Statistica Sinica Preprint No: SS-2020-0005	
Title	Time-Varying Mixture Copula Models with Copula Selection
Manuscript ID	SS-2020-0005
URL	http://www.stat.sinica.edu.tw/statistica/
DOI	10.5705/ss.202020.0005
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Notice: Accepted version subject to English editing.	

Time-Varying Mixture Copula Models with Copula Selection

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Abstract: Modeling the joint tails of multiple financial time series has many important implications for risk management. Classical models for dependence often encounter a lack of fit in the joint tails, calling for additional flexibility. This paper introduces a new semiparametric time-varying mixture copula model, in which both weights and dependence parameters are deterministic and unspecified functions of time. We propose penalized time-varying mixture copula models with group smoothly clipped absolute deviation penalty functions to do the estimation and copula selection simultaneously. Monte Carlo simulation results suggest that the shrinkage estimation procedure performs well in selecting and estimating both constant and time-varying mixture copula models. Using the proposed model and method, we analyze the evolution of the dependence among four international stock markets, and find substantial changes in the levels and patterns of the dependence, in particular around crisis periods.

Key words and phrases: Copula Selection; EM Algorithm; Mixture Copula; SCAD; Time-Varying Distribution.

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1. Introduction

Copulas have received considerable attention because they offer great flexibility to model multivariate distributions and to characterize nonlinear dependence and tail dependency. A copula function glues various types of marginal distributions, including symmetric, skewed and heavy-tailed distributions, into a multivariate distribution, and by Sklar's theorem (1959), this is always possible. The variety of dependence patterns is of great importance for financial and macroeconomic time series, which leads to many applications, such as volatility clustering (Ning, Xu and Wirjanto, 2015), real-time density forecasting (Smith and Vahey, 2016), stock returns modeling under non-stationarity (Wollschläger and Schäfer, 2016), systemic risk (Mensi et al., 2016), and so on.

Among various applications of copula models, studying co-movements of returns across international equity markets is one of the most popular topics. For example, Hu (2006), Cai and Wang (2014), and Liu et al. (2019) employ time-invariant mixture copula models and find that international equity markets usually show lower tail dependence, which implies that the markets are more likely to crash together than to boom together. However, the dependence structures among international equity markets are likely to change substantially over time because international financial markets change from time to time so that a time-invariant copula model is incapable of capturing the evolution of the dependence structures. Therefore, a time-varying copula model is needed to solve this problem.

In the literature, time-varying copulas or dynamic copulas have been extensively used to model multiple financial time series. For example, Patton (2006) uses a symmetrized Joe-Clayton copula in which the dependence structure follows an autoregressive moving average (ARMA)-type process to capture asymmetric dependence between mark-dollar and

yen-dollar exchange rates. Hafner and Reznikova (2010) propose a semiparametric dynamic copula (SDC) model in which the copula parameter changes over time in a nonparametric way. Other dynamic copulas include dynamic stochastic copula models (Hafner and Manner, 2012), stochastic copula autoregressive models (Almeida and Czado, 2012), generalized autoregressive score models (Creal, Koopman and Lucas, 2013), variational mode decomposition methods (Mensi et al., 2016), single-index copula models (Fermanian and Lopez, 2018), and semiparametric copula models under non-stationarity (Nasri, Rémillard and Bouezmarni, 2019), among others. For a comprehensive survey of dynamic copulas and their applications in financial time series analysis, readers are referred to the survey paper by Patton (2012a).

Although many time-varying copula models have been proposed in the literature, most of them rely on a single copula instead of a mixture copula. Thus, although the copula dependence parameters which reflect the levels of dependence can change with time, the copula function which represents the pattern of dependence is still time-invariant. However, international equity markets may exhibit different dependence patterns during different time periods, i.e., in tranquil periods and in crisis periods, and hence a single copula model is inadequate. Therefore, in this paper, we contribute to the literature by proposing a time-varying mixture copula model with both time-varying dependence parameters and weights (or coefficients of component copulas in a mixture copula model) to analyze the co-movements across international equity markets. Indeed, the weights in a mixture copula model summarize the dependent patterns or shapes and their magnitudes signify the importance of the corresponding copulas. By allowing the weights to be dynamic, we can recognize different dependence patterns during different time periods. Furthermore, we do not specify any parametric form

for the weights and dependence parameters in a time-varying mixture copula model, and use a data-driven method for their specification. In this way, we alleviate typical misspecification problems in copulas. The proposed model can be considered as an ideal copula model, as described in Patton (2012b), in the sense that it accommodates dependence of either positive or negative sign, captures both symmetric and asymmetric dependence, and allows for the possibility of non-zero tail dependence. The proposed model is different from prior studies which focus exclusively on single copula models (Hafner and Reznikova, 2010; Acar, Craiu and Yao, 2011). It also differs from previous mixture copula models that assume either the weights or dependence parameters do not change with time (Garcia and Tsafack, 2011; Liu et al., 2019). Finally, it generalizes the time-varying optimal copula model of Liu, Ji and Fan (2017) which assumes a single copula at each time point.

An important issue is that the range of both copula dependence parameters and their corresponding weights are restricted, e.g. $\theta \in (-1, 1)$ for a Gaussian copula, $\theta \in (0, \infty)$ for a Clayton copula, and the weights satisfying $\lambda_k \in [0, 1]$ and $\sum_k \lambda_k = 1$. To overcome this difficulty in the nonparametric estimation, Abegaz, Gijbels and Veraverbeke (2012) and Acar, Craiu and Yao (2011) use some known inverse functions to ensure that the copula parameters are properly defined and employ a local polynomial framework to estimate the dependence parameters. However, in the asymptotic properties both the bias and variance depend on the choice of the inverse link function, see, e.g., Theorem 2 of Abegaz, Gijbels and Veraverbeke (2012) and Corollary 1 of Acar, Craiu and Yao (2011). It is nontrivial to find an optimal inverse link function in a large functional space. In this study, we employ a local constant (Nadaraya-Watson) kernel method without choosing any inverse link function and show that the local constant estimators have the same asymptotic behavior as the local

linear estimators at the interior points: both have the same bias and variance terms as well as the same convergence rate.

To reduce the risk of over-fitting and efficiency loss, we propose a penalized time-varying mixture copula model with the group smoothly clipped absolute deviation (SCAD) penalty term as in Fan and Li (2001) to do the estimation and copula selection simultaneously. The functional norms of the weight functions are penalized so that we can shrink them to zeros if contributions of corresponding copulas are small. To facilitate the estimation, we propose a semiparametric version of the expectation maximization (EM) algorithm for the estimation of the weights and dependence parameters in the penalized local copula log-likelihood function. Furthermore, we discuss other important practical issues including the bandwidth and tuning parameter selection, and confidence intervals. In a simulation study, we consider mixture copulas with both constant and time-varying weights and dependence parameters. The results show that the proposed method can correctly select the appropriate copulas and accurately estimate the unknown parameters in both cases.

In the application, we employ the proposed model and method to investigate the evolution of the dependence structures among four international stock markets (the United States, the United Kingdom, Hong Kong and South Korea), using 28 years of weekly returns on the main equity indices. Interestingly, based on the analysis using the proposed model, we find that all pairs of markets show lower tail dependence but no upper tail dependence, since the Clayton and Frank copulas are always selected while the Gumbel is always filtered out. We also observe that the dependence exhibits quite different levels and patterns during different periods, e.g., in tranquil periods and in crisis periods. The detailed results for analyzing this empirical example can be found in Section 4.

The remainder of the paper is organized as follows. Section 2 introduces the proposed time-varying mixture copula models. In the same section, we introduce penalized time-varying mixture copula models. Three practical issues are discussed including a semiparametric EM algorithm, the bandwidth and tuning parameter selection, and the construction of pointwise confidence intervals by using Bootstrap. Section 3 reports the Monte Carlo simulation results. Section 4 applies the model and method to examine the evolution of the dependence among four international stock markets. The final section provides some concluding comments, and the mathematical proofs are gathered in the online supplement.

2. Time-Varying Mixture Copula Models

2.1 Model setup

In this section, we model the time-varying mixture copula in a semiparametric way so that the dynamics in both the weights and dependence parameters are simultaneously captured. Let $\{\mathbf{X}_i\}_{i=1}^T = \{X_{1i}, X_{2i}, \cdots, X_{Ni}\}_{i=1}^T$ be a N-dimensional time series sequence and $\mathbf{Z}_i$ a vector of predetermined or exogenous variables. Denote by $\mathcal{F}_{i-1}$ the σ-field generated by $\{\mathbf{X}_{i-1}, \mathbf{X}_{i-2}, ...; \mathbf{Z}_i, \mathbf{Z}_{i-1}, ...\}$. We assume that $\mathbf{X}_i$ follows

$$\mathbf{X}_i = \mu_i(\psi_{01}) + \Sigma_i(\psi_{02})^{1/2}\epsilon_i, \quad i = 1, \ldots, T, \tag{2.1}$$

where $\mu_i(\psi_{01}) = E\{\mathbf{X}_i|\mathcal{F}_{i-1}\}$ with ψ_{01} being the parameters for the conditional mean, $\Sigma_i(\psi_{02}) = \mathrm{diag}(\sigma_{1i}^2(\psi_{02}), \cdots, \sigma_{Ni}^2(\psi_{02}))$ with ψ_{02} being the parameters for the conditional variance, and $\sigma_{si}^2(\psi_{02}) = E\{(X_{si} - \mu_{si}(\psi_{01}))^2|\mathcal{F}_{i-1}\}$ for $s = 1, \cdots, N$. ψ_{01} and ψ_{02} are unknown parameters with fixed dimensions.

We further assume that the standardized innovations $\{\epsilon_i \equiv (\epsilon_{1i}, \cdots, \epsilon_{Ni})^\intercal\}$ are independent of $\mathcal{F}_{i-1}$. For each $s \in \{1, \cdots, N\}$, $\{\epsilon_{si}\}_{i=1}^T$ have zero means and one unit standard deviation. The settings specified here cover many commonly used specifications such as ARCH, GARCH, vector autoregressions (VAR), and so on (see Chen and Fan, 2006 for a detailed discussion).

For example, for $s = 1, \cdots, N$, X_{si} may follow an AR-GARCH model with exogenous variables as

$$X_{si} = \varphi_{s0} + \varphi_{s1} X_{s,i-1} + \cdots + \varphi_{sp} X_{s,i-p} + \phi_s Z_{si} + \sigma_{si}\epsilon_{si},$$

$$\sigma_{si}^2 = \alpha_{s0} + \alpha_{s1}(X_{s,i-1} - \varphi_{s0} - \varphi_{s1} X_{s,i-2} - \cdots - \varphi_{sp} X_{s,i-p-1} - \phi_s Z_{s,i-1})^2 + \beta_{s1}\sigma_{s,i-1}^2$$

where the parameters satisfy $\alpha_{s0} > 0, \alpha_{s1} > 0, \beta_{s1} > 0$, and $(\alpha_{s1} + \beta_{s1}) < 1$. More examples about the specifications of the conditional mean and conditional variance in (2.1) can be found in Chen and Fan (2006). Note that for simplicity, we only consider AR models (without using exogenous variables $\mathbf{Z}_i$) for the conditional mean in the simulation and application sections.

The goal of this paper is to estimate the joint distribution of $\{\epsilon_i \equiv (\epsilon_{1i}, \cdots, \epsilon_{Ni})^\intercal\}$ based on a time-varying mixture copula model. Theoretically, the time-varying mixture copula model can be written as a linear combination of infinite single copula terms as

$$C(F_1(y_1), \cdots, F_N(y_N); \delta(t_i)) = \sum_{k=1}^{\infty} \lambda_k(t_i) C_k(F_1(y_1), \cdots, F_N(y_N); \theta_k(t_i)),$$

where $\{C_k(\cdot;\cdot)\}_{k=1}^{\infty}$ is a set of candidate copulas, $y_s, s = 1, \cdots, N$, denote the realizations of innovations, and $F_s(\cdot)$ are the marginal distribution functions. We rescale time t_i by $t_i = i/T$ to provide the asymptotic justification for nonparametric smoothing estimators. The under-

lying assumption is that there is an increasingly intense sampling of data points that can be used to derive consistent estimation, see e.g., Robinson (1991) and Cai (2007). $\{C_k(\cdot;\cdot)\}_{k=1}^{\infty}$ can be regarded as known basis copula functions so that $C(F_1(y_1),\cdots,F_N(y_N);\delta(t_i))$ can be regarded as a series expansion based on the basis copula functions $\{C_k(\cdot;\cdot)\}_{k=1}^{\infty}$. In real applications, we use finite number of d single copulas to approximate the true one

$$C(F_1(y_1),\cdots,F_N(y_N);\delta(t_i)) = \sum_{k=1}^{d} \lambda_k(t_i) C_k(F_1(y_1),\cdots,F_N(y_N);\theta_k(t_i)), \tag{2.2}$$

where $\{C_1(\cdot;\cdot),...,C_d(\cdot;\cdot)\}$ is a set of candidate copulas. Further, $\delta(t_i) = (\theta(t_i)^\intercal, \lambda(t_i)^\intercal)^\intercal$ is a vector of $(p_1+\cdots+p_d)$-dimensional dependence parameters $\theta(t_i) = (\theta_1(t_i)^\intercal,...,\theta_d(t_i)^\intercal)^\intercal$ and d-dimensional weights $\lambda(t_i) = (\lambda_1(t_i),...,\lambda_d(t_i))^\intercal$. For simplicity of presentation, we set $p_1 = \cdots = p_d = 1$. The weight $\lambda_k(t_i)$ controls the contribution of the copula C_k and satisfies both $0 \le \lambda_k(t_i) \le 1$ and $\sum_{k=1}^{d} \lambda_k(t_i) = 1$ for all $t_i \in [0,1]$. The parameter $\theta_k(t_i)$ represents the level of the dependence corresponding to the copula C_k at time t_i. The above mixture copula model implies that the joint cumulative distribution function of a random vector $(\epsilon_{1i},\cdots,\epsilon_{Ni})$ is given by a linear combination of $C_k(F_1(\cdot),\cdots,F_N(\cdot);\theta_k(t_i))$ with time-varying weights $\lambda_k(t_i)$.

When using (2.2) to approximate the true model, we may have a misspecification problem since some true single copulas might be excluded. To avoid this problem, we can first consider a large candidate copula set and then employ the copula model selection procedures discussed in Section 2.2 to filter out the "insignificant" component copulas. Furthermore, even if some true single copulas are not included in this approximation so that the model is misspecified, we can still estimate and select the closest mixture copula model by the model selection criterion described in Section 2.2 (see Cai and Wang, 2014). Therefore, the model in (2.2)

is flexible enough to capture a true copula in real applications.

Remark 1. In models (2.1)-(2.2), one can allow ψ_{01}, ψ_{02}, and $F_1(\cdot), \cdots, F_N(\cdot)$ to depend on time. However, for simplicity, here we assume that the marginals in (2.2) and ψ_{01} and ψ_{02} in (2.1) do not depend on time, because our main focus is on time-varying weights and dependence parameters in a mixture copula and its copula selection. It would be an interesting research topic to investigate the case with time-varying ψ_{01}, ψ_{02} and marginals $F_1(\cdot), \cdots, F_N(\cdot)$.

Next, to discuss the identification issue of the model in (2.2), similar to the time-invariant model proposed in Cai and Wang (2014) (see Definition 1 in Cai and Wang, 2014), we let $\mathbf{u} = (u_1, \cdots, u_N)^\intercal$ with $u_s = F_s(y_s), s = 1, \cdots, N$, and define two time-varying mixture copulas $C(\mathbf{u}; \delta(t_i)) = \sum_{k=1}^{d} \lambda_k(t_i) C_k(\mathbf{u}; \theta_k(t_i))$ and $C^*(\mathbf{u}; \delta^*(t_i)) = \sum_{k=1}^{d^*} \lambda_k^*(t_i) C_k^*(\mathbf{u}; \theta_k^*(t_i))$ to be identified, i.e. $C(\mathbf{u}; \delta(t_i)) \equiv C^*(\mathbf{u}; \delta^*(t_i))$, if and only if $d = d^*$ and we can order the summations such that $\lambda_k(t_i) = \lambda_k^*(t_i)$ and $C_k(\mathbf{u}; \theta_k(t_i)) = C_k^*(\mathbf{u}; \theta_k^*(t_i))$ for all possible values of $\mathbf{u}$, $k = 1, \cdots, d$ and $t_i \in [0, 1]$. Without loss of generality, we assume that the time-varying mixture copula model under investigation is identified.

In the following, we propose a three-step estimation procedure.

Step 1: We estimate ψ_{01} and ψ_{02} in model (2.1) by specifying conditional mean models for $\mu(\psi_{01})$, such as ARMA model, and conditional volatility models for $\Sigma(\psi_{02})$, such as GARCH model. To see the various conditional mean models and conditional volatility models, as well as their corresponding estimation methods, the reader is referred to Chapters 2 & 3 in Tsay (2010).

Step 2: After obtaining the estimates $\hat{\psi}_1$ and $\hat{\psi}_2$ from the first step, we calculate

$$\hat{\epsilon}_i = \Sigma_i^{-1/2}(\hat{\psi}_2)(\mathbf{X}_i - \mu_i(\hat{\psi}_1)),$$

and then the marginal distribution functions can be estimated by the rescaled empirical distribution of the residuals

$$\hat{F}_s(y_s) = \frac{1}{T+1}\sum_{j=1}^{T} I(\hat{\epsilon}_{sj} \leq y_s), \quad s = 1, \cdots, N,$$

with $I(\cdot)$ being an indicator function. For $i = 1, \cdots, T$, denote $\hat{u}_i = (\hat{u}_{1i}, \cdots, \hat{u}_{Ni})^\intercal$ with $\hat{u}_{si} = \hat{F}_s(\hat{\epsilon}_{si})$.

Step 3: Given the estimators $\hat{u}_i$ from the second step, we employ the profile likelihood method and calculate the nonparametric estimator $\hat{\delta}(\tau) = (\hat{\theta}(\tau)^\intercal, \hat{\lambda}(\tau)^\intercal)^\intercal$ at a given point $\tau \in (0, 1)$ by maximizing the local copula log-likelihood function as

$$\hat{\delta}(\tau) = \arg\max_{(\theta(\tau),\lambda(\tau))} \sum_{i=1}^{T} \log\left(\sum_{k=1}^{d} \lambda_k(\tau) c_k(\hat{u}_i, \theta_k(\tau))\right) K_h(t_i - \tau), \tag{2.3}$$

where $c_k(\cdot)$ is the density function of copula $C_k(\cdot)$, $K_h(\cdot) = K(\cdot/h)/h$ with $K(\cdot)$ being a kernel function and h a bandwidth that tunes the smoothness of the kernel estimator. In our simulation and empirical study, the commonly adopted Epanechnikov kernel function $K(z) = 3/4(1 - z^2)I(|z| \leq 1)$ is used.

Remark 2. Lemma A.1 in Chen and Fan (2006) shows that $\sup_{y_s} |\hat{F}_s(y_s) - F_s(y_s)| = O_p(T^{-1/2})$ holds for the rescaled empirical distribution function. It implies that the estimator $\hat{F}_s(\cdot)$ has the $\sqrt{T}$ convergence rate so that it has little effect on the nonparametric

estimators $\hat{\delta}(\tau)$ in large samples.

In the following, we present the copula selection procedure and its asymptotic properties. The regularity conditions and asymptotic properties for unpenalized estimators are given in the online supplement.

2.2 Penalized time-varying mixture copula models

When many candidate copula families are included in the proposed time-varying mixture copula model, there is a risk of overfitting and efficiency loss, which motivates us to do the estimation and copula selection simultaneously. For this purpose, we define a $T\times(2d)$ matrix $\delta = (\delta(t_1),\cdots,\delta(t_T))^\intercal = (\theta_{\cdot 1},\cdots,\theta_{\cdot d},\lambda_{\cdot 1},\cdots,\lambda_{\cdot d})$, where $\delta(t_j) = (\theta_1(t_j),\cdots,\theta_d(t_j),\lambda_1(t_j),\cdots,\lambda_d(t_j))^\intercal$ for $j = 1,\cdots,T$, and $\theta_{\cdot k} = (\theta_k(t_1),\cdots,\theta_k(t_T))^\intercal$ and $\lambda_{\cdot k} = (\lambda_k(t_1),\cdots,\lambda_k(t_T))^\intercal$ for $k = 1,\cdots,d$. We follow the idea of the group least absolute shrinkage and selection operator (LASSO) as in Yuan and Lin (2006) and propose the following penalized local log-likelihood function as

$$Q^P(\delta) = \sum_{j=1}^{T}\sum_{i=1}^{T}\ell(\hat{u}_i,\delta(t_j))K_h(t_i-t_j) - T\sum_{k=1}^{d}P_{\gamma_k}(\|\lambda_{\cdot k}\|) + \sum_{j=1}^{T}\rho_{t_j}\left(1-\sum_{k=1}^{d}\lambda_k(t_j)\right) \quad (2.4)$$

where $\ell(\hat{u}_i,\delta(t_j)) = \log(\sum_{k=1}^{d}\lambda_k(t_j)c_k(\hat{u}_i,\theta_k(t_j)))$, $\hat{u}_i = (\hat{u}_{1i},\cdots,\hat{u}_{Ni})^\intercal$ are obtained from Steps 1-2 as in Section 2.1, $P_{\gamma_k}(\cdot)$ is a penalty function with the tuning parameter γ_k, $\|\lambda_{\cdot k}\|=(\lambda_k^2(t_1)+\cdots+\lambda_k^2(t_T))^{1/2}$, and ρ_{t_j} is a Lagrange multiplier for the constraint $\sum_{k=1}^{d}\lambda_k(t_j) = 1$. The norm of $\lambda_{\cdot k}$, i.e. $\|\lambda_{\cdot k}\|$, is penalized so that we can shrink the weight function $\lambda_k(\cdot)$ to zero if the contribution of copula $C_k(\cdot)$ is small. We do not penalize the dependence parameters $\theta_k(\cdot)$ since our main focus is on the copula selection. Clearly, the purpose of using the penalized locally weighted log-likelihood function is to select important copula families.

Various penalty functions have been proposed over the last decades. As pointed out by Fan and Li (2001), a good penalty function should satisfy the following three properties: unbiasedness for the non-zero coefficients, sparsity, and continuity of the resulting estimators to avoid instability in model prediction. Here, we propose to use the SCAD penalty function proposed by Fan and Li (2001) that enjoys all three properties, although many other penalty functions are applicable including the classical LASSO in Tibshirani (1996) and adaptive LASSO in Zou (2006). The first-order derivative $P'_{\gamma_k}(z)$ of the continuous SCAD penalty function $P_{\gamma_k}(z)$ is given by

$$P'_{\gamma_k}(z) = \gamma_k I(z \leq \gamma_k) + \frac{(\varrho\gamma_k - z)_+}{(\varrho - 1)} I(z > \gamma_k)$$

for some $\varrho > 2$, where $(\varrho\gamma_k - z)_+ = \max(\varrho\gamma_k - z, 0)$. For simplicity of presentation, we assume that the tuning parameters γ_k are the same for all $k = 1, \cdots, d$ by taking $\gamma_k = \gamma_T$. We select $\varrho = 3.7$ from a Bayesian risk point of view as suggested by Fan and Li (2001) who state that this choice provides a good practical performance for various model selection problems.

To find the asymptotic properties of the penalized estimator, we assume that the first d_0 functional weights are nonzero and the remaining $d - d_0$ functional weights are zero. That is, $\lambda_0(\tau) = [\lambda_{0a}^\intercal(\tau), \lambda_{0b}^\intercal(\tau)]^\intercal$, where $\lambda_{0a}(\tau) = [\lambda_{01}(\tau), \cdots, \lambda_{0d_0}(\tau)]^\intercal$ with $\|\lambda_{\cdot 0k}\| \neq 0$ for $1 \leq k \leq d_0$ and $\lambda_{0b}(\tau) = [\lambda_{0(d_0+1)}(\tau), \cdots, \lambda_{0d}(\tau)]^\intercal$ with $\|\lambda_{\cdot 0k}\| = 0$ for $d_0 + 1 \leq k \leq d$. Similarly, we let $\theta_0(\tau) = [\theta_{0a}^\intercal(\tau), \theta_{0b}^\intercal(\tau)]^\intercal$ with $\theta_{0a}(\tau) = [\theta_{01}(\tau), \cdots, \theta_{0d_0}(\tau)]^\intercal$ and $\theta_{0b}(\tau) = [\theta_{0(d_0+1)}(\tau), \cdots, \theta_{0d}(\tau)]^\intercal$, in which $\theta_{0b}(\tau)$ can be arbitrary since the corresponding weights are zeros. Moreover, we define $\delta_0(\tau) = [\theta_0^\intercal(\tau), \lambda_0^\intercal(\tau)]^\intercal$ and $\delta_{0a}(\tau) = [\theta_{0a}^\intercal(\tau), \lambda_{0a}^\intercal(\tau)]^\intercal$, and their corresponding penalized estimators $\hat{\delta}_{\gamma_T}(\tau) = [\hat{\theta}_{\gamma_T}^\intercal(\tau), \hat{\lambda}_{\gamma_T}^\intercal(\tau)]^\intercal$ and $\hat{\delta}_{a,\gamma_T}(\tau) =$

$[\hat{\theta}^{\intercal}_{a,\gamma_T}(\tau), \hat{\lambda}^{\intercal}_{a,\gamma_T}(\tau)]^{\intercal}$, respectively. One can partition $\delta_0(\tau)$ into an identified subset $[\theta^{\intercal}_{0a}(\tau), \lambda^{\intercal}_{0a}(\tau), \lambda^{\intercal}_{0b}(\tau)]^{\intercal}$ and an unidentified subset $\theta_{0b}(\tau)$ in which the former is unique and the latter is a vector of arbitrary fixed points. Furthermore, we include the following additional technical conditions:

(B1) $\lim_{T\to\infty} \inf_{z\to 0^+} P'_{\gamma_T}(z)/\gamma_T > 0$, $h \propto T^{-1/5}$ and $T^{-1/10}\gamma_T \to 0$, as $T \to \infty$.

The condition $\lim_{T\to\infty} \inf_{z\to 0^+} P'_{\gamma_T}(z)/\gamma_T > 0$ can be found in Lemma 1 of Fan and Li (2001). The last condition in B1 imples that the order of the tuning parameter γ_T needs to be smaller than $T^{1/10}$, which will be crucial for the consistency result in Theorem 1 and the oracle property in Theorem 2.

Theorem 1. *Let $\{X_{1i}, \cdots, X_{Ni}\}_{i=1}^T$ be a strictly stationary α-mixing sequence following the proposed models (2.1)-(2.2). For a fixed point $\tau \in (0,1)$, under Conditions A1-A6 in the online supplement and B1, there exists a $\sqrt{Th}$-consistent estimator $\hat{\delta}_{\gamma_T}(\tau)$ that maximizes (2.4) satisfying $\|\hat{\delta}_{\gamma_T}(\tau) - \delta_0(\tau)\| = O_p(1/\sqrt{Th})$.*

Remark 3. Theorem 1 shows the consistency for the nonparametric kernel-based estimator $\hat{\delta}_{\gamma_T}(\tau)$ at a given point $\tau \in (0,1)$.

Theorem 2. *(Oracle Property). Let $\{X_{1i}, \cdots, X_{Ni}\}_{i=1}^T$ be a strictly stationary α-mixing sequence following the proposed models (2.1)-(2.2). For a fixed point $\tau \in (0,1)$, under Conditions A1-A6 in the online supplement and B1, we have*

(a) Sparsity: $\|\hat{\lambda}_{\cdot k}\| = 0$ for $k = d_0 + 1, \cdots, d$,

(b) Asymptotic normality:

$$\sqrt{Th}(\hat{\delta}_{a,\gamma_T}(\tau) - \delta_{0a}(\tau) - h^2 B_a(\tau)) \to N(0, v_0 \Sigma_a(\tau)^{-1} \Omega_a(\tau) \Sigma_a(\tau)^{-1}),$$

where $v_0 = \int K^2(z)dz$, $\Sigma_a(\tau) = -E\{\ell''(u_i, \delta_{0a}(\tau))|t_i = \tau\}$, $\Omega_a(\tau) = \sum_{s=-\infty}^{\infty} \Gamma_{a,s}(\tau)$ *with* $\Gamma_{a,s}(\tau) = E\{\ell'(u_i, \delta_{0a}(\tau))\ell'(u_{i+s}, \delta_{0a}(\tau))^\intercal|t_i = \tau\}$ *and the bias term* $h^2 B_a(\tau) = \frac{h^2}{2}\delta''_{0a}(\tau)\mu_2$ *with* $\mu_2 = \int z^2 K(z)dz$. $\ell'(u_i, \delta_{0a}(\tau))$ *and* $\ell''(u_i, \delta_{0a}(\tau))$ *respectively denote the first and second derivatives of* $\ell(u_i, \delta_{0a}(\tau))$ *with respect to* $\delta_{0a}(\tau)$.

Sparsity is an important statistical property in high-dimensional statistics. By assuming that only a small subset of copula families is important, it can reduce complexity so that it improves interpretability and predictability of the model. The sparsity property from Theorem 2 demonstrates that the penalized time-varying mixture copula model shrinks superfluous components of the weight vector exactly to zero with probability one as the sample size T goes to infinity.

2.3 Practical issues

A. A semiparametric EM algorithm. One possible way to optimize the penalized local log-likelihood copula function in (2.4) is to use a coordinate descent approach. However, it is usually not easy to obtain the explicit forms of the first and second derivatives of the object function, especially when the number of copulas is large and there exist some constraints on the weights and dependence parameters. As stated in Cai and Wang (2014), the expectation maximization (EM) algorithm is one of the most popular algorithms for finding the maximum likelihood estimation of the finite mixture model. Hence in this section, we propose a semiparametric version of the EM algorithm to estimate the weights and dependence parameters, which dramatically reduces the computational complexity. It iteratively alternates between an expectation step (E-step) and a maximization step (M-step). The E-step updates the weights of each copula with given dependence parameters, and the M-step maximizes

the local log-likelihood with respect to the dependence parameters for given copula weights. For details of the EM algorithm and its applications in parametric mixture copula models, see Cai and Wang (2014).

To develop a semiparametric version of the EM algorithm for the proposed model, we follow Fan and Li (2001) and Cai, Juhl and Yang (2015) and approximate equation (2.4) by

$$Q^P(\delta) \approx \sum_{j=1}^{T}\left[\sum_{i=1}^{T}\ell(\hat{u}_i,\delta(t_j))K_h(t_i-t_j) - T\sum_{k=1}^{d}\frac{P'_{\gamma_k}(\|\hat{\lambda}_{\cdot k}^{(0)}\|)}{2\|\hat{\lambda}_{\cdot k}^{(0)}\|}\lambda_k^2(t_j) + \rho_{t_j}\left(1-\sum_{k=1}^{d}\lambda_k(t_j)\right)\right]$$
$$+\{\text{terms unrelated to } \delta\}$$

where $\hat{\lambda}_{\cdot k}^{(0)}$ are the estimates from the previous iteration. At the first iteration, $\hat{\lambda}_{\cdot k}^{(0)}$ denotes a set of starting values for the weights.

Then the estimator $\hat{\delta}(t_j)$ at a given iteration step can be obtained by maximizing the criterion function

$$Q^P(\delta(t_j)) = \sum_{i=1}^{T}\ell(\hat{u}_i,\delta(t_j))K_h(t_i-t_j) - T\sum_{k=1}^{d}\frac{P'_{\gamma_k}(\|\hat{\lambda}_{\cdot k}^{(0)}\|)}{2\|\hat{\lambda}_{\cdot k}^{(0)}\|}\lambda_k^2(t_j) + \rho_{t_j}\left(1-\sum_{k=1}^{d}\lambda_k(t_j)\right).$$

We take the first derivative of $Q^P(\delta(t_j))$ with respect to $\lambda_k(t_j)$, and multiply both sides by $\lambda_k(t_j)$, which leads to

$$\sum_{i=1}^{T}\frac{\lambda_k(t_j)c_k(\hat{u}_{1i},\cdots,\hat{u}_{Ni},\theta_k(t_i))}{c_c(\hat{u}_{1i},\cdots,\hat{u}_{Ni},\delta_k(t_i))}K_h(t_i-t_j) - T\frac{P'_{\gamma_k}(\|\hat{\lambda}_{\cdot k}^{(0)}\|)}{\|\hat{\lambda}_{\cdot k}^{(0)}\|}\lambda_k^2(t_j) - \rho_{t_j}\lambda_k(t_j) = 0, \quad k=1,\ldots,d,$$

where $c_c(\hat{u}_{1i},\cdots,\hat{u}_{Ni},,\delta_k(t_i)) = \sum_{k=1}^{d}\lambda_k(t_i)c_k(\hat{u}_{1i},\cdots,\hat{u}_{Ni},\theta_k(t_i))$.

We next introduce the expectation and maximization steps.

Expectation step

Let $\lambda_k^{(0)}(\tau)$ and $\theta_k^{(0)}(\tau)$ be the initial estimators in each iterative step. Given a grid point τ, we update the new weight parameters $\lambda_k^{(1)}(\tau)$ as

$$\lambda_k^{(1)}(\tau) = \Big(\sum_{i=1}^{T} \frac{\lambda_k^{(0)}(\tau) c_k(\hat{u}_{1i}, \cdots, \hat{u}_{Ni}, \theta_k^{(0)}(\tau))}{c_c(\hat{u}_{1i}, \cdots, \hat{u}_{Ni}, \delta_k^{(0)}(\tau))} K_h(t_i - \tau) - T D_k^{(0)}\Big) \Big/ \Big(\sum_{i=1}^{T} K_h(t_i - \tau) - T \sum_{k=1}^{d} D_k^{(0)}\Big),$$

for $k = 1, \ldots, d$, where $D_k^{(0)} = \frac{P'_{\gamma_k}(\|\hat{\lambda}_{\cdot k}^{(0)}\|)}{\|\hat{\lambda}_{\cdot k}^{(0)}\|} \lambda_k^{(0)2}(\tau)$.

Maximization step

After updating the weight $\lambda_k^{(0)}(\tau)$ with $\lambda_k^{(1)}(\tau)$ from the above E-step, we obtain the dependence estimator $\theta^{(1)}(\tau)$ by maximizing the objective function $Q^P(\delta(\tau))$ with respect to the dependence parameter θ. Note that the penalty and constraint terms of $Q^P(\delta(\tau))$ do not depend on θ, so that it is equivalent to maximize $Q(\delta(\tau)) = \sum_{i=1}^{T} \ell(\hat{u}_i, \delta(\tau)) K_h(t_i - \tau)$. We use a one-step Newton-Raphson method:

$$\theta^{(1)}(\tau) = \theta^{(0)}(\tau) - \frac{Q'_\theta(\delta^{(0)}(\tau))}{Q''_\theta(\delta^{(0)}(\tau))},$$

where $Q'_\theta(\delta(\tau))$ and $Q''_\theta(\delta(\tau))$ are the first and second derivatives of $Q_\theta(\delta(\tau))$ with respect to θ, respectively. It may not be easy to find explicit expressions for $Q'_\theta(\delta(\tau))$ and $Q''_\theta(\delta(\tau))$, in which case one can use numerical derivatives as

$$Q'_{\theta_k}(\delta(\tau)) \approx \frac{Q(\delta(\tau)+\varsigma\iota_k) - Q(\delta(\tau)-\varsigma\iota_k)}{2\varsigma} \qquad \text{and} \qquad Q''_{\theta_k}(\delta(\tau)) \approx \frac{Q'_{\theta_k}(\delta(\tau)+\varsigma\iota_k) - Q'_{\theta_k}(\delta(\tau)-\varsigma\iota_k)}{2\varsigma}$$

where ς is a small positive real number and ι_k is a $(2d)$-dimensional vector with the k-th element being one and the others being zero.

B. Bandwidth and tuning parameter selection. The bandwidth h determines the trade-off between the bias and variance of the nonparametric estimators, while the tuning parameter

γ_T adjusts the weight for the penalty term. We need to choose suitable regularization parameters to do the nonparametric estimation and variable selection simultaneously. Various methods for the selection of bandwidths and tuning parameters have been proposed in the variable selection literature, such as cross-validation, AIC- and BIC-type criteria, among others. Due to the time series nature of the sequence $\{X_{1i}, \cdots, X_{Ni}\}_{t=1}^T$, we propose using a *forward leave-one-out* cross-validation to select both the bandwidth h and tuning parameter γ_T in the penalty term simultaneously.

Define $\hat{\delta}(h, \gamma_T)$ as the nonparametric estimators for the penalized time-varying mixture copula models in (2.4) with a known bandwidth h and tuning parameter γ_T. For each data point $i_0 + 1 \leq i^* \leq T$, we use the data $\{X_{1i}, \cdots, X_{Ni}, i < i^*\}$ to construct the estimate $\hat{\delta}_{t^*}(h, \gamma_T)$ at the sample point $\{x_{1i^*}, \cdots, x_{Ni^*}\}$, where i_0 is the minimum window size used to estimate $\hat{\delta}_{i_0+1}(h, \gamma_T)$. Under this forward recursive scheme, we obtain the sequential estimators $\{\hat{\delta}_{i^*}(h, \gamma_T)\}_{i^*=i_0+1}^T$. The optimal bandwidth h^* and tuning parameter γ_T^* can be obtained by maximizing the objective function

$$(h^*, \gamma_T^*) = \arg\max_{(h,\gamma_T)} \sum_{i^*=i_0+1}^{T} \{\ell(\hat{u}_{i^*}, \hat{\delta}(t_i^*))|\hat{\delta}_{i^*}(h, \gamma_T)\}, \tag{2.5}$$

and (h^*, γ_T^*) is the *forward leave-one-out* cross-validation estimator in terms of the log-likelihood.

C. Confidence intervals. For inference, i.i.d bootstrap approaches are not applicable here because most of the financial/economic data are dependent. Patton (2012a) suggests a block bootstrap to construct the pointwise confidence intervals on copula dependence parameters for serially dependent data although its theoretical properties require formal justification. The intuition behind this method is that, by dividing the data into several blocks, it can

preserve the original time series structure within a block. A simple block bootstrap for calculating confidence intervals can be implemented as follows:

i. Generate a sample sequence $\{x^*_{1,i}, \cdots, x^*_{N,i}\}_{i=1}^T$ from the original data $\{x_{1,i}, \cdots, x_{N,i}\}_{i=1}^T$ using a stationary bootstrap technique as described in the online supplement;

ii. Obtain $\hat{u}^*_{1i}, \cdots, \hat{u}^*_{Ni}$ by Steps 1-2 described in Section 2.1;

iii. Calculate new local constant estimators $\hat{\delta}^*(\tau)$ at the grid point τ by equation (2.4) with estimators $\{\hat{u}^*_{1i}, \cdots, \hat{u}^*_{Ni}\}_{i=1}^T$;

iv. Repeat Steps i-iii M times (say, M=1000), and get M values of the estimators $\hat{\delta}^*(\tau)$ as an empirical sample at each grid point τ. Let the $\alpha/2$-th and $(1-\alpha/2)$-th percentiles of the sample sequence $\{\hat{\delta}^*(\tau)\}$ be $q_{\alpha/2}$ and $q_{1-\alpha/2}$, respectively; and

v. The empirical $100(1-\alpha)\%$ confidence interval for $\hat{\delta}(\tau)$ is $[q_{\alpha/2}, q_{1-\alpha/2}]$.

3. Monte Carlo Simulation Studies

This section illustrates the finite-sample performance of the proposed estimation and selection method through a series of simulation studies. We consider the bivariate case where the data are generated by AR(1)-GARCH(1,1) processes:

$$x_{si} = \varphi_s x_{s,i-1} + e_{si}, \qquad s = 1, 2;\ i = 2, ..., T,$$

where $\varphi_1 = 0.1$, $\varphi_2 = 0.05$, $e_{si} = \sigma_{si}\epsilon_{si}$, ϵ_{si} has a standard normal marginal distribution and

$$\sigma^2_{si} = \alpha_{s0} + \alpha_{s1} e^2_{s,i-1} + \beta_{s1}\sigma^2_{s,i-1},$$

where $\alpha_{10} = 0.0001$, $\alpha_{11} = 0.02$, $\beta_{11} = 0.93$ for the first margin, and $\alpha_{20} = 0.0001$, $\alpha_{21} = 0.03$, $\beta_{21} = 0.92$ for the second margin. Our working mixture copula model consists of three copulas: the Gumbel, Frank and Clayton copulas, which are widely used in empirical studies. The Frank copula shows a symmetric dependence structure, while the Clayton and Gumbel copulas are asymmetric. In particular, the Clayton copula displays strong lower tail dependence, while the Gumbel copula exhibits strong upper tail dependence. The dependence structure between ϵ_{1i} and ϵ_{2i} is governed by a time-varying mixture copula models as

$$(\epsilon_{1i}, \epsilon_{2i}) \sim \sum_{k=1}^{3} \lambda_k(t_i) C_k(u_1, u_2; \theta_k(t_i)),$$

where one of the three weight parameters (λ_1, λ_2 and λ_3) is zero.

We consider two cases for the weights and dependence parameters. First, the weights and dependence parameters are set to constants. Second, they are time-varying according to some given functions. In each case, we simulate three mixture copulas with two components. The sample size $T = 400$ and 800, and each simulation is repeated 1000 times ($M = 1000$). For each sample we calculate the estimated weights and dependence parameters on a grid of 50 equally spaced points $\tau_j = -0.01 + 0.02j$ for $j \in \{1, 2, ..., 50\}$.

3.1 Case I simulations

We first consider the scenario where data are generated from mixture copulas with constant weights and dependence parameters. Let λ_1, λ_2 and λ_3 denote the weights of the Gumbel, Frank and Clayton copulas respectively, and θ_1, θ_2 and θ_3 denote the corresponding dependence parameters. We consider the following models for the weights and dependence parameters:

- Model 1: $\lambda_1 = 1/2$, $\lambda_2 = 1/2$, $\lambda_3 = 0$, $\theta_1 = 6$, $\theta_2 = 4$;
- Model 2: $\lambda_1 = 1/2$, $\lambda_2 = 0$, $\lambda_3 = 1/2$, $\theta_1 = 6$, $\theta_3 = 5$;
- Model 3: $\lambda_1 = 0$, $\lambda_2 = 1/2$, $\lambda_3 = 1/2$, $\theta_2 = 4$, $\theta_3 = 5$.

We summarize the estimation results for the weights and dependence parameters in the Case I simulations in Tables 1-2, Panel A. Table 1, Panel A presents the percentages corresponding to the correctly and incorrectly (in parentheses) selected copulas. From Table 1, Panel A, the proposed method performs very well in selecting appropriate copulas from mixture copula models with constant parameters, although our method is designed for time-varying mixture copula models. For all three models, the correct component copulas are selected with 100% probability. Moreover, the probability that the incorrect copulas are chosen is small. There is zero probability of selecting the incorrect copulas for the mixture of Gumbel and Frank, and the mixture of Clayton and Frank. For the mixture model consisting of the Gumbel and Clayton copulas, the chance to incorrectly select the Frank copula is also small and decreases with T.

To examine the performance of the proposed method in estimating the unknown parameters, we calculate the mean square errors (MSEs) of the estimated weights and dependence parameters for the mixture copula models under Case I simulations. The MSEs are calculated as

$$\begin{aligned} MSE(\widehat{\theta}_k) &= \frac{1}{M}\frac{1}{50}\sum_{m=1}^{M}\sum_{j=1}^{50}\left(\widehat{\theta}_{mk}(\tau_j) - \theta_k\right)^2, \text{ and} \\ MSE(\widehat{\lambda}_k) &= \frac{1}{M}\frac{1}{50}\sum_{m=1}^{M}\sum_{j=1}^{50}\left(\widehat{\lambda}_{mk}(\tau_j) - \lambda_k\right)^2, \text{ for } k = 1, 2, 3, \end{aligned}$$

Table 1: Percentages of correctly (incorrectly) chosen copulas for models in Case I (Panel A) and Case II (Panel B) simulations, when the working mixture copula model is time-varying

Model	T	Gumbel	Frank	Clayton
Panel A: Case I				
Gumbel+Frank	400	1.000	1.000	(0.000)
	800	1.000	1.000	(0.000)
Gumbel+Clayton	400	1.000	(0.115)	1.000
	800	1.000	(0.073)	1.000
Clayton+Frank	400	(0.000)	1.000	1.000
	800	(0.000)	1.000	1.000
Panel B: Case II				
Gumbel+Frank	400	1.000	1.000	(0.000)
	800	1.000	1.000	(0.000)
Gumbel+Clayton	400	1.000	(0.143)	1.000
	800	1.000	(0.058)	1.000
Clayton+Frank	400	(0.011)	1.000	1.000
	800	(0.000)	1.000	1.000

NOTE: Values without parentheses are the percentages that copulas in the mixture copulas are chosen correctly. Values with parentheses are the percentages that copulas not in the mixture copulas are chosen incorrectly.

where 50 is the number of grid points and $M = 1000$ is the replication time.

The results are shown in Table 2, Panel A. As expected, the MSEs decrease when the sample size increases for all three models.

Table 2: Mean squared errors of the estimated weights and dependence parameters for models in Case I (Panel A) and Case II (Panel B) simulations, when the working mixture copula model is time-varying

Model	T	(λ_1, θ_1)	(λ_2, θ_2)	(λ_3, θ_3)
Panel A: Case I				
Gumbel+Frank	400	(0.007, 0.513)	(0.007, 0.858)	
	800	(0.003, 0.239)	(0.003, 0.380)	
Gumbel+Clayton	400	(0.006, 0.570)		(0.008, 0.727)
	800	(0.003, 0.326)		(0.004, 0.365)
Clayton+Frank	400		(0.006, 0.829)	(0.006, 0.653)
	800		(0.001, 0.426)	(0.001, 0.395)
Panel B: Case II				
Gumbel+Frank	400	(0.009, 2.085)	(0.009, 1.509)	
	800	(0.004, 0.630)	(0.004, 0.497)	
Gumbel+Clayton	400	(0.006, 0.920)		(0.008, 2.971)
	800	(0.002, 0.416)		(0.003, 0.985)
Clayton+Frank	400		(0.009, 1.978)	(0.009, 1.763)
	800		(0.004, 0.716)	(0.004, 0.855)

In the online supplement, we further evaluate the quality of the estimators graphically. Figures S1-S3 in the supplement respectively display simulation results of the weights and dependence parameters for Models 1-3 under Case I simulations. In each figure, the black solid line denotes true parameters (the weight or dependence parameter), and two curves respectively represent medians (blue) and means (red) of the 1000 simulation parameter function estimates at the grid points. The two green dashed lines represent the 5% and 95% percentiles of the parameter estimates at the grid points. To save space, we only present the results for $T = 800$. In all three models, the median and mean curves are close to the true parameter paths, which are constant in this case.

3.2 Case II simulations

In the Case II simulations, the weights and dependence parameters are dynamic according to the following functions:

- Model 1: $\lambda_1(\tau) = 0.7 - 0.4\sin^2(\frac{\pi}{2}\tau)$, $\lambda_2(\tau) = 1-\lambda_1(\tau)$, $\lambda_3(\tau) = 0$, $\theta_1(\tau) = e^{2\tau} + 3$, $\theta_2(\tau) = 6\tau^2 + 4$;
- Model 2: $\lambda_1(\tau) = 0.7 - 0.4\sin^2(\frac{\pi}{2}\tau)$, $\lambda_2(\tau) = 0$, $\lambda_3(\tau) = 1-\lambda_1(\tau)$, $\theta_1(\tau) = e^{2\tau} + 3$, $\theta_3(\tau) = \ln(1 + \tau T) + 3$;
- Model 3: $\lambda_1(\tau) = 0$, $\lambda_2(\tau) = 0.7 - 0.4\sin^2(\frac{\pi}{2}\tau)$, $\lambda_3(\tau) = 1-\lambda_2(\tau)$, $\theta_2(\tau) = 6\tau^2 + 4$, $\theta_3(\tau) = \ln(1 + \tau T) + 3$;

where $\lambda_k(\tau)$ and $\theta_k(\tau)$, $k = 1, 2, 3$, respectively represent the weights and dependence parameters of the Gumbel, Frank and Clayton copulas.

Tables 1-2, Panel B show the estimation results for this case. We use Table 1, Panel B to examine whether the proposed method can efficiently select appropriate copulas from

different time-varying mixture copula models. As in Table 1, Panel A, the values without parentheses correspond to the percentages that copulas in the mixture models are selected correctly, and the values within parentheses are the percentages that copulas not in the mixture models are selected incorrectly. For all three time-varying mixture copula models, the correct copulas are selected in all replications. For the mixture of the Gumbel and Frank, the probability of choosing the incorrect copula (Clayton) is zero. For the other two mixtures, the chance to select incorrect copulas is also small. For example, when $T = 800$, there is only 5.8% to select Frank when data are generated from a mixture of Gumbel and Clayton. Therefore, Table 1, Panel B demonstrates the good performance of the proposed method in copula selection for mixture copulas with dynamic parameters.

We now use Table 2, Panel B to check whether the proposed method can accurately estimate the unknown parameters under the Case II simulations. Again, we omit the results of the marginal parameters to save space. In Table 2, Panel B, we calculate the MSEs of the estimated weights and dependence parameters for the mixture copulas with dynamic parameters. Similar to the Case I simulations, the MSEs are calculated as $MSE(\widehat{\theta}_k) = \frac{1}{M}\frac{1}{50}\sum_{m=1}^{M}\sum_{j=1}^{50}\left(\widehat{\theta}_{mk}(\tau_j) - \theta_k(\tau_j)\right)^2$ and $MSE(\widehat{\lambda}_k) = \frac{1}{M}\frac{1}{50}\sum_{m=1}^{M}\sum_{j=1}^{50}\left(\widehat{\lambda}_{mk}(\tau_j) - \lambda_k(\tau_j)\right)^2$, for $k = 1, 2, 3$. We note two observations from Table 2, Panel B. First, as the sample size increases from 400 to 800, the MSEs decrease and the estimates become more accurate. Second, compared to the results in Table 2, Panel A, the MSEs in Panel B are larger in most cases. This is not surprising because the true models in Case II are time-varying mixture copulas with dynamic parameters, which are more difficult to estimate than the true models in Case I (mixture copulas with constant parameters).

Figures S4-S6 in the online supplement present the estimated and the true parameter

paths for different time-varying mixture copulas models ($T = 800$). We can observe from Figures S4-S6 that the median and mean paths are still close to the true parameter functions in all models.

We next employ time-invariant mixture copula models for comparision (we thank one anonymous reviewer for suggesting this.). That is, the working mixture copula model still consists of the Gumbel, Frank, and Clayton copulas, but now the weights and dependence parameters are assumed to be constants. We use the penalized likelihood method of Cai and Wang (2014) for estimation, and the mean square errors (MSEs) of the estimates are presented in Table 3 (Panel A for Case I, and Panel B for Case II). Comparing the results in Table 3 with the results in Table 2, we observe that when the true models are time-invariant (Case I), employing time-invariant working mixture copula models may indeed gain some efficiency, but the differences between the two types of working models are minor. However, when the true models are dynamic (Case II), using time-invariant mixture copula models show much worse estimation results than those yielded by time-varying working mixture copula models.

Finally, to check the robustness of the proposed method, we consider two additional candidate copulas: rotated Gumbel and rotated Clayton, and both of them can capture negative dependence. That is, our working mixture copula model now consists of five copulas: the Gumbel, Frank, Clayton, rotated Gumbel, and rotated Clayton copulas. The ture model is still set to be a mixture of two copulas. Therefore, three of the five weight parameters are equal to zero in this case. Denote λ_4 and λ_5 as the weights of rotated Gumbel and rotated Clayton respectively. To save space, we only consider Model 3 of Case II (Clayton+Frank) as the true mixture copula: $\lambda_1(\tau) = 0$, $\lambda_2(\tau) = 0.7 - 0.4\sin^2(\frac{\pi}{2}\tau)$, $\lambda_3(\tau) = 1-\lambda_2(\tau)$, $\lambda_4(\tau) = 0$,

Table 3: Mean squared errors of the estimated weights and dependence parameters for models in Case I (Panel A) and Case II (Panel B) simulations, when the working mixture copula model is time-invariant

Model	T	(λ_1, θ_1)	(λ_2, θ_2)	(λ_3, θ_3)
Panel A: Case I				
Gumbel+Frank	400	(0.005, 0.313)	(0.005, 0.596)	
	800	(0.002, 0.161)	(0.002, 0.279)	
Gumbel+Clayton	400	(0.005, 0.405)		(0.006, 0.603)
	800	(0.003, 0.183)		(0.003, 0.290)
Clayton+Frank	400		(0.004, 0.713)	(0.004, 0.551)
	800		(0.001, 0.325)	(0.001, 0.272)
Panel B: Case II				
Gumbel+Frank	400	(0.052, 10.664)	(0.047, 9.023)	
	800	(0.029, 5.026)	(0.026, 4.357)	
Gumbel+Clayton	400	(0.038, 8.428)		(0.055, 20.799)
	800	(0.016, 3.343)		(0.023, 8.098)
Clayton+Frank	400		(0.062, 11.370)	(0.070, 12.385)
	800		(0.027, 5.839)	(0.031, 7.648)

Table 4: Selection and estimation performance of the proposed method when considering five candidate copulas: Gumbel, Frank, Clayton, rotated Gumbel, and rotated Clayton

Panel A: Selection	Gumbel	Frank	Clayton	Rotated Gumbel	Rotated Clayton
$T = 400$	(0.000)	1.000	0.986	(0.000)	(0.000)
$T = 800$	(0.000)	1.000	1.000	(0.000)	(0.000)

Panel B: Estimation	λ_2	θ_2	λ_3	θ_3
$T = 400$	0.014	2.873	0.014	3.115
$T = 800$	0.006	1.066	0.006	1.287

NOTE: The true model is a mixture copula of Clayton and Frank: $\lambda_1(\tau) = 0$, $\lambda_2(\tau) = 0.7 - 0.4\sin^2(\frac{\pi}{2}\tau)$, $\lambda_3(\tau) = 1-\lambda_2(\tau)$, $\lambda_4(\tau) = 0$, $\lambda_5(\tau) = 0$, $\theta_2(\tau) = 6\tau^2 + 4$, $\theta_3(\tau) = \ln(1 + \tau T) + 3$.

$\lambda_5(\tau) = 0$, $\theta_2(\tau) = 6\tau^2 + 4$, $\theta_3(\tau) = \ln(1 + \tau T) + 3$.

Table 4 presents the selection and estimation performance of the proposed method when employing a larger candidate copula set. The results in Panel A of Table 4 once again support the finding that the proposed method can accurately select the appropriate copulas. Even for $T = 400$, the probability of correctly selecting Clayton and Frank is very close or equal to 100%. Furthermore, the percentage that the redundant copulas (Gumbel, rotated Gumbel, and rotated Clayton) are selected is zero. In Panel B, we report the MSEs of the estimated weights and dependence parameters for Frank (λ_2 and θ_2) and Clayton (λ_3 and

θ_3). Compared with the estimation results in Table 2, Panel B, we find that increasing the number of the candidate copulas from three to five only moderately lower the accuracy of the estimation. Figure S7 in the online supplement provides more graphical evidence of the performance of the proposed method in estimating unknown parameters in this case.

4. An Empirical Study

In this section, we apply the proposed model and method to analyze the co-movements of returns among international stock markets during different periods. Specifically, we consider weekly returns of the Morgan Stanley Capital International (MSCI) equity indices of four economies (in U.S. Dollars): the United States (US), the United Kingdom (UK), Hong Kong (HK), and South Korea (KR). These four economies are much affected by the Asian crisis of 1997 and/or the global financial crisis of 2008. By analyzing the evolution of the dependence structures among these four markets, we can examine how these markets are related, for example, in tranquil periods and in crisis periods.

4.1 Data

The data we use span the period of over 28 years from January 1990 until July 2018, with a total of 1488 observations for each economy. We first obtain the equity indices from Datastream and then calculate their log-returns by $r_{s,i} = \log(P_{s,i}) - \log(P_{s,i-1})$, where $P_{s,i}$ is the stock index of the i-th market at time i. We use weekly data instead of daily data to remove the effect of different trading hours for international stock markets (Chollete, Heinen and Valdesogo, 2009; Hafner and Reznikova, 2010). Descriptive statistics are presented in Table 5, Panel A. The United States market exhibits the highest mean and median returns.

Table 5: Summary statistics and correlations

	US	UK	HK	KR
Panel A: Summary statistics				
Mean (%)	0.140	0.061	0.136	0.061
Median (%)	0.284	0.193	0.277	0.218
Min (%)	-16.75	-15.21	-16.79	-40.25
Max (%)	10.34	11.56	14.03	30.02
Std. Dev.	0.022	0.026	0.032	0.047
Skewness	-0.669	-0.425	-0.482	-0.515
Kurtosis	7.892	6.117	5.813	10.556
JB statistic	1595	647	548	3606
JB p-value	0.000	0.000	0.000	0.000

Panel B: Correlations

	UK	KR	HK
US	0.681 (0.451)	0.389 (0.249)	0.487 (0.330)
UK		0.409 (0.280)	0.515 (0.351)
KR			0.495 (0.350)

NOTE: Panel A presents the summary statistics of weekly index returns for the United States (US), the United Kingdom (UK), Hong Kong (HK) and South Korea (KR). All returns are expressed in U.S. dollars from January, 1990 to July, 2018, which correspond to a sample of 1488 observations. JB statistic and JB p-value refer to Jarque-Bera test of normality. Panel B reports the linear correlation coefficients and the Kendall's τs (Kendall's τs are in parentheses) across the US, UK, HK and KR markets.

The Korea market shows the largest volatility of returns. We employ the Jarque-Bera test for normality and the test strongly rejects the null hypothesis for all series.

Table 5, Panel B reports the unconditional correlation coefficients and Kendall's τs (in parentheses) across the four markets. We observe that the US and UK markets display the highest correlation, based on both the correlation coefficient (0.681) and the Kendall's τ (0.451). The US-HK, UK-HK, and HK-KR pairs show similar dependence of moderate size (around 0.5 for the correlation coefficients and around 0.35 for Kendall's τs). The least dependent pairs are US-KR (0.389 for correlation and 0.249 for Kendall's τ) and UK-KR (0.409 for correlation and 0.280 for Kendall's τ).

4.2 The models for the marginal distributions

First of all, we model the marginal distributions of the data. We employ AR(p)-GARCH(1,1) models, a special case of model (2.1) presented in Section 2, to capture possible autocorrelation and conditional heteroscedasticity in returns. The Bayesian information criterion (BIC) is used to select the appropriate number of lags p of the AR(p) models. Specifically, we use the following models for the marginal distributions:

$$\begin{aligned} X_{si} &= \varphi_{s0} + \sum_{k=1}^{p} \varphi_{sk} X_{s,i-k} + e_{si}, \quad e_{si} = \sigma_{si}\epsilon_{si}, \\ \sigma_{si}^2 &= \alpha_{s0} + \alpha_{s1} e_{s,i-1}^2 + \beta_{s1}\sigma_{s,i-1}^2, \end{aligned}$$

where X_{si} denotes the return of the s-th market at time i. The innovations ϵ_{si} are assumed to be i.i.d for $i = 1, \cdots, T$ with a fixed s, and the distribution is estimated through the rescaled empirical distribution of the residuals. Model diagnostics such as portmanteau type tests for the mean and the variance confirm our model specifications. To economize on space they are not reported here but available upon request.

Table 6: Estimation results and tests of the marginal distribution models

	AR(p)	GARCH(1,1)			LB	
	φ_1	α_0	α_1	β_1	4	16
	(s.e.)	(s.e.)	(s.e.)	(s.e.)		
US	-0.118	0.007E-3	0.100	0.890	0.617	0.643
	(0.026)	(0.004E-3)	(0.019)	(0.019)		
UK	-0.098	0.023E-3	0.127	0.840	0.892	0.577
	(0.027)	(0.009E-3)	(0.028)	(0.036)		
HK		0.020E-3	0.098	0.885	0.112	0.155
		(0.008E-3)	(0.020)	(0.023)		
KR		0.045E-3	0.115	0.862	0.120	0.337
		(0.015E-3)	(0.020)	(0.023)		

NOTE: The second to fifth columns report parameter estimates of AR(p)-GARCH(1,1) models. Values in parentheses are corresponding standard errors. The sixth and seventh columns report the p-values of the Ljung-Box (LB) tests for autocorrelation of the residuals using 4 and 16 lags, respectively.

4.3 The models for the copula

We focus on studying the dependence structures of four pairs (US-UK, US-HK, UK-HK, and HK-KR) which have relatively large correlation coefficients. Scatter plots (omitted here) of four pairs of standardized returns show violations of elliptical multivariate distributions, because asymmetry and a large number of outliers can be observed in all pairs. Therefore, we employ a mixture copula model including the Clayton, Frank and Gumbel copulas to implement copula selection and estimation. In such a way, we can capture various dependence structures in the data such as lower or upper tail dependence, or a symmetric but non-elliptical dependence structure.

We first fit the data to a time-invariant mixture copula model to examine the overall dependence structures during the period of 28 years. The penalized likelihood method of Cai and Wang (2014) is employed to select and estimate the model. The results are reported in Table 7. We have two findings from Table 7. First, the Gumbel copula is excluded from the mixture model for all pairs of data, implying that no pairs exhibit upper tail dependence. Second, the Clayton copula is selected and the weight and dependence parameters are statistically significant away from zero for all pairs. This indicates that lower tail dependence can be found for all pairs of markets. These two findings are similar to those in Cai and Wang (2014).

Although the time-invariant model can tell us that overall the pairs of markets show lower tail dependence, it can neither capture the evolution of the dependence structures, nor distinguish between the dependence structures in tranquil periods and those in crisis periods. Therefore, we next employ the time-varying mixture copula model proposed in this paper to analyze the dependence structures of the international stock markets. Figures 1-4

Table 7: Estimation results of the time-invariant mixture copula models for international markets

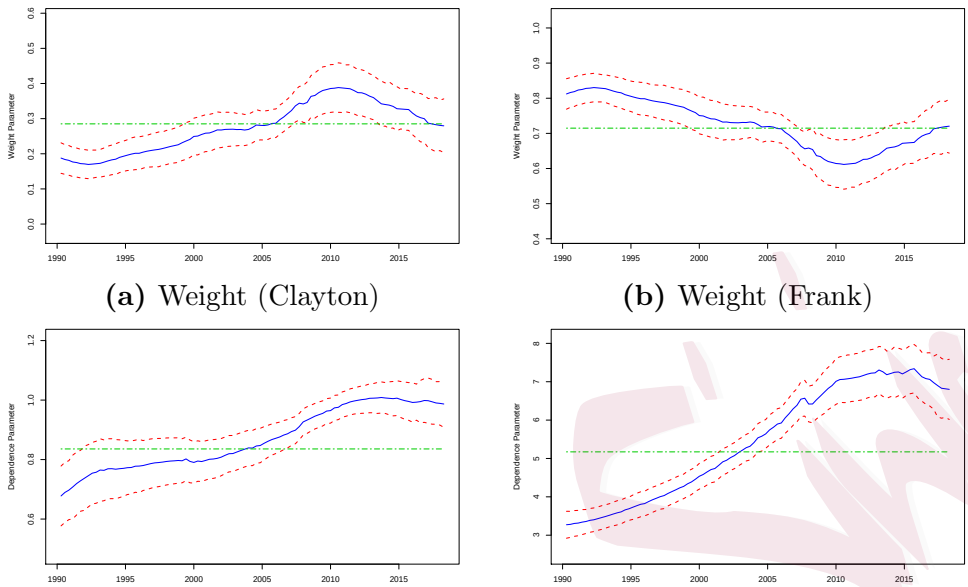
	Markets	Clayton	Gumbel	Frank
λ	US-UK	0.285(0.241,0.329)	0	0.715(0.671,0.759)
	US-HK	0.314(0.268,0.360)	0	0.686(0.640,0.732)
	UK-HK	0.352(0.304,0.399)	0	0.648(0.601,0.696)
	HK-KR	0.208(0.150,0.267)	0	0.792(0.733,0.850)
θ	US-UK	0.836(0.781,0.890)		5.172(4.823,5.521)
	US-HK	0.594(0.536,0.652)		3.761(3.565,3.956)
	UK-HK	0.657(0.585,0.729)		4.148(3.862,4.433)
	HK-KR	0.918(0.848,0.987)		3.208(2.923,3.493)

NOTE: This table presents estimates of the weights (λ) and dependence parameters (θ) of time-invariant mixture copula models using the penalized likelihood method of Cai and Wang (2014). Values in parentheses are the 90% confidence interval of the estimates.

respectively present the estimation results and the 90% confidence intervals of all nonzero weights and dependence parameters for the US-UK, US-HK, UK-HK, and HK-KR pairs. In each figure, the path of the estimated parameter (the weight or dependence parameter) is represented by a blue solid line. The two red dashed curves show the 90% confidence intervals of each estimated parameter. The green two-dashed line (horizontal line) is the estimate using the time-invariant mixture copula model. We have several interesting results from these figures.

First, for all pairs of markets, the Clayton and Frank copulas are selected at any time period of the 28 years. The confidence intervals for the weights on Clayton and Frank do not cover zeros, showing that they are always statistically significant. On the other hand, the weight on Gumbel is always zero during the 28 years for all pairs. Therefore, the four pairs of markets show significantly lower tail dependence, but no upper tail dependence from 1990 to 2018. Second, we observe notable fluctuations of both the weights and dependence parameters during the 28-year period for all four pairs of markets, implying the limitation of time-invariant copula models.

For the US-UK pair presented in Figure 1, the weight and dependence parameter of



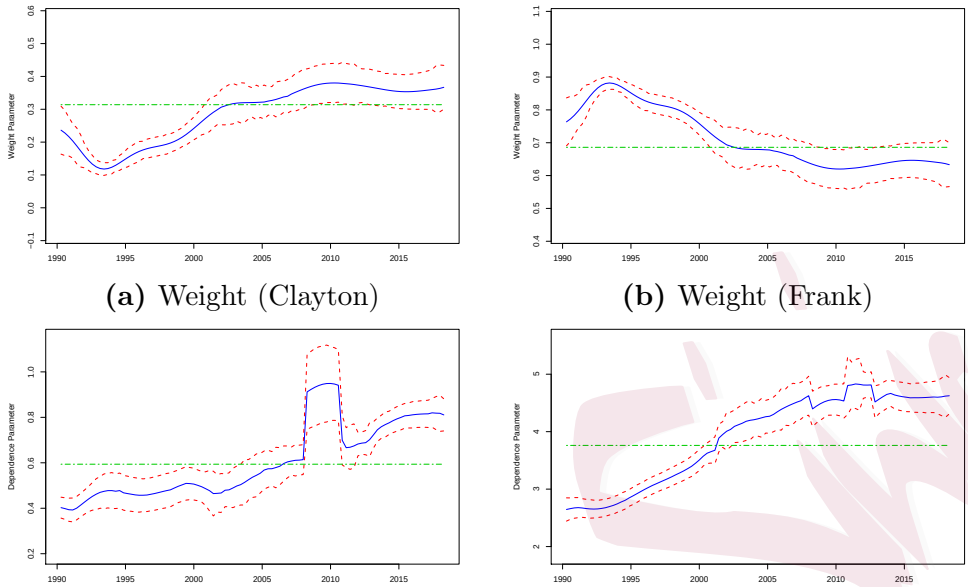
(a) Weight (Clayton) **(b)** Weight (Frank)

(c) Dependence parameter (Clayton) **(d)** Dependence parameter (Frank)

Figure 1: Estimation results of nonzero weights and dependence parameters (blue solid lines) along with the 90% confidence intervals (red dashed curves) for the US-UK pair. The green two-dashed lines (horizontal lines) show the estimates using the time-invariant mixture copula model.

the Clayton copula are both relatively small in the early 1990s. Meanwhile, the dependence parameter of the Frank copula is also small during this period. These findings show that both the lower tail dependence and the overall dependence are weak at the beginning of the 1990s. The dependence parameter of Clayton increases sharply after the events of September 11, 2001. At the same time, the weight of Clayton also reaches a relatively high value. During the financial crisis of 2008, the weight and dependence parameter of the Clayton copula, and the dependence parameter of the Frank copula all increase sharply, reaching their maxima around 2010. This implies that the lower tail and general dependence of the two markets attains high levels in crisis periods.

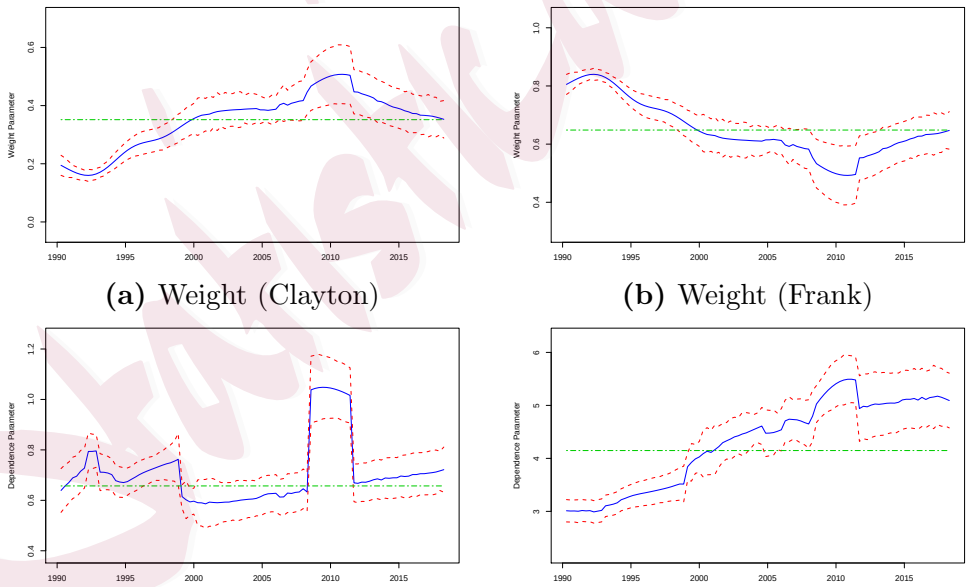
Turning to the US-HK and UK-HK pairs, the dependence structures display similar evolution paths (see Figures 2-3). Both pairs show relatively weak lower tail dependence and overall dependence during the 1990s. A notable jump in the Clayton parameter took



(a) Weight (Clayton) **(b)** Weight (Frank)

(c) Dependence parameter (Clayton) **(d)** Dependence parameter (Frank)

Figure 2: Estimation results of nonzero weights and dependence parameters (blue solid lines) along with the 90% confidence intervals (red dashed curves) for the US-HK pair. The green two-dashed lines (horizontal lines) show the estimates using the time-invariant mixture copula model.



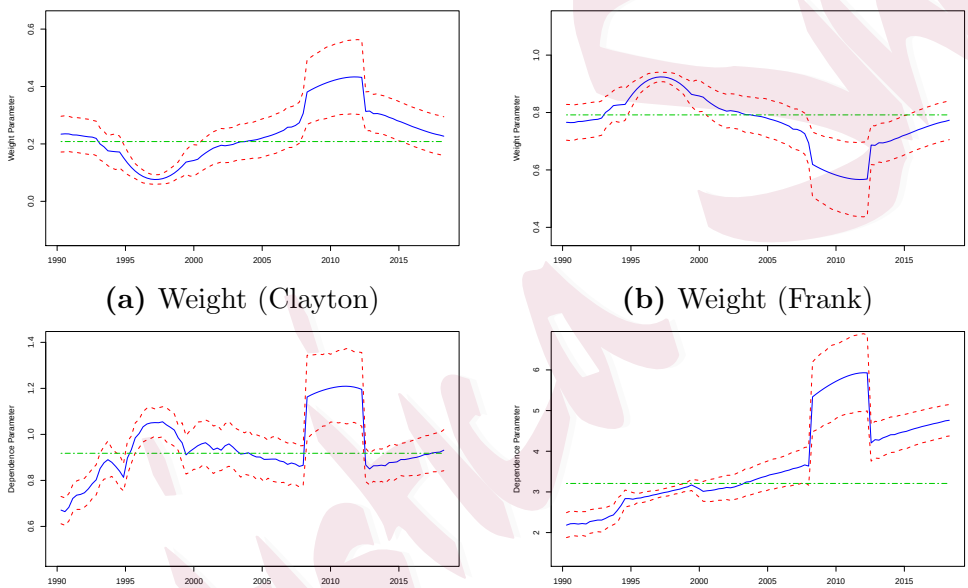
(a) Weight (Clayton) **(b)** Weight (Frank)

(c) Dependence parameter (Clayton) **(d)** Dependence parameter (Frank)

Figure 3: Estimation results of nonzero weights and dependence parameters (blue solid lines) along with the 90% confidence intervals (red dashed curves) for the UK-HK pair. The green two-dashed lines (horizontal lines) show the estimates using the time-invariant mixture copula model.

place in 2008 for both pairs due to the financial crisis. An increase in the weight on Clayton can be observed during the same time period.

The last figure (Figure 4) exhibits the dependence structure of the HK-KR pair. These two markets are strongly affected by the Asian crisis of 1997. Therefore, we can observe a relatively high level of the Clayton parameter, and a quick increase in the weight on Clayton in 1997. During the periods of the financial crisis of 2008, a significant increase and a remarkable jump in the weight and dependence parameter of the Clayton copula are also detected for this pair of markets.



(a) Weight (Clayton) **(b)** Weight (Frank)

(c) Dependence parameter (Clayton) **(d)** Dependence parameter (Frank)

Figure 4: Estimation results of nonzero weights and dependence parameters (blue solid lines) along with the 90% confidence intervals (red dashed curves) for the HK-KR pair. The green two-dashed lines (horizontal lines) show the estimates using the time-invariant mixture copula model.

Finally, we check the goodness-of-fit of the estimated time-varying mixture copula model with the Kolmogorov-Smirnov (KS) test, the Cramer-von Mises (CM) test, and the Anderson-Darling (AD) test for correct copula specification. The procedures of doing the goodness-of-fit can be found in the online supplement. Table 8 reports the bootstrap p-values of the three tests. All models pass these three tests with large p-values.

Table 8: Goodness-of-fit tests for the time-varying mixture copula models

Markets	KS	CM	AD
US-UK	0.256	0.244	0.250
US-HK	0.670	0.470	0.686
UK-HK	0.468	0.474	0.530
HK-KR	0.362	0.372	0.592

NOTE: This table reports the p-values from three goodness-of-fit tests including the Kolmogorov-Smirnov (KS) test, the Cramer-von Mises (CM) test and the Anderson-Darling (AD) test.

5. Conclusion

In this paper, we introduce a time-varying mixture copula model, in which both the weights and dependence parameters are deterministic functions of time. To reduce the risk of over-fitting and efficiency loss, we propose a penalized time-varying mixture copula model with the SCAD penalty term to do the estimation and copula selection simultaneously. Based on α-mixing conditions, large sample properties of the penalized and unpenalized estimators have been established. Meanwhile, we study and discuss a semiparametric EM algorithm, the bandwidth selection, and the construction of pointwise confidence intervals. Monte Carlo simulations demonstrate the good performance of the proposed method in copula selection and estimation for time-varying mixture copulas. The proposed methodology has been applied to study the evolution of the dependence among four international stock markets. All pairs of markets present strong dependence at the lower tail that fluctuates significantly over time. Furthermore, all pairs exhibit the highest levels of both the lower tail and overall dependence during the financial crisis of 2008.

Future researches include employing the proposed model and method to other fields in finance and economics, such as exchange rate, bond, and crude oil. Moreover, studying higher dimensional dependence structures among financial markets by using the proposed

model and method is another interesting research topic.

Supplementary Materials

The supplement provides the stationary bootstrap resampling scheme, the regularity conditions and asymptotic properties for unpenalized estimators, the mathematical proofs, the procedures of doing the goodness-of-fit, and some additional figures.

Acknowledgements

The authors thank the Co-Editor, Professor Yazhen Wang, and referees for their helpful suggestions for improvements to this paper. Yang's research is partly supported by Humanity and Social Science Youth foundation of Ministry of Education of China (19C10421028), the Fundamental Research Funds for the Central Universities(19wkpy61), the Major Program of the National Social Science Foundation of China (17ZDA073) and the Major Program of NSFC (71991474). Cai's research is partially supported by the National Natural Science Foundation of China (Grant Nos. 71631004 and 72033008). Liu's research is supported by the National Natural Science Foundation of China (Grant No. 71803160). The corresponding author is Guannan Liu, MOE Key Laboratory of Econometrics, School of Economics and Wang Yanan Institute for Studies in Economics, Xiamen University, Xiamen, China; email: gliuecon@gmail.com.

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