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Complete List of Authors	Kai Xu and Liping Zhu
Corresponding Author	Liping Zhu
E-mail	zhulp1@hotmail.com
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POWER ANALYSIS OF PROJECTION-PURSUIT INDEPENDENCE TESTS

Kai Xu and Liping Zhu

Anhui Normal University and Renmin University of China

Abstract: Three important projection-pursuit correlations, namely, distance correlation, projection correlation and the multivariate Blum-Kiefer-Rosenblatt (BKR) correlation, have been proposed in the literature to test independence between two random vectors in arbitrary dimensions. In this paper we compare the asymptotic power performance of independence tests built upon these three projection-pursuit correlations in a uniform sense. We show that, in the presence of outliers, the projection correlation test and the multivariate BKR correlation test are still powerful, whereas the distance correlation test may lose power. We also analyze the minimax optimality of these independence tests. We show that their minimum separation rates are of order n^{-1}, where n stands for the sample size, and this minimax optimal rate is tight in terms of projection correlation, distance correlation and multivariate BKR correlation, respectively.

Key words and phrases: distance correlation; independence test; minimax optimality; projection correlation; power function; robustness

1. INTRODUCTION

Many important applications require to quantify the degree of nonlinear dependence between two random vectors. For example, in genomics research, one may be interested to test whether certain diseases are associated with mutations of a particular group of genes. In economic studies, one may wish to evaluate nonlinear dependence between the stock market and real estate returns. In brain sciences, one may expect to discover whether two sets of voxels measured over time at different parts of brain are functionally related. We formulate these applications into the problems of testing independence. In symbols, let $\mathbf{x} = (X_1, \ldots, X_p)^{\mathrm{T}} \in \mathbb{R}^p$ and $\mathbf{y} = (Y_1, \ldots, Y_q)^{\mathrm{T}} \in \mathbb{R}^q$ be two random vectors. We assume throughout that $p > 1$ and $q > 1$ unless stated otherwise. The goal of independence tests is to test

$$H_0 : \mathbf{x} \text{ and } \mathbf{y} \text{ are statistically independent; } H_1 : \mathbf{x} \text{ and } \mathbf{y} \text{ are dependent.} \quad (1.1)$$

Testing for independence has a long history in the literature. Pearson correlation is perhaps the first and one of the most important metrics to test for independence between two univariate random variables (i.e. $p = q = 1$). Extensions within the univariate case include, but not limited to, Hoeffding (1948), Blum et al. (1961) and Bergsma and Dassios (2014). These extensions are based on ranks of observations and thus are not able to be used if either $\mathbf{x}$ or $\mathbf{y}$ is multivariate (i.e. $p > 1$ or $q > 1$). In the multivariate case where both $\mathbf{x}$ and $\mathbf{y}$ follow jointly normal or elliptically symmetric distributions, testing for independence amounts to testing whether they are linearly uncorrelated (Oja, 2010). Important examples along this line include likelihood ratio test (Wilks, 1935) and canonical correlation coefficient (Hotelling, 1936). Interested readers may refer to Puri and Sen (1971), Hettmansperger and Oja (1994) and Taskinen et al. (2005) for extensions of likelihood ratio test.

In the past two decades, there has been much effort to relax the distributional assumptions. See, for example, Kankainen (1995) and Bakirov et al. (2006). Gretton et al. (2005) proposed an independence criterion based on the entire eigen-spectrum of covariance operators in reproducing kernel Hilbert spaces. Székely et al. (2007) and Székely and Rizzo (2009) made important advances through proposing distance correlation to test independence between two random vectors in arbitrary dimensions. Distance correlation is well defined by assuming the first moments of both $\mathbf{x}$ and $\mathbf{y}$ are finite, and is generalized by Sejdinovic et al. (2013), Pan et al. (2019) and Shen et al. (2019) from different perspectives. Heller et al. (2013) pointed out that, if the moment conditions are violated, say, if the underlying distribution of either $\mathbf{x}$ or $\mathbf{y}$ is heavy-tailed or the observations contain outliers, the distance correlation test may suffer from low power. Given that outlying observations arise frequently in practice with high-dimensional data, it is highly desirable to develop robust counterparts of distance correlation. Towards this goal, Zhu et al. (2017) proposed an alternative projection correlation, which completely removes the moment conditions required by distance correlation. The projection correlation is in spirit a multivariate version of Hoeffding (1948). Kim et al. (2018) suggested a projection-averaging approach to the classic two-sample test problems, and stated that their approach can be readily generalized to test independence between two random vectors. In this paper, we follow Kim et al. (2018) through extending the Blum-Kiefer-Rosenblatt (BKR) correlation to the multivariate case. Neither projection correlation nor the multivariate BKR correlation requires moment condition on either $\mathbf{x}$ or $\mathbf{y}$. We shall show that, both distance correlation and projection correlation are based on the integrated squared distance between the joint distribution of the projections and the product of their marginal distributions over unit spheres. The independence tests built upon distance correlation, projection correlation and the multivariate BKR correlation are indeed all of projection-pursuit type.

In this paper we compare power performance of the aforementioned three projection-pursuit independence tests because they share many similarities. In particular, projection correlation, distance correlation and the multivariate BKR correlation have closed-form expressions and require no tuning parameters, and all tests are consistent against all fixed alternatives. More importantly, all three tests can be represented by integrals of the distance between the joint distribution function of $(\mathbf{x}, \mathbf{y})$ and the product of the marginal distribution functions of $\mathbf{x}$ and $\mathbf{y}$. They differ only in the weights. To elaborate, we define $\mathcal{S}^{d-1} \stackrel{\text{def}}{=} \{\boldsymbol{\alpha} \in \mathbb{R}^d : \|\boldsymbol{\alpha}\| = 1\}$, where $\|\cdot\|$ is Euclidean norm. $F_{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x}}(s) \stackrel{\text{def}}{=} \mathrm{pr}(\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x} \le s)$, $F_{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y}}(t) \stackrel{\text{def}}{=} \mathrm{pr}(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y} \le t)$ and $F_{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x},\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y}}(s,t) \stackrel{\text{def}}{=} \mathrm{pr}(\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x} \le s, \boldsymbol{\beta}^{\mathrm{T}}\mathbf{y} \le t)$, for $\boldsymbol{\alpha} \in \mathcal{S}^{p-1}$, $\boldsymbol{\beta} \in \mathcal{S}^{q-1}$, $s \in \mathbb{R}^1$ and $t \in \mathbb{R}^1$. Both $(\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x})$ and $(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y})$ are the respective projections of $\mathbf{x}$ and $\mathbf{y}$. In the Supplementary Material, we shall show that the squared distance covariance can be represented as

$$\begin{aligned}\mathrm{DC}(\mathbf{x}, \mathbf{y}) &= (c_p c_q)^{-1} \int_{\boldsymbol{\alpha} \in \mathcal{S}^{p-1}} \int_{\boldsymbol{\beta} \in \mathcal{S}^{q-1}} \int_{t \in \mathbb{R}^1} \int_{s \in \mathbb{R}^1} \\ &\quad \left\{F_{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x},\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y}}(s,t) - F_{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x}}(s) F_{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y}}(t)\right\}^2 (ds\ dt) d\boldsymbol{\beta} d\boldsymbol{\alpha}, \end{aligned} \tag{1.2}$$

and the squared projection covariance can be represented as

$$\begin{aligned}\mathrm{PC}(\mathbf{x}, \mathbf{y}) &= (\gamma_p \gamma_q)^{-1} \int_{\boldsymbol{\alpha} \in \mathcal{S}^{p-1}} \int_{\boldsymbol{\beta} \in \mathcal{S}^{q-1}} \int_{t \in \mathbb{R}^1} \int_{s \in \mathbb{R}^1} \\ &\quad \left\{F_{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x},\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y}}(s,t) - F_{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x}}(s) F_{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y}}(t)\right\}^2 dF_{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x},\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y}}(s,t) d\boldsymbol{\beta} d\boldsymbol{\alpha}. \end{aligned} \tag{1.3}$$

Kim et al. (2018) wrote the multivariate BKR correlation coefficient as

$$\begin{aligned}\mathrm{mBKR}(\mathbf{x},\mathbf{y}) &= (\gamma_p\gamma_q)^{-1}\int_{\boldsymbol{\alpha}\in\mathcal{S}^{p-1}}\int_{\boldsymbol{\beta}\in\mathcal{S}^{q-1}}\int_{t\in\mathbb{R}^1}\int_{s\in\mathbb{R}^1} \\ &\quad \left\{F_{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x},\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y}}(s,t)-F_{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x}}(s)F_{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y}}(t)\right\}^2 dF_{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x}}(s)dF_{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y}}(t)d\boldsymbol{\beta}d\boldsymbol{\alpha}. \quad (1.4)\end{aligned}$$

In the above three displays, $c_p \stackrel{\text{def}}{=} \left\{2\pi^{(p-1)/2}/(p-1)\right\}/\Gamma\{(p-1)/2\}$, $\gamma_p \stackrel{\text{def}}{=} \pi^{p/2-1}/\Gamma(p/2)$ and $\Gamma(\cdot)$ is gamma function. These displays differ at how we average over s and t. In particular, in (1.2) the uniform weights are given on the $\mathbb{R}^1\otimes\mathbb{R}^1$ space, and in (1.3) and (1.4) more weights are given on higher density regions. It is thus anticipated that the projection correlation test and the multivariate BKR correlation test are more robust to extreme observations than the distance correlation test. The projection correlation uses the joint density of $(\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x})$ and $(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{y})$ as a weight function, whereas the multivariate BKR correlation uses the product of their marginal densities.

The asymptotic null distributions of the above projection-pursuit independence tests depend on the joint distribution of $\mathbf{x}$ and $\mathbf{y}$ which are however generally unknown in practice. To approximate the asymptotic null distributions, random permutations are widely used in these independence tests. However, the consistency of random permutations is rarely explored in the literature. In the present context, we shall show that, the permutation procedure provides a reasonable approximation of the asymptotic null distributions without exhausting all possible permutations. As a by-product, this allows us to carry out power analysis of projection-pursuit independence tests. We shall show that, in the presence of outliers, the permutation test based on either projection correlation or the multivariate BKR correlation is very powerful while the permutation test based on distance correlation may lose power. To gain more insights on their asymptotic behaviors, we analyze the minimax optimality of these projection-pursuit independence tests over a wide class of distributions using the Le Cam's Lemma (Baraud, 2002). We show that their minimum separation rates are all of order n^{-1}, where n stands for the sample size. The minimum separation rate is a lower bound that characterizes the separation boundary between the testable and non-testable regions. The rate n^{-1} is indeed tight in terms of projection correlation, distance correlation and the multivariate BKR correlation, respectively.

2. SOME PRELIMINARIES

2.1 The Computational Complexities

We provide explicit forms for (1.2), (1.3) and (1.4) first. Suppose $\{(\mathbf{x}_i, \mathbf{y}_i), i = 1, \ldots, 6\}$ are six independent copies of $(\mathbf{x}, \mathbf{y})$. Let $\mathbf{z}$ be either $\mathbf{x}$ or $\mathbf{y}$. We define $a(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4, \mathbf{z}_5) \stackrel{\text{def}}{=} \text{ang}(\mathbf{z}_1 - \mathbf{z}_5, \mathbf{z}_2 - \mathbf{z}_5) + \text{ang}(\mathbf{z}_3 - \mathbf{z}_5, \mathbf{z}_4 - \mathbf{z}_5) - \text{ang}(\mathbf{z}_1 - \mathbf{z}_5, \mathbf{z}_3 - \mathbf{z}_5) - \text{ang}(\mathbf{z}_2 - \mathbf{z}_5, \mathbf{z}_4 - \mathbf{z}_5)$, where $\text{ang}(\mathbf{a}, \mathbf{b}) \stackrel{\text{def}}{=}$

$\arccos\{(\mathbf{a}^{\mathrm{T}}\mathbf{b})/(\|\mathbf{a}\|\|\mathbf{b}\|)\}$ stands for the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$ and $\arccos(\cdot)$ is the inverse cosine function. If $\mathbf{z}_i$, $\mathbf{z}_j$ and $\mathbf{z}_k$ are all distinctive, $\mathrm{ang}(\mathbf{z}_i - \mathbf{z}_k, \mathbf{z}_j - \mathbf{z}_k)$ is well defined and ranges from 0 to π. Following Escanciano (2006) and Zhu et al. (2017), we define $\mathrm{ang}(\mathbf{z}_i-\mathbf{z}_k, \mathbf{z}_j-\mathbf{z}_k) = 0$, if $\mathbf{z}_i = \mathbf{z}_j \neq \mathbf{z}_k$, or $\mathbf{z}_i = \mathbf{z}_k \neq \mathbf{z}_j$, or $\mathbf{z}_j = \mathbf{z}_k \neq \mathbf{z}_i$, and $\mathrm{ang}(\mathbf{z}_i-\mathbf{z}_k, \mathbf{z}_j-\mathbf{z}_k) = -\pi$ if $\mathbf{z}_i = \mathbf{z}_j = \mathbf{z}_k$. We further define $b(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4) \stackrel{\mathrm{def}}{=} \|\mathbf{z}_1 - \mathbf{z}_2\| + \|\mathbf{z}_3 - \mathbf{z}_4\| - \|\mathbf{z}_1 - \mathbf{z}_3\| - \|\mathbf{z}_2 - \mathbf{z}_4\|$. Székely et al. (2007) and Székely and Rizzo (2009) showed that $\mathrm{DC}(\mathbf{x}, \mathbf{y}) = E\left\{b(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) b(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4)\right\}/4$. By Theorem 1 of Zhu et al. (2017), the explicit form of projection correlation is given by $\mathrm{PC}(\mathbf{x}, \mathbf{y}) = E\{a(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5) a(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_5)\}/4$. Kim et al. (2018, Theorem 7.2) derived that the multivariate BKR correlation has the form of $\mathrm{mBKR}(\mathbf{x}, \mathbf{y}) = E\{a(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5) a(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_6)\}/4$. With a random sample of size n, say, $\{(\mathbf{x}_i, \mathbf{y}_i), i = 1, \ldots, n\}$, we estimate $\mathrm{DC}(\mathbf{x}, \mathbf{y})$, $\mathrm{PC}(\mathbf{x}, \mathbf{y})$ and $\mathrm{mBKR}(\mathbf{x}, \mathbf{y})$ with U-statistic theory. In particular,

$$\widehat{\mathrm{DC}}(\mathbf{x}, \mathbf{y}) \stackrel{\mathrm{def}}{=} \{4(n)_4\}^{-1} \sum_{(i,j,k,l)}^{n} b(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k, \mathbf{x}_l) b(\mathbf{y}_i, \mathbf{y}_j, \mathbf{y}_k, \mathbf{y}_l),$$

$$\widehat{\mathrm{PC}}(\mathbf{x}, \mathbf{y}) \stackrel{\mathrm{def}}{=} \{4(n)_5\}^{-1} \sum_{(i,j,k,l,r)}^{n} a(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k, \mathbf{x}_l, \mathbf{x}_r) a(\mathbf{y}_i, \mathbf{y}_j, \mathbf{y}_k, \mathbf{y}_l, \mathbf{y}_r),$$

and

$$\widehat{\mathrm{mBKR}}(\mathbf{x}, \mathbf{y}) \stackrel{\mathrm{def}}{=} \{4(n)_6\}^{-1} \sum_{(i,j,k,l,r,s)}^{n} a(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k, \mathbf{x}_l, \mathbf{x}_r) a(\mathbf{y}_i, \mathbf{y}_j, \mathbf{y}_k, \mathbf{y}_l, \mathbf{y}_s),$$

where $(n)_m \stackrel{\text{def}}{=} n(n-1)\cdots(n-m+1)$. The summations

$$\sum_{(i,j,k,l)}^{n}, \sum_{(i,j,k,l,r)}^{n} \quad \text{and} \quad \sum_{(i,j,k,l,r,s)}^{n}$$

are taken over the indexes that are different from each other.

Next we compare the computational complexity of calculating $\widehat{\mathrm{PC}}(\mathbf{x},\mathbf{y})$, $\widehat{\mathrm{DC}}(\mathbf{x},\mathbf{y})$ and $\widehat{\mathrm{mBKR}}(\mathbf{x},\mathbf{y})$. The sample distance covariance is a U-statistic of order four, the sample projection covariance is a U-statistic of order five, and the sample multivariate BKR correlation is a U-statistic of order six. Székely and Rizzo (2013) and Yao et al. (2018) stated that

$$\begin{aligned}\widehat{\mathrm{DC}}(\mathbf{x},\mathbf{y}) &= \{n(n-3)\}^{-1}\Big[\mathrm{tr}(\widetilde{\mathbf{A}}\widetilde{\mathbf{B}}) \\ &\quad +\{(n-1)_2\}^{-1}\mathbf{1}_n^{\mathrm{T}}\widetilde{\mathbf{A}}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}\widetilde{\mathbf{B}}\mathbf{1}_n - 2(n-2)^{-1}\mathbf{1}_n^{\mathrm{T}}\widetilde{\mathbf{A}}\widetilde{\mathbf{B}}\mathbf{1}_n\Big]\end{aligned} \tag{2.5}$$

where $\mathbf{1}_n \in \mathbb{R}^n$ is a vector of ones, $\widetilde{\mathbf{A}} = (\|\mathbf{x}_i - \mathbf{x}_j\|)_{n\times n} \in \mathbb{R}^{n\times n}$ and $\widetilde{\mathbf{B}} = (\|\mathbf{y}_i - \mathbf{y}_j\|)_{n\times n} \in \mathbb{R}^{n\times n}$. That is, the computational complexity of $\widehat{\mathrm{DC}}(\mathbf{x},\mathbf{y})$ is of order $O(n^2)$. To calculate $\widehat{\mathrm{PC}}(\mathbf{x},\mathbf{y})$ and $\widehat{\mathrm{mBKR}}(\mathbf{x},\mathbf{y})$, we define $\mathbf{A}_k \stackrel{\text{def}}{=} (a_{ijk}) \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\mathbf{B}_k \stackrel{\text{def}}{=} (b_{ijk}) \in \mathbb{R}^{(n-1)\times(n-1)}$, where $a_{ijk} \stackrel{\text{def}}{=} \mathrm{ang}(\mathbf{x}_i - \mathbf{x}_k, \mathbf{x}_j - \mathbf{x}_k)$, $b_{ijk} \stackrel{\text{def}}{=} \mathrm{ang}(\mathbf{y}_i - \mathbf{y}_k, \mathbf{y}_j - \mathbf{y}_k)$, for $i \neq k$, $j \neq k$ and $k = 1, \ldots, n$.

With some straightforward algebraic calculations, it can be verified that

$$
\begin{aligned}
\sum_{(i,j,k)}^{n} a_{ijk} b_{ijk} &= \sum_{k=1}^{n} \mathrm{tr}(\mathbf{A}_k \mathbf{B}_k), \\
\sum_{(i,j,k,l)}^{n} a_{ijl} b_{ikl} &= \sum_{l=1}^{n} \{ \mathbf{1}_{(n-1)}^{\mathrm{T}} \mathbf{A}_l \mathbf{B}_l \mathbf{1}_{(n-1)} - \mathrm{tr}(\mathbf{A}_l \mathbf{B}_l) \}, \\
\sum_{(i,j,k,l,r)}^{n} a_{ijr} b_{klr} &= \sum_{r=1}^{n} \{ \mathbf{1}_{(n-1)}^{\mathrm{T}} \mathbf{A}_r \mathbf{1}_{(n-1)} \mathbf{1}_{(n-1)}^{\mathrm{T}} \mathbf{B}_r \mathbf{1}_{(n-1)} \\
&\qquad - 4 \mathbf{1}_{(n-1)}^{\mathrm{T}} \mathbf{A}_r \mathbf{B}_r \mathbf{1}_{(n-1)} + 2\mathrm{tr}(\mathbf{A}_r \mathbf{B}_r) \}.
\end{aligned}
$$

Collecting these results, we have

$$
\begin{aligned}
\widehat{\mathrm{PC}}(\mathbf{x}, \mathbf{y}) &= \{n(n-1)(n-4)\}^{-1} \sum_{r=1}^{n} \Big[\mathrm{tr}(\mathbf{A}_r \mathbf{B}_r) \\
&\quad + \{(n-2)_2\}^{-1} \mathbf{1}_{(n-1)}^{\mathrm{T}} \mathbf{A}_r \mathbf{1}_{(n-1)} \mathbf{1}_{(n-1)}^{\mathrm{T}} \mathbf{B}_r \mathbf{1}_{(n-1)} \\
&\quad - 2(n-3)^{-1} \mathbf{1}_{(n-1)}^{\mathrm{T}} \mathbf{A}_r \mathbf{B}_r \mathbf{1}_{(n-1)} \Big]. \qquad (2.6)
\end{aligned}
$$

Thus, the computational complexity of $\widehat{\mathrm{PC}}(\mathbf{x}, \mathbf{y})$ is of order $O(n^3)$. Similarly, we can verify that

$$
\begin{aligned}
\widehat{\mathrm{mBKR}}(\mathbf{x}, \mathbf{y}) &= \{n(n-1)(n-2)(n-5)\}^{-1} \sum_{r \neq s}^{n} \Big[\mathrm{tr}(\mathbf{A}_r \mathbf{B}_s) \\
&\quad + \{(n-3)_2\}^{-1} \mathbf{1}_{(n-1)}^{\mathrm{T}} \mathbf{A}_r \mathbf{1}_{(n-1)} \mathbf{1}_{(n-1)}^{\mathrm{T}} \mathbf{B}_s \mathbf{1}_{(n-1)} \\
&\quad - 2(n-4)^{-1} \mathbf{1}_{(n-1)}^{\mathrm{T}} \mathbf{A}_r \mathbf{B}_s \mathbf{1}_{(n-1)} \Big], \qquad (2.7)
\end{aligned}
$$

indicating that estimating the multivariate BKR correlation requires $O(n^4)$ operations. Calculating distance correlation has the smallest complexity.

2.2 The Permutation Procedure

Zhu et al. (2017) and Székely et al. (2007) showed that the U-statistic estimates, $\widehat{\mathrm{DC}}(\mathbf{x},\mathbf{y})$ and $\widehat{\mathrm{PC}}(\mathbf{x},\mathbf{y})$, are n-consistent under H_0 and root-n-consistent under fixed alternatives, respectively. Consequently, $n\,\widehat{\mathrm{DC}}(\mathbf{x},\mathbf{y})$ and $n\,\widehat{\mathrm{PC}}(\mathbf{x},\mathbf{y})$ converge in distribution to their respective nondegenerate limits under H_0 and diverge to infinity under fixed alternatives. Following Zhu et al. (2017), we can establish the distribution theory for $\widehat{\mathrm{mBKR}}$ under both the null and the alternative hypotheses. To be precise, $\widehat{\mathrm{mBKR}}$ is n-consistent under H_0 and root-n-consistent under fixed alternatives. Therefore, we reject H_0 when $n\,\widehat{\mathrm{DC}}(\mathbf{x},\mathbf{y})$, $n\,\widehat{\mathrm{PC}}(\mathbf{x},\mathbf{y})$ and $n\,\widehat{\mathrm{mBKR}}$ are greater than or equal to certain critical values. However, the asymptotic null distributions of $n\,\widehat{\mathrm{DC}}(\mathbf{x},\mathbf{y})$, $n\,\widehat{\mathrm{PC}}(\mathbf{x},\mathbf{y})$ and $n\,\widehat{\mathrm{mBKR}}$ are not tractable when $p > 1$ or $q > 1$. To address this issue, Zhu et al. (2017) and Székely et al. (2007) suggested to approximate the critical values adaptively using the following random permutation approach.

1. Suppose $\{i_1, i_2, \ldots, i_n\}$ and $\{j_1, j_2, \ldots, j_n\}$ are two random permutations of $\{1, 2, \ldots, n\}$. Define $\mathbf{x}_k^b \stackrel{\mathrm{def}}{=} \mathbf{x}_{i_k}$ and $\mathbf{y}_k^b \stackrel{\mathrm{def}}{=} \mathbf{y}_{j_k}$, for $k = 1, \ldots, n$. Re-estimate $\mathrm{DC}(\mathbf{x},\mathbf{y})$, $\mathrm{PC}(\mathbf{x},\mathbf{y})$ and $\mathrm{mBKR}(\mathbf{x},\mathbf{y})$ using $\{(\mathbf{x}_k^b, \mathbf{y}_k^b), k = 1, \ldots, n\}$. Denote the resulting estimates by $\widehat{\mathrm{DC}}(\mathbf{x}^b,\mathbf{y}^b)$, $\widehat{\mathrm{PC}}(\mathbf{x}^b,\mathbf{y}^b)$ and $\widehat{\mathrm{mBKR}}(\mathbf{x}^b,\mathbf{y}^b)$, respectively. Replicate this permutation proce-

dure B times, say, $B = 1000$, to approximate the asymptotic null distributions of $\widehat{\mathrm{DC}}(\mathbf{x}^b, \mathbf{y}^b)$, $\widehat{\mathrm{PC}}(\mathbf{x}^b, \mathbf{y}^b)$ and $\widehat{\mathrm{mBKR}}(\mathbf{x}^b, \mathbf{y}^b)$.

2. Denote the observations $\mathcal{D}_n \overset{\text{def}}{=} \{(\mathbf{x}_i, \mathbf{y}_i), i = 1, \cdots, n\}$. We define the critical values at the significance level α by

$$q_{\alpha,n}^{DC} \overset{\text{def}}{=} \inf\left[t \in \mathbb{R} : 1-\alpha \le \mathrm{pr}\{n\, \widehat{\mathrm{DC}}(\mathbf{x}^b, \mathbf{y}^b) \le t \mid \mathcal{D}_n\}\right], \quad (2.8)$$

$$q_{\alpha,n}^{PC} \overset{\text{def}}{=} \inf\left[t \in \mathbb{R} : 1-\alpha \le \mathrm{pr}\{n\, \widehat{\mathrm{PC}}(\mathbf{x}^b, \mathbf{y}^b) \le t \mid \mathcal{D}_n\}\right], \quad (2.9)$$

$$q_{\alpha,n}^{mBKR} \overset{\text{def}}{=} \inf\left[t \in \mathbb{R} : 1-\alpha \le \mathrm{pr}\{n\, \widehat{\mathrm{mBKR}}(\mathbf{x}^b, \mathbf{y}^b) \le t \mid \mathcal{D}_n\}\right], \quad (2.10)$$

We approximate $\mathrm{pr}\{n\, \widehat{\mathrm{DC}}(\mathbf{x}^b, \mathbf{y}^b) \le t \mid \mathcal{D}_n\}$, $\mathrm{pr}\{n\, \widehat{\mathrm{PC}}(\mathbf{x}^b, \mathbf{y}^b) \le t \mid \mathcal{D}_n\}$ and $\mathrm{pr}\{n\, \widehat{\mathrm{mBKR}}(\mathbf{x}^b, \mathbf{y}^b) \le t \mid \mathcal{D}_n\}$ with empirical probabilities

$$B^{-1}\sum_{b=1}^{B} I\left\{n\, \widehat{\mathrm{DC}}(\mathbf{x}^b, \mathbf{y}^b) \le t\right\}, \quad B^{-1}\sum_{b=1}^{B} I\left\{n\, \widehat{\mathrm{PC}}(\mathbf{x}^b, \mathbf{y}^b) \le t\right\},$$

and

$$B^{-1}\sum_{b=1}^{B} I\left\{n\, \widehat{\mathrm{mBKR}}(\mathbf{x}^b, \mathbf{y}^b) \le t\right\}.$$

This is in spirit to approximate the asymptotic null distributions of $n\, \widehat{\mathrm{DC}}(\mathbf{x}^b, \mathbf{y}^b)$, $n\, \widehat{\mathrm{PC}}(\mathbf{x}^b, \mathbf{y}^b)$ and $n\, \widehat{\mathrm{mBKR}}(\mathbf{x}^b, \mathbf{y}^b)$, respectively.

This random permutation procedure is intuitively valid and thus widely used in multiple testing problems and independence tests. A random permutation procedure is said to be consistent if it provides a reasonable approximation to the asymptotic null distribution. The consistency of random permutation has been extensively studied by Romano and Wolf (2005) in the context of multiple testing problems. However, its consistency is rarely discussed in the context of independence tests. In Theorem 1 we show that this permutation procedure is consistent in all three independence tests. The detailed proofs are given in the Supplementary Material. Throughout $\mathrm{pr}(\cdot \mid H_0)$ and $\mathrm{pr}(\cdot \mid H_1)$ stand for the respective probabilities of a random event occurs under H_0 and H_1. They are not conditional probabilities.

Theorem 1. As $n \to \infty$, both

$$\sup_{t\in\mathbb{R}} \left| \mathrm{pr}\{n\ \widehat{\mathrm{PC}}(\mathbf{x}^b, \mathbf{y}^b) \le t \mid \mathcal{D}_n\} - \mathrm{pr}\{n\ \widehat{\mathrm{PC}}(\mathbf{x}, \mathbf{y}) \le t \mid H_0\} \right|$$

and

$$\sup_{t\in\mathbb{R}} \left| \mathrm{pr}\{n\ \widehat{\mathrm{mBKR}}(\mathbf{x}^b, \mathbf{y}^b) \le t \mid \mathcal{D}_n\} - \mathrm{pr}\{n\ \widehat{\mathrm{mBKR}}(\mathbf{x}, \mathbf{y}) \le t \mid H_0\} \right|$$

converge in probability to 0. If we assume $E(\|\mathbf{x}\|^2) + E(\|\mathbf{y}\|^2) < \infty$, then

$$\sup_{t\in\mathbb{R}} \left| \mathrm{pr}\{n\ \widehat{\mathrm{DC}}(\mathbf{x}^b, \mathbf{y}^b) \le t \mid \mathcal{D}_n\} - \mathrm{pr}\{n\ \widehat{\mathrm{DC}}(\mathbf{x}, \mathbf{y}) \le t \mid H_0\} \right|$$

converges in probability to 0 as $n \to \infty$.

We require the condition $E(\|\mathbf{x}\|^2) + E(\|\mathbf{y}\|^2) < \infty$ to ensure that the kernel of the U-statistic estimate $\widehat{\mathrm{DC}}(\mathbf{x}, \mathbf{y})$ is uniformly integrable. Theorem 1 guarantees that this random permutation procedure approximates the asymptotic null distributions precisely as long as the sample size n is sufficiently large. In other words, the type-I error rates of all projection-pursuit independence tests are asymptotically controllable. This allows us to analyze statistical power of these projection-pursuit independence tests.

Exhausting all possible permutations is usually computationally prohibitive and practically infeasible. Therefore, we provide a random approximation in the above permutation procedure. Proposition 1 states that, as long as the number of random permutations, B, is sufficiently large, the random approximation is asymptotically valid.

Proposition 1. Given the data $\mathcal{D}_n$,

$$\sup_{t\in\mathbb{R}}\left|B^{-1}\sum_{b=1}^{B} I\left\{n\ \widehat{\mathrm{PC}}(\mathbf{x}^b,\mathbf{y}^b)\le t\right\}-\mathrm{pr}\{n\ \widehat{\mathrm{PC}}(\mathbf{x}^b,\mathbf{y}^b)\le t\mid \mathcal{D}_n\}\right|,$$

$$\sup_{t\in\mathbb{R}}\left|B^{-1}\sum_{b=1}^{B} I\left\{n\ \widehat{\mathrm{mBKR}}(\mathbf{x}^b,\mathbf{y}^b)\le t\right\}-\mathrm{pr}\{n\ \widehat{\mathrm{mBKR}}(\mathbf{x}^b,\mathbf{y}^b)\le t\mid \mathcal{D}_n\}\right|,$$

and

$$\sup_{t\in\mathbb{R}}\left|B^{-1}\sum_{b=1}^{B} I\left\{n\ \widehat{\mathrm{DC}}(\mathbf{x}^b,\mathbf{y}^b)\le t\right\}-\mathrm{pr}\{n\ \widehat{\mathrm{DC}}(\mathbf{x}^b,\mathbf{y}^b)\le t\mid \mathcal{D}_n\}\right|$$

converge in probability to 0, as $B\to\infty$.

3. ROBUSTNESS STUDY

We first highlight the robustness of the projection correlation test and the multivariate BKR correlation test in a Huber contamination model. The following is an ϵ-contamination model:

$$(\mathbf{x}, \mathbf{y}) \sim F_{\mathbf{x},\mathbf{y}} = (1-\epsilon)F^{(1)}_{\mathbf{x},\mathbf{y}} + \epsilon H^{(n)}_{\mathbf{x},\mathbf{y}}, \tag{3.11}$$

where $F^{(1)}_{\mathbf{x},\mathbf{y}}$ and $H^{(n)}_{\mathbf{x},\mathbf{y}}$ are two distributional functions, $F^{(1)}_{\mathbf{x},\mathbf{y}}$ is fixed yet $H^{(n)}_{\mathbf{x},\mathbf{y}}$ may vary with n, and $0 < \epsilon < 1$. We remark here that $\mathbf{x}$ and $\mathbf{y}$ are dependent if $(\mathbf{x}, \mathbf{y}) \sim F^{(1)}_{\mathbf{x},\mathbf{y}}$ and independent if $(\mathbf{x}, \mathbf{y}) \sim H^{(n)}_{\mathbf{x},\mathbf{y}}$. We use the ϵ-contamination model (3.11) to evaluate whether an independence test can maintain adequate power when $H^{(n)}_{\mathbf{x},\mathbf{y}}$ has an adverse impact on its power performance. The test functions using distance correlation, projection correlation and the multivariate BKR correlation are defined, respectively, by

$$\Phi^{DC}_{\alpha} \stackrel{\text{def}}{=} I\{n\,\widehat{\mathrm{DC}}(\mathbf{x}, \mathbf{y}) \geq q^{DC}_{\alpha,n}\}, \quad \Phi^{PC}_{\alpha} \stackrel{\text{def}}{=} I\{n\,\widehat{\mathrm{PC}}(\mathbf{x}, \mathbf{y}) \geq q^{PC}_{\alpha,n}\},$$
$$\Phi^{mBKR}_{\alpha} \stackrel{\text{def}}{=} I\{n\,\widehat{\mathrm{mBKR}}(\mathbf{x}, \mathbf{y}) \geq q^{mBKR}_{\alpha,n}\},$$

where $q^{DC}_{\alpha,n}$, $q^{PC}_{\alpha,n}$ and $q^{mBKR}_{\alpha,n}$ are the critical values defined in (2.8), (2.9) and (2.10) through random permutations, and $I(A)$ is an indicator function, which equals one if A is true and zero otherwise. For all three projection-pursuit independence tests, we reject H_0 at the significance level α when the estimates of projection-pursuit correlations are larger than their critical values, that is, when $n\ \widehat{\mathrm{DC}}(\mathbf{x},\mathbf{y}) \geq q_{\alpha,n}^{DC}$, $n\ \widehat{\mathrm{PC}}(\mathbf{x},\mathbf{y}) \geq q_{\alpha,n}^{PC}$ and $n\ \widehat{\mathrm{mBKR}}(\mathbf{x},\mathbf{y}) \geq q_{\alpha,n}^{mBKR}$. We study the robustness of projection-pursuit independence tests through comparing their power performance in that Theorem 1 ensures that one can always use random permutations to control the type-I error rate.

Theorem 2 states that the independence tests built upon projection correlation and the multivariate BKR correlation are uniformly powerful over different types of contaminations. By contrast, the distance correlation test becomes asymptotically powerless against certain contaminations.

Theorem 2. Suppose $\{(\mathbf{x}_i,\mathbf{y}_i), i=1,\ldots,n\}$ are generated independently from model (3.11) with the contamination ratio $\epsilon = cn^{-1/2}$, where c is a small positive constant not depending on n, and there exist three positive constants, ϖ, ϖ' and ϖ'', such that $\mathrm{PC}(\mathbf{x},\mathbf{y}) \geq \varpi$, $\mathrm{DC}(\mathbf{x},\mathbf{y}) \geq \varpi'$ and $\mathrm{mBKR}(\mathbf{x},\mathbf{y}) \geq \varpi''$ for sufficiently large n.

1. The projection correlation test and the multivariate BKR correlation test are asymptotically powerful uniformly over $H_{\mathbf{x},\mathbf{y}}^{(n)}$ in the sense that

$$\lim_{n\to\infty} \inf_{H_{\mathbf{x},\mathbf{y}}^{(n)}} \mathrm{pr}(\Phi_\alpha^{PC} = 1 \mid H_1) = 1 \text{ and } \lim_{n\to\infty} \inf_{H_{\mathbf{x},\mathbf{y}}^{(n)}} \mathrm{pr}(\Phi_\alpha^{mBKR} = 1 \mid H_1) = 1.$$

2. Assume $E(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2) < \infty$ if $(\mathbf{x}, \mathbf{y}) \sim F^{(1)}_{\mathbf{x},\mathbf{y}}$, and if $(\mathbf{x}, \mathbf{y}) \sim H^{(n)}_{\mathbf{x},\mathbf{y}}$, $n\{\mathrm{var}(\|\mathbf{x}\|)\ \mathrm{var}(\|\mathbf{y}\|)\}^{-1/2} = o(1)$. The distance correlation test is asymptotically powerless against such choices of $H^{(n)}_{\mathbf{x},\mathbf{y}}$ in the sense of

$$\lim_{n\to\infty} \inf_{H^{(n)}_{\mathbf{x},\mathbf{y}}} \mathrm{pr}(\Phi^{DC}_\alpha = 1 \mid H_1) \leq \alpha.$$

The first assertion of Theorem 2 implies that the projection correlation test and the multivariate BKR correlation test are insensitive to the presence of outliers. In the second statement of Theorem 2, we assume $n\{\mathrm{var}(\|\mathbf{x}\|)\ \mathrm{var}(\|\mathbf{y}\|)\}^{-1/2} = o(1)$ if $(\mathbf{x}, \mathbf{y}) \sim H^{(n)}_{\mathbf{x},\mathbf{y}}$, which allows $\mathrm{var}(\|\mathbf{x}\|)$ and $\mathrm{var}(\|\mathbf{y}\|)$ to be divergent, and accordingly, model (3.11) to yield outliers. We impose this condition to demonstrate that the distance correlation test might lose power in the presence of outliers.

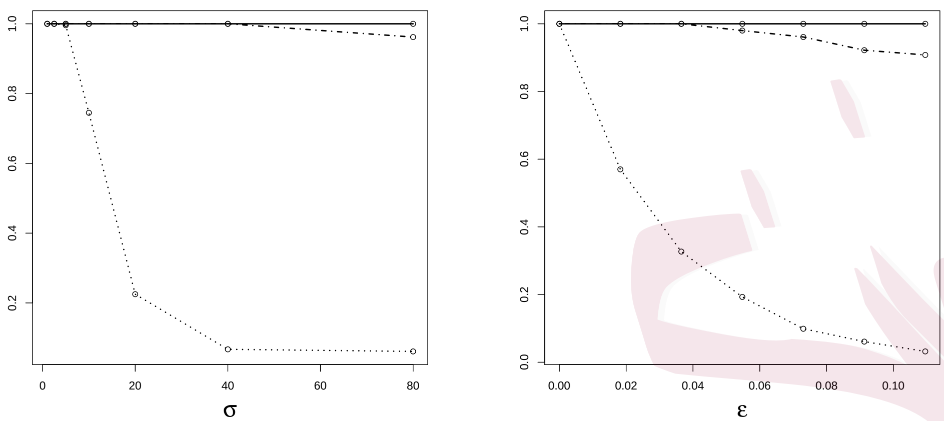
We conduct simulations to illustrate Theorem 2 with finite sample size. Following Davison and Hinkley (1997), we set $B = 1000$ throughout our numerical studies.

Example 1. In the ϵ-contamination model (3.11), we consider an extreme case for $F^{(1)}_{\mathbf{x},\mathbf{y}}$: $\mathbf{x}$ follows multivariate standard normal distribution, and $\mathbf{y}$ equals $\mathbf{x}$ exactly. This ensures that $\mathbf{x}$ and $\mathbf{y}$ are dependent. In other words, the observations are drawn under H_1. In addition, we set

$$H^{(n)}_{\mathbf{x},\mathbf{y}} = (2\pi\sigma^2)^{-p/2} \exp\{-(\mathbf{x}^{\mathrm{T}}\mathbf{x})^2/(2\sigma^2)\} \prod_{k=1}^{p} I(0 \leq Y_k \leq 1).$$

We consider two scenarios for (ϵ, σ). In the first scenario, $\epsilon = 0.5n^{-1/2}$ and $\sigma = \{1, 2.5, 5, 10, 20, 40, 80\}$. In the second scenario, $\sigma = 100$ and $\epsilon = cn^{-1/2}$, for $c = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$. Both σ and ϵ control the degree of heavy-tailedness. As σ and c increase, the distance between H_0 and H_1 is smaller and the probabilities of observing extreme values from $H^{(n)}_{\mathbf{x},\mathbf{y}}$ increase as well. We fix $p = q = 10$, $n = 30$, and decide the critical values with permutations at the significance level $\alpha = 0.05$. The simulations are replicated 1000 times. The empirical powers of the projection correlation test, the distance correlation test and the multivariate BKR correlation test are summarized in Figure 1. It can be clearly seen that, the empirical powers of the projection correlation test and the multivariate BKR correlation test are very close to one throughout, indicating that the projection correlation test and the multivariate BKR correlation test are consistently robust to the changes of σ and ϵ. By contrast, the empirical power of the distance correlation test drops down very quickly as σ and ϵ increase. The distance correlation test is completely powerless when σ or ϵ is sufficiently large.

Example 2. In the ϵ-contamination model (3.11), we set $H^{(n)}_{\mathbf{x},\mathbf{y}}$ to be the product of $(p+q)$ independent t distributions with one degree of freedom, and $F^{(1)}_{\mathbf{x},\mathbf{y}}$ to be the Dirac measure of the form $F^{(1)}_{\mathbf{x},\mathbf{y}} = I(\mathbf{x} = \kappa \mathbf{1}_p) I(\mathbf{y} = \mathbf{x})$, for $\kappa = \{5, 15\}$, where $\mathbf{1}_p$ is a p-vector of ones. Let $\epsilon = cn^{-1/2}$, for $c =$



(A): $\epsilon = 0.5n^{-1/2}$ and $\sigma = \{1, 2.5, 5, 10, 20, 40, 80\}$

(B): $\sigma = 100$ and $\epsilon = cn^{-1/2}$, $c = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$

Figure 1: The empirical powers of the projection correlation test (solid line), the distance correlation test (dotted line) and the multivariate BKR correlation test (dotdash line) when the random sample is drawn from the ϵ-contamination model (3.11) with different ϵ and σ.

$\{0.2, 0.4, 0.6, 0.8, 1.0\}$. As c increases, the probabilities of observing extreme values from $H^{(n)}_{\mathbf{x},\mathbf{y}}$ increase as well, which, as stated in Theorem 2, may affect the power performance of independence tests. Let $p = q = \{5, 10, 20\}$ and $n = 30$. The significance level is set to be $\alpha = 0.05$.

The empirical powers are summarized in Tables 1 and 2 based on 1000 replications. Following the suggestion of an anonymous reviewer, we also include the distance correlation based t-test (Székely and Rizzo, 2013) into our comparison. We denote this test by SR, the initials of the authors' last names. The SR test is asymptotically distribution-free. Therefore, we use its asymptotic null distribution directly to decide the critical values. It is expected that, the projection correlation test and the multivariate BKR correlation test are significantly more powerful than the distance correlation test and the distance correlation t-test across all scenarios. When c decreases from 1 to 0.2, p and q increase from 5 to 20, or κ increase from 5 to 15, the deviation from H_0 is accumulating. The powers of the projection correlation test and the multivariate BKR correlation test increase significantly. By contrast, the distance correlation test loses its power completely when $\kappa = 5$. Since the SR test was specifically developed for large dimensions, it is more powerful than the distance correlation test, especially when $p = 20$. However, the SR test is still inferior significantly to the projection correlation test and the multivariate BKR correlation test in terms of power performance, particularly when p and c are relatively small.

4. MINIMAX OPTIMALITY

Next we study the minimax optimality of the projection correlation test, the distance correlation test and the multivariate BKR correlation test. To

Table 1: The empirical powers of the projection correlation test ("PC"), the distance correlation test ("DC"), the multivariate BKR correlation test ("mBKR") and the distance correlation t-test ("SR") in Example 2 with three different settings of dimension when $\kappa = 5$ and the nominal level is 0.05.

		$c = 1.0$	$c = 0.8$	$c = 0.6$	$c = 0.4$	$c = 0.2$
$p = 5$	PC	0.152	0.391	0.635	0.808	0.907
	DC	0.056	0.069	0.104	0.158	0.219
	mBKR	0.114	0.303	0.534	0.725	0.841
	SR	0.059	0.102	0.247	0.386	0.457
$p = 10$	PC	0.198	0.458	0.701	0.851	0.946
	DC	0.051	0.063	0.083	0.123	0.160
	mBKR	0.164	0.397	0.616	0.798	0.884
	SR	0.058	0.143	0.295	0.479	0.542
$p = 20$	PC	0.231	0.544	0.777	0.897	0.955
	DC	0.053	0.061	0.078	0.089	0.109
	mBKR	0.182	0.410	0.647	0.841	0.916
	SR	0.074	0.221	0.346	0.552	0.703

simplify subsequent illustration, let Φ_α be a level-α test function, which equals 1 if one rejects H_0 and 0 otherwise. Denote by $\mathrm{pr}(\cdot \mid H_0)$ and $\mathrm{pr}(\cdot \mid H_1)$ the probabilities evaluated under H_0 and H_1, respectively. Accordingly, $\mathrm{pr}(\Phi_\alpha = 1 \mid H_0)$ is the type-I error rate and $\mathrm{pr}(\Phi_\alpha = 0 \mid H_1)$ is the type-II error rate. We define the class of level-α test functions by $\mathcal{T}_\alpha \overset{\text{def}}{=} \{\Phi_\alpha : \mathrm{pr}(\Phi_\alpha = 1 \mid H_0) \leq \alpha\}$. We measure the dependence between $\mathbf{x}$ and $\mathbf{y}$

Table 2: The empirical powers of the projection correlation test ("PC"), the distance correlation test ("DC"), the multivariate BKR test ("mBKR") and the distance correlation t-test ("SR") in Example 2 with three different settings of dimension when $\kappa = 15$ and the nominal level is 0.05.

		$c = 1.0$	$c = 0.8$	$c = 0.6$	$c = 0.4$	$c = 0.2$
$p = 5$	PC	0.216	0.508	0.749	0.876	0.948
	DC	0.112	0.239	0.394	0.512	0.620
	mBKR	0.195	0.482	0.711	0.804	0.896
	SR	0.132	0.384	0.508	0.615	0.732
$p = 10$	PC	0.272	0.592	0.791	0.913	0.971
	DC	0.082	0.177	0.282	0.401	0.512
	mBKR	0.266	0.514	0.750	0.875	0.914
	SR	0.190	0.455	0.682	0.796	0.889
$p = 20$	PC	0.318	0.655	0.852	0.940	0.977
	DC	0.061	0.110	0.188	0.264	0.354
	mBKR	0.287	0.568	0.796	0.918	0.962
	SR	0.242	0.544	0.751	0.885	0.951

by projection correlation, distance correlation and the multivariate BKR correlation, respectively. Define

$$\mathcal{U}^{PC}(c) \stackrel{\text{def}}{=} \{(\mathbf{x}, \mathbf{y}) : \mathrm{PC}(\mathbf{x}, \mathbf{y}) \geq cn^{-1}\}, \quad \mathcal{U}^{DC}(c) \stackrel{\text{def}}{=} \{(\mathbf{x}, \mathbf{y}) : \mathrm{DC}(\mathbf{x}, \mathbf{y}) \geq cn^{-1}\},$$

$$\mathcal{U}^{mBKR}(c) \stackrel{\text{def}}{=} \{(\mathbf{x}, \mathbf{y}) : \mathrm{mBKR}(\mathbf{x}, \mathbf{y}) \geq cn^{-1}\}.$$

If the degree of dependence between $\mathbf{x}$ and $\mathbf{y}$ is weak, it may be difficult to distinguish between H_0 and H_1. Theorem 3 states that, for all level-α tests, there exist $(\mathbf{x}, \mathbf{y}) \in \mathcal{U}^{PC}(c_0)$ for the projection correlation test, $(\mathbf{x}, \mathbf{y}) \in \mathcal{U}^{DC}(c_0)$ for the distance correlation test and $(\mathbf{x}, \mathbf{y}) \in \mathcal{U}^{mBKR}(c_0)$ for the multivariate BKR correlation test, such that their type-II error rates, $\mathrm{pr}(\Phi_\alpha = 0 \mid H_1)$, are not asymptotically negligible even when $n \to \infty$. The specified constant c_0 is quantifies the degree of deviation from H_0.

Theorem 3. For any $0 < \xi < 1 - \alpha$, there exists $c_0 > 0$ such that the minimax type-II error rates are lower bounded as $n \to \infty$, namely

$$\lim_{n\to\infty} \inf_{\Phi_\alpha \in \mathcal{T}_\alpha} \sup_{(\mathbf{x},\mathbf{y}) \in \mathcal{U}^{PC}(c_0)} \mathrm{pr}(\Phi_\alpha^{PC} = 0 \mid H_1) \geq \xi,$$
$$\lim_{n\to\infty} \inf_{\Phi_\alpha \in \mathcal{T}_\alpha} \sup_{(\mathbf{x},\mathbf{y}) \in \mathcal{U}^{mBKR}(c_0)} \mathrm{pr}(\Phi_\alpha^{mBKR} = 0 \mid H_1) \geq \xi,$$
$$\lim_{n\to\infty} \inf_{\Phi_\alpha \in \mathcal{T}_\alpha} \sup_{(\mathbf{x},\mathbf{y}) \in \mathcal{U}^{DC}(c_0)} \mathrm{pr}(\Phi_\alpha^{DC} = 0 \mid H_1) \geq \xi.$$

Theorem 3 indicates that, the projection-pursuit independence tests can not maintain adequate power even if the dependence between $\mathbf{x}$ and $\mathbf{y}$ are cn^{-1} far apart in terms of $\mathrm{PC}(\mathbf{x}, \mathbf{y})$, $\mathrm{mBKR}(\mathbf{x}, \mathbf{y})$ or $\mathrm{DC}(\mathbf{x}, \mathbf{y})$, respectively, for an arbitrarily small c. However, if we allow c to diverge to infinity, the story will be different. To be specific, the type-II error rates of these independence tests shrink to zero as $n \to \infty$. This is formulated in Theorem

4. Define

$$\Phi_\alpha^{DC} \overset{\text{def}}{=} I\{n\,\widehat{\mathrm{DC}}(\mathbf{x},\mathbf{y}) \geq q_{\alpha,n}^{DC}\}, \quad \Phi_\alpha^{PC} \overset{\text{def}}{=} I\{n\,\widehat{\mathrm{PC}}(\mathbf{x},\mathbf{y}) \geq q_{\alpha,n}^{PC}\}$$

$$\Phi_\alpha^{mBKR} \overset{\text{def}}{=} I\{n\,\widehat{\mathrm{mBKR}}(\mathbf{x},\mathbf{y}) \geq q_{\alpha,n}^{mBKR}\},$$

where $q_{\alpha,n}^{DC}$, $q_{\alpha,n}^{PC}$ and $q_{\alpha,n}^{mBKR}$ are defined in (2.8), (2.9) and (2.10).

Theorem 4. The minimax type-II error rate of the projection correlation test tends to zero uniformly over $\mathcal{U}^{PC}(c_n)$ with $c_n \to \infty$ as $n \to \infty$, namely

$$\lim_{n\to\infty} \sup_{(\mathbf{x},\mathbf{y})\in\mathcal{U}^{PC}(c_n)} \mathrm{pr}(\Phi_\alpha^{PC} = 0 \mid H_1) = 0.$$

The minimax type-II error rate of the multivariate BKR correlation test tends to zero uniformly over $\mathcal{U}^{mBKR}(c_n)$ with $c_n \to \infty$ as $n \to \infty$, namely

$$\lim_{n\to\infty} \sup_{(\mathbf{x},\mathbf{y})\in\mathcal{U}^{mBKR}(c_n)} \mathrm{pr}(\Phi_\alpha^{mBKR} = 0 \mid H_1) = 0.$$

Furthermore, if $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are squared-integrable, the minimax type-II error rate of the distance correlation test tends to zero uniformly over $\mathcal{U}^{DC}(c_n)$ with $c_n \to \infty$ as $n \to \infty$, namely

$$\lim_{n\to\infty} \sup_{(\mathbf{x},\mathbf{y})\in\mathcal{U}^{DC}(c_n)} \mathrm{pr}(\Phi_\alpha^{DC} = 0 \mid H_1) = 0.$$

Theorem 4, together with Theorem 3, indicates that the minimax lower bound of the minimum separation rate is n^{-1}. This lower bound is asymptotically tight for the projection correlation test and the multivariate BKR correlation test. If $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are squared-integrable, this lower bound is also asymptotically tight for the distance correlation test.

5. DISCUSSION

We consider three projection-pursuit correlations, namely, distance correlation, projection correlation and the multivariate BKR correlation. All these correlations quantify the difference between the joint distribution function and the product of the marginal distribution functions. These three projection-pursuit correlations differ only in the weight function. We investigate their robustness, and compare the power performance of independence tests built upon these projection-pursuit correlations under a minimax framework. We also seek for conditions under which these projection-pursuit independence tests are minimax rate optimal.

It is practically interesting yet theoretically challenging to characterize the exact value of c in $\mathcal{U}^{DC}(c)$, $\mathcal{U}^{PC}(c)$ and $\mathcal{U}^{mBKR}(c)$ that separates the testable region from the non-testable one. This is because the class of alternatives we are targeting is very huge owing to the existence of nonlinear dependence. This issue is beyond the scope of the present context though, it deserves further investigations.

Supplementary Materials The supplementary materials contain proofs of (1.2), Proposition 1 and Theorems 1-4.

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Kai Xu

School of Mathematics and Statistics, Anhui Normal University, Wuhu 241002, China

E-mail: tjxxukai@163.com

Liping Zhu

Center for Applied Statistics and Institute of Statistics and Big Data, Renmin University of China, Beijing 100872, China

E-mail: zhu.liping@ruc.edu.cn