde{J}\}}] \\
 &= \sum_{j \geq 0} \mathbb{E} \left[1_{\{\min\{\tilde{J}, m\} < j \leq \max\{\tilde{J}, m\}\}} \right] - \Pr(\tilde{J} = -1) \\
 &= \mathbb{E} \left[\max\{\tilde{J}, m\} - \min\{\tilde{J}, m\} \right] - 0.5 \Pr(J = 0) \\
 &= \mathbb{E}|\tilde{J} - m| - 0.5 \Pr(J = 0). \tag{2.14}
 \end{aligned}$$

It is clear from (2.14) that the optimal m must be an integer median of \tilde{J} and we choose the smallest one, $m_{\tilde{J}}$. With this choice of $\vec{\eta}$, substituting (2.14) into (2.7) yields the expression

$$B_{k,L} = \mathbb{E}|\tilde{J} - m_{\tilde{J}}| - 0.5 \Pr(J = 0) + 0.5 \Pr(J > 0) \tag{2.15}$$

$$= \Pr(J > 0) + \mathbb{E}|\tilde{J} - m_{\tilde{J}}| - 0.5, \tag{2.16}$$

which proves (2.11).

In order to prove (2.12) we start from (2.10). Let

$$G(j) = \Pr(J \leq j) - \Pr(J \geq j) = \Pr(J < j) - \Pr(J > j), \tag{2.17}$$

for $j \geq 0$. Then it is easy to verify that $G(j)$ is a monotone increasing function, so $G(j) - G(m_J)$ share the same sign with $j - m_J$, for all $j \neq m_J$. It follows that the sum in (2.10) can be decomposed into three parts $A = \sum_{j=1}^{m_J-1} \Pr(J \leq j)$,

2. THEORETICAL GAINS FROM INCORPORATING CONTROL VARIATES

$B = 1_{\{m_J > 0\}} \min\{\Pr(J \leq m_J), \Pr(J \geq m_J)\}$, and $C = \sum_{j \geq m_J+1} \Pr(J \geq j)$. When $m_J = 0$, $C = E[J]$, $B = 0$ because $1_{\{m_J > 0\}} = 0$ and $A = 0$ by convention since $m_J - 1 < 1$. If $p_j = \Pr(J = j)$, then it is easy to see that whenever $m_J \geq 1$,

$$\begin{aligned} A &= \sum_{j=1}^{m_J-1} \sum_{h=1}^j p_h + (m_J - 1)p_0 = \sum_{h=1}^{m_J-1} \sum_{j=h}^{m_J-1} p_h + (m_J - 1)p_0 \\ &= \sum_{h=1}^{m_J-1} (m_J - h)p_h + (m_J - 1)p_0 = \sum_{h=0}^{m_J-1} (m_J - h)p_h - p_0; \end{aligned} \quad (2.18)$$

$$C = \sum_{j=m_J+1}^{\infty} \sum_{h=j}^{\infty} p_h = \sum_{h=m_J+1}^{\infty} \sum_{j=m_J+1}^h p_h = \sum_{h=m_J+1}^{\infty} (h - m_J)p_h. \quad (2.19)$$

Noting that $(m_J - h)p_h = 0$ when $h = m_J$, we see that when $m_J \geq 1$

$$\begin{aligned} A + B + C &= E|J - m_J| - p_0 + \min\{\Pr(J \geq m_J), \Pr(J \leq m_J)\} \\ &= E|J - m_J| + \Pr(J > 0) + \min\{\Pr(J \geq m_J), \Pr(J \leq m_J)\} - 1 \\ &= E|J - m_J| + \Pr(J > 0) - \max\{\Pr(J < m_J), \Pr(J > m_J)\}. \end{aligned} \quad (2.20)$$

When $m_J = 0$, $A = B = 0$, and $C = \sum_{h \geq 1} h p_h = E[J]$, which is (2.12) because $S_{k,L} = \Pr(J > 0)$, cancelling exactly the $\Pr(J > 0)$ term. This completes the proof of (2.12).

The proof of (2.13) follows also from (2.10), using the identities $\max\{a, b\} = 0.5[a + b + |a - b|]$ and $\Pr(J \geq j) + \Pr(J \leq j) = 1 + \Pr(J = j)$ for any j . This leads

2. THEORETICAL GAINS FROM INCORPORATING CONTROL VARIATES

to

$$\begin{aligned}
 & \sum_{j \geq 1} \min \{ \Pr(J \geq j), \Pr(J \leq j) \} \\
 = & 0.5 \sum_{j \geq 1} [1 + \Pr(J = j) - | \Pr(J \geq j) - 1 + \Pr(J > j) |] \\
 = & 0.5 \sum_{j \geq 1} [1 - | \Pr(J > j) + \Pr(J > j - 1) - 1 |] + 0.5 \Pr(J > 0)
 \end{aligned}$$

Expression (2.13) then follows because $\{J > j\} = \{\tau > k + (j + 1)L\}$ \square

The above result tells us that, whenever $m_J = 0$, our bound is identical to the one given by Biswas et al. (2019). From (2.10), the two bounds are the same if and only if $G(1) \geq 0$, which is the same as $2p_0 \geq 1 - p_1$, where $p_k = \Pr(J = k)$. Clearly, this inequality is satisfied when $m_J = 0$, that is, when $p_0 \geq 1/2$. It also implies that $m_J \leq 1$, because for any $j < m_J$

$$G(j) = \Pr(J < j) + \Pr(J \leq j) - 1 \leq 2\Pr(J < m_J) - 1 < 0, \quad (2.21)$$

as $\Pr(J \leq m_J - 1) < 0.5$ since m_J is the smallest integer median. Therefore we have

Theorem 2. *Under the same regularity conditions of Theorem 1, a sufficient and necessary condition for the bound in Theorem 1 to equal $E[J]$ is $2p_0 \geq 1 - p_1$.*

Remark 3 Theorem 2 implies that $m_J = 0$ is a sufficient condition, and $m_J \leq 1$ is a necessary condition for the two bounds to be the same. But the condition $m_J = 1$ itself is not sufficient.

3. ESTIMATION AND PRACTICAL IMPLEMENTATION

Remark 4 An intriguing new insight provided by bound (2.12) is that not only the average coupling time matters, the variation of the coupling time is important too. The $S_{k,L}$ term also suggests that even the symmetry matters, since $S_{k,L}$ achieves its maximum when the distribution is symmetrical locally around the median.

Let $\zeta = \tau - t$, which is the number of steps needed after time t in order to couple (assuming the coupling has not already happened by time t). Then the sufficient and necessary condition in Theorem 2 is the same as

$$\Pr(\zeta \leq L) \geq \Pr(\zeta > 2L), \quad (2.22)$$

suggesting that the new bound is more useful when the distribution of ζ places more mass on the right side of the coupling interval $(L, 2L]$ than on its left side, that is, when (2.22) is violated. The implication is that the improvement of the new bound, if any, will more likely come from those situations where either t is small or τ is large (for fixed L), that is, when the mixing is poor.

3. Estimation and Practical Implementation

We assume that $Q > 1$ coupled processes $\{(X_t^{(q)}, Y_t^{(q)}) : 1 \leq q \leq Q\}$ are run in parallel and that, for all $1 \leq q \leq Q$, $\mathcal{X}^{(q)} = \{X_k^{(q)}\}_{k \geq 1}$ and $\mathcal{Y}^{(q)} = \{Y_k^{(q)}\}_{k \geq 1}$ have been successfully L -coupled. The latter implies that the chains $\mathcal{X}^{(q)}$ are run L more steps than the $\mathcal{Y}^{(q)}$ chains and there exists stopping time $\{\tau^{(q)} : q = 1, \dots, Q\}$ such that $X_{t+L}^{(q)} = Y_t^{(q)}$ for all $t \geq \tau^{(q)}$.

3. ESTIMATION AND PRACTICAL IMPLEMENTATION

3.1 Control Variate Estimators

We will work with a modified version of (2.4) that incorporates control variates:

$$\begin{aligned}
 H_{k,L}^{*(q)}(\mathcal{X}^{(q)}, \mathcal{Y}^{(q)}) &= h\left(X_{k+J_{k,L}^{(q)}}^{(q)}\right) + \sum_{j=0}^{J_{k,L}^{(q)}-1} \left[h(X_{k+jL}^{(q)}) - h(Y_{k+jL}^{(q)})\right] \\
 &\quad - \sum_{j=0}^{m_{\tilde{J}_{k,L}^{(q)}}} \left[h(X_{k+jL}^{(q)}) - h(Y_{k+jL}^{(q)})\right], \tag{3.1}
 \end{aligned}$$

where $m_{\tilde{J}}$ denotes the smallest integer median of \tilde{J} . Henceforth, in order to simplify notation, we will use $m_{k,L}^{(q)}$ and $\tilde{m}_{k,L}^{(q)}$ to denote $m_{J_{k,L}^{(q)}}$, respectively, $m_{\tilde{J}_{k,L}^{(q)}}$. An unbiased estimator of $H_{k,L}^{*(q)}(\mathcal{X}^{(q)}, \mathcal{Y}^{(q)})$ is straightforward to produce, but additional care must be paid to maintain the independence between the estimator for $\tilde{m}_{k,L}^{(q)}$ (or $m_{k,L}^{(q)}$) and $(\mathcal{X}^{(q)}, \mathcal{Y}^{(q)})$. To satisfy the latter we construct the unbiased estimators $m_{k,L}^{(q)}$ and $\tilde{m}_{k,L}^{(q)}$ from all coupled processes but the q -th one, as described in Algorithm 1.

Algorithm 1: Algorithm for computing $m_{k,L}^{(q)}$ and $\tilde{m}_{k,L}^{(q)}$ for a fixed k and all $q \in \{1, 2, \dots, Q\}$.

1. Compute $J_{k,L}^{(q)}$;
2. Sample independently $\zeta^{(q)} \sim \text{Bernoulli}(0.5)$ and set $\tilde{J}_{k,L}^{(q)} = J_{k,L}^{(q)} - \zeta^{(q)}$;
3. Set $m_{k,L}^{(q)} = \lfloor \text{med}(\{J_{k,L}^{(h)} : 1 \leq h \leq Q, h \neq q\}) \rfloor$ and $\tilde{m}_{k,L}^{(q)} = \lfloor \text{med}(\{\tilde{J}_{k,L}^{(h)} : 1 \leq h \leq Q, h \neq q\}) \rfloor$, where $\text{med}(A)$ denotes the median of the values in set A and $\lfloor \cdot \rfloor$ is the floor function.

When $L = 1$, in order to reduce the variance of the unbiased estimator H_k in

3. ESTIMATION AND PRACTICAL IMPLEMENTATION

(1.1), Jacob et al. (2020) recommend using the time-averaging estimator

$$H_{k:r}(\mathcal{X}, \mathcal{Y}) = \frac{1}{r-k+1} \sum_{t=k}^r H_t(\mathcal{X}, \mathcal{Y}).$$

We follow the same strategy and consider the time-averaging version of (2.4)

$$\begin{aligned} H_{k:r;L}^{(q)}(\mathcal{X}^{(q)}, \mathcal{Y}^{(q)}) &= \frac{1}{r-k+1} \sum_{t=k}^r h(X_{t+J_{t,L}^{(q)}}^{(q)}) \\ &+ \frac{1}{r-k+1} \sum_{t=k}^r \sum_{j=0}^{J_{t,L}^{(q)}-1} \left[h(X_{t+jL}^{(q)}) - h(Y_{t+jL}^{(q)}) \right] \end{aligned} \quad (3.2)$$

and the average estimator that includes the control-variate swindle is then:

$$H_{k:r;L}^{*(q)}(\mathcal{X}^{(q)}, \mathcal{Y}^{(q)}) = H_{k:r;L}^{(q)}(\mathcal{X}^{(q)}, \mathcal{Y}^{(q)}) - \frac{1}{r-k+1} \sum_{t=k}^r \left\{ \sum_{j=0}^{\tilde{m}_{k,L}^{(q)}} \left[h(X_{t+jL}^{(q)}) - h(Y_{t+jL}^{(q)}) \right] \right\}. \quad (3.3)$$

Note that the original versions are obtained when $k = r$. Because each term in the control-variate term above, $h(X_{t+jL}^{(q)}) - h(Y_{t+jL}^{(q)})$, has mean zero, we expect that the gain from the control variate swindle diminishes when r increases because the law of large numbers would kick in, leading the overall control-variate term approaching zero. We will see this phenomenon in action in Section 4.

3.2 Estimating the Total Variation Bound

When estimating $B_{k,L}$ we can use either (2.11), (2.12), or (2.13). In all our numerical experiments we have used (2.12) using the steps described in Algorithm 2.

3. ESTIMATION AND PRACTICAL IMPLEMENTATION

Algorithm 2: Algorithm for estimation of $B_{k,L}$ for any given k and L .

1. Compute $J_{k,L}^{(q)}$ and $m_{k,L}^{(q)}$ for all $q = 1, \dots, Q$;
2. Compute the empirical means

$$e_{k,L} = \frac{1}{Q} \sum_{q=1}^Q \left| J_{k,L}^{(q)} - m_{k,L}^{(q)} \right|, \quad p_{k,L} = \frac{1}{Q} \sum_{q=1}^Q \mathbf{1}_{\{J_{k,L}^{(q)} > 0\}}$$
$$a_{k,L} = \frac{1}{Q} \sum_{q=1}^Q \mathbf{1}_{\{J_{k,L}^{(q)} > m_{k,L}^{(q)}\}}, \quad b_{k,L} = \frac{1}{Q} \sum_{q=1}^Q \mathbf{1}_{\{J_{k,L}^{(q)} < m_{k,L}^{(q)}\}};$$

3. Compute

$$\hat{B}_{k,L} = e_{k,L} + p_{k,L} - a_{k,L} \vee b_{k,L},$$

where $a \vee b$ denotes the maximum between a and b .

4. EXAMPLES AND ILLUSTRATIONS

In the next section we investigate the performance of the control variate swindle and compare the new total variation bound with (2.1) provided by Biswas et al. (2019).

4. Examples and Illustrations

4.1 A Theoretical Comparison of the Bounds: The Geometric Case

The distribution of the coupling time τ_L is, generally, unknown. However, there is one instance in which the distribution of τ_L is exactly Geometric. Specifically, when coupling two independent Metropolis samplers, the maximal coupling procedure uses the same proposal for both transition kernels and coupling occurs when both chains accept it. The tractability of derivations in the Geometric case allow us to better understand theoretically the relationship between the bound in Biswas et al. (2019) and ours.

We consider the case in which $\delta = \tau - (L - 1) \sim Geo(p)$. Because $J = \max\{0, \lceil \frac{\delta - k - 1}{L} \rceil\}$, we see that

$$\begin{aligned}\Pr(J = 0) &= \Pr(\delta \leq k + 1) = 1 - q^{k+1}, \\ \Pr(J > j) &= \Pr(\delta > k + 1 + Lj) = q^{k+1+Lj}, \quad j = 0, 1, \dots,\end{aligned}\tag{4.1}$$

where $q = 1 - p$. That is, the distribution of J is a mixture of (i) the Dirac point measure $\delta_{\{0\}}$ with mixture proportion $1 - q^{k+1}$, and (ii) a geometric distribution with

4. EXAMPLES AND ILLUSTRATIONS

probability of success $1 - q^L$ with weight q^{k+1} . This implies immediately that the bound given in Biswas et al. (2019) has the expression (2.2).

For our new bound, in this case it is easier to use expression (2.10) directly. Let m be the largest integer such that $\Pr(J \geq m) \geq \Pr(J \leq m)$, that is, m is the largest integer that ensures

$$q^{k+1+L(m-1)} + q^{k+1+Lm} \geq 1 \iff m = \left\lfloor \frac{L - k - 1}{L} - \frac{\log[1 + q^L]}{L \log(q)} \right\rfloor. \quad (4.2)$$

Clearly, when $m \leq 0$, our $B_{k,L}(p)$ is the same as the old bound (2.2). When $m \geq 1$, we have

$$\begin{aligned} B_{k,L}(p) &= \sum_{j=1}^m \Pr(J \leq j) + \sum_{j=m+1}^{\infty} \Pr(J \geq j) \\ \{\text{by (4.1)}\} &= \sum_{j=1}^m [1 - q^{k+1+Lj}] + \sum_{j=m+1}^{\infty} q^{k+1+L(j-1)} \\ &= m - \frac{q^{k+1+L}[1 - q^{mL}]}{1 - q^L} + \frac{q^{k+1+mL}}{1 - q^L} \\ &= m - \frac{q^{k+1+L}[1 - q^{mL} - q^{(m-1)L}]}{1 - q^L}. \end{aligned} \quad (4.3)$$

We can also compute $B_{k,L}(p)$ directly from (2.13) as an infinite sum

$$B_{k,L}(p) = 0.5 \sum_{j \geq 1} [1 - |q^{k+1+L(j-1)} + q^{k+1+Lj} - 1|] + 0.5q^{k+1}. \quad (4.4)$$

Figure 1 compares the bounds (2.2) in dashed line, and (4.4), in solid line, for different values of p , L and t . One can see that the new bound is sharper, and only for larger values of p , corresponding to fast-mixing chains, the two bounds are

4. EXAMPLES AND ILLUSTRATIONS

indistinguishable. The horizontal line in Figure 1 marks the obvious bound since $d_{TV} \leq 1$. We also notice that for very small values of p both bounds are vacuous, but the new bound has a larger range for being non-vacuous.

The simulations in the remaining two examples rely on the `unbiasedmcmc` package of Pierre Jacob that is available here:

<https://github.com/pierrefjacob/unbiasedmcmc/tree/master/vignettes>. Additional programs for implementing the new ideas in this paper are available as supplemental materials from the authors.

4.2 An Empirical Comparison of the Bounds: Ising Model

The Ising model example follows the setup in Biswas et al. (2019). We consider a 32×32 square lattice of pixels with values in $\{-1, 1\}$, and with periodic boundaries. A state of the system is then $x \in \{-1, 1\}^{32 \times 32}$ and the target probability is defined as $\pi_\beta(x) \propto \exp(\beta \sum_{i \sim j} x_i x_j)$, where $i \sim j$ means that x_i and x_j are pixel values in neighboring sites. This illustration uses the parallel tempering algorithm (PT, see Swendsen and Wang, 1986) coupled with a single site Gibbs (SSG) updating. It is known that larger values of β increase the dependence between neighboring sites and this “stickiness” leads to slow mixing of the SSG. The target of interest corresponds to $\beta_0 = 0.46$ and we use 12 chains, each corresponding to a different $\pi_\beta(x)$ with β values equally spaced between 0.3 and $\beta_0 = 0.46$. Figure 2 shows the total variation bounds,

4. EXAMPLES AND ILLUSTRATIONS

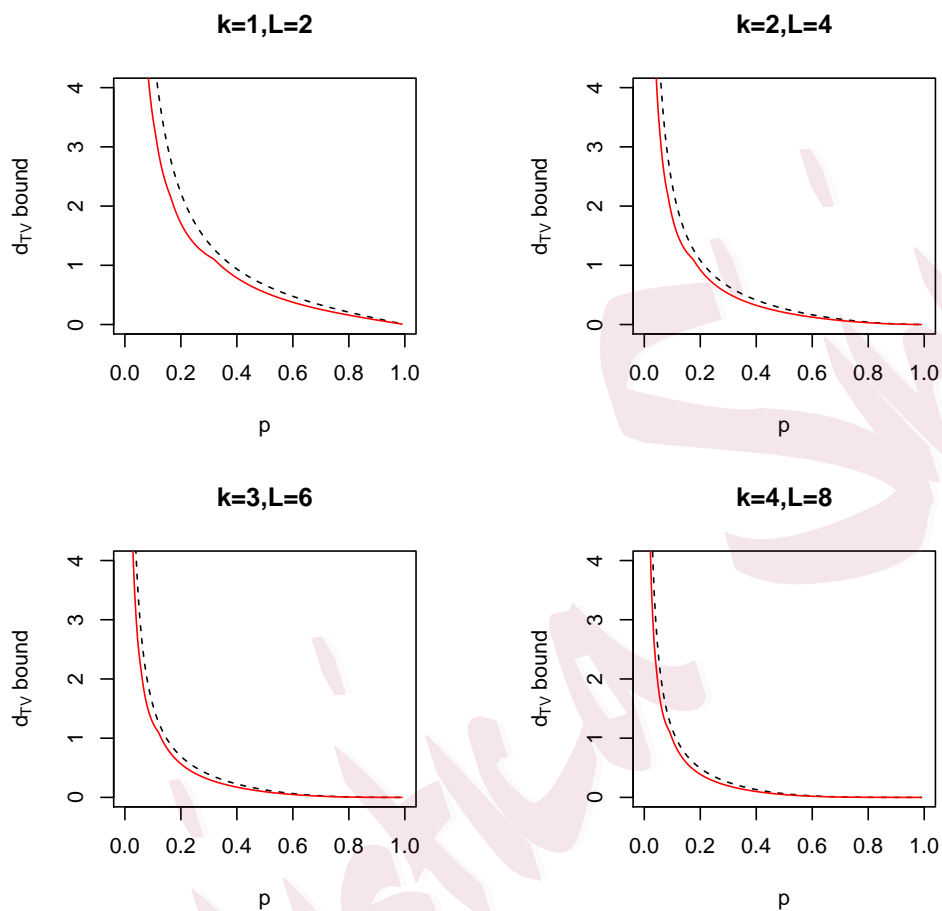


Figure 1: Comparison of bound (2.1) provided by Biswas et al. (2019) (dashed line) and the new bound given in (2.16) (solid line). Note that for small values of p both bounds are vacuous.

4. EXAMPLES AND ILLUSTRATIONS

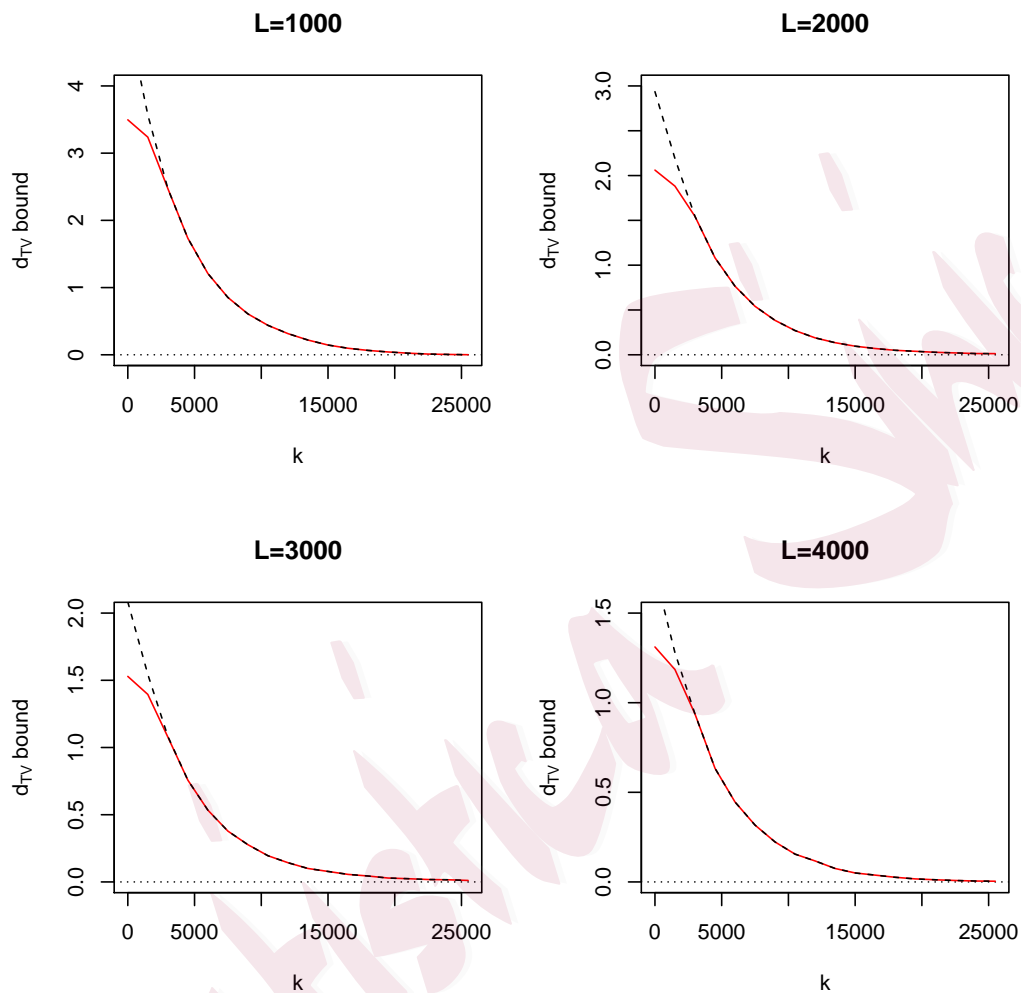


Figure 2: Ising Model: Comparison of TV bounds for the PT algorithm for SGS for $L \in \{1000, 2000, 3000, 4000\}$. The dashed line shows the bound (2.1) derived in Biswas et al. (2019) and the solid line shows the new bound given in (2.16).

4. EXAMPLES AND ILLUSTRATIONS

the one provided by (2.1) is shown in dashed lines and the (2.16) is in solid line. The bounds are derived for $1 \leq k < 25,000$ and $L \in \{1000, 2000, 3000, 4000\}$. For smaller values of L the patterns are similar but TV bounds are larger for smaller values of k . The new bound, (2.16), is computed from $Q = 50$ parallel runs and is averaged over 20 independent replicates, while (2.1) is averaged over 1000 independent replicates of a single coupled process.

Although the numerical results confirm that our new bound never exceeds the bound of Biswas et al. (2019), unfortunately, in this case the improvement from our bound is visible only when it is not needed, that is, when both bounds exceed one. Whereas this is a disappointment for our effort to improve the bound with real gain, it is a piece of good news for practitioners since the bound in Biswas et al. (2019) is a bit simpler to use.

4.3 Comparing Bounds and Estimators: A Logistic Regression Example

To compare both the bounds and the unbiased estimators, we follow Biswas et al. (2019) and consider a Bayesian logistic regression model for the German Credit data from Lichman (2013). The data consist of $n = 1000$ binary responses, $\{Y_i : 1 \leq i \leq n\}$ and $d = 49$ covariates, $\{x_i \in \mathbf{R}^d; 1 \leq i \leq n\}$. The response Y_i indicates whether the i -th individual is fit to receive credit ($Y_i = 1$) or not ($Y_i = 0$). The logistic regression model frames the probabilistic dependence between the response

4. EXAMPLES AND ILLUSTRATIONS

and covariate as $\Pr(Y_i = 1|x_i) = [1 + \exp(-x_i^T \beta)]^{-1}$. The prior is set to $\beta \sim N(0, 10\mathbf{I}_d)$. Sampling from the posterior distribution is done via the Polya-Gamma sampler of Polson et al. (2013), using the R programs made available by Biswas et al. (2019) at <https://github.com/niloyb/LlagCouplings>. In Figure 3 we compare Biswas et al.'s bound (2.1) (dashed line) with our bound (2.16) (solid line). The bound in (2.1) is averaged over 2000 independent replicates. The new bound is computed from running 50 coupled processes in parallel and averaged over 40 replicates, thus totalling the same number of runs. The difference between the two bounds is apparent for smaller values of L when the new bound is sharper for small values of k , but the gain diminishes quickly as L increases.

Of interest are also the gains in efficiency for the Monte Carlo estimators when implementing the control variate swindle. Using 500 independent replicates of a single coupled process with lag $L = 5$ we obtain Monte Carlo estimates of the posterior means for the regression coefficients. In Figure 4 we present the relative reduction in variance (RRV) computed as $\text{RRV} = \frac{\text{Var}_{MCCV}(\hat{\beta})}{\text{Var}_{MC}(\hat{\beta})}$ where $\hat{\beta}$ is the posterior mean of the regression coefficients, $\beta \in \mathbf{R}^{49}$ and Var_{MC} , Var_{MCCV} denote the estimated Monte Carlo variances of $\hat{\beta}$ obtained without and, respectively, with control variates. The left panel shows the RRV when using the single run estimators (2.4) and (3.1), while the right panel plots RRV for the mean estimators (3.2) and (3.3). We see clearly that the gain is rather significant for $r = k = 5$ (left panels), but diminishes when

4. EXAMPLES AND ILLUSTRATIONS

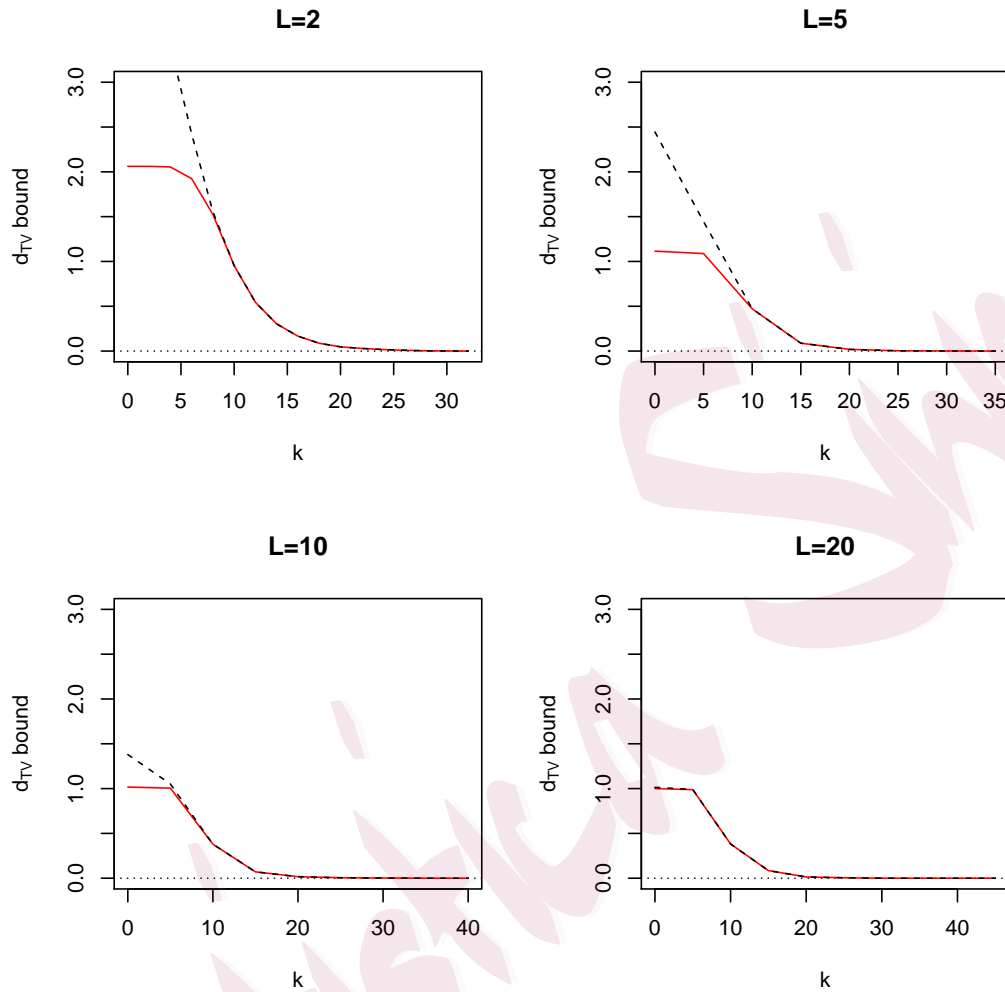


Figure 3: German Credit Data: Comparison of TV bounds for the Polya-Gamma sampler for $L \in \{2, 5, 10, 20\}$. The dashed line shows the bound (2.1) derived in Biswas et al. (2019) and the solid line shows the new bound given in (2.16). The bound (2.16) is obtained from running 50 coupled chains in parallel and averaging over 40 independently replicated experiments. The bound (2.1) is averaged over 2000 independent replicates.

$k = 5$ and $r = 30$ (right panels), as we discussed in Section 3.

5. Can We Do Even Better?

The idea of L-lag coupling clearly has opened multiple avenues for future research. The use of control variates is just one of them. Although the practical gain is small or possibly even negative when we take into account the increased computation for computing the control variates, the theoretical gain is very intriguing, because it is truly a free lunch. That is, we obtain a theoretically superior bound without imposing any additional assumption. This naturally raises a question: is our bound the best possible without further conditions? We do not know. We do not even know how to study theoretically such a question, because as far as we are aware of, this is the first time that a tighter *theoretical bound* is obtained by a better *empirical estimator*. Whereas seeking other more efficient estimators seems to be a natural direction, we must keep in mind that they would likely incur additional computational cost. Therefore, there is no logical implication that a free lunch in theory guarantees a free lunch in practice.

One plausible direction is to go beyond linearly combining mean-zero control variates, though we have not been able to land anything useful. We certainly invite interested readers to seek better (theoretical) free lunch. But, even without seeking better bounds, our current bounds already offer the opportunity to investigate fresh

5. CAN WE DO EVEN BETTER?

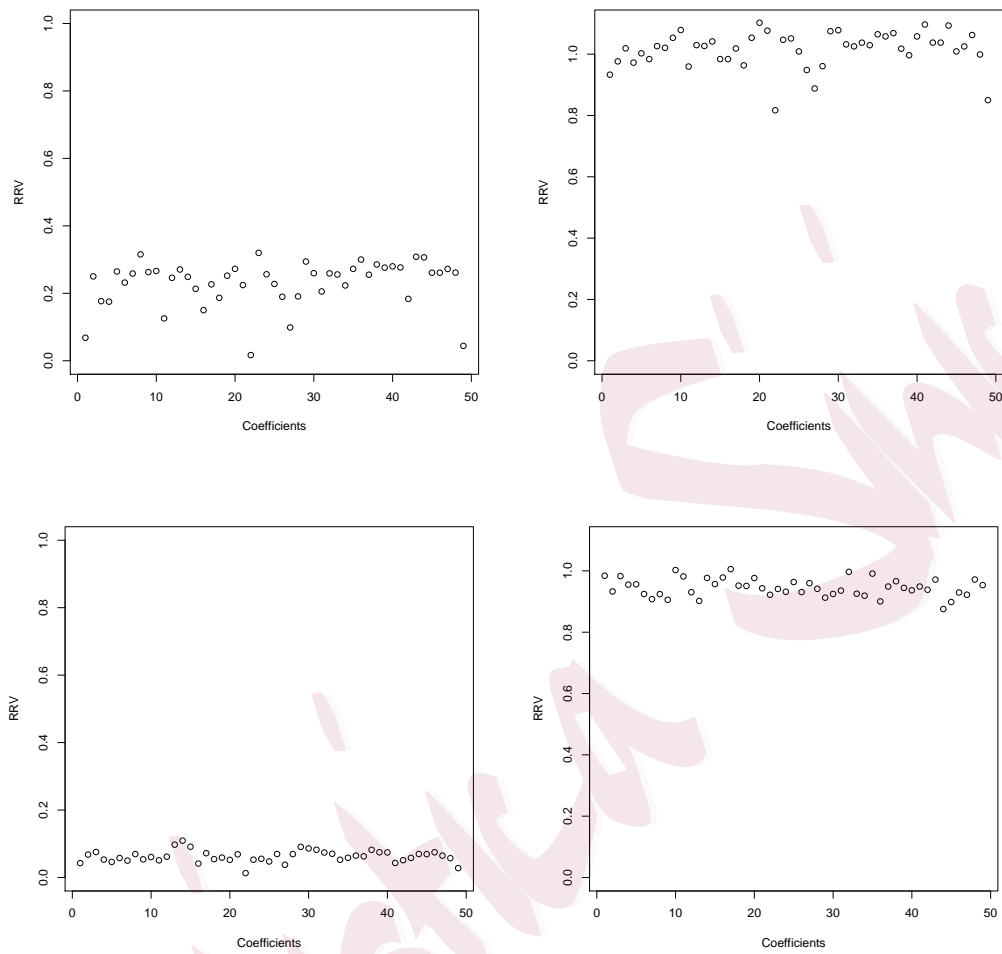


Figure 4: German Credit Data. Relative reduction in variance (RRV) for the 49 regression coefficients. *Top panels:* the lag is $L = 5$. *Bottom panels:* the lag is $L = 20$. *Left panels:* RRV is obtained from the single estimators without and, respectively, with control variates using $k = 5$ in (2.4) and (3.1). *Right panels:* RRV is obtained from the average estimators without and, respectively, with control variates using $k = 5$ and $r = 30$ in (3.2) and (3.3).

perspectives for optimizing an MCMC kernel using adaptive ideas and we intend to pursue these in our second tour.

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