Statistica Sinica Preprint No: SS-2019-0265	
Title	A stable and more efficient doubly robust estimator
Manuscript ID	SS-2019-0265
URL	http://www.stat.sinica.edu.tw/statistica/
DOI	10.5705/ss.202019.0265
Complete List of Authors	Min Zhang and Baqun Zhang
Corresponding Author	Min Zhang
E-mail	mzhangst@umich.edu
Notice: Accepted version subject to English editing.	

Statistica Sinica

A stable and more efficient doubly robust estimator

Min Zhang[1] and Baqun Zhang[2]

[1] *University of Michigan, and* [2] *Shanghai University of Finance and Economics*

Abstract: Under the assumption of missing at random, doubly robust (DR) estimators are consistent when either the propensity score or the outcome model is correctly specified. However, despite its appealing theoretic properties, Kang and Schafer (2007) show that the usual augmented inverse probability weighted (AIPW) DR estimator may sometimes exhibit unsatisfying behavior. We propose an alternative DR method for mean estimation. In this method, we do not directly weight outcomes by the inverse of estimated propensity scores and instead a nonparametric kernel regression is used to model residuals from an outcome regression model as a function of propensity scores. The proposed method does not suffer from the instability issue of the usual inverse propensity weighted (IPW) estimator resulted from small estimated propensities. We show that asymptotically the new estimator has the double robustness property. Moreover, we show that it is guaranteed to be more efficient than the usual AIPW DR estimator when the propensity score model is correct but the outcome model is incorrect. Our simulation studies show that it has improved finite sample performance compared to existing DR estimators.

Key words and phrases: Causal inference; Comparative effectiveness; Inverse probability weighting; Kernel regression; Missing data; Propensity score.

1 Introduction

Missing data is a common problem in many settings. It is well accepted that not properly accounting for missing data can lead to severely biased estimation and invalid inference. Missing data problems have been an area of active research (e.g., Rosenbaum, Rubin, 1984; Scharfstein, et al., 1999; Lunceford and Davidian, 2004). Much of the literature has been focused on the situation where missingness can be assumed missing at random (MAR), i.e., conditional on the observed variables, the probability of missingness does not depend on variables that are missing (Little and Rubin, 2002). Methods for dealing with missing data when MAR holds generally fall into three categories where one models the outcome as a function of covariates, or models the probability of missingness, namely, the propensity score, as a function of covariates (Rosenbaum and Rubin, 1984; Rosenbaum, 1987; Rubin and Thomas, 1996) or both (Lunceford and Davidian, 2004; Bang and Robins, 2005). These different approaches each have their own advantages and disadvantages in terms of bias, efficiency, robustness and numerical stability. In general, in methods where only the outcome or propensity score is modeled, the valid statistical inference depends on the correct specification of the corresponding model and an incorrect model may lead to inconsistent estimation and invalid inference. In this sense, these methods are not robust. The so-called doubly robust (DR) methods, where both outcomes and propensities are modeled, lead to consistent estimation as long as one of the models, but not necessarily both, is correctly specified (Bang and Robins, 2005), overcoming the issue of nonrobustness to some degree. In addition, DR methods usually have good efficiency properties and achieve the semiparametric efficiency bound if the outcome regression model is correct. DR methods seem to combine the strength of methods which model either outcomes or propensities, at least theoretically based on asymptotic theory, and are very appealing.

DR estimators have received a lot of attention in literature and many different versions have been proposed. In an attempt to demystifying double robustness, Kang and Schafer (2007) reviewed several versions of DR estimators and compared them to alternative methods for estimating the population mean when outcomes are subject to missingness. They found that, although theoretically appealing, DR estimators may have "disastrous" performance when some estimated propensities are small. Kang and Schafer (2007) was followed by several discussion papers, including Robins et al (2007). Following the work of Kang and Schafer (2007), there is considerable interest on improving the usual DR estimators (Tan, 2006, 2007; Cao, Tsiatis and Davidian, 2009; Tsiatis, et al., 2011; Imai and Ratkovic, 2014; Zubizarreta,2015), in particular, for the setting that was originally designed by Kang and Schafer (2007). Most of the efforts for improvement have been focused on the problem of estimating the population mean when some responses are missing. Methods by Tan (2006, 2007) and Cao, Tsiatis and Davidian (2009) focus on modifying estimation of the outcome-regression model used in the augmentation term. Alternatively, methods by Imai and Ratkovic (2014) and Zubizarreta (2015) attempt to improve performance by seeking better and more stable way to estimate the weight used in the doubly robust augmened IPW estimator. These methods can be viewed as modified versions of augmented inverse probability (or propensity) weighted (IPW) methods along the line of the original DR estimator.

Approaching the issue from a different perspective, in this article we propose an alternative DR method for mean estimation, where estimated propensity scores are not directly used for weighting. In contrast to previous methods, we do not change the way how the outcome-regression model or the propensity score model/weights are estimated. We show that asymptotically the new estimator has the double robustness property and in addition it has improved efficiency and finite sample performance compared to existing DR estimators. Interestingly, although the proposed method is not motivated from the perspective of an augmented IPW estimator, we show that the estimator has a close connection with the usual augmented IPW estimator. It has long been noted in the literature that IPW estimator, including the usual DR estimators as demonstrated by Kang and Schafer (2007), may have unsatisfying performances as a result of small estimated propensity scores and huge weights. Efforts have been made to improve the stability of IPW estimators by trimming or smoothing using Bayesian methods (Elliot, 2000; 2008; Austin and Stuart, 2015), although not in the context of DR estimators. We will see that the proposed estimator can also be viewed as a principled way to smooth over inverse of estimated propensities, therefore reducing the impact of huge weights.

In addition to references discussed earlier, there has been numerous work on DR estimators and IPW methods in general. For example, Gruber and van der Laan (2010) studied DR estimators using targeted maximum likelihood estimation and Zhou, Elliot and Little (2019) used penalized spline method to achieve double-robustness. For censored data, Chen, Lu and Zhao (2018) used a kernel-based weighting approach to estimate the survival function of medical cost data subject to censoring. Moreover, efforts have been made to improve IPW-based methods by better estimating the propensity scores using machine learning methods; for example, Pirracchio, Peterson and van der Laan (2014) used super learner and other machine learning methods to estimate propensity scores, which can improve robustness to model misspecification of the propensity score.

2 Method

2.1 Notation and background

Consider a study with a random sample of n units from an intended population. Ideally the full data are $(Y_i, X_i), i = 1, \ldots, n$, independent and identically distributed (i.i.d.) across i, where Y_i and X_i are respectively the response and a vector of auxiliary covariates for subject i. Suppose the outcome is subject to missingness. Let R_i be the indicator for observing Y_i with $R_i = 1$ if Y_i is observed and $R_i = 0$ if missing. Then the actually observed data are (R_iY_i, R_i, X_i), i.i.d. across i. Interest focuses on estimation of the population mean, $E(Y) \equiv \mu$. We assume the missingness is MAR, denoted by $Y \perp\!\!\!\perp R|X$; that is, missingness is independent of outcomes given the observed covariates. When interest is in making causal inference on treatment effect from observational data, then even the outcome variable Y is observed for all subjects, it can still be cast as a missing data problem using the framework of counterfactual outcomes. Hereafter, we only discuss about estimating μ, recognizing that the proposed method can be directly applied for comparing treatment effects for observational data.

Various methods are available to adjust for missingness, as reviewed by Kang and Schafer (2007). In general, these methods involve modeling of either the outcome or missingness given covariates, or both. Specifically, the outcome regression-based method builds a model for the outcome Y using covariates X from the observed data, then estimates μ using the average of predicted values across all subjects from the fitted model. The consistency of the outcome regression estimator relies on the correct specification of the model for $E(Y|X) = m(X;\beta)$. In contrast, another broad class of methodologies involve modeling the probability of nonmissingness given covariates, i.e., $P(R = 1|X)$, referred to as propensity scores. Propensity scores can be estimated by positing, for example, a logistic regression model which specifies $P(R = 1|X) = \exp(X^T\theta)/\{1 + \exp(X^T\theta)\} \equiv \pi(X;\theta)$. After obtaining the estimated propensity scores, then one can weight the contribution of each observed outcome by the inverse of the estimated propensity score, referred to as the IPW estimator. In propensity-score based methods, the consistency of estimation requires the model for the propensity score be correctly specified.

In a DR estimator, both outcome and propensity scores are modeled and it remains consistent if either one of the models is correctly specified (Bang and Robins, 2005). Hence, the DR estimator affords protection against misspecification of one of the models and this property is referred to as double robustness. The usual augmented inverse probability weighted DR estimator is given by

$$n^{-1}\sum_{i=1}^{n}\Big\{\frac{R_iY_i}{\widehat{\pi}_i}-\frac{R_i-\widehat{\pi}_i}{\widehat{\pi}_i}m(X_i;\widehat{\beta})\Big\}, \tag{2.1}$$

where $\widehat{\pi}_i=\pi(X_i;\widehat{\theta})$ and $\pi(X_i;\theta)$ and $m(X_i;\beta)$ are the specified models for the propensity and outcome respectively. Hereafter, we use $\widehat{\pi}_i$ for brevity or $\pi(X_i;\widehat{\theta})$ if we would like to emphasize dependence on X_i and $\widehat{\theta}$. The unknown parameters β and θ are usually estimated by the maximum likelihood (ML) method. The first term in (2.1) is an IPW estimator with the inverse of propensity serving as the weight and the second part is an augmentation term. The DR estimator also enjoys good efficiency properties. It is usually more efficient than IPW estimators (Lunceford and Davidian, 2004). Moreover, if $m(X_i;\beta)$ is correctly specified, it has the smallest asymptotic variance among all estimators that are consistent and asymptotically normal when the propensity model is correct; i.e., it is semiparametric efficient.

Despite the appealing theoretical properties based on asymptotics, empirical studies show that the usual DR estimator may exhibit poor performance under some situations in practice (Kang and Schafer, 2007). They note that the usual DR estimator may be severely biased when both specified models are close to the truth but are not completely correct, and may have "disastrous" performance when some of the estimated propensities are small, even if the propensity score model is correctly specified. Alternative DR estimators are developed and some are directly targeted at improving the performance of the usual DR estimator. For example, Kang and Schafer (2007) identified some alternative DR estimators and Tan (2006) studied a likelihood estimator that possesses the DR property and may alleviate some of the problems associated with the usual DR estimator.

As a follow-up paper to Kang and Schafer (2007) and Tan (2007), Cao, Tsiatis and Davidian (2009), abbreviated as CTD below, studied alternative DR estimators that aim to improve efficiency and robustness of existing DR estimators. The main idea of the CTD projection method is based on the novel observation that when the propensity score model is correct but outcome model is incorrect then the usual DR estimator coupled with the ML estimate of β does not achieve the minimal asymptotic variance among all estimators in (2.1) with an augmentation term $m(X_i; \beta^*)$ for any β^*. They then sought to identify estimator of β such that it will lead to an estimator of μ that is both doubly robust and achieves the minimal asymptotic variance when propensity score model is correct but outcome model is incorrect. As a result, the corresponding estimator would be more efficient than the usual DR estimator when propensity score model is correct but outcome model is misspecified. They proposed to estimate β instead by solving

$$\sum_{i=1}^{n}\Big[\frac{R_i}{\widehat{\pi}_i}\frac{1-\widehat{\pi}_i}{\widehat{\pi}_i}\Big\{\begin{array}{c} m_\beta(X_i;\beta) \\ \frac{\pi_\theta(X_i;\widehat{\theta})}{1-\widehat{\pi}_i}\end{array}\Big\}\Big\{Y_i - m(X_i;\beta) - c^T\frac{\pi_\theta(X_i;\widehat{\theta})}{1-\widehat{\pi}_i}\Big\}\Big] = 0, \quad (2.2)$$

where c is a vector that needs to be solved jointly with β , $m_\beta = \partial/\partial\beta\{m(X;\beta)\}$ and $\pi_\theta(X;\theta) = \partial/\partial\theta\{\pi(X;\theta)\}$. Suppose the dimension of β and θ are p and q respectively, then m_β and $\pi_\theta(X;\theta)$ are column vectors with dimension p and q respectively and both sides of (2.2) are of dimension $(p+q)$. Simulation studies using scenarios designed by Kang and Schafer (2007) and Tan (2007) demonstrated that this method does not suffer the difficulties of the usual DR estimators observed by Kang and Schafer (2007) and achieves comparable or improved performance relative to existing methods including method of Tan (2006), which is actually closely related to the CTD method as discussed by Cao, Tsiatis and Davidian (2009). Nevertheless, these nice properties are again based on large sample theory, which may not necessarily translate into good performance in practice. Taking a closer look at (2.2), the estimation of β in CTD method is intertwined with the estimated propensities $\pi(X_i; \widehat{\theta})$ and huge weights are given to subjects with small $\pi(X_i; \widehat{\theta})$ (propensities). Therefore, we conjecture that the good properties of the CTD estimator when propensity model is correct or close to correct but outcome model is wrong might be achieved at the expense of worse performance when the outcome model is correct in finite samples.

Other efforts have been made to improve DR estiamtors by improving estimation of the weights. Imai and Ratkovic (2014) exploited the dual characteristics of the propensity score as a conditional probability and covariate balancing score and proposed to estimate propensity scores using generalized method-of-moments or empirical likelihood, referred to as covariate balancing propensity score (CBPS) method. Zubizarreta (2015) proposed methods to directly estimate weights by finding the weights of minimum variance that balance the empirical distribution of the observed covariates up to prespecified levels and the method is referred to as stable balancing weights (SBW) method. Both methods were applied to the augmented DR estimator and evaluated using the Kang and Schafer (2007) scenarios by their authors.

2.2 Proposed method

In contrast to previous methods, which are based on the usual augmented IPW framework, we propose an alternative and improved DR estimator from a different perspective. In our method, we focus on directly addressing the issue of sensitivity to small estimated propensity score resulted from inverse propensity score weighting and our strategy is not to directly inverse weight by propensity scores, at least not explicitly. The proposed estimator for μ is given by

$$\widehat{\mu} = n^{-1} \sum_{i=1}^{n} \Big[\frac{\sum_{j=1}^{n} R_j \{Y_j - m(X_j; \widehat{\beta})\} K\big(\frac{\widehat{\pi}_j - \widehat{\pi}_i}{h_n}\big)}{\sum_{j=1}^{n} R_j K\big(\frac{\widehat{\pi}_j - \widehat{\pi}_i}{h_n}\big)} + m(X_i; \widehat{\beta}) \Big], \tag{2.3}$$

where $K(\cdot)$ is a symmetric kernel function in $\mathcal{R}$, h_n is a bandwidth and $\widehat{\beta}$ is the usual estimator of β in the outcome regression method, which is different from the one used in CTD.

To offer some intuition and motivation behind this estimator we provide a heuristic argument why the proposed estimator is expected to possess the double-robustness property. The proposed estimator is motivated from estimating the following quantity

$$E\big[E\big\{Y - m(X; \beta) | R = 1, \pi(X; \theta)\big\} + m(X; \beta)\big]. \tag{2.4}$$

If $\pi(X; \theta)$ is the true model for propensity score, Rosenbaum and Rubin (1983) showed that, conditional on the propensity score, missingness is independent of confounders and outcomes, i.e., $R \perp\!\!\!\perp X | \pi(X;\theta_0)$ and $Y \perp\!\!\!\perp R | \pi(X;\theta_0)$, where θ_0 is the truth such that $P(R=1|X) = \pi(X;\theta_0)$. It then follows that

$$\begin{aligned} (2.4) &= E\big[E\big\{Y - m(X;\beta)|\pi(X;\theta_0)\big\} + m(X;\beta)\big] \\ &= E(Y) - E\{m(X;\beta)\} + E\{m(X;\beta)\} \\ &= E(Y) \equiv \mu, \end{aligned}$$

where the first equality is due to $R \perp\!\!\!\perp X | \pi(X;\theta_0)$ and $Y \perp\!\!\!\perp R | \pi(X;\theta_0)$. The above result holds regardless of whether $m(X;\beta)$ is the correct model for Y. This suggests that, if the propensity score can be correctly estimated, then estimating the quantity (2.4) may lead to a valid estimator for μ even if the model for the outcome may be possibly wrong.

It is also easy to see that

$$\begin{aligned} &E\big\{Y - m(X;\beta)|R=1, \pi(X;\theta)\big\} \\ &= E\big[E\big\{Y - m(X;\beta)|R=1, X, \pi(X;\theta)\big\}|R=1, \pi(X;\theta)\big]. \quad (2.5) \end{aligned}$$

If $m(X;\beta)$ is the correct specification of $E(Y|X) = E(Y|R=1,X)$, then we have

$$(2.5) = E\big\{m(X;\beta) - m(X;\beta)|R=1, \pi(X;\theta)\big\} = 0,$$

and as a result the targeting quantity satisfies

$$(2.4) \quad = \quad E\big\{0 + m(X;\beta)\big\} = E(Y) \equiv \mu.$$

Therefore, (2.4) equals the target μ if either one of the models is correct and, if we could estimate (2.4), then this estimator is expected to be doubly robust. The proposed estimator (2.3) substitutes unknown parameters β and θ by their estimates and replaces the outer expectation in (2.4) by the sample average and $E\big\{Y - m(X;\beta)|R = 1, \pi(X;\theta)\big\}$ by the nonparametric Nadaraya-Watson kernel estimator (Fan and Gijbels, 1996). Specifically, the Nadaraya-Watson kernel estimator for $E\big\{Y - m(X;\beta)|R = 1, \pi(X;\theta)\big\}$ is $\sum_{j=1}^n R_j\{Y_j - m(X_j;\beta)\}K\big(\frac{\pi(X_j;\theta)-\pi(X_i;\theta)}{h_n}\big)/\sum_{j=1}^n R_j K\big(\frac{\pi(X_j;\theta)-\pi(X_i;\theta)}{h_n}\big)$ if β and θ are known. Under standard conditions usually assumed for $K(u)$, including $\int K(u)du = 1$, $\int uK(u)du = 0$, $h_n \to 0$, and $nh_n \to \infty$, it can be shown that $n^{-1}\sum_{i=1}^n \frac{1}{h_n}K(\frac{x-X_i}{h_n}) \xrightarrow{p} f_X(x)$ and $n^{-1}\sum_{i=1}^n\{Y_i\frac{1}{h_n}K(\frac{x-X_i}{h_n})\}$ $\xrightarrow{p} E(Y|x)f_X(x)$, where $f_X(x)$ is the density of X. Therefore, the Nadaraya-Watson kernel estimator $\frac{\sum_{i=1}^n\{Y_iK(\frac{x-X_i}{h_n})\}}{\sum_{i=1}^n K(\frac{x-X_i}{h_n})}$ estimates $E(Y|x)$ and similarly one can obtain the Nadaraya-Watson kernel estimator for $E\big\{Y - m(X;\beta)|R = 1, \pi(X;\theta)\big\}$ detailed above.

In contrast, the usual augmented IPW DR estimator, as well as the estimators of CTD and some other alternatives, is based on or equivalent to directly estimating this quantity instead

$$E\Big[\frac{R\{Y-m(X;\beta)\}}{\pi(X;\theta)}+m(X;\beta)\Big], \tag{2.6}$$

since the usual IPW DR estimator in (2.1) can be written equivalently as

$$n^{-1}\sum_{i=1}^{n}\Big[\frac{R_i\{Y_i-m(X_i;\widehat{\beta})\}}{\pi(X_i;\widehat{\theta})}+m(X_i;\widehat{\beta})\Big]. \tag{2.7}$$

Comparing the quantity in (2.6) with (2.4), the two quantities differ in their first terms inside the expectation. The first term inside the expectation of (2.6) weights the residual for subjects with observed outcomes by the inverse of his/her propensity $\pi(X;\theta)$. In estimation, even though $\pi(X_i;\theta)$ is bounded away from zero, the estimated $\pi(X_i;\theta)$ can be close to zero, putting huge weights on those individuals. This leads to numeric instability of estimators based on this quantity. The proposed estimator may alleviate this issue as propensities are not directly used as weights. Finally, unlike the CTD method, the estimator does not change the way β is estimated in the outcome regression model and, therefore, we anticipate that it will not suffer from degraded performance in finite samples when the outcome model is correctly specified.

Our discussion above focuses on explaining the difference between the proposed estimator and the various versions of augmented IPW DR estimators, i.e., they are motivated from directly estimating (2.4) or (2.7) respectively. However, taking a closer look at the proposed estimator, we also see a connection with the augmented IPW DR estimators. By interchanging the order of summation over i and j, the proposed estimator can also be written equivalently as

$$\widehat{\mu} = n^{-1}\sum_{j=1}^{n}\Big[R_j\{Y_j - m(X_j;\widehat{\beta})\}\Big\{\sum_{i=1}^{n}\frac{K\big(\frac{\widehat{\pi}_j-\widehat{\pi}_i}{h_n}\big)}{\sum_{j=1}^{n}R_jK\big(\frac{\widehat{\pi}_j-\widehat{\pi}_i}{h_n}\big)}\Big\} + m(X_j;\widehat{\beta})\Big].$$

Comparing it with the usual augmented IPW DR estimator in (2.7), we see that the only difference between the two estimators are in the weight for $R_j\{Y_j - m(X_j;\widehat{\beta})\}$. Specifically, in the usual IPW DR estimator the weight is directly the inverse of the estimated propensity score $1/\pi(X_j;\widehat{\theta})$, whereas the weight in the proposed estimator is $\sum_{i=1}^{n}\frac{K\big(\frac{\widehat{\pi}_j-\widehat{\pi}_i}{h_n}\big)}{\sum_{j=1}^{n}R_jK\big(\frac{\widehat{\pi}_j-\widehat{\pi}_i}{h_n}\big)}$, which can be shown to converge in probability to $1/E\{R|\pi(X;\theta^*)\}$, where θ^* is the limit of $\widehat{\theta}$. When $\pi(X;\theta)$ is the correct model for $R|X$, then θ^* equals the truth, θ_0, and $E\{R|\pi(X;\theta^*)\}$ equals $\pi(X;\theta_0)$. In this sense, the proposed estimator resembles an inverse probability weighted DR estimator, although the motivation for this estimator is quite different. It can be viewed as an inverse probability weighted estimator where the propensity is being smoothed to achieve more stability since the weight for a particular subject now depends on all estimated propensities and their absolute difference (or distance) with the propensity score of the subject, instead of depending only on the propensity for a single subject. It is easier to intuitively see how the stability is achieved by comparing (2.3) with (2.7). For the usual DR estimator in (2.7), each $R_i\{Y_i - m(X_i;\widehat{\beta})\}$ is weighted by $1/\pi(X_i;\widehat{\theta})$ and if the estimated propensity is close to zero, then the huge weight on $R_i\{Y_i - m(X_i;\widehat{\beta})\}$ will lead to unstable estimate of μ. Ignoring the second term on $m(X_i;\widehat{\beta})$, visually we can view this estimator as a summation of many spikes around each observed $Y_i - m(X_i;\widehat{\beta})$. For the proposed estimator in (2.3), however, for each i the first term is a mountain (all observed $Y_j - m(X_j;\widehat{\beta})$, $j = 1,\ldots,n$, receives positive weight) concentrated around the observed $Y_i - m(X_i;\widehat{\beta})$ and the closer $\pi(X_j;\widehat{\theta})$ to $\pi(X_i;\widehat{\theta})$ the larger the weight on $Y_j - m(X_j;\widehat{\beta})$. Visually, ignoring the second term on $m(X_i;\widehat{\beta})$, the proposed estimator is a summation of many mountains and as a result it is less sensitive to small estimated propensity scores.

As discussed above, one way to intuitively understand the proposed estimator is that it estimates $E\big\{Y - m(X;\beta)|R = 1, \pi(X;\theta)\big\}$ in (2.4) using the nonparametric Nadaraya-Watson kernel estimator. Nadraya-Watson kernel estimator is a special case of local polynomial estimator with the order of polynomial being 0, i.e., the local average kernel estimator. It is well-known in the kernel regression literature that local linear kernel estimator (or high order local polynomial regression) can reduce asymptotic bias especially at the boundary relative to the local average estimator. Then naturally, one may expect that if we instead estimate $E\{Y - m(X;\beta)|R = 1, \pi(X;\theta)\}$ using more refined estimators, say, the local linear kernel estimator, it may lead to better performance. However, this conjecture is not true. We implemented a similar estimator as the proposed estimator but replacing the Nadaraya-Watson kernel estimator by a local linear estimator. Our simulations (not shown) show that the performance of the version with local linear estimator is considerably worse across all scenarios and the performance is very sensitive to the choice of the bandwidth, in contrast to the proposed estimator. Therefore, we did not pursue this estimator further.

2.3 Asymptotic Results

We show in the supplementary material that under some mild regularity conditions, if either the working model for outcome or for propensity score is correctly specified, but not necessarily both, then $\widehat{\mu}$ is consistent for μ and $\sqrt{n}(\widehat{\mu} - \mu)$ converges in distribution to a normal distribution. We assume the standard regularity conditions required for convergence of $\widehat{\beta}$ and $\widehat{\theta}$ under possibly misspecified models (Tsiatis, 2006) and for consistency and asymptotic normality of the nonparametric kernel estimator (Fan and Gijbels, 1996). We show that $\widehat{\mu}$ is asymptotically linear and derive its influence function. Define P and P_n as a probability measure and an empirical measure respectively, i.e., $Pf(X) = \int f(x)P(dx)$ and $P_n f(X) = n^{-1}\sum_{i=1}^n f(X_i)$, and we denote $G_n = n^{\frac{1}{2}}(P_n - P)$. When at least one of the working models is correctly specified, we have $n^{\frac{1}{2}}(\widehat{\mu} - \mu)$

$$
\begin{aligned}
= & \; G_n\Big\{\frac{E[R\{Y - m(X;\beta^*)\}|\pi(X;\theta^*)]}{E\{R|\pi(X;\theta^*)\}} + \frac{R\{Y - m(X;\beta^*)\}}{E\{R|\pi(X;\theta^*)\}} \\
- & \; \frac{RE[R\{Y - m(X;\beta^*)\}|\pi(X;\theta^*)]}{E^2\{R|\pi(X;\theta^*)\}} + m(X;\beta^*) - \mu\Big\} \\
+ & \; \frac{d}{d\theta}|_{\theta=\theta^*} E\Big[\frac{R\{Y - m(X;\beta^*)\}}{E\{R|\pi(X;\theta)\}}\Big] n^{\frac{1}{2}}(\widehat{\theta} - \theta^*) \\
+ & \; \frac{d}{d\beta}|_{\beta=\beta^*} E\Big[\frac{R\{Y - m(X;\beta)\}}{E\{R|\pi(X;\theta^*)\}} + m(X;\beta)\Big] n^{\frac{1}{2}}(\widehat{\beta} - \beta^*) + o_p(1),
\end{aligned}
$$

where β^* and θ^* are the limiting value of $\widehat{\beta}$ and $\widehat{\theta}$ respectively. Under suitable regularity conditions and by standard M-estimation theory, $\widehat{\theta}$ and $\widehat{\beta}$ will be asymptotic normal. Therefore, $\widehat{\mu}$ is asymptotically normal with mean zero. Suppose working logistic regression model and linear model are specified for the outcome and propensity score respectively, then $n^{\frac{1}{2}}(\widehat{\beta} - \beta^*) = \frac{1}{\sqrt{n}E(X_i X_i^T)}\sum_{i=1}^n X_i(Y_i - X_i^T\beta^*) + o_p(1)$, and $n^{\frac{1}{2}}(\widehat{\theta} - \theta^*) = \frac{1}{\sqrt{n}E[X_i X_i^T \pi(X_i,\theta^*)\{1-\pi(X_i,\theta^*)\}]}$ $\sum_{i=1}^n X_i\{R_i - \text{expit}(X_i^T\theta^*)\} + o_p(1)$. We then obtain the influence function of $\widehat{\mu}$ and we have $\sqrt{n}(\widehat{\mu} - \mu) = G_n h(R, X, Y; \beta^*, \theta^*) + o_p(1)$, where the influence function

$$h(R,X,Y;\beta^*,\theta^*) = \frac{E[R\{Y-m(X;\beta^*)\}|\pi(X;\theta^*)]}{E\{R|\pi(X;\theta^*)\}} + \frac{R\{Y-m(X;\beta^*)\}}{E\{R|\pi(X;\theta^*)\}} - \frac{RE[R\{Y-m(X;\beta^*)\}|\pi(X;\theta^*)]}{E^2\{R|\pi(X;\theta^*)\}} + m(X;\beta^*) - \mu$$
$$+\frac{d}{d\theta}|_{\theta=\theta^*} E\Big[\frac{R\{Y-m(X;\beta^*)\}}{E\{R|\pi(X;\theta)\}}\Big] \frac{X\{R-\pi(X,\theta^*)\}}{E[XX^T\pi(X,\theta^*)\{1-\pi(X,\theta^*)\}]}$$
$$+\frac{d}{d\beta}|_{\beta=\beta^*} E\Big[\frac{R\{Y-m(X;\beta)\}}{E\{R|\pi(X;\theta^*)\}} + m(X;\beta)\Big] \frac{X(Y-X^T\beta^*)}{E(XX^T)}.$$

If one or both working models are correctly specified, the influence function can be further simplified. Specifically, if the working model for $E(Y|R=1,X) = m(X;\beta)$ is correctly specified, the influence function $h(R,X,Y;\beta^*,\theta^*) =$

$$\frac{RY}{E\{R|\pi(X;\theta^*)\}} - \frac{R-E\{R|\pi(X;\theta^*)\}}{E\{R|\pi(X;\theta^*)\}} m(X;\beta_0) - \mu$$
$$+\frac{d}{d\beta}|_{\beta=\beta_0} E\Big[\frac{R\{Y-m(X;\beta)\}}{E\{R|\pi(X;\theta^*)\}} + m(X;\beta)\Big] \frac{X(Y-X^T\beta_0)}{\{E(XX^T)\}}.$$

If the model for $P(R=1|X) = \pi(X;\theta)$ is correctly specified, the influence function $h(R,X,Y;\beta^*,\theta^*) =$

$$\Big\{\frac{RY}{\pi(X;\theta_0)} - \frac{R-\pi(X;\theta_0)}{\pi(X;\theta_0)}\big[E\{Y-m(X;\beta^*)|\pi(X;\theta_0)\} + m(X;\beta^*)\big] - \mu\Big\}$$
$$+\frac{d}{d\theta}|_{\theta=\theta_0} E\Big[\frac{R\{Y-m(X;\beta^*)\}}{E\{R|\pi(X;\theta)\}}\Big] \frac{X\{R-\pi(X,\theta_0)\}}{E[XX^T\pi(X,\theta_0)\{1-\pi(X,\theta_0)\}]}.$$

When both working models are correct, the influence function is

$$\Big\{\frac{RY}{\pi(X;\theta_0)} - \frac{R-\pi(X;\theta_0)}{\pi(X;\theta_0)} m(X;\beta_0) - \mu\Big\},$$

where $\pi(X;\theta_0) = P(R=1|X)$, and it is the semiparametric efficient influence function. Theorems 1 and 2 summarize the results described above.

The variance of $\widehat{\mu}$ can be estimated by n^{-1} times the sample variance of $\widehat{h}(R_i, X_i, Y_i; \widehat{\beta}, \widehat{\theta})$, $i = 1, \ldots, n$, where $\widehat{h}(R_i, X_i, Y_i; \widehat{\beta}, \widehat{\theta})$ is defined as above except that we replace all marginal expectations by sample averages and all conditional expectations by the corresponding Nadaraya-Watson kernel estimator. For example, we replace $E(XX^T)$, $E\{R_i|\pi(X_i;\theta^*)\}$ and $E[R_i\{Y_i - m(X_i;\beta^*)\}|\pi(X_i;\theta^*)]$ by $n^{-1}\sum_{j=1}^n X_j X_j^T$, $\frac{\sum_{j=1}^n R_j K\left(\frac{\widehat{\pi}_j - \widehat{\pi}_i}{h_n}\right)}{\sum_{j=1}^n K\left(\frac{\widehat{\pi}_j - \widehat{\pi}_i}{h_n}\right)}$, $\frac{\sum_{j=1}^n R_j K\left(\frac{\widehat{\pi}_j - \widehat{\pi}_i}{h_n}\right)}{\sum_{j=1}^n K\left(\frac{\widehat{\pi}_j - \widehat{\pi}_i}{h_n}\right)}$, and $\frac{\sum_{j=1}^n R_j \{Y_j - m(X_j;\widehat{\beta})\} K\left(\frac{\widehat{\pi}_j - \widehat{\pi}_i}{h_n}\right)}{\sum_{j=1}^n K\left(\frac{\widehat{\pi}_j - \widehat{\pi}_i}{h_n}\right)}$ respectively.

Theorem 1. *When at least one of the working models for propensity score or the outcome is correctly specified, then $\widehat{\mu}$ is consistent for μ and is asymptotically normal with an influence function defined above.*

Theorem 2. *When both working models for propensity score and the outcome are correctly specified, then $\widehat{\mu}$ attains the semiparametric efficiency bound.*

Asymptotically, the proposed estimator is equivalent to the usual DR estimator when the outcome regression is correct, regardless of whether the propensity score model is correct. As Cao, Tsiatis and Davidian (2009) commented, when the outcome regression model is correct, it will be fruitless to attempt to further improve efficiency; see Tsiatis and Davidian (2007) for detailed explanation. Therefore, we focus on the case when the propensity score model is correct but the outcome regression model is incorrect in discussing efficiency. For simplicity, we first assume the propensity score is known and is not estimated, denoted by $\pi_0(X)$. The asymptotical variance of an estimator is proportional to the variance of its influence function. Following the same argument as in Cao, Tsiatis and Davidian (2009, page 726), the asymptotical variance of the usual DR estimator is proportional to $\mathrm{var}\Big\{\frac{RY}{\pi_0(X)} - \frac{R-\pi_0(X)}{\pi_0(X)}m(X;\beta^*)\Big\}$ which equals $E\Big\{\frac{1-\pi_0(X)}{\pi_0(X)}\{Y - m(X;\beta^*)\}^2\Big\} + \mathrm{var}(Y)$, and the asymptotical variance of the proposed estimator is proportional to

$$\begin{aligned}
& \mathrm{var}\Big\{\frac{RY}{\pi_0(X)} - \frac{R-\pi_0(X)}{\pi_0(X)}\big[m(X;\beta^*) + E\{Y - m(X;\beta^*)|\pi_0(X)\}\big]\Big\} \\
= \; & E\Big\{\frac{1-\pi_0(X)}{\pi_0(X)}\big[Y - m(X;\beta^*) - E\{Y - m(X;\beta^*)|\pi_0(X)\}\big]^2\Big\} + \mathrm{var}(Y).
\end{aligned}$$

The first term of the above expression is equal to

$$\begin{aligned}
& E\Big\{\frac{1-\pi_0(X)}{\pi_0(X)}\{Y - m(X;\beta^*)\}^2\Big\} + E\Big\{\frac{1-\pi_0(X)}{\pi_0(X)}E^2\{Y - m(X;\beta^*)|\pi_0(X)\}\Big\} \\
& -2E\Big\{\frac{1-\pi_0(X)}{\pi_0(X)}\{Y - m(X;\beta^*)\}E\{Y - m(X;\beta^*)|\pi_0(X)\}\Big\}.
\end{aligned}$$

Regarding the last term, by using the fact $E(\cdot) = E[E\{\cdot|\pi_0(X)\}]$, it is equal to $-2E\Big\{\frac{1-\pi_0(X)}{\pi_0(X)}E^2\{Y - m(X;\beta^*)|\pi_0(X)\}\Big\}$. Therefore the asymptotic variance of the proposed estimator is proportional to $E\Big\{\frac{1-\pi_0(X)}{\pi_0(X)}\{Y -$

$m(X;\beta^*)\}^2\Big\} + \mathrm{var}(Y) - E\Big\{\frac{1-\pi_0(X)}{\pi_0(X)} E^2\{Y - m(X;\beta^*)|\pi_0(X)\}\Big\}$, which is always less than the variance of the usual doubly robust estimator when the outcome regression model is incorrect. When the propensity scores are not known but the model is correctly specified, then the influence function of both the original DR estimator and the proposed one each has an additional term representing the effect of estimating θ. It is straightforward to check that these two additional terms are equal. Then by the same argument as above, we can show that the asymptotic variance of the proposed estimator is that of the original DR estimator minus a nonnegative term. We summarize the result below.

Theorem 3. *When the model for the propensity score is correct but the model for the outcome is misspecified, the asymptotic variance of $\widehat{\mu}$ is no greater than the asymptotic variance of the usual AIPWE-based doubly robust estimator.*

4 Simulation Studies

We conducted simulation studies to evaluate the performance of the proposed method and compare it with the usual augmented inverse probability weighted DR estimator, the modified DR method (referred to as CTD method) proposed by Cao, Tsiatis and Davidian (2009), the CBPS method (Imai and Ratkovic, 2014), and the SBW method (Zubizarreta, 2015). For comparison we also included the usual outcome regression method, i.e., the average of predictions from outcome regression model fitted using the observed data. For the proposed method, the bandwidth needs to satisfy $h_n \to 0$ and $nh_n \to \infty$. To assess how sensitive the method is to different choices of bandwidth, we implemented it with bandwidth $h_n = n^{-1/3}, n^{-1/4}$, and $n^{-1/5}$, where n is the sample size. Bootstraping with 100 bootstrap samples was used to obtain standard errors. We chose to use bootstrapping to obtain standard errors for several considerations. For example, some of the comparison methods (CBPS and SBW) did not provide standard error estimates in their paper and also bootstrapping allows one to easily account for variability in estimating propensity scores and fitting outcome regression models regardless of the methods used for fitting the models. The latter point is especially important because flexible nonparametric and machine learning methods are becoming popular in practice to model propensity scores and outcomes and in this case implementing the usual standard error estimates based on directly estimating asymptotic variance becomes rather complicated.

In the first set of simulations, we duplicated the scenario that was originally designed by Kang and Schafer (2007), which has become a standard scenarios to compare DR estimators (Cao, Tsiatis and Davidian, 2009; Imai and Ratkovic, 2014; Zubizarreta, 2015). Under this scenario, Kang and Schafer (2007) demonstrated that when both outcome regression model and propensity score model are incorrect but nearly perfect in the sense that they look trustworthy based on model diagnostics, the usual doubly robust estimator may be severely biased and unstable. In this scenario, $Z = (Z_1, \ldots, Z_4)^T$ were generated from independent standard normal distributions, and $X = (X_1, \ldots, X_4)^T$ were defined as $X_1 = \exp(Z_1/2), X_2 = Z_2/\{1+\exp(Z_1)\}+10, X_3 = (Z_1 Z_3/25+0.6)^3, X_4 = (Z_2+Z_4+20)^2$. That is, Z and X can be expressed with each other. Outcomes were generated as $Y = 210 + 27.4Z_1 + 13.7Z_2 + 13.7Z_3 + 13.7Z_4 + \epsilon$, where ϵ is standard normal, and non-missingness indicator R were generated according to the true propensity score $P(R = 1|Z) = \text{expit}(-Z_1 + 0.5Z_2 - 0.25Z_3 - 0.1Z_4)$. In real data, covariates that are seen by data analysts are X. Naturally a data analyst that only sees X would fit a linear regression model for Y given X and a logistic regression model for R given X. As illustrated by Kang and Schafer (2007), although misspecified these models would appear trustworthy and are nearly as correct. To be specific, the misspecified outcome model is $m(X;\beta) = \beta^T(1, X)$ and the misspecified propensity score model is $\pi(X;\theta) = \text{expit}\{\theta^T(1, X)\}$. As previous work, we considered sample size $n = 200$ and $n = 1000$.

Results on the first set of simulations are shown in Tables 1 and 2. Under this scenario when both working models are strictly speaking incorrect but are nearly perfect, the usual doubly robust estimator is extremely unstable and has huge variability as demonstrated by Monte Carlo standard deviation and root mean square error. All the other DR estimators do not exhibit this type of "disastrous" behavior and among them the CTD estimator has the best performance. The proposed estimators perform comparably with other improved DR estimators and the performance is not sensitive to different choices of bandwidth. We note that, when the propensity score model is correct but outcome model is incorrect, the SBW method has relatively larger bias than that of other methods and lower coverage probability, which is not surprising because this method controls for bias under the condition that covariates are related to the outcome through a generalized additive form and this condition is not satisfied under this scenario. For the CTD projection method, when $n = 200$ and when both outcome regression model and propensity score model are misspecified, the bootstrap method cannot reliably estimate the uncertainty and the average of standard error is much larger than the Monte Carlo standard deviation. This appears to be a finite sample problem and when $n = 1000$ the bootstrap standard error performs as expected. This phenomenon was also observed for the second set of simulations discussed below.

The Kang and Schafer scenario was specifically constructed such that the usual doubly robust estimator may have "disastrous" performance and results on this scenario may not generalize to other scenarios. For example, the outcome regression model, when incorrect, is only mildly misspecified and as a result the outcome regression method only results in slight bias. To supplement the above simulation study, we compared various methods under the second set of simulations, where model misspecifications are due to ignoring some important variables. Four covariates were generated, where X_1 was generated as uniform (0, 1), X_2 as standard normal, X_3 as Bernoulli (0.3) and X_4 as lognormal (0,1). Outcomes were generated according to $Y = 2.5 + X_1/2 + X_2 + X_3 + X_4 + \epsilon$, where ϵ follows a standard normal distribution, and R was generated with propensity score $\text{expit}(-1 - X_1/2 + X_2 - X_3 + X_4)$. The proportion of missingness is about 60%. In methods considered the misspecified outcome regression and propensity score models are fit by ignoring X_4.

Results on the second scenario are shown in Tables 3 and 4. Both the usual doubly robust estimator and the proposed estimator are consistent when at least one working model is correctly specified, offering more protection against model misspefication than the outcome regression method. When both working models are incorrect, the usual and proposed doubly robust estimators are all biased but do not show extreme variability as observed in the first scenario. Except for the situation when propensity score model is correct but outcome regression model is incorrect, the proposed method performs very similarly to the usual doubly robust method. Consistent with result summarized in Theorem 3, when propensity score model is correct but outcome regression model is incorrect, the proposed method is more efficient than the usual DR estimator and the relative efficiency (the ratio of mean squared error) of the proposed method relative to the usual DR estimator is 1.50 and 1.79 for $n = 200$ and $n = 1000$ respectively. Again, the performance is not sensitive to different choices of bandwidth.

The CTD method exhibits unexpected results. When sample size is small (n=200), compared with other methods, it has considerably larger variability, especially when the model for outcome regression is correct regardless of whether the propensity score model is correct or not. Moreover, the bootstrap method cannot estimate the standard error well for the CTD method and the average of the standard error estimates are significantly greater than the corresponding Monte Carlo standard deviation. Some of

Table 1: Simulation setting 1 (n=200). BIAS: average Monte Carlo bias; MCSD: Monte Carlo standard deviation; RMSE: Root Mean Squared error; SE: average of standard error; CP: coverage probability of 95% confidence interval. Correct or incorrect refer to corresponding specified model (OR: outcome regression; PS: propensity score). Proposed a, b and c correspond to a bandwidth of $n^{-1/3}, n^{-1/4}$ and $n^{-1/5}$ respectively.

METHODS	BIAS	MCSD	RMSE	SE	CP	BIAS	MCSD	RMSE	SE	CP
	OR correct, PS correct					OR correct, PS incorrect				
OR	-0.054	2.59	2.59	2.55	0.95	-0.054	2.59	2.59	2.55	0.95
Usual DR	-0.055	2.59	2.59	2.55	0.95	-0.067	2.58	2.58	2.62	0.95
CTD pj	-0.049	2.60	2.60	2.64	0.95	-0.053	2.59	2.58	2.57	0.95
CBPS	-0.055	2.59	2.59	2.54	0.94	-0.056	2.59	2.59	2.55	0.94
SBW	-0.054	2.59	2.59	2.54	0.95	-0.055	2.59	2.59	2.55	0.95
Proposed a	-0.055	2.59	2.59	2.56	0.95	-0.055	2.59	2.59	2.56	0.95
Proposed b	-0.055	2.59	2.59	2.56	0.95	-0.056	2.59	2.59	2.56	0.95
Proposed c	-0.055	2.59	2.59	2.56	0.95	-0.056	2.59	2.59	2.56	0.95
	OR incorrect, PS correct					OR incorrect, PS incorrect				
OR	-0.65	3.42	3.48	3.28	0.92	-0.65	3.42	3.48	3.28	0.92
Usual DR	0.27	3.53	3.54	3.48	0.95	-5.69	19.78	20.57	14.92	0.94
CTD pj	-0.028	2.66	2.65	2.78	0.95	-0.85	5.01	5.09	10.71	0.996
CBPS	-0.26	3.29	3.30	3.18	0.94	-2.19	3.58	4.20	3.39	0.88
SBW	1.58	3.10	3.48	3.01	0.91	-0.74	3.39	3.47	3.23	0.92
Proposed a	0.49	3.38	3.41	3.32	0.95	-1.79	3.33	3.78	3.26	0.90
Proposed b	0.56	3.29	3.33	3.23	0.94	-1.68	3.30	3.70	3.22	0.90
Proposed c	0.63	3.23	3.28	3.18	0.95	-1.56	3.28	3.63	3.20	0.90

Table 2: Simulation setting 1 (n=1000). Entries are as in Table 1.

METHODS	BIAS	MCSD	RMSE	SE	CP	BIAS	MCSD	RMSE	SE	CP
	OR correct, PS correct					OR correct, PS incorrect				
OR	0.04	1.15	1.15	1.14	0.95	0.04	1.15	1.15	1.14	0.95
Usual DR	0.04	1.15	1.15	1.14	0.95	0.056	1.41	1.41	1.29	0.95
CTD pj	0.04	1.15	1.15	1.15	0.95	0.041	1.15	1.15	1.15	0.95
CBPS	0.04	1.15	1.15	1.14	0.95	0.04	1.15	1.15	0.14	0.95
SBW	0.04	1.15	1.15	1.14	0.95	0.04	1.15	1.15	1.14	0.95
Proposed a	0.042	1.15	1.15	1.14	0.95	0.04	1.15	1.15	1.14	0.95
Proposed b	0.042	1.15	1.15	1.14	0.95	0.04	1.15	1.15	1.14	0.95
Proposed c	0.04	1.15	1.15	1.14	0.95	0.04	1.15	1.15	1.14	0.95
	OR incorrect, PS correct					OR incorrect, PS incorrect				
OR	-0.77	1.50	1.68	1.48	0.91	-0.77	1.50	1.68	1.48	0.91
Usual DR	0.11	1.65	1.65	1.55	0.95	-17.92	166.73	167.6	27.04	0.69
CTD pj	0.08	1.15	1.15	1.16	0.95	-1.36	1.28	1.87	1.35	0.83
CBPS	0.14	1.53	1.54	1.46	0.94	-3.61	2.26	4.25	1.74	0.45
SBW	1.58	1.36	2.09	1.34	0.78	-0.80	1.48	1.69	1.47	0.91
Proposed a	0.27	1.50	1.53	1.49	0.94	-2.13	1.42	2.57	1.39	0.66
Proposed b	0.35	1.44	1.48	1.44	0.94	-2.03	1.41	2.47	1.38	0.68
Proposed c	0.45	1.40	1.47	1.41	0.94	-1.89	1.40	2.35	1.38	0.71

Table 3: Simulation setting 2 (n=200). Entries are as in Table 1.

METHODS	BIAS	MCSD	RMSE	SE	CP	BIAS	MCSD	RMSE	SE	CP
	OR correct, PS correct					OR correct, PS incorrect				
OR	0.003	0.18	0.18	0.18	0.94	0.003	0.18	0.18	0.18	0.94
Usual DR	0.003	0.20	0.20	0.20	0.94	0.003	0.19	0.19	0.18	0.94
CTD pj	0.011	0.50	0.50	1.3	0.997	0.006	0.28	0.28	0.45	0.99
CBPS	0.003	0.19	0.19	0.19	0.94	0.003	0.18	0.18	0.18	0.94
SBW	0.003	0.18	0.18	0.18	0.93	0.003	0.18	0.18	0.18	0.94
Proposed a	0.002	0.20	0.20	0.19	0.93	0.003	0.19	0.18	0.18	0.95
Proposed b	0.002	0.19	0.19	0.19	0.93	0.003	0.18	0.18	0.18	0.94
Proposed c	0.002	0.19	0.19	0.18	0.93	0.002	0.18	0.18	0.18	0.94
	OR incorrect, PS correct					OR incorrect, PS incorrect				
OR	0.54	0.23	0.58	0.22	0.33	0.54	0.23	0.58	0.22	0.33
Usual DR	0.023	0.24	0.24	0.27	0.94	0.56	0.25	0.61	0.24	0.36
CTD pj	0.032	0.27	0.27	0.63	0.98	0.56	0.40	0.69	0.69	0.89
CBPS	0.10	0.21	0.23	0.22	0.92	0.55	0.24	0.60	0.24	0.36
SBW	0.004	0.18	0.18	0.18	0.94	0.54	0.23	0.59	0.22	0.35
Proposed a	0.029	0.20	0.20	0.20	0.94	0.55	0.25	0.60	0.24	0.36
Proposed b	0.038	0.20	0.20	0.19	0.94	0.55	0.24	0.60	0.23	0.35
Proposed c	0.048	0.19	0.20	0.19	0.94	0.55	0.24	0.60	0.23	0.33

Table 4: Simulation setting 2 (n=1000). Entries are as in Table 1.

METHODS	BIAS	MCSD	RMSE	SE	CP	BIAS	MCSD	RMSE	SE	CP
	OR correct, PS correct					OR correct, PS incorrect				
OR	-0.002	0.080	0.080	0.078	0.95	-0.002	0.080	0.080	0.078	0.95
Usual DR	-0.001	0.09	0.09	0.086	0.93	-0.001	0.083	0.082	0.080	0.94
CTD pj	0.0002	0.10	0.10	0.11	0.96	-0.001	0.10	0.10	0.097	0.93
CBPS	-0.001	0.087	0.087	0.083	0.94	-0.001	0.082	0.082	0.079	0.95
SBW	-0.002	0.080	0.080	0.078	0.94	-0.002	0.080	0.080	0.078	0.95
Proposed a	-0.002	0.087	0.087	0.082	0.94	-0.001	0.083	0.083	0.080	0.94
Proposed b	-0.002	0.085	0.085	0.081	0.94	-0.001	0.082	0.081	0.079	0.94
Proposed c	-0.002	0.084	0.084	0.081	0.94	-0.001	0.081	0.081	0.079	0.94
	OR incorrect, PS correct					OR incorrect, PS incorrect				
OR	0.54	0.10	0.55	0.10	0	0.54	0.10	0.55	0.10	0
Usual DR	-0.0004	0.12	0.12	0.11	0.93	0.56	0.11	0.75	0.11	0.00
CTD pj	0.009	0.096	0.096	0.095	0.94	0.56	0.12	0.57	0.12	0.002
CBPS	0.036	0.103	0.109	0.095	0.91	0.56	0.11	0.57	0.10	0.001
SBW	-0.0003	0.080	0.080	0.078	0.95	0.55	0.10	0.55	0.10	0.001
Proposed a	0.006	0.088	0.089	0.086	0.94	0.56	0.11	0.57	0.11	0
Proposed b	0.013	0.086	0.087	0.084	0.94	0.56	0.11	0.57	0.10	0
Proposed c	0.021	0.085	0.088	0.083	0.93	0.56	0.11	0.57	0.10	0

the surprising results are due to finite sample performance. When the sample size is 1000, we see that the bootstrap standard error estimates for the projection method work well and, when the outcome regression model is correct, the loss of efficiency of the CTD method relative to the usual doubly robust estimator is less. In addition, consistent with the asymptotic results, when n=1000, if propensity score is correct but outcome model is incorrect, the projection estimator is more efficient than the usual estimator, but are still slightly less efficient than the proposed method. We note that weighting the estimating equation for outcome regression model using the inverse of the square of propensity score, as shown in (2.2) may lead to quite unstable estimates of parameters in outcome regression model and this may explain the unsatisfying finite sample performance of the CTD method. Unlike scenario 1 where the SBW method shows relatively larger bias, the SBW method seems to have the best performance under scenario 2, which is expected according to proposition 4.1 in (Zubizarreta, 2015) as in scenario 2 covariates have additive effect on outcomes. Compared with other methods, CBPS exhibits relatively larger bias and lower coverage probability when the propensity score model is correct but the outcome regression model is incorrect. Overall, the proposed methods have comparable or superior performance under all cases.

Finally, we add that we also implemented the usual AIPW DR estimator after trimming the estimated propensity score at and smaller than 0.1. Results and discussions are given in the supplementary materials.

5 Discussion

Our work follows up that of Tan (2006, 2007), Robins et al. (2007), Cao, Davidian and Tsitis (2008), Imai and Ratkovic (2014), Zubizarreta (2015) and other related work and is an effort to improve the original DR estimators so that it does not exhibit the "disastrous" behaviors observed by Kang and Schafer (2007) but enjoys the appealing double robustness property.

As clear from (4), the proposed estimator is motivated from the usual outcome regression approach. However, instead of averaging predictors from the fitted outcome regression model alone, we further model the expectation of residuals from the outcome regression model conditional on propensity scores and take average of predictions from the residual model. In contrast to the usual augmented inverse probability weighted DR estimator, where the inverse of propensity scores are used as weights, in the proposed approach the propensity score is viewed as a predictor and being conditioned on. Because of this, the proposed method does not suffer from the instability problem in the presence of some very small estimated propensity scores. In terms of stability and bias, our simulation studies show the proposed estimator is behaving similarly to the outcome regression method, an estimator that is typically thought to be stable, whereas at least in the Kang and Schafer (2007) setting the usual DR estimator exhibits extreme variability when both working models are only mildly misspecified. However, unlike the outcome regression method, the proposed estimator enjoys the double robustness property as shown by asymptotic theory and simulation studies. Interestingly, although the proposed estimator is not developed within the framework of augmented inverse propensity weighted estimators, asymptotically it has an influence function that belongs to the class of augmented IPW estimators and to the best our knowledge it is the first time such a connection is established explicitly. Because of the connection with augmented IPW estimators, an alternative way to understand the proposed method is to view it as an augmented IPW estimator with smoothed weights. Although this perspective offers an intuitive way to understand the proposed estimator, we comment that such as connection is not obvious and it would be difficult to directly come up with ways to smooth over weights. Also quite interestingly, although the proposed estimator is not developed from the the perspective of improving efficiency, it enjoys a nice property that is similar to that of Cao, Davidian and Tsiatis (2008). Specifically we show by asymptotic theory and simulations that the proposed estimator is more efficient than the usual DR estimator when the outcome regression model is incorrect but the propensity score model is correct and the improvement in efficiency can be considerable as demonstrated by our simulations. In terms of performance in finite samples, our simulation studies show that overall it has quite nice and stable performance under different sample sizes and across scenarios, whereas other existing modified doubly robust estimators may exhibit relatively large bias and/or less satisfying finite sample performances. Finally, we comment that the proposed method is very easy to implement and an example code for implementing the method is available at https://github.com/MinZhangUMBiostat/DoubleRobust.

Supplementary Materials

In the online supplementary materials, we provide proofs for asymptotic results described in Section 2.3, and we briefly summarize simulation results using the trimming method.

References

Barrett, J.K., Henderson, R.& Rosthøj, S. (2014). Doubly robust estimation of optimal dynamic treatment regimes. *Statistics in Biosciences,* **6**, 244–260.

Austin, P.C. & Stuart, E.A. (2015). Moving towards best practice when using inverse probability of treatment weighting (IPTW) using the propensity score to estimate causal treatment effects in observational studies. *Statistics in Medicine* **34,** 3661–79.

Bang, H. & Robins, J.M. (2005). Doubly robust estimation in missing data and causal inference models. *Biometrics* **61,** 962-72.

Cao, W., Tsiatis, A.A. & Davidian, M. (2009). Improving efficiency and robustness of the doubly robust estimator for a population mean with incomplete data. *Biometrika* **96,** 732–734.

Chen, S., Lu, W. & Zhao, H. (2018). An improved survival estimator for censored medical costs with a kernel approach *Communications in Statistics - Theory and Methods* **47,** 5702–5716.

Fan, J. & Gijbels, I. (1996). Local polynomial modeling and its application. Chapman and Hall/CRC.

Elliott M.R. (2008). Model Averaging Methods for Weight Trimming. *Journal of Official Statistics* **24,** 517540.

Elliott M & Little R. (2000). Model-based alternatives to trimming survey weights.*Journal of Official Statistics* **16,** 191-209.

Imai, K., and Ratkovic, M. (2014). Covariate balancing propensity score. *Journal of the Royal Statistical Society B* **76,** 243–263.

Little, R.J.A. & Rubin, D.B. (2002). Statistical Analysis with Missing Data, 2nd edition, New

York: John Wiley.

Lunceford, J.K. & Davidian, M. (2004). Stratification and weighting via the propensity score in estimation of causal treatment effects: A comparative study. *Statistics in Medicine* **23,** 2937–2960.

Moodie, E.E.M., Richardson, T.S. & Stephens, D.A. (2007). Demystifying optimal dynamic treatment regimes, *Biometrics* **63,** 447–455.

Kang, D.Y. J. & Schafer, J.L. (2007). Demystifying double robustness: a comparison of alternative strategies for estimating a population mean from incomplete data (with discussion and rejoinder) *Statistical Science* **22,** 523–580.

Robins, J., Orellana, L. & Rotnitzky, A. (2008). Estimation and extrapolation of optimal treatment and testing strategies, *Statistics in Medicine* **27,** 4678-4721.

Pirracchio, R., Petersen, M.L. & van der Laan, M.(2015). Improving propensity score estimators' robustness to model misspecification using super learner, *American journal of epidemiology* **181,** 108–119.

Robins, J.M., Rotnitzky & Zhao, L.P. (1994). Estimation of regression coefficients when some regressors are not always observed. *Journal of American Statistical Association* **89,** 846-66.

Robins, J., Sued, M., Lei-Gomez, Q., & Rotnitzky, A. (2007). Comment: Performance of double-robust estimators when" inverse probability" weights are highly variable. *Statistical Science* **22(4),** 544–559.

Rosenbaum, P.R. (1987). Model-based direct adjustment. *Journal of American Statistical Association* **82,** 387-94.

Rosenbaum, P.R. & Rubin, D.B. (1983). The central role of the propensity score in observational studies for causal effects. *Biometrika* **70,** 41-55.

Rosenbaum, P.R. & Rubin, D.B. (1984). Reducing bias in observational studies using subclassification on the propensity score. *Journal of American Statistical Association* **79,** 516-24.

Rubin, DR. (1974). Estimating causal effects of treatments in randomized and nonrandomized studies. *Journal of Educational Psychology* **66,** 688–701.

Rubin DB. (1980). Discussion of "Randomization Analysis of Experimental Data in the Fisher Randomization Test," by D. Basu. *Journal of the American Statistical Association* **75,** 591-593.

Rubin, D.B. & Thomas, N. (1996). Matching using estimated propensity scores: relating theory to practice. *Biometrics* **52,** 249-64.

Scharfstein, D.O., Rotnitzky, A. & Robins, J.M. (1999). Adjusting for nonignorable drop-out using semiparametric nonresponse models. (with discussion and rejoinder). *Journal of American Statistical Association* **94,** 1096-146.

Tan, Z. (2006). A distributional approach for causal inference using propensity scores. *Journal of American Statistical Association* **101,** 1619-37.

Tan, Z. (2007). Understanding OR, PS and DR. *Statistical Science* **22,** 560-8.

Tsiatis, A. A. (2006). Semiparametric Theory and Missing Data. New York: Springer.

Tsiatis, A.A. & Davidian, M. (2007) Comment on the paper "Demystifying Double Robustness: A Comparison of Alternative Strategies for Estimating a Population Mean from Incomplete Data" by Joseph D.Y. Kang and Joseph L. Schafer. *Statistical Science* **22,** 569–573.

Gruber, S. & van der Laan, M.J. (2010) A targeted maximum likelihood estimator of a causal effect on a bounded continuous outcome. *The International Journal of Biostatistics* **6,** article 26.

Zhou, T., Elliot, M.R. & Little, R.J.A. (2019) Penalized Spline of Propensity Methods for Treatment Comparison. *Journal of the American Statistical Association* **114,** 1–19.

Zubizarreta, J.R. (2015). Stable Weights that Balance Covariates for Estimation With Incomplete Outcome Data. *Journal of the American Statistical Association* **110,** 910-922.

Department of Biostatistics,

University of Michigan, Ann Arbor, USA.

E-mail: mzhangst@umich.edu

School of Statistics and Management,

Shanghai University of Finance and Economics

E-mail: zhang.baqun@mail.shufe.edu.cn