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Network Influence Analysis

Tao Zou, Ronghua Luo, Wei Lan and Chih-Ling Tsai

The Australian National University,

Southwestern University of Finance and Economics, and

University of California, Davis

Abstract: Due to the rapid development of social networking sites, the spatial autoregressive (SAR) model has played an important role in social network studies. However, the underlying structure of SAR implicitly assumes that all nodes (or actors or users) within the network have the same influential power measured by the common autocorrelation parameter. Hence, the classical SAR is unable to identify influential nodes. This paper proposes the adaptive SAR model by introducing the network influence index, which includes the classical SAR model as a special case. Using this proposed model without imposing any specific error distribution, we apply Lee's (2004) quasi-maximum likelihood approach to estimate the unknown parameters of the index, which can then be used to characterize the influential power of each node. The asymptotic properties of parameter estimates are established and three test statistics for assessing the homogeneity of the network influence indices are presented. The usefulness of the adaptive SAR model and its associated network index are illustrated via simulation studies and an empirical investigation of the spillover effects in Chinese mutual fund cash

flows.

Key words and phrases: Network influence; Quasi-maximum likelihood estimation; Spatial autoregressive model; Weighted chi-squared test.

1. INTRODUCTION

In the last three decades, online social network sites (SNS) have been developing rapidly across different disciplines and professions. Accordingly, many SNS, such as Facebook, Twitter and Weibo, possess a large amount of data encompassing both users' personal information and network relationships. These important and valuable types of data have attracted considerable attention from both industry practitioners and academic researchers. For example, Wang et al. (2012) demonstrated that advertising agencies can effectively promote new products through social network sites; Kass-Hout and Alhinnawi (2013) found that social network sites allow researchers to investigate the person-to-person spread of communicable diseases and behaviors; Ozsoylev et al. (2014) employed network information to study the trading behavior of investors and found that central investors earn higher returns; Fracassi (2017) indicated that managers' social networks can affect their corporate policy decisions. The above examples are illustrative of how extensively social networks have been applied in practice.

To understand the network structure, we construct a network with n nodes, and denote $a_{ij} = 1$ if a direct connection leads from node i to node j and $a_{ij} = 0$ otherwise. For the sake of completeness, define $a_{jj} = 0$ for any $1 \leq j \leq n$. Accordingly, the matrix $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ with $i, j = 1, \cdots, n$, describes the network relationships among the n nodes. In social network studies, A is called the adjacency matrix and presents useful information relating any two adjacent nodes (see, e.g., Zhu et al., 2017; Zou et al., 2017). For node i, let Y_i be its associated response variable. To assess the influential power of each node, one needs to understand the relationship among the Y_is via the network structure. Hence, we first consider the spatial autoregressive process below, which has been commonly used for modeling social network information,

$$Y_i = \lambda \sum_{j=1}^{n} w_{ij} Y_j + \varepsilon_i, \tag{1.1}$$

where $\lambda > 0$ is the autocorrelation (or influence) parameter, $w_{ij} = a_{ij} / \sum_{j=1}^{n} a_{ij}$, and ε_i is the random error for $i = 1, \cdots, n$. Some useful references for model (1.1) can be found in Whittle (1954), Ord (1975), LeSage and Pace (2009), and Zhou et al. (2017).

Model (1.1) basically decomposes Y_i into two parts: (i) the total amount

of information allocated to node i from nodes $j \neq i$ in the network, which is $\sum_{j=1}^{n} w_{ij} Y_j$ together with the influence parameter λ; (ii) information from the outside of the network, denoted by ε_i.

Although model (1.1) is widely used in extant literature to characterize the relationship among the Y_is, it is unable to identify influential nodes. The reason is that model (1.1) simply assumes all the nodes have the same influential power measured by the parameter λ. In practice, however, node i can have more (or less) influence than node j for any two connected nodes i and j. Accordingly, the influence parameter can be different across nodes. To this end, let λ_j be the influence measure of node j for $j = 1, \cdots, n$ in the network. Then the information of node i received from node j is $Y_j w_{ij} \lambda_j$. Accordingly, we propose the following model,

$$Y_i = \sum_{j=1}^{n} Y_j \lambda_j w_{ij} + \varepsilon_i. \tag{1.2}$$

This model allows us to identify influential nodes via their associated influence measures λ_js, which is an interesting problem in applications; for example, Anagnostopoulos et al. (2008) stated that "A marketing firm, for example, can use this information to design viral marketing campaigns or give out coupons to influential nodes in the network."

From model (1.2), the influence of Y_j on Y_i is $\lambda_j w_{ij}$. Accordingly, it includes two components: (i) λ_j, which characterizes the influential power of node j; (ii) w_{ij}, which describes the interaction between nodes i and j. When all the λ_is are equal, model (1.2) reduces to the classical spatial autoregressive (SAR) model (1.1) (e.g., see Lee, 2004; LeSage and Pace, 2009). Since model (1.2) is able to characterize the influential power of each node, we refer to it as the adaptive SAR model, and name its associated vector $(\lambda_1, \cdots, \lambda_n)^\top \in \mathbb{R}^n$ the network influence index.

It is worth noting that Dou et al. (2016) proposed the model $Y_i = \lambda_i \sum_{j=1}^n w_{ij} Y_j + \varepsilon_i$, and they also studied influential effects. However, the λ_i in their model measures the magnitude of node i being influenced by its connected nodes. In contrast, λ_j in model (1.2) denotes node j's own influential power, which can affect its connected nodes.

The aim of this paper is to demonstrate the novelty and usefulness of the proposed adaptive SAR model. To this end, we study parameter estimators and their properties in the proposed model without imposing any specific error distribution, and then we make inferences on the influence index and illustrate its usefulness. We find that the adaptive SAR model can play an important role in identifying the most influential nodes, which is a key

problem in social network analysis.

The rest of this paper is organized as follows. Section 2 presents the detailed adaptive SAR model structure, applies the quasi-maximum likelihood approach of Lee (2004) to estimate unknown parameters, and explores asymptotic properties. In addition, Section 2 provides three test statistics, (quasi-likelihood ratio test, quasi-score test and quasi-Wald test) to examine the significance of the adaptive SAR model versus the classical SAR model. This allows us to determine the contribution of the influence index. Monte Carlo studies and an empirical analysis of the Chinese mutual fund market are given in Sections 3 and 4, respectively. A short discussion and some concluding remarks are presented in Section 5. The Appendix presents five useful conditions to establish the theoretical results. The technical materials, additional simulation studies and empirical results are relegated to the Supplementary Material.

2. MODELS AND METHODOLOGY

2.1 Models with Parametrization

In addition to the network effect in model (1.2), the response Y_i can also be affected by node i's own attributes. Accordingly, we extend model

(1.2) as follows,

$$Y_i = \sum_{j=1}^{n} Y_j \lambda_j w_{ij} + X_i^\top \alpha + \varepsilon_i \text{ (i.e., } \mathbb{Y} = W\Lambda\mathbb{Y} + \mathbb{X}\alpha + \mathcal{E}), \tag{2.1}$$

where $X_i = (x_{i1}, \cdots, x_{ip})^\top \in \mathbb{R}^p$ represents the p-dimensional covariates associated with their corresponding attributes, $\alpha = (\alpha_1, \cdots, \alpha_p)^\top$ is the $p \times 1$ unknown regression vector, $\mathbb{Y} = (Y_1, \cdots, Y_n)^\top$, $W = (w_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, $\mathcal{E} = (\varepsilon_1, \cdots, \varepsilon_n)^\top$, $\mathbb{X} = (X_1, \cdots, X_n)^\top$, and $\Lambda = \text{diag}\{\lambda_1, \cdots, \lambda_n\}$ denotes the diagonal matrix with $\lambda_1, \cdots, \lambda_n$ being its diagonals. The error components ε_is of $\mathcal{E}$ are assumed to be independent and identically distributed with mean 0 and finite variance σ^2.

In the adaptive SAR model (2.1), one needs to estimate n parameters of λ and p parameters of α, which is infeasible with only n observations. Note that λ_i measures the node i's influential power, which should be affected by its own attributes. For example, a movie star in the Weibo network often has larger influential power than normal users. That is, the influential power of node i is affected by its vocation. To this end, let $Z_i = (z_{i1}, \cdots, z_{id})^\top \in \mathbb{R}^{d\times 1}$, $z_{i1} \equiv 1$, and $Z_{-1,i} = (z_{i2}, \cdots, z_{id})^\top$ be the $d-1$ possible attributes that may affect the influential power of node i. In addition, we assume that $\mathbb{Z}_{-1} = (Z_{-1,1}, \cdots, Z_{-1,n})^\top \in \mathbb{R}^{n\times(d-1)}$ is of full rank. Then, we parameterize

the network influence index λ_i by $\lambda_i(\beta) = F(Z_i^\top \beta)$, where $F(\cdot)$ is a strictly monotone and known function and $\beta = (\beta_1, \cdots, \beta_d)^\top \in \mathbb{R}^{d\times 1}$ is an unknown influence coefficient vector. Accordingly, $z_{i1} \equiv 1$ is associated with the intercept β_1 for $i = 1, \cdots, n$. If $\beta_2 = \cdots = \beta_d = 0$, then $\lambda_i = F(\beta_1)$, that is, all the λ_is are equal. This implies that the classical SAR model is a special case of the adaptive SAR model. Since Λ is a function of β, we further express (2.1) as

$$\mathbb{Y} = W\Lambda(\beta)\mathbb{Y} + \mathbb{X}\alpha + \mathcal{E}. \tag{2.2}$$

In the above equation, the parameter vector α is associated with the covariate matrix $\mathbb{X}$. Analogous to classical regression models, α can be interpreted as the effect of covariate matrix $\mathbb{X}$on the mean of vector $\{I_n - W\Lambda(\beta)\}\mathbb{Y}$. On the other hand, the vector β is the effect of attributes $\mathbb{Z}$ on influence indices, $\lambda_1, \cdots, \lambda_n$.

To make the proposed model (2.2) practically useful, one needs to specify the link function $F(\cdot)$. One often assumes the influence parameter λ satisfies $|\lambda| < 1$ in the SAR model setting to ensure the invertibility of $I_n - \lambda W$ for any weighting matrix W (see, e.g., LeSage and Pace 2009), where I_n is the $n \times n$ identity matrix. Recently, Zhou et al. (2017) further indicated that non-negative λ could provide more precise interpretation in

social network analysis. This motivates us to consider the following three known link functions, which are often considered in binary regression models: logistic, inverse of the probit, and inverse of the log-log. In fact, the parameter λ in the SAR model can be any value as long as $I_n - \lambda W$ is invertible, as mentioned in Lee (2004). Hence, we adopt the inverse of the canonical link function from the Poisson regression model and propose the exponential link function, which can be larger than 1 in our adaptive SAR model by requiring instead that $I_n - W\Lambda(\beta)$ in (2.2) be invertible.

The four link functions mentioned above can be summarized as follows: LINK I (logistic), $F(Z_i^\top \beta) = e^{Z_i^\top \beta}/(1 + e^{Z_i^\top \beta})$; LINK II (inverse of the probit), $F(Z_i^\top \beta) = \Phi(Z_i^\top \beta)$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution; LINK III (inverse of the log-log), $F(Z_i^\top \beta) = 1 - e^{-e^{Z_i^\top \beta}}$; and LINK IV (exponential), $F(Z_i^\top \beta) = e^{Z_i^\top \beta}$. We next study parameter estimators of Model (2.2) under a given link function.

2.2 Quasi-Maximum Likelihood Estimation

We follow Lee's (2004) approach and employ the quasi-maximum likelihood estimation (QMLE) method to estimate the unknown parameters in model (2.2). Specifically, the estimator is derived from a normal likelihood but the random errors in model (2.2) are not required to be normally

distributed and the corresponding assumptions are stated below equation (2.1).

Define $S(\beta) = I_n - W\Lambda(\beta)$. We then have $\mathcal{E} = S(\beta)\mathbb{Y} - \mathbb{X}\alpha$. Based on the Jacobian transformation, the normal log-likelihood function of (2.2) is

$$\ell(\theta) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\left\{S(\beta)\mathbb{Y} - \mathbb{X}\alpha\right\}^\top \left\{S(\beta)\mathbb{Y} - \mathbb{X}\alpha\right\} + \log\left|\det\{S(\beta)\}\right|,$$

where $\theta = (\alpha^\top, \beta^\top, \sigma^2)^\top$. Define $\mathcal{E}(\alpha, \beta) = S(\beta)\mathbb{Y} - \mathbb{X}\alpha$, which is a function of α and β. It is worth noting that $\mathcal{E}$ is $\mathcal{E}(\alpha, \beta)$ evaluated at the true parameter values of α and β. We then adopt Lee's (2004) concentrated quasi-likelihood approach and estimate the parameters. Specifically, given β, we maximize $\ell(\theta)$ with respect to α and σ^2, which leads to

$$\hat{\alpha}(\beta) = \left(\mathbb{X}^\top\mathbb{X}\right)^{-1}\mathbb{X}^\top S(\beta)\mathbb{Y}, \text{ and}$$

$$\hat{\sigma}^2(\hat{\alpha}(\beta), \beta) = \frac{1}{n}\mathcal{E}(\hat{\alpha}(\beta), \beta)^\top \mathcal{E}(\hat{\alpha}(\beta), \beta) = \frac{1}{n}\mathbb{Y}^\top S(\beta)^\top \mathcal{M}_{\mathbb{X}} S(\beta)\mathbb{Y},$$

where $\mathcal{E}(\hat{\alpha}(\beta), \beta) = \mathcal{M}_{\mathbb{X}} S(\beta)\mathbb{Y}$ and $\mathcal{M}_{\mathbb{X}} = I_n - \mathbb{X}\left(\mathbb{X}^\top\mathbb{X}\right)^{-1}\mathbb{X}^\top$. Accordingly, the resulting concentrated quasi-log-likelihood is

$$\ell_c(\beta) = \ell(\hat{\alpha}(\beta), \beta, \hat{\sigma}^2(\hat{\alpha}(\beta), \beta)) = -\frac{n}{2}\log(2\pi) - \frac{n}{2} - \frac{n}{2}\log\hat{\sigma}^2(\hat{\alpha}(\beta), \beta) + \log\left|\det\{S(\beta)\}\right|.$$

Maximize the above equation with respect to β, which yields the QMLE $\hat{\beta} = \arg\max_\beta \ell_c(\beta)$. We then obtain the QMLEs of α and σ^2, and they are $\hat{\alpha} = \hat{\alpha}(\hat{\beta})$ and $\hat{\sigma}^2 = \hat{\sigma}^2(\hat{\alpha}, \hat{\beta})$. We next introduce notations and equations that will be used in developing the asymptotic distribution of $\hat{\theta} = (\hat{\alpha}^\top, \hat{\beta}^\top, \hat{\sigma}^2)^\top$.

Let $\Lambda_{\beta_k}(\beta) := \partial\Lambda(\beta)/\partial\beta_k = \mathrm{diag}\{z_{1k}F'(Z_1^\top\beta), \cdots, z_{nk}F'(Z_n^\top\beta)\}$ for $k = 1, \cdots, d$. In the following, we use a generic notation $(g_{k_1k_2})_{K_1\times K_2}$ to denote a matrix that has dimension $K_1 \times K_2$ and whose (k_1, k_2)-th element is $g_{k_1k_2}$ for $k_1 = 1, \cdots, K_1$ and $k_2 = 1, \cdots, K_2$. After algebraic simplification, the Fisher information matrix of the quasi-log-likelihood $\ell(\theta)$ is

$$\mathcal{I}_n(\theta) := -n^{-1}\mathrm{E}\left\{\frac{\partial\ell^2(\theta)}{\partial\theta\partial\theta^\top}\right\} = \begin{pmatrix} \sigma^{-2}n^{-1}\mathbb{X}^\top\mathbb{X} & \mathcal{I}_{\alpha\beta,n} & 0_{p\times 1} \\ \mathcal{I}_{\beta\alpha,n} & \mathcal{I}_{\beta\beta,n} & \mathcal{I}_{\beta\sigma^2,n} \\ 0_{1\times p} & \mathcal{I}_{\sigma^2\beta,n} & 2^{-1}\sigma^{-4} \end{pmatrix}, \text{ where} \tag{2.3}$$

$$\mathcal{I}_{\alpha\beta,n} = \frac{1}{n\sigma^2}\left(\mathbb{X}^\top W\Lambda_{\beta_1}(\beta)S^{-1}(\beta)\mathbb{X}\alpha, \ \cdots, \ \mathbb{X}^\top W\Lambda_{\beta_d}(\beta)S^{-1}(\beta)\mathbb{X}\alpha\right),$$

$$\begin{aligned} \mathcal{I}_{\beta\beta,n} \ = \ & n^{-1}\Bigg(\mathrm{tr}\left\{W\Lambda_{\beta_{k_1}}(\beta)S^{-1}(\beta)W\Lambda_{\beta_{k_2}}(\beta)S^{-1}(\beta)\right\} \\ & +\mathrm{tr}\left\{W\Lambda_{\beta_{k_1}}(\beta)S^{-1}(\beta)S^{-1}(\beta)^\top\Lambda_{\beta_{k_2}}(\beta)W^\top\right\} \\ & +\frac{1}{\sigma^2}\alpha^\top\mathbb{X}^\top S^{-1}(\beta)^\top\Lambda_{\beta_{k_1}}(\beta)W^\top W\Lambda_{\beta_{k_2}}(\beta)S^{-1}(\beta)\mathbb{X}\alpha\Bigg)_{d\times d}, \end{aligned}$$

$$\mathcal{I}_{\beta\sigma^2,n} = \frac{1}{n\sigma^2}\left(\mathrm{tr}\left\{W\Lambda_{\beta_1}(\beta)S^{-1}(\beta)\right\},\cdots,\mathrm{tr}\left\{W\Lambda_{\beta_d}(\beta)S^{-1}(\beta)\right\}\right)^\top,$$

$\mathcal{I}_{\beta\alpha,n} = \mathcal{I}_{\alpha\beta,n}^\top$ and $\mathcal{I}_{\sigma^2\beta,n} = \mathcal{I}_{\beta\sigma^2,n}^\top$.

Let $\circ$ be the Hadamard product of matrices, $l_n = (1,\cdots,1)^\top \in \mathbb{R}^{n\times 1}$, and $\mathbb{X}_j = (x_{1j},\cdots,x_{nj})^\top \in \mathbb{R}^n$ for $j = 1,\cdots,p$. Since the random error vector $\mathcal{E} = (\varepsilon_1,\cdots,\varepsilon_n)^\top$ in model (2.2) is not required to be normally distributed, the third order moment $\mu^{(3)} = \mathrm{E}(\varepsilon_i^3)$ and the fourth order moment $\mu^{(4)} = \mathrm{E}(\varepsilon_i^4)$ will be involved in the asymptotic distribution of $\hat{\theta}$. We then denote the matrix $\mathcal{J}_n(\theta,\mu^{(3)},\mu^{(4)})$ as follows:

$$\mathcal{J}_n(\theta,\mu^{(3)},\mu^{(4)}) = \begin{pmatrix} 0_{p\times p} & \mathcal{J}_{\alpha\beta,n} & \frac{\mu^{(3)}\mathbb{X}^\top l_n}{2n\sigma^6} \\ \mathcal{J}_{\beta\alpha,n} & \mathcal{J}_{\beta\beta,n} & \mathcal{J}_{\beta\sigma^2,n} \\ \frac{\mu^{(3)}l_n^\top\mathbb{X}}{2n\sigma^6} & \mathcal{J}_{\sigma^2\beta,n} & \frac{\mu^{(4)}-3\sigma^4}{4\sigma^8} \end{pmatrix}, \text{ where}$$

$$\mathcal{J}_{\alpha\beta,n} = \frac{\mu^{(3)}}{n\sigma^4}\left(\mathrm{tr}\left[\left(\mathbb{X}_j l_n^\top\right)\circ\left\{W\Lambda_{\beta_k}(\beta)S^{-1}(\beta)\right\}\right]\right)_{p\times d},\ \mathcal{J}_{\beta\alpha,n} = {\mathcal{J}_{\alpha\beta,n}}^\top,$$

$$\begin{aligned}\mathcal{J}_{\beta\beta,n} &= \frac{\mu^{(4)}-3\sigma^4}{n\sigma^4}\left(\mathrm{tr}\left[\left\{W\Lambda_{\beta_{k_1}}(\beta)S^{-1}(\beta)\right\}\circ\left\{W\Lambda_{\beta_{k_2}}(\beta)S^{-1}(\beta)\right\}\right]\right)_{d\times d}\\ &\quad+\frac{\mu^{(3)}}{n\sigma^4}\left(\mathrm{tr}\left[\left\{W\Lambda_{\beta_{k_1}}(\beta)S^{-1}(\beta)\mathbb{X}\alpha l_n^\top\right\}\circ\left\{W\Lambda_{\beta_{k_2}}(\beta)S^{-1}(\beta)\right\}\right]\right)_{d\times d}\\ &\quad+\frac{\mu^{(3)}}{n\sigma^4}\left(\mathrm{tr}\left[\left\{W\Lambda_{\beta_{k_2}}(\beta)S^{-1}(\beta)\mathbb{X}\alpha l_n^\top\right\}\circ\left\{W\Lambda_{\beta_{k_1}}(\beta)S^{-1}(\beta)\right\}\right]\right)_{d\times d},\end{aligned}$$

$$\mathcal{J}_{\beta\sigma^2,n}=\frac{\mu^{(4)}-3\sigma^4}{2n\sigma^6}\left(\mathrm{tr}\left\{W\Lambda_{\beta_k}(\beta)S^{-1}(\beta)\right\}\right)_{d\times 1}+\frac{\mu^{(3)}}{2n\sigma^6}\left(l_n^\top W\Lambda_{\beta_k}(\beta)S^{-1}(\beta)\mathbb{X}\alpha\right)_{d\times 1},$$

$\mathcal{J}_{\sigma^2\beta,n}=\mathcal{J}_{\beta\sigma^2,n}{}^\top$, and $l_n=(1,\cdots,1)^\top\in\mathbb{R}^n$. The asymptotic distribution of $\hat{\theta}$ is given in the following theorem.

Theorem 1. *Under Conditions (C1)-(C5) in Appendix, $\sqrt{n}(\hat{\theta}-\theta)$ is asymptotic normal with mean 0 and covariance matrix $\mathcal{I}^{-1}(\theta)+\mathcal{I}^{-1}(\theta)\mathcal{J}(\theta,\mu^{(3)},\mu^{(4)})\mathcal{I}^{-1}(\theta)$, where $\mathcal{I}(\theta)$ and $\mathcal{J}(\theta,\mu^{(3)},\mu^{(4)})$ are stated in Condition (C5) and they are the convergences of matrices $\mathcal{I}_n(\theta)$ and $\mathcal{J}_n(\theta,\mu^{(3)},\mu^{(4)})$, respectively.*

In practice, both $\mathcal{I}(\theta)$ and $\mathcal{J}(\theta,\mu^{(3)},\mu^{(4)})$ are unknown. To make the above theorem practically useful, one needs to find their consistent estimators. Using the fact that $\mathcal{I}_n(\theta)\to\mathcal{I}(\theta)$ and $\mathcal{J}_n(\theta,\mu^{(3)},\mu^{(4)})\to\mathcal{J}(\theta,\mu^{(3)},\mu^{(4)})$, we can show that the asymptotic covariance matrix $\mathcal{I}^{-1}(\theta)+\mathcal{I}^{-1}(\theta)\mathcal{J}(\theta,\mu^{(3)},\mu^{(4)})\mathcal{I}^{-1}(\theta)$ can be consistently estimated by $\mathcal{I}_n^{-1}(\hat{\theta})+\mathcal{I}_n^{-1}(\hat{\theta})\mathcal{J}_n(\hat{\theta},\hat{\mu}^{(3)},\hat{\mu}^{(4)})\mathcal{I}_n^{-1}(\hat{\theta})$, where $\hat{\mu}^{(s)}=n^{-1}\sum_{i=1}^n\hat{\varepsilon}_i^s$ for $s=3,4$ and $(\hat{\varepsilon}_1,\cdots,\hat{\varepsilon}_n)^\top=\mathcal{E}(\hat{\alpha},\hat{\beta})$.

2.3 Homogeneous Influence Test

After obtaining the parameter estimator $\hat{\theta}$ and its asymptotic property, we next assess the homogeneity of the influence in (2.2) by testing the effect of different influence indices λ_i. To this end, we consider the following null and alternative hypotheses:

$$\mathrm{H}_{0,\lambda} : \lambda_1 = \cdots = \lambda_n = \lambda \ \text{ v.s. } \ \mathrm{H}_{1,\lambda} : \lambda_{i_1} \neq \lambda_{i_2} \text{ for some } i_1 \neq i_2.$$

According to the definition $\lambda_i(\beta) = F(Z_i^\top \beta)$ for $i = 1, \cdots, n$, the above hypotheses are equivalent to

$$\mathrm{H}_0 : \beta_2 = \cdots = \beta_d = 0 \ \text{ v.s. } \ \mathrm{H}_1 : \ \text{at least one of } \beta_2, \cdots, \beta_d \text{ is not zero}, \tag{2.4}$$

under the assumptions that the link function $F(\cdot)$ is strictly monotone and the covariate matrix $\mathbb{Z}_{-1}$ is of full rank. If one does not reject the null hypothesis, then the SAR model and its associated estimators and properties can be considered (e.g., see Lee, 2004).

Within the maximum likelihood framework, there are three commonly used tests for making inferences about β. They are likelihood ratio test,

Wald test, and score (i.e., Lagrange multiplier) test. This motivates us to employ them to test (2.4). Since we consider the quasi-likelihood function and QMLE, we name them quasi-likelihood ratio test, quasi-Wald test, and quasi-score test. We first consider the quasi-likelihood ratio test. Given $\hat{\theta} = (\hat{\alpha}^\top, \hat{\beta}^\top, \hat{\sigma}^2)^\top$, we obtain the estimated quasi-log-likelihood function $\ell(\hat{\theta}) = \ell(\hat{\alpha}, \hat{\beta}, \hat{\sigma}^2)$. Under the null hypothesis of $\mathrm{H}_0 : \beta_2 = \cdots = \beta_d = 0$, we can also obtain the constrained QMLE, $\hat{\theta}^{(r)}$, and its associated quasi-log-likelihood function $\ell(\hat{\theta}^{(r)})$. Accordingly, the quasi-likelihood ratio test statistic is

$$T_{lr} = -2\left\{\ell(\hat{\theta}^{(r)}) - \ell(\hat{\theta})\right\}.$$

To show the theoretical property of T_{lr}, we introduce additional notations and equations as below. Let

$$\Delta_c = \begin{pmatrix} I_p & 0_{p\times 1} & 0_{p\times(d-1)} & 0_{p\times 1} \\ 0_{1\times p} & 1 & 0_{1\times(d-1)} & 0 \\ 0_{1\times p} & 0 & 0_{1\times(d-1)} & 1 \end{pmatrix} \in \mathbb{R}^{(p+2)\times(p+d+1)},$$

where $0_{K_1\times K_2}$ denotes a $K_1 \times K_2$ matrix with all the elements being zeros.

Let $\mathcal{I}_{11}(\theta) = \Delta_c \mathcal{I}(\theta) \Delta_c^\top$ and

$$\mathcal{I}_{11}^{-1}(\theta) = \left(\Delta_c \mathcal{I}(\theta) \Delta_c^\top\right)^{-1} =: \begin{pmatrix} \iota_{11}(\theta) & \iota_{12}(\theta) \\ \iota_{21}(\theta) & \iota_{22}(\theta) \end{pmatrix},$$

where $\iota_{11}(\theta) \in \mathbb{R}^{(p+1)\times(p+1)}$, $\iota_{12}(\theta) \in \mathbb{R}^{(p+1)\times 1}$, $\iota_{21}(\theta) \in \mathbb{R}^{1\times(p+1)}$ and $\iota_{22}(\theta) \in \mathbb{R}^{1\times 1}$. In addition, let

$$\mathcal{I}_1(\theta) = \begin{pmatrix} \iota_{11}(\theta) & 0_{(p+1)\times(d-1)} & \iota_{12}(\theta) \\ 0_{(d-1)\times(p+1)} & 0_{(d-1)\times(d-1)} & 0_{(d-1)\times 1} \\ \iota_{21}(\theta) & 0_{1\times(d-1)} & \iota_{22}(\theta) \end{pmatrix}, \tag{2.5}$$

and denote $\mathcal{K}(\theta, \mu^{(3)}, \mu^{(4)}) = \mathcal{I}(\theta) + \mathcal{J}(\theta, \mu^{(3)}, \mu^{(4)})$. Then the asymptotic distribution of T_{lr} is given below.

Theorem 2. *Assume Conditions (C1)-(C5) in Appendix hold. Under the null hypothesis* H_0*, the quasi-likelihood ratio test statistic* T_{lr} *is asymptotically distributed as* $\sum_{l=1}^{p+d+1} \lambda_l(\theta, \mu^{(3)}, \mu^{(4)}) \chi_l^2(1)$ *as* $n \to \infty$*, where* $\lambda_l(\theta, \mu^{(3)}, \mu^{(4)})$ *is the* l*-th largest eigenvalue of the matrix* $\mathcal{K}^{1/2}(\theta, \mu^{(3)}, \mu^{(4)})\{\mathcal{I}^{-1}(\theta) - \mathcal{I}_1(\theta)\}\mathcal{K}^{1/2}(\theta, \mu^{(3)}, \mu^{(4)})$*, and* $\chi_l^2(1)$ *are independent chi-squared random variables with degree of freedom 1 for* $l = 1, \cdots, (p+d+1)$*. Furthermore, under the normal assumption of* $\mathcal{E}$*,* T_{lr} *is asymptotically* $\chi^2(d-1)$*.*

In practice, $\lambda_l(\theta, \mu^{(3)}, \mu^{(4)})$ is unknown, and it can be estimated by $\lambda_{n,l}(\hat{\theta}^{(r)}, \hat{\mu}^{(3,r)}, \hat{\mu}^{(4,r)})$, where $\lambda_{n,l}(\hat{\theta}^{(r)}, \hat{\mu}^{(3,r)}, \hat{\mu}^{(4,r)})$ is the l-th largest eigenvalue of the $(p+d+1) \times (p+d+1)$ matrix $\mathcal{K}_n^{1/2}(\hat{\theta}^{(r)}, \hat{\mu}^{(3,r)}, \hat{\mu}^{(4,r)})\{\mathcal{I}_n^{-1}(\hat{\theta}^{(r)}) - \mathcal{I}_{n,1}(\hat{\theta}^{(r)})\}\mathcal{K}_n^{1/2}(\hat{\theta}^{(r)}, \hat{\mu}^{(3,r)}, \hat{\mu}^{(4,r)})$. Note first that $\mathcal{K}_n(\hat{\theta}^{(r)}, \hat{\mu}^{(3,r)}, \hat{\mu}^{(4,r)}) = \mathcal{I}_n(\hat{\theta}^{(r)}) + \mathcal{J}_n(\hat{\theta}^{(r)}, \hat{\mu}^{(3,r)}, \hat{\mu}^{(4,r)})$ is a consistent estimator of $\mathcal{K}^{1/2}(\theta, \mu^{(3)}, \mu^{(4)})$, second that $\mathcal{I}_{n,1}(\hat{\theta}^{(r)})$ is a consistent estimator of $\mathcal{I}_1(\theta)$, $\hat{\theta}^{(r)} = (\hat{\alpha}^{(r)\top}, \hat{\beta}^{(r)\top}, \{\hat{\sigma}^{(r)}\}^2)^\top$, and finally that $\hat{\mu}^{(s,r)} = n^{-1}\sum_{i=1}^n \{\hat{\varepsilon}_i^{(r)}\}^s$ for $s = 3, 4$ with $(\hat{\varepsilon}_1^{(r)}, \cdots, \hat{\varepsilon}_n^{(r)})^\top = \mathcal{E}(\hat{\alpha}^{(r)}, \hat{\beta}^{(r)})$.

An alternative approach for testing H_0 is the quasi-Wald test. Let

$$\Delta = \begin{pmatrix} 0_{(d-1)\times p} & 0_{(d-1)\times 1} & I_{d-1} & 0_{(d-1)\times 1} \end{pmatrix} \in \mathbb{R}^{(d-1)\times(p+d+1)}.$$

Then, the quasi-Wald test statistic for testing H_0 can be constructed as follows:

$$T_w = (\Delta\hat{\theta})^\top \left[\Delta \left\{ n^{-1}\mathcal{I}_n^{-1}(\hat{\theta})\mathcal{K}_n(\hat{\theta}, \hat{\mu}^{(3)}, \hat{\mu}^{(4)})\mathcal{I}_n^{-1}(\hat{\theta}) \right\} \Delta^\top \right]^{-1} \Delta\hat{\theta},$$

and its asymptotic distribution is below.

Corollary 1. *Assume Conditions (C1)-(C5) in Appendix hold. Then, under the null hypothesis* H_0*, we have* $T_w \xrightarrow{d} \chi^2(d-1)$ *as* $n \to \infty$.

We lastly consider the quasi-score test. The advantage of this test is that we only need to obtain the constrained estimator $\hat{\theta}^{(r)}$ under the null hypothesis of $\mathrm{H}_0 : \beta_2 = \cdots = \beta_d = 0$. Specifically, the quasi-score test can be constructed by

$$T_s = n^{-1} \left\{ \frac{\partial \ell(\hat{\theta}^{(r)})}{\partial \theta} \right\}^{\top} \mathcal{I}_n^{-1}(\hat{\theta}^{(r)}) \frac{\partial \ell(\hat{\theta}^{(r)})}{\partial \theta},$$

where the detailed expression of $\partial \ell(\theta)/\partial \theta$ can be found in (S.6) of the Supplementary Material. The asymptotic distribution of T_s is given in the following corollary.

Corollary 2. *Assume Conditions (C1)-(C5) in Appendix hold. Under the null hypothesis* H_0*, the test statistic* $T_s = T_{lr} + o_{\mathrm{P}}(1)$ *as* $n \to \infty$.

The above corollary indicates that the quasi-score test and the quasi-likelihood ratio test are asymptotically equivalent with weighted chi-squared distribution. Based on our understanding, such an asymptotic result obtained without imposing any specific error distribution has not been rigorously discussed in the SAR literature. It is also known that under the normal assumption, the quasi-likelihood ratio test, the quasi-Wald test and the quasi-score test are all asymptotically equivalent as $n \to \infty$, whereas

this may not be true under the non-normal assumption. A good review paper about these three tests can be found in Rao (2005). Since these three tests can be different in terms of finite sample performance, we evaluate them in the following simulation studies.

3. SIMULATION STUDIES

To demonstrate the finite sample performance of our proposed adaptive SAR model, we conduct simulation studies with various settings. Let the diagonals of the adjacency matrix A be zeros and the off-diagonals of A be independent and identically generated from the Bernoulli distribution with probability $5/n$. Then, the weighting matrix is set to be $W = (w_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ with $w_{ij} = a_{ij}/\sum_{j=1}^{n} a_{ij}$ for $i, j = 1, \cdots, n$. Consider the 2×1 covariate vector $X_i = (x_{i1}, x_{i2})^\top$ with $x_{i1} \equiv 1$ and x_{i2} being independent and identically generated from the standard normal distribution $N(0, 1)$, and their corresponding regression parameters are $\alpha = (\alpha_1, \alpha_2)^\top = (2, 1)^\top$. In addition, consider the 3×1 influential covariates $Z_i = (z_{i1}, z_{i2}, z_{i3})^\top$, where $z_{i1} \equiv 1$, and z_{i2} and z_{i3} are independent and identically generated from the uniform distribution $U(-0.25, 0.25)$ and the normal distribution $N(0, 0.2^2)$, respectively. Six sets of parameters $\beta = (\beta_1, \beta_2, \beta_3)^\top \in \mathbb{R}^3$ are

associated with the influential covariates Z_i: $(\beta_1, \beta_2, \beta_3) = (-1, 5\varrho, -2\varrho)$, where $\varrho = 0.0$, 0.2, 0.4, 0.6, 0.8 and 1.0 measure the signal strengths of the covariates, and $\varrho = 0.0$ corresponds to the classical SAR model. As a result, the network influence matrix is $\Lambda = \text{diag}\{F(Z_1^\top \beta), \cdots, F(Z_n^\top \beta)\}$, where the link functions $F(\cdot)$s are LINKs I - IV presented in Section 2.1. It is worth noting that the above model settings satisfy our technical Conditions (C1)-(C5) in Appendix. Finally, the response vector $\mathbb{Y}$ is generated from model (2.2) with the above setting, and its associated random error terms ε_i $(i = 1, \cdots, n)$ are independent and identically generated from the four distributions: the normal distribution $N(0, \sigma^2)$ and $\sigma\zeta$, where ζ follows a mixture normal distribution $0.9N(0, 5/9) + 0.1N(0, 5)$, a standardized t_3 distribution, and a standardized exponential distribution, respectively, with $\sigma^2 = 1$. This allows us to examine the robustness of parameter estimates with respect to the error distributions.

For each setting, we consider three sample sizes: n=200, 500 and 1,000. In addition, all simulations are conducted via 1,000 realizations. To assess the performance of parameter estimators, we define $\hat{\theta}^{(k)} = (\hat{\alpha}_1^{(k)}, \hat{\alpha}_2^{(k)}, \hat{\beta}_1^{(k)}, \hat{\beta}_2^{(k)}, \hat{\beta}_3^{(k)}, \hat{\sigma}^{2(k)})^\top \in \mathbb{R}^6$ as the vector estimate of θ obtained via the QMLE approach in the k-th realization. For each component of θ, say θ_j, the aver-

aged bias of $\hat{\theta}_j^{(k)}$, $k = 1, \cdots, 1{,}000$, is BIAS $= 1000^{-1} \sum_k (\hat{\theta}_j^{(k)} - \theta_j)$, and the standard deviation of $\hat{\theta}_j^{(k)}$ is SD $= \left\{1000^{-1} \sum_k (\hat{\theta}_j^{(k)} - 1000^{-1} \sum_k \hat{\theta}_j^{(k)})^2\right\}^{1/2}$. Thus, the root mean squared error is RMSE $= \sqrt{\text{SD}^2 + \text{BIAS}^2}$.

For normal random errors, Table S.1 in the Supplementary Material reports the BIAS, SD and RMSE of the QMLEs via 1,000 realizations across the four link functions with three sample sizes. To save space, we only present the results for the setting with coefficients $(\beta_1, \beta_2, \beta_3) = (-1, 5, -2)$, since the setting with coefficients $(\beta_1, \beta_2, \beta_3) = (-1, 5\varrho, -2\varrho)$ yields similar findings for $\varrho = 0.0$, 0.2, 0.4, 0.6 and 0.8. According to Table S.1, we find that the absolute values of BIAS and SD generally become smaller for all parameter estimates and for all four link functions when n gets large. It is not surprising that RMSE shows the same pattern.

We further study the performance of QMLE when the random errors are mixture normal, standardized t_3, and standardized exponential. Tables S.2-S.4 in the Supplementary Material indicate that the resulting estimators yield qualitatively similar conclusions to those obtained from the Gaussian errors. Hence, our estimates still exhibit nice properties under these three non-normal cases. The above findings support our theoretical result that the QMLEs are consistent and asymptotically normal.

We next assess the finite sample performance of the quasi-likelihood ratio test, the quasi-Wald test and the quasi-score test. It is worth noting that both the quasi-likelihood ratio test statistic T_{lr} and the quasi-score test statistic T_s are asymptotically weighted chi-squared distributed with the weights $\lambda_l(\theta, \mu^{(3)}, \mu^{(4)})$ under the null hypothesis. In order to conduct these two tests, we independently and identically generate $\{\chi^2_{l,m} : l = 1, \cdots, (p+d+1), \text{ and } m = 1, \cdots, 10,000\}$ from the chi-squared distribution with 1 degree of freedom. Let T be either of these two test statistics T_{lr} or T_s. We can compute the p-values of the quasi-likelihood ratio test and the quasi-score test approximately by $p\text{-value}_1 = 10000^{-1} \sum_m I\{T > \sum_{l=1}^{p+d+1} \lambda_l(\theta, \mu^{(3)}, \mu^{(4)}) \chi^2_{l,m}\}$ and $p\text{-value}_2 = 10000^{-1} \sum_m I\{T > \sum_{l=1}^{p+d+1} \lambda_{n,l}(\hat{\theta}^{(r)}, \hat{\mu}^{(3,r)}, \hat{\mu}^{(4,r)}) \chi^2_{l,m}\}$, respectively, where $\lambda_{n,l}(\hat{\theta}^{(r)}, \hat{\mu}^{(3,r)}, \hat{\mu}^{(4,r)})$ is a consistent estimator of $\lambda_l(\theta, \mu^{(3)}, \mu^{(4)})$ under the null hypothesis stated below Theorem 2, and $I\{\cdot\}$ is the indicator function. Based on our simulation studies, we find that $p\text{-value}_1$ and $p\text{-value}_2$ yield very similar results under the null hypothesis. In addition, $p\text{-value}_1$ is not applicable since θ, $\mu^{(3)}$ and $\mu^{(4)}$ are unknown. As a result, we use $p\text{-value}_2$ to assess the performance of the quasi-likelihood ratio test and the quasi-score test in the rest of studies.

We evaluate the empirical sizes of the quasi-likelihood ratio test, the

quasi-Wald test and the quasi-score test with the significance levels ranging from 0.01 to 0.30 and examine their empirical powers with the significance level 0.05. For the exponential link function under the mixture normal distribution, Figures 1 and 2 depict sizes and powers, respectively, when n = 200, 500, and 1,000. The testing results of the other three link functions under the mixture normal, as well as under the other three random error distributions, yield similar findings, so we do not present them here. The empirical size and power are the percentages of rejections under H_0 and H_1, respectively, via the hypothesis test (2.4) with 1,000 realizations. Specifically, the empirical size is the percentage of rejections under the setting of $(\beta_1, \beta_2, \beta_3) = (-1, 0, 0)$, while the empirical power is the percentage of rejections under the settings of $(\beta_1, \beta_2, \beta_3) = (-1, 5\varrho, -2\varrho)$, where the signal strength $\varrho > 0$.

From Figures 1 and 2, we obtain four interesting findings. The first is that the empirical sizes of the three tests are almost identical to the predetermined significance levels as n=1,000. The second is that the empirical powers of the three tests tend to 100% when the sample size n or the signal strength ϱ gets larger. These two findings indicate that the three homogeneous influence tests perform well when n is large. The third is that the

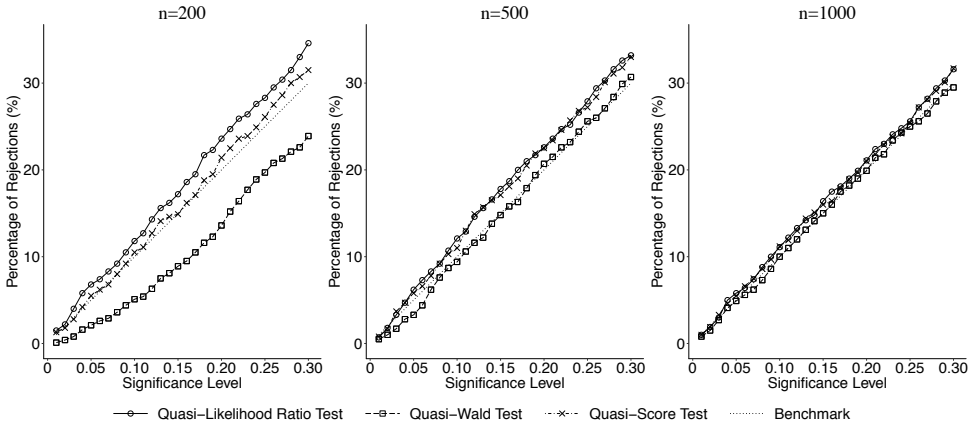

Figure 1: The empirical sizes of the three homogeneous influence tests for the significance levels ranging from 0.01 to 0.30 under the setting of the exponential link function. The benchmark represents the ideal case when the percentage of rejections from 1,000 realizations is equal to the significance level. The independent and identically distributed random errors are simulated from $\sigma\zeta$, where ζ follows a mixture normal distribution $0.9N(0, 5/9) + 0.1N(0, 5)$.

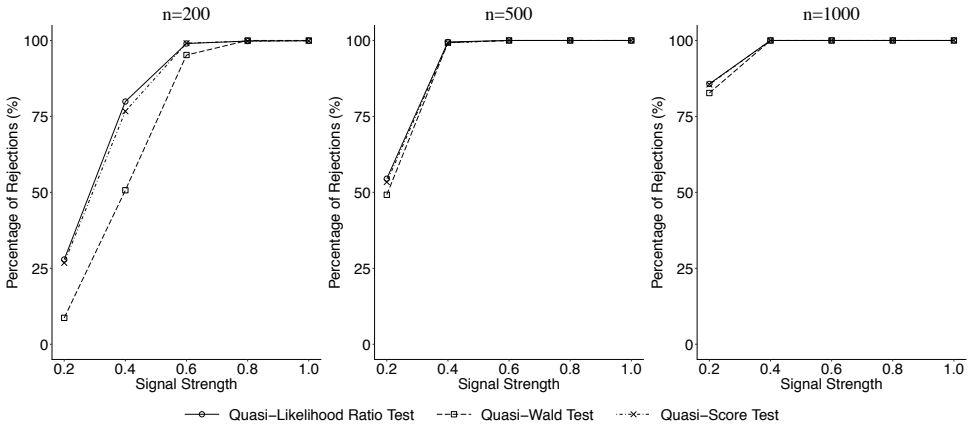

Figure 2: The empirical powers of the three homogeneous influence tests at a nominal level of 0.05 under the exponential link function with 1,000 realizations. The signal strengths $\varrho = 0.0$, 0.2, 0.4, 0.6, 0.8 and 1.0, which correspond to the settings of $(\beta_1, \beta_2, \beta_3) = (-1, 5\varrho, -2\varrho)$, respectively. The independent and identically distributed random errors are simulated from $\sigma\zeta$, where ζ follows a mixture normal distribution $0.9N(0, 5/9)+0.1N(0, 5)$.

quasi-likelihood ratio test is sometimes oversized (anticonservative) and the quasi-Wald test is undersized (conservative) when n is not large enough (see Figure 1). In contrast, the quasi-score test enables us to control the size reasonably well, especially at the significance level 0.05. Lastly, Figure 2 shows that the powers of the quasi-score and quasi-likelihood ratio tests are very close to each other for different ns and ϱs. However, the quasi-Wald test is not powerful when the signal strength ϱ is small. Based on the above four findings, we conclude that the quasi-score test performs best at the significance level 0.05. In addition, the calculation of the quasi-score test only involves the constrained QMLE under H_0, which is easier to compute than the other two tests. Consequently, we recommend using the quasi-score test to compare between the SAR and adaptive SAR models in practice, particularly when the sample size is not large enough.

4. REAL DATA ANALYSIS

4.1 Network and Covariates

To demonstrate the usefulness of the proposed adaptive SAR model, we present a real example of the spillover effect using Chinese mutual fund cash flows, where this effect is crucial for both fund managers and general

investors (Spitz, 1970; Sirri and Tufano, 1998; Zheng, 1999; Nanda et al., 2004). For example, for fund managers, these cash flow are usually compensated from the management fees that are charged as a fixed proportion of the total net assets under management. To explore the mechanism of cash flows, existent studies (see e.g., Spitz, 1970; Sirri and Tufano, 1998; Zheng, 1999; Nanda et al., 2004; Brown and Wu, 2016) address the characteristics of the mutual funds themselves, and do not consider the influence of mutual funds on cash flows from the network perspective, i.e., the spillover effect. The proposed adaptive SAR model enables us to discuss this mechanism of influence from one mutual fund to another via cash flows by combining the characteristics of fund itself and network structure among mutual funds together.

To this end, we collect data on actively managed open-ended mutual funds in the second semiannual period of 2015 from the WIND financial database, which is one of the most authoritative databases regarding Chinese financial market. After removing funds in existence for less than one year, there are totally 420 mutual funds in our study. To assess the network influence of mutual funds, we construct the network as follows. Define the funds i and j being connected (i.e., $a_{ij} = a_{ji} = 1$) if these two funds allo-

cate at least 2.5% of their portfolios to the same stock (see Pareek, 2012). Otherwise, we consider these two funds are disconnected, i.e., $a_{ij} = 0$. For the sake of a robustness check, the allocations of funds at 1% and 5% are also considered, and they yield similar results.

We next define the response variable, cash flow, as follows. The cash flow of fund i at time t, C_{it}, is calculated from the equation (Zheng, 1999; Nanda et al., 2004)

$$C_{it} = \frac{TA_{it} - TA_{i,t-1}(1 + r_{it})}{TA_{it}},$$

where TA_{it} is the total net asset of fund i at time t and r_{it} is the fund return at time t. To avoid the impact of outliers induced by cash flow, we remove the top 2.5% observations, i.e., 11 funds, from the data set so there remain 409 observations in our study, and the resulting network density for these 409 funds is 20.9%. Removing the top percentage of observations is not uncommon in finance applications; for instance, Choi et al. (2016) proposed removing the top 2.5% mutual funds by cash flow and Li and Schürhoff (2019) suggested eliminating the top percentage of observations in studying financial networks. In addition, after removing those observations,

the distribution of the remaining cash flow is not heavy-tailed. Thus, the moment assumption in Condition (C1) can be satisfied.

In the spirit of the pioneering work of Spitz (1970), we include four control variables as X covariates to account for their effect on cash flow. (i) Size: the logarithm of the total net asset of fund i at time $t-1$; (ii) Age: the logarithm of the age of fund i at the end of $t-1$; (iii) Return: the raw return of fund i at time $t-1$; (iv) Alpha: the risk-adjusted return of fund i measured by the intercept of Carhart's (1997) four-factor model. To quantify the influential power on the spillover effect on cash flow, we include three variables as Z covariates. (1) Size defined above; (2) Volatility: the standard deviation of the weekly returns of fund i at time $t-1$; (3) Degree: the number of funds connected to fund i. It seems natural that both volatility and size can be influential. We also include the degree in Z covariates. This is motivated by the empirical work of Ozsoylev et al. (2014), who found that the central investor not only performs better, but also yields larger impact on its neighbor investors. Finally, both X and Z covariates have been standardized to have zero mean and unit standard deviation in our study.

4.2 Empirical Results

We fit the data with the proposed adaptive SAR model under four different link functions: exponential, logistic, inverse of the probit and inverse of the log-log link. Their corresponding quasi-loglikelihood values evaluated at their associated QMLEs are -460.731, -463.549, -463.674, and -463.549. Motivated by Vuong (1989), we apply the exponential link function in the rest of study since it has the largest estimated quasi-loglikelihood value. Based on this link function, Table 1 reports the resulting parameter estimators and their associated standard errors and t-statistics as well as the p-values of the three homogeneous influence tests.

Table 1: The regression results of the adaptive SAR model with the exponential link function.

		Estimate	Standard-Error	t-statistic	p-value
X	Intercept	-0.1584	0.0111	-14.3178	0.0000
	Size	0.0180	0.0083	2.1717	0.0299
	Age	0.0101	0.0084	1.2053	0.2281
	Return	0.0633	0.0083	7.5817	0.0000
	Alpha	0.0107	0.0094	1.1371	0.2555
Z	Intercept	-15.2207	8.7928	-1.7310	0.0835
	Degree	4.4926	2.1233	2.1158	0.0344
	Size	0.6434	0.7159	0.8987	0.3688
	Volatility	-5.4266	2.7665	-1.9616	0.0498
	σ^2	0.0233	0.0030	7.8225	0.0000

For the X covariates, we find that the cash flow after adjusting for influential effects is positively and significantly related to the past size and

raw return at the 5% significance level. For example, $\hat{\alpha}_{\text{Size}} = 0.0180$ on Table 3 implies that fund i's size has a significantly positive effect on the corresponding response Y_i (cash flow) after removing the influence of other connected cash flows. As for fund age and alpha, their coefficients are positive but not significant. The above findings are consistent with existing research (see, e.g., Brown et al., 1996; Sirri and Tufano, 1998; Zheng, 1999). This implies that investors tend to invest in big funds. In addition, investors pay more attention on the raw return than the risk adjusted return since the former one is more easily observed.

For Z covariates, we employ the quasi-score test to assess the influential effect. The resulting p-value is 0.019, which indicates that the influential power of the spillover on cash flow among mutual funds indeed depends on funds' influential characteristics. For more details, Table 1 shows three interesting findings. Firstly, the influential power is positively and significantly related to degree at the 5% significance level. Specifically, $\hat{\beta}_{\text{Degree}} = 4.4926$ on Table 3 indicates that there is a significantly positive effect of fund i's degree on its influential power λ_i. This finding is not surprising since more connections yield bigger influential power after controlling the size and volatility. It is also consistent with the results of

Ozsoylev and Walden (2011) and Sirri and Tufano (1998). Secondly, the coefficient of size is positive, but not significant. It means that influential powers may not strongly depend on funds sizes. Lastly, the coefficient of volatility is negative and significant at the 5% significance level. This finding is consistent with the intuition that a stable fund would have a larger impact on other funds.

To further illustrate the usefulness of the adaptive SAR model, we compute the estimated influence index $\hat{\lambda}_i = \exp(Z_i^\top \hat{\beta})$ for $i = 1, \cdots, n$. We then sort the $\hat{\lambda}_i$s and obtain $\hat{\lambda}_{(1)} \geq \cdots \geq \hat{\lambda}_{(n)}$. Figure 3 depicts the sorted influence indices. We next conduct k-means clustering analysis based on these sorted $\hat{\lambda}_i$s via the R package `NbClust`. Accordingly, the best number of clusters is 4, as shown in Figure 3. Cluster I only consists of the mutual fund with the largest influence index. Cluster II consists of the mutual funds with the second and third largest influence indices. Cluster III consists of the mutual funds with the fourth, fifth, sixth, seventh, and eighth largest influence indices. The other mutual funds, whose influence indices are all close to zeros, are categorized into Cluster IV.

To visualize the influential power, Figure 4 depicts the four clusters in the network of 409 mutual funds. Each node in Figure 4 is a mutual fund,

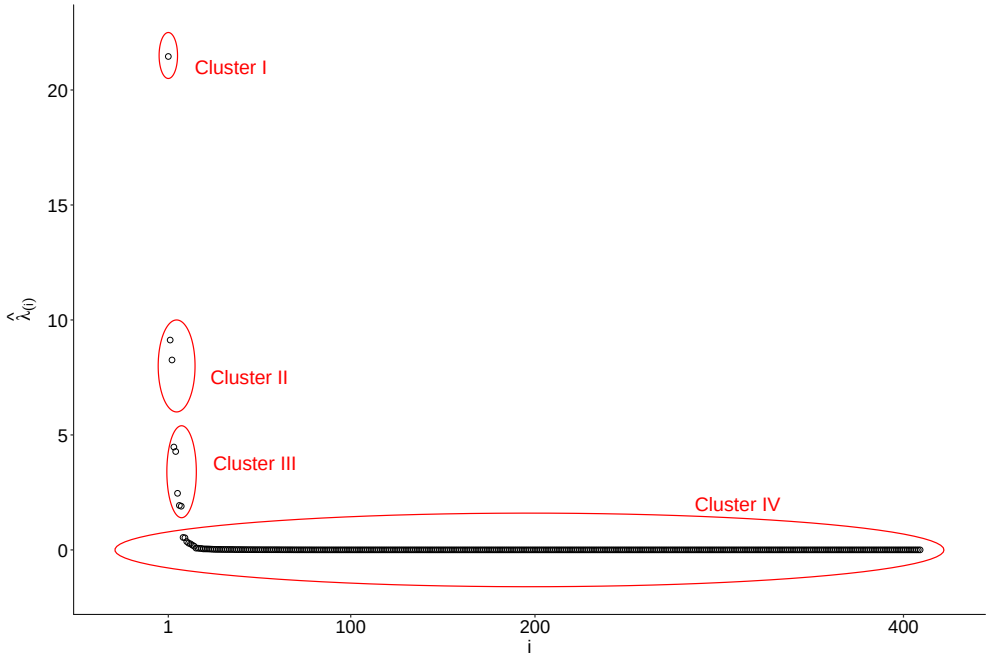

Figure 3: The sorted influence indices $(\hat{\lambda}_{(i)})$ of the $i = 1, \cdots, 409$ mutual funds.

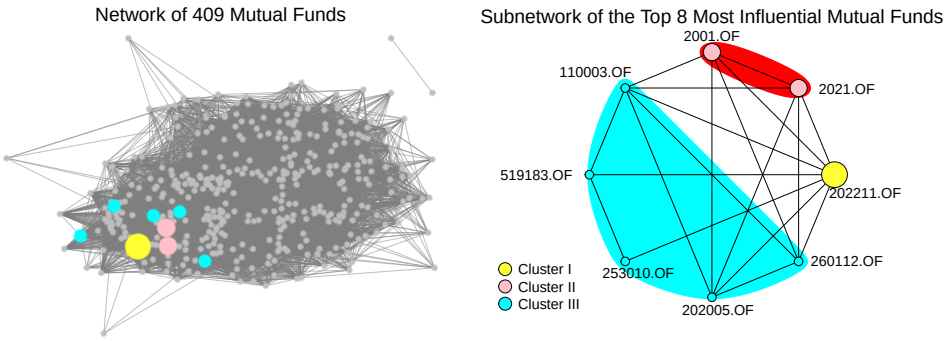

Figure 4: The network of 409 mutual funds and the subnetwork of the top 8 most influential mutual funds with their associated codes.

and we configure node sizes from large to small to represent Clusters I – IV, respectively. Specifically, the left panel of Figure 4 reveals the whole

network structure of the 409 mutual funds and the top 8 most influential mutual funds in Clusters I – III are marked in colors. The detailed subnetwork structure among these eight most influential mutual funds is presented on the right panel of Figure 4. Note that the location of each node in the left panel is constructed based on the number of connections of each node (i.e., the degree of each node). As a result, the greater connections a node has, the closer to the center of the network. However, none of the top eight most influential mutual funds is located in the center. This indicates that a larger degree does not necessarily lead to greater influence. This is because volatility also plays a significant role in constructing the influence index. For the sake of illustration, we present the eight largest influence indices along with their two significant covariates, degree and volatility, in Table S.9 of the Supplementary Material. It shows that although the second, third, fourth and seventh influential funds have more connections than the most influential fund, their volatilities are higher. Accordingly, the fund with the largest influence index does not have the highest number of connections. It is also of interest to note that the right panel of Figure 4 indicates that the fund 202211.OF with the largest influential power is connected to the other top seven influential funds. We also observe that these

top eight influential funds are almost all connected to each other within the network constructed by the 409 Chinese mutual funds. In sum, we have used the adaptive SAR model to effectively identify influential funds with valuable findings.

5. CONCLUDING REMARKS

In this article, we propose the adaptive SAR model and then introduce an influence index for identifying influential nodes in a large network. In addition, we obtain the asymptotic properties of parameter estimates, which allow us to make inferences on the network influence index. The usefulness of the adaptive SAR model and its associated network influence index are demonstrated via Monte Carlo studies and an application from the Chinese mutual fund market. We believe empirical finance researchers can apply the proposed model in order to investigate other possible factors (e.g., centrality) that can determine the influential power of individual mutual funds.

In practical applications, one usually considers positive influence parameters (e.g., Zhou et al., 2017). However, using the fact that $F(Z_i^\top \beta) \in (0,1)$ for LINKs I-III, the transformation $G(Z_i^\top \beta) = 2F(Z_i^\top \beta) - 1$, can lead to $\lambda_i(\beta) \in (-1,1)$ if we specify $\lambda_i(\beta) = G(Z_i^\top \beta)$ in model (2.2). Thus,

one can assume a negative influence index if it is needed to broaden the application of the adaptive SAR model. We next identify four avenues for future research. The first avenue is employing a non-parametric approach to constructing network influence indices. The second avenue is using the screening or regularization method to obtain the sparse solution for constructing n influence indices λ_i (e.g., see Zhu et al. 2019a) or to develop the test statistic for testing a subset of λ_is being equal. The third avenue is proposing a computationally feasible estimation approach (e.g., the least squares method in Huang et al. 2019 and Zhu et al. 2019b), to overcome the computational challenge of QMLE under large scale networks (see numerical illustrations in Section S.4 of the Supplementary Material). The fourth avenue is motivated by an anonymous referee's comment, which extends the adaptive SAR model (1.2) to $Y_i = \sum_{j=1}^{n} \lambda_{ij} Y_j + \varepsilon_i$ so that the closeness between node i and node j can be characterized via the influence parameter λ_{ij}. We believe that these efforts would further increase the application of the adaptive SAR model.

Appendix

This Appendix introduces five useful conditions. As defined in details

in Section S.1 of the Supplementary Material, $\|\cdot\|_s$ denotes the vector s-norm or the matrix s-norm for $1 \leq s \leq \infty$ and $|G|_\infty = \|\mathrm{vec}(G)\|_\infty$ denotes the element-wise ℓ_∞ norm for any generic matrix G. The discussions of the following conditions are presented in Section S.2 of the Supplementary Material.

(C1) Assume that the random errors ε_i are independent and identically distributed with mean 0, and there exists some $\eta > 0$ such that $\mathrm{E}|\varepsilon_i|^{4+\eta} < \infty$.

(C2) Assume $\sup_{n\geq 1} \|W\|_1 < \infty$, $\sup_{n\geq 1} \|W\|_\infty < \infty$ and $\sup_{n\geq 1} |\mathbb{X}|_\infty < \infty$.

(C3) Assume that $S(\beta) = I_n - W\Lambda(\beta)$ is nonsingular uniformly over β in a compact parameter space $\mathcal{B}$ and the true parameter β is in the interior of $\mathcal{B}$. In addition, assume $\sup_{\beta\in\mathcal{B}} \sup_{n\geq 1} \|S^{-1}(\beta)\|_1 < \infty$ and $\sup_{\beta\in\mathcal{B}} \sup_{n\geq 1} \|S^{-1}(\beta)\|_\infty < \infty$.

(C4) Assume, for the true parameter β,

$$\sup_{n\geq 1} \max_{1\leq i\leq n} |z_{ik_1} F'(Z_i^\top \beta)| < \infty, \sup_{n\geq 1} \max_{1\leq i\leq n} |z_{ik_1} z_{ik_2} F''(Z_i^\top \beta)| < \infty, \text{ and}$$

$$\sup_{\beta\in\mathcal{B}}\sup_{n\geq 1}\max_{1\leq i\leq n}|z_{ik_1}z_{ik_2}z_{ik_3}F'''(Z_i^\top\beta)|<\infty$$

for any $k_1, k_2, k_3 \in \{1, \cdots, d\}$, where the link function F is assumed to be three times differentiable.

(C5) Assume $\mathcal{I}_n(\theta) \to \mathcal{I}(\theta)$ and $\mathcal{J}_n(\theta, \mu^{(3)}, \mu^{(4)}) \to \mathcal{J}(\theta, \mu^{(3)}, \mu^{(4)})$ as $n \to \infty$, where $\mathcal{I}_n(\theta)$ and $\mathcal{J}_n(\theta, \mu^{(3)}, \mu^{(4)})$ are defined above Theorem 1. We further assume $\mathcal{I}(\theta)$ and $\mathcal{I}(\theta) + \mathcal{J}(\theta, \mu^{(3)}, \mu^{(4)})$ are finite and positive definite.

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