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Testing the Linear Mean and Constant Variance Conditions in Sufficient Dimension Reduction

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Abstract: Sufficient dimension reduction (SDR) methods characterise the relationship between the response Y and the covariates $\mathbf{x}$, through a few linear combinations of the covariates. Extensive techniques are developed, among which the inverse regression based methods are perhaps the most appealing in practice because they do not involve multi-dimensional smoothing and are easy to implement. However, these inverse regression based methods require two distributional assumptions on the covariates. In particular, the first-order methods, such as the sliced inverse regression, require the linear conditional mean (LCM) assumption, while the second-order methods, such as the sliced average variance estimation, require additionally the constant conditional variance (CCV) assumption. We propose to check the validity of the LCM and the CCV conditions through mean independence tests, which are facilitated by the martingale difference divergence. We suggest a consistent bootstrap procedure to decide critical values of the test. Monte Carlo simulations as well as an application to the horse mussels dataset are conducted to demonstrate the finite sample performance of our proposal.

Key words and phrases: Constant variance; dimension reduction; inverse regression; linear mean; mean independence.

1. Introduction

Sufficient dimension reduction (SDR) (Li, 1991; Cook, 1998) has received considerable attention in the past two decades. As a useful tool to reduce dimensionality, SDR can be combined with many other multivariate analysis methods for building regression models. SDR methods have also been widely used for exploratory data analysis and data visualization. Let Y be the response variable and $\mathbf{x}$ be the p dimensional predictor. When the goal is to make inference about the conditional distribution of Y given $\mathbf{x}$, SDR aims to find $\boldsymbol{\beta} \in \mathbb{R}^{p\times d}$ with $d < p$, such that

$$Y \perp\!\!\!\perp \mathbf{x} \mid \boldsymbol{\beta}^{\mathrm{T}}\mathbf{x}, \tag{1.1}$$

where "$\perp\!\!\!\perp$" means statistical independence. Under (1.1), the conditional distribution of Y given $\mathbf{x}$ is the same as that of Y given $\boldsymbol{\beta}^{\mathrm{T}}\mathbf{x}$. The column space of $\boldsymbol{\beta}$ is referred to as the dimension reduction space. If the intersection of all dimension reduction spaces exists and satisfies (1.1), this minimum subspace of $\mathbb{R}^p$ is named the central space (Cook, 1998; Chiaromonte and Cook, 2002). When the goal is to make inference about the conditional

mean $E(Y \mid \mathbf{x})$, SDR considers

$$Y \perp\!\!\!\perp E(Y \mid \mathbf{x}) \mid \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x}. \tag{1.2}$$

The column space of $\boldsymbol{\alpha}$ is referred to as the mean dimension reduction space. The smallest mean dimension reduction space that satisfies (1.2) is called the central mean space (Cook and Li, 2002).

There are many inverse regression based methods in the literature to estimate the central space or the central mean space. Estimators of the central space include, among others, sliced inverse regression (SIR) (Li, 1991), sliced average variance estimation (SAVE) (Cook and Weisberg, 1991), directional regression (Li and Wang, 2007), and cumulative slicing estimation (Zhu et al., 2010). On the other hand, ordinary least squares (OLS), principal Hessian directions (PHD) (Li, 1992) and contour regression (Li et al., 2005) are perhaps the most popular methods to estimate the central mean space. The aforementioned methods fall into two categories. In the first category, SIR and OLS involve linear functions of $\mathbf{x}$, such as $E(\mathbf{x}Y)$ and $E(\mathbf{x} \mid Y)$, and will be called the first-order methods. In the second category, SAVE, PHD, directional and contour regression involve quadratic functions of $\mathbf{x}$, such as $E(Y\mathbf{x}\mathbf{x}^{\mathrm{T}})$ and $E(\mathbf{x}\mathbf{x}^{\mathrm{T}}|Y)$, and will be called the second-order methods. Unlike other nonparametric and semiparametric methods, the inverse regression based SDR methods do not involve multi-dimensional

smoothing regardless of p. This feature, together with the fact that they are easy to implement, makes the inverse regression based SDR methods very appealing in practice.

Two assumptions about the distribution of $\mathbf{x}$ are required for the inverse regression based SDR methods to properly recover the central space or the central mean space. To ease subsequent presentation, we use $\mathbf{B} \in \mathbb{R}^{p\times d}$ to denote either the basis of the central space or that of the central mean space. The first-order methods require that

$$E(\mathbf{x} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x}) \text{ is a linear function of } \mathbf{B}^{\mathrm{T}}\mathbf{x}, \tag{1.3}$$

which is referred to as the linear conditional mean (LCM) condition. In addition to LCM, the second-order methods require that

$$\mathrm{var}(\mathbf{x} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x}) \text{ is a constant matrix}, \tag{1.4}$$

which is known as the constant conditional variance (CCV) condition. When (1.3) holds for all possible $\mathbf{B} \in \mathbb{R}^{p\times d}$, $\mathbf{x}$ must have an elliptically-contoured distribution. When both (1.3) and (1.4) hold for all possible $\mathbf{B} \in \mathbb{R}^{p\times d}$, $\mathbf{x}$ has to be multivariate normal.

The LCM and the CCV conditions have motivated many important developments in the SDR literature. To achieve these conditions, Cook and Nachtsheim (1994) proposed elliptically-contoured reweighting, and Cook

(1998) suggested marginal predictor transformations. To relax these conditions, Xia et al. (2002) proposed the minimum average variance estimation based on semiparametric models, Fukumizu et al. (2009) developed a contrast function by using operators on reproducing kernel Hilbert spaces to estimate the subspaces, Li and Dong (2009) and Dong and Li (2010) introduced the concept of central solution space and modeled $E(\mathbf{x} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x})$ parametrically. More recently, Ma and Zhu (2012) proposed the semiparametric approach, where $E(\mathbf{x} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x})$ and $\mathrm{var}(\mathbf{x} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x})$ are estimated through nonparametric smoothing. These methods avoid the common assumptions of linear mean and constant variance on the covariates, but they are computationally intensive compared with the classical SIR and SAVE.

We provide a novel treatment of the LCM and the CCV conditions in this paper. Based on a root-n-consistent estimator of $\mathbf{B}$, we formally test the validity of the LCM and the CCV conditions through hypotheses testing. It turns out that (1.3) and (1.4) are equivalent to statements about mean independence. Thus testing the validity of the LCM and the CCV conditions becomes equivalent to testing mean independence. There exists extensive literature on testing consistently the correct specification of a particular regression model, which involves testing mean independence. Most of these approaches can be divided into two classes: the local smoothing approach

(Zheng (1996),Li (1999),Koul and Ni (2004) and Guo et al. (2016)) and the global smoothing approach (Stute (1997),Li et al. (2003),Escanciano (2006)). The local approach requires nonparametric smoothing and thus its finite-sample performance depends heavily on the choice of the bandwidth, while the global approach generally turns the mean independence into an infinite number of unconditional constraints.

To formally measure the departure of mean independence between two random variables U and V, Shao and Zhang (2014) extended the distance correlation proposed by Székely et al. (2007) and Székely and Rizzo (2009) and introduced a novel metric called martingale difference divergence (MDD). They found that the MDD of V given U is always nonnegative and equal to 0 if and only if the conditional mean of V given U is independent of U. We observe that, testing the LCM and the CCV conditions is equivalent to testing mean independence. Therefore, our test procedure is thus facilitated by the martingale difference divergence (MDD) originally proposed in Shao and Zhang (2014).

The rest of this paper is organised as follows. In Section 2 we explain the rationale of our test for the LCM and the CCV conditions. Then we investigate the sample level properties of our proposal in Section 3. An extension to high dimensional case is discussed in Section 4. Numerical

studies are conducted in Section 5 with Monte Carlo simulations and an application to the horse mussels dataset. All technical proofs are collected in the Supplement.

2. The principle of testing LCM and CCV

To simplify the discussions in this section, we assume without loss of generality that $E(\mathbf{x}) = \mathbf{0}$ and $\mathrm{var}(\mathbf{x}) = \mathbf{I}_p$, where $\mathbf{I}_p$ is the identity matrix. This is a valid assumption due to the invariance property (Cook, 1998) of the central space and the central mean space. Let "$\otimes$" be the kronecker product and denote $\mathbf{P}_{\mathbf{B}} = \mathbf{B}\left(\mathbf{B}^{\mathrm{T}}\mathbf{B}\right)^{-1}\mathbf{B}^{\mathrm{T}}$ as the projection matrix onto the column space of $\mathbf{B} \in \mathbb{R}^{p\times d}$. We have the following key observation.

Proposition 1. *Suppose* $E(\mathbf{x}) = \mathbf{0}$ *and* $\mathrm{var}(\mathbf{x}) = \mathbf{I}_p$. *Then*

1. *The LCM condition (1.3) holds if and only if* $E(\boldsymbol{\varepsilon} \mid \mathbf{B}^T\mathbf{x}) = E(\boldsymbol{\varepsilon})$ *almost surely, where* $\boldsymbol{\varepsilon} \stackrel{\mathrm{def}}{=} \mathbf{x} - \mathbf{P}_{\mathbf{B}}\mathbf{x}$.

2. *Suppose the LCM condition is true. Then the CCV condition (1.4) holds if and only if* $E(\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon} \mid \mathbf{B}^T\mathbf{x}) = E(\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon})$ *almost surely.*

3. *The LCM condition (1.3) and the CCV condition (1.4) hold simultaneously if and only if* $E(\boldsymbol{\zeta} \mid \mathbf{B}^T\mathbf{x}) = E(\boldsymbol{\zeta})$ *almost surely, where* $\boldsymbol{\zeta} \stackrel{\mathrm{def}}{=} \{\boldsymbol{\varepsilon}^T, (\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon})^T\}^T$.

Consider two random vectors $\mathbf{u} \in \mathbb{R}^q$ and $\mathbf{v} \in \mathbb{R}^t$, Proposition 1 suggests that the LCM condition and the CCV condition have the same form as $E(\mathbf{v} \mid \mathbf{u}) = E(\mathbf{v})$ almost surely. This motivates us to consider testing $E(\mathbf{v} \mid \mathbf{u}) = E(\mathbf{v})$ almost surely for any $\mathbf{u} \in \mathbb{R}^q$ and $\mathbf{v} \in \mathbb{R}^t$, which can then be used to facilitate the tests for the LCM and the CCV conditions.

We note $E(\mathbf{v} \mid \mathbf{u}) = E(\mathbf{v})$ means that the conditional mean of $\mathbf{v}$ given $\mathbf{u}$ is independent of $\mathbf{u}$. We refer to this property as the mean independence, which measures the relationship between two random vectors $\mathbf{v}$ and $\mathbf{u}$, and lies between independence and uncorrelatedness. Specifically, $\mathbf{v} \perp\!\!\!\perp \mathbf{u}$ implies $E(\mathbf{v} \mid \mathbf{u}) = E(\mathbf{v})$ almost surely, which implies $\mathrm{cov}(\mathbf{v}, \mathbf{u}) = \mathbf{0}$. Therefore, to measure the mean independence, the concept of martingale difference divergence (Shao and Zhang, 2014, MDD) can be readily used. Although they only considered the case of scalar response $\mathbf{v} \in \mathbb{R}$, the definition of MDD can be generalized to the case with vector response $\mathbf{v} \in \mathbb{R}^t$.

Let $|\mathbf{c}|_q \stackrel{\text{def}}{=} (\mathbf{c}^{\mathrm{T}}\mathbf{c})^{1/2}$ be the Euclidean norm of $\mathbf{c} \in \mathbb{R}^q$. For $\mathbf{u} \in \mathbb{R}^q$ and $\mathbf{v} \in \mathbb{R}^t$, denote $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{u}})$ as an independent copy of $(\mathbf{v}, \mathbf{u})$. From part (1) of Theorem 1 in Shao and Zhang (2014), the square of MDD is equivalent to $m(\mathbf{v} \mid \mathbf{u})$, which is defined as

$$m(\mathbf{v} \mid \mathbf{u}) \stackrel{\text{def}}{=} -E\left[\{\mathbf{v} - E(\mathbf{v})\}^{\mathrm{T}}\{\widetilde{\mathbf{v}} - E(\widetilde{\mathbf{v}})\}|\mathbf{u} - \widetilde{\mathbf{u}}|_q\right]. \tag{2.1}$$

The next result is similar to Theorem 1 of Shao and Zhang (2014).

Proposition 2. *If* $E(|\mathbf{u}|_q^2 + |\mathbf{v}|_t^2) < \infty$*, then* $m(\mathbf{v} \mid \mathbf{u}) \geq 0$*, and the equality holds if and only if* $E(\mathbf{v} \mid \mathbf{u}) = E(\mathbf{v})$ *almost surely.*

Proposition 1 and Proposition 2 together provide the basic principle of testing the LCM and the CCV conditions in this paper. For the first-order methods such as OLS and SIR, only the LCM condition is required. Motivated by part 1 of Proposition 1, we consider the following hypotheses,

$$\begin{aligned} &H_0 : E(\boldsymbol{\varepsilon} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x}) = E(\boldsymbol{\varepsilon}) \text{ a.s. } \text{ for some } \mathbf{B} \in \mathbb{R}^{p\times d} \text{ v.s.} \\ &H_1 : E(\boldsymbol{\varepsilon} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x}) \neq E(\boldsymbol{\varepsilon}) \text{ a.s. } \text{ for all } \mathbf{B} \in \mathbb{R}^{p\times d}. \end{aligned} \tag{2.2}$$

where "a.s." means almost surely. Hypotheses (2.2) is to test the mean independence between $\boldsymbol{\varepsilon}$ and $\mathbf{B}^{\mathrm{T}}\mathbf{x}$. We will refer to it as the LCM hypotheses. To test hypotheses (2.2), Proposition 2 suggests that we consider the following pivotal quantity

$$m(\boldsymbol{\varepsilon} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x}) \stackrel{\text{def}}{=} -E\left[\{\boldsymbol{\varepsilon} - E(\boldsymbol{\varepsilon})\}^{\mathrm{T}}\{\widetilde{\boldsymbol{\varepsilon}} - E(\widetilde{\boldsymbol{\varepsilon}})\}|\mathbf{B}^{\mathrm{T}}(\mathbf{x} - \widetilde{\mathbf{x}})|_d\right], \tag{2.3}$$

where $\widetilde{\mathbf{x}}$ is an independent copy of $\mathbf{x}$, $\boldsymbol{\varepsilon} = \mathbf{x} - \mathbf{P}_{\mathbf{B}}\mathbf{x}$, and $\widetilde{\boldsymbol{\varepsilon}} \stackrel{\text{def}}{=} \widetilde{\mathbf{x}} - \mathbf{P}_{\mathbf{B}}\widetilde{\mathbf{x}}$.

For second-order methods such as SAVE, PHD and directional regression, both the LCM and the CCV conditions are required. Motivated by part 3 of Proposition 1, we consider the following hypotheses,

$$\begin{aligned} &H_0 : E(\boldsymbol{\zeta} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x}) = E(\boldsymbol{\zeta}) \text{ a.s. } \text{ for some } \mathbf{B} \in \mathbb{R}^{p\times d} \text{ v.s.} \\ &H_1 : E(\boldsymbol{\zeta} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x}) \neq E(\boldsymbol{\zeta}) \text{ a.s. } \text{ for all } \mathbf{B} \in \mathbb{R}^{p\times d}. \end{aligned} \tag{2.4}$$

Hypotheses (2.4) is to test the conditional mean independence between the response $\boldsymbol{\zeta}$ and the predictor $\mathbf{B}^{\mathrm{T}}\mathbf{x}$, and we will refer to it as the joint hypotheses. To test hypotheses (2.4), Proposition 2 suggests that we consider

$$m(\boldsymbol{\zeta} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x}) \stackrel{\mathrm{def}}{=} -E\left[\{\boldsymbol{\zeta} - E(\boldsymbol{\zeta})\}^{\mathrm{T}}\{\widetilde{\boldsymbol{\zeta}} - E(\widetilde{\boldsymbol{\zeta}})\}|\mathbf{B}^{\mathrm{T}}(\mathbf{x} - \widetilde{\mathbf{x}})|_d\right], \tag{2.5}$$

where $\widetilde{\boldsymbol{\zeta}} \stackrel{\mathrm{def}}{=} \{\widetilde{\boldsymbol{\varepsilon}}^{\mathrm{T}}, (\widetilde{\boldsymbol{\varepsilon}} \otimes \widetilde{\boldsymbol{\varepsilon}})^{\mathrm{T}}\}^{\mathrm{T}}$ is an independent copy of $\boldsymbol{\zeta} = \{\boldsymbol{\varepsilon}^{\mathrm{T}}, (\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon})^{\mathrm{T}}\}^{\mathrm{T}}$.

3. The sample level properties

We focus on testing the LCM hypotheses (2.2) in this section. The properties of sample level test for the joint hypotheses (2.4) are similar, and are thus omitted for ease of presentation. Let $\{(\mathbf{x}_j, Y_j) : j = 1, \ldots, n\}$ be an i.i.d. sample of $(\mathbf{x}, Y)$. Our main idea is to test (2.2) through the sample estimator of $m(\boldsymbol{\varepsilon} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x})$. Let $\widehat{\mathbf{B}}$ be a sample estimator of $\mathbf{B}$, which depends on $\mathbf{x}_j$ and Y_j, $j = 1, \ldots, n$. Let $\mathbf{P}_{\widehat{\mathbf{B}}} \stackrel{\mathrm{def}}{=} \widehat{\mathbf{B}}\big(\widehat{\mathbf{B}}^{\mathrm{T}}\widehat{\mathbf{B}}\big)^{-1}\widehat{\mathbf{B}}^{\mathrm{T}}$, $\mathbf{Q}_{\widehat{\mathbf{B}}} \stackrel{\mathrm{def}}{=} \mathbf{I}_p - \mathbf{P}_{\widehat{\mathbf{B}}}$, $\widehat{\boldsymbol{\varepsilon}}_j \stackrel{\mathrm{def}}{=} \mathbf{Q}_{\widehat{\mathbf{B}}}\mathbf{x}_j$, and

$$\overline{\boldsymbol{\varepsilon}} \stackrel{\mathrm{def}}{=} n^{-1}\sum_{j=1}^{n}\widehat{\boldsymbol{\varepsilon}}_j.$$

The sample estimator of $m(\boldsymbol{\varepsilon} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x})$ becomes

$$\widehat{\omega}_n \stackrel{\mathrm{def}}{=} -n^{-2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left\{(\widehat{\boldsymbol{\varepsilon}}_j - \overline{\boldsymbol{\varepsilon}})^{\mathrm{T}}(\widehat{\boldsymbol{\varepsilon}}_k - \overline{\boldsymbol{\varepsilon}})|\widehat{\mathbf{B}}^{\mathrm{T}}(\mathbf{x}_j - \mathbf{x}_k)|_d\right\}. \tag{3.1}$$

We follow Ma and Zhu (2013) to ensure the identifiability of $\mathbf{B}$. To be specific, for an arbitrary basis matrix $\mathbf{B}_t \in \mathbb{R}^{p\times d}$ of the central (mean) space,

we write $\mathbf{B}_t \stackrel{\text{def}}{=} (\mathbf{B}_u^{\mathrm{T}}, \mathbf{B}_l^{\mathrm{T}})^{\mathrm{T}}$, where $\mathbf{B}_u$ is a $d \times d$ upper submatrix, $\mathbf{B}_l$ is a $(p-d) \times d$ lower submatrix and the subscript " t" stands for total. We assume, without loss of generality, $\mathbf{B}_u$ is invertible. In case $\mathbf{B}_u$ is not invertible we can always rotate the order of $\mathbf{x}$ to ensure that $\mathbf{B}_u$ is invertible because the rank of $\mathbf{B}_t$ is d. As long as $\mathbf{B}_u$ is invertible, the column spaces of $\mathbf{B}_t$ and $\mathbf{B}_t\mathbf{B}_u^{-1}$ are identical. We define $\mathbf{B} \stackrel{\text{def}}{=} \mathbf{B}_t\mathbf{B}_u^{-1}$, the upper $d \times d$ submatrix of $\mathbf{B}$ is an identity matrix. This uniquely defines the true parameter. e At the sample level, we apply a certain SDR method to estimate $\mathbf{B}$. The resultant estimate is denoted as $\widehat{\mathbf{B}}_t$, which is of form $\widehat{\mathbf{B}}_t \stackrel{\text{def}}{=} (\widehat{\mathbf{B}}_u^{\mathrm{T}}, \widehat{\mathbf{B}}_l^{\mathrm{T}})^{\mathrm{T}}$, where $\widehat{\mathbf{B}}_u$ is a $d \times d$ upper submatrix, and $\widehat{\mathbf{B}}_l$ is a $(p-d) \times d$ lower submatrix. We then define $\widehat{\mathbf{B}} \stackrel{\text{def}}{=} \widehat{\mathbf{B}}_t\widehat{\mathbf{B}}_u^{-1}$ as the sample estimator of $\mathbf{B}$.

Some notations are needed before we state the main theorem. Let $i = (-1)^{1/2}$ be the imaginary unit. Let $c_p \stackrel{\text{def}}{=} \pi^{(1+p)/2}/\Gamma\{(1+p)/2\}$, where $\Gamma(\cdot)$ is the Gamma function. For a complex-valued function $\boldsymbol{\gamma}$: $\mathbb{R}^q \to \mathbb{C}^p$, we define its norm as

$$\|\boldsymbol{\gamma}(\mathbf{s})\|^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^q} |\boldsymbol{\gamma}(\mathbf{s})|_p^2 (c_q|\mathbf{s}|_q^{1+q})^{-1} d\mathbf{s}, \text{ where } |\boldsymbol{\gamma}(\mathbf{s})|_p^2 \stackrel{\text{def}}{=} \sum_{j=1}^{p} \nu_j(\mathbf{s})\overline{\nu_j(\mathbf{s})},$$

with $\nu_j(\mathbf{s}) \in \mathbb{C}$ being the jth element of $\boldsymbol{\gamma}(\mathbf{s}) \in \mathbb{C}^p$ and $\overline{\nu_j(\mathbf{s})}$ being the conjugate of $\nu_j(\mathbf{s})$, $j = 1, \ldots, p$. Similar notations have been introduced in Shao and Zhang (2014). Let "$\xrightarrow{d}$" stand for "convergence in distribution", and "$\xrightarrow{p}$" stand for "converge in probability". The following technical condition

about $\widehat{\mathbf{B}}$ is needed for the main result.

(C1). Suppose

$$\begin{aligned}\widehat{\mathbf{B}} - \mathbf{B} &= n^{-1}\sum_{j=1}^{n} \boldsymbol{\ell}_1(\mathbf{x}_j, Y_j) + o_p(n^{-1/2}), \text{ and} \\ \mathbf{P}_{\widehat{\mathbf{B}}} - \mathbf{P}_{\mathbf{B}} &= n^{-1}\sum_{j=1}^{n} \boldsymbol{\ell}_2(\mathbf{x}_j, Y_j) + o_p(n^{-1/2}).\end{aligned}$$

Assume $E\{\boldsymbol{\ell}_k(\mathbf{x}, Y)\} = \mathbf{0}$ and the elements of $\mathrm{var}\{\mathrm{vec}(\boldsymbol{\ell}_k(\mathbf{x}, Y))\}$ are bounded, for $k = 1, 2$, where $\mathrm{vec}(\mathbf{M})$ is the vector formed by concatenating the columns of matrix $\mathbf{M}$.

Theorem 1. *Suppose $E(\mathbf{x}) = \mathbf{0}$, $\mathrm{var}(\mathbf{x}) = \mathbf{I}_p$, and condition (C1) holds. Let $\boldsymbol{\phi} : \mathbb{R}^d \to \mathbb{C}^p$ be a complex-valued zero-mean Gaussian process with covariance function*

$$\begin{aligned}\mathrm{cov}_{\boldsymbol{\phi}}(\mathbf{s}, \mathbf{s}_0) \stackrel{\mathrm{def}}{=} E\Big[&\big\{\boldsymbol{\varepsilon}\exp(i\mathbf{s}^T\mathbf{B}^T\mathbf{x}) - \boldsymbol{\ell}_2(\mathbf{x}, Y)\mathbf{g}(\mathbf{s}) + \mathbf{h}(\mathbf{s})\boldsymbol{\ell}_1(\mathbf{x}, Y)\mathbf{s}\big\} \\ &\big\{\boldsymbol{\varepsilon}\exp(-i\mathbf{s}_0^T\mathbf{B}^T\mathbf{x}) - \boldsymbol{\ell}_2(\mathbf{x}, Y)\mathbf{g}(-\mathbf{s}_0) - \mathbf{h}(-\mathbf{s}_0)\boldsymbol{\ell}_1(\mathbf{x}, Y)\mathbf{s}_0\big\}^T\Big],\end{aligned} \tag{3.2}$$

where $\mathbf{g}(\mathbf{s}) \stackrel{\mathrm{def}}{=} E\{\mathbf{x}\exp(i\mathbf{s}^T\mathbf{B}^T\mathbf{x})\}$ and $\mathbf{h}(\mathbf{s}) \stackrel{\mathrm{def}}{=} E[\boldsymbol{\varepsilon}\{i\cos(\mathbf{s}^T\mathbf{B}^T\mathbf{x}) - \sin(\mathbf{s}^T\mathbf{B}^T\mathbf{x})\}\mathbf{x}^T]$.

1. *Under $H_0 : E(\boldsymbol{\varepsilon} \mid \mathbf{B}^T\mathbf{x}) = E(\boldsymbol{\varepsilon})$ a.s., we have $n\widehat{\omega}_n \xrightarrow{d} \|\boldsymbol{\phi}(\mathbf{s})\|^2$ as n goes to infinity.*

2. *Under $H_1 : E(\boldsymbol{\varepsilon} \mid \mathbf{B}^T\mathbf{x}) \neq E(\boldsymbol{\varepsilon})$ a.s., we have $n\widehat{\omega}_n \xrightarrow{p} \infty$ as n goes to infinity.*

Theorem 1 is similar to Theorem 5 of Székely et al. (2007) and Theorem 4 of Shao and Zhang (2014). We reject H_0 in (2.2) when $\widehat{\omega}_n$ is sufficiently large. The exact form of $\|\boldsymbol{\phi}(\mathbf{s})\|^2$ is very complicated and difficult to use in practice. To approximate the asymptotic distribution of $\widehat{\omega}_n$, we propose the following bootstrap procedure.

S0. Based on i.i.d. sample $\{(\mathbf{x}_j, Y_j) : j = 1, \ldots, n\}$, use a chosen SDR method to estimate $\mathbf{B} \in \mathbb{R}^{p\times d}$ as $\widehat{\mathbf{B}}$. Compute $\mathbf{P}_{\widehat{\mathbf{B}}} = \widehat{\mathbf{B}}\big(\widehat{\mathbf{B}}^{\mathrm{T}}\widehat{\mathbf{B}}\big)^{-1}\widehat{\mathbf{B}}^{\mathrm{T}}$, $\mathbf{Q}_{\widehat{\mathbf{B}}} = \mathbf{I}_p - \mathbf{P}_{\widehat{\mathbf{B}}}$ and $\widehat{\boldsymbol{\varepsilon}}_j = \mathbf{Q}_{\widehat{\mathbf{B}}}\mathbf{x}_j$, $j = 1, \ldots, n$. Calculate the test statistic $\widehat{\omega}_n$ in (3.1).

S1. In the (t)th iteration, let $\{W_j^{(t)} : j = 1, \ldots, n\}$ be i.i.d. Bernoulli random variables such that $\Pr(W_j^{(t)} = 1) = \Pr(W_j^{(t)} = -1) = 0.5$. Set $\mathbf{x}_j^{(t)} \stackrel{\text{def}}{=} \mathbf{P}_{\widehat{\mathbf{B}}}\mathbf{x}_j + W_j^{(t)}\widehat{\boldsymbol{\varepsilon}}_j$, $j = 1, \ldots, n$.

S2. Based on $\{(\mathbf{x}_j^{(t)}, Y_j) : j = 1, \ldots, n\}$, use the same SDR method as in step S0 to estimate $\mathbf{B}$. Denote the corresponding estimator as $\widehat{\mathbf{B}}^{(t)}$.

S3. Compute $\mathbf{P}_{\widehat{\mathbf{B}}^{(t)}} \stackrel{\text{def}}{=} \widehat{\mathbf{B}}^{(t)}\big\{(\widehat{\mathbf{B}}^{(t)})^{\mathrm{T}}\widehat{\mathbf{B}}^{(t)}\big\}^{-1}(\widehat{\mathbf{B}}^{(t)})^{\mathrm{T}}$ and $\widehat{\boldsymbol{\varepsilon}}_j^{(t)} \stackrel{\text{def}}{=} \mathbf{x}_j^{(t)} - \mathbf{P}_{\widehat{\mathbf{B}}^{(t)}}\mathbf{x}_j^{(t)}$, $j = 1, \ldots, n$. Let

$$\overline{\boldsymbol{\varepsilon}}^{(t)} \stackrel{\text{def}}{=} n^{-1}\sum_{j=1}^{n}\widehat{\boldsymbol{\varepsilon}}_j^{(t)}$$

and calculate

$$\widehat{\omega}_n^{(t)} \stackrel{\text{def}}{=} -n^{-2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left\{(\widehat{\boldsymbol{\varepsilon}}_j^{(t)}-\overline{\boldsymbol{\varepsilon}}^{(t)})^{\mathrm{T}}(\widehat{\boldsymbol{\varepsilon}}_k^{(t)}-\overline{\boldsymbol{\varepsilon}}^{(t)}) \mid (\widehat{\mathbf{B}}^{(t)})^{\mathrm{T}}(\mathbf{x}_j^{(t)}-\mathbf{x}_k^{(t)}) \mid_d\right\}.$$

S4. Repeat S1-S3 for T times. Calculate the p-value defined as

$$T^{-1}\sum_{t=1}^{T}\mathbf{1}(\widehat{\omega}_n<\widehat{\omega}_n^{(t)}),$$

where $\mathbf{1}(\cdot)$ stands for the indicator function. For a given significance level α, reject $H_0: E(\boldsymbol{\varepsilon} \mid \mathbf{B}^{\mathrm{T}}\mathbf{x}) = E(\boldsymbol{\varepsilon})$ if the p-value is less than α.

The validity of the bootstrap procedure is guaranteed by the next theorem. Define $\mathbf{x}^* \stackrel{\text{def}}{=} \mathbf{P}_{\widehat{\mathbf{B}}}\mathbf{x} + W^*\mathbf{Q}_{\widehat{\mathbf{B}}}\mathbf{x}$, where W^* is a Bernoulli random variable such that $\Pr(W^*=1)=\Pr(W^*=-1)=0.5$. It follows that $\{(\mathbf{x}_j^{(t)}, Y_j) : j = 1,\ldots,n\}$ is an i.i.d. sample of $(\mathbf{x}^*, Y)$. The following technical conditions are needed before we state the main result.

(C2). Suppose

$$\begin{aligned}\widehat{\mathbf{B}}^{(t)}-\widehat{\mathbf{B}} &= n^{-1}\sum_{j=1}^{n}\boldsymbol{\ell}_1(\mathbf{x}_j^{(t)}, Y_j)+o_p(n^{-1/2}) \text{ and}\\ \mathbf{P}_{\widehat{\mathbf{B}}^{(t)}}-\mathbf{P}_{\widehat{\mathbf{B}}} &= n^{-1}\sum_{j=1}^{n}\boldsymbol{\ell}_2(\mathbf{x}_j^{(t)}, Y_j)+o_p(n^{-1/2}),\end{aligned}$$

Assume $E\{\boldsymbol{\ell}_k(\mathbf{x}^*, Y)\} = \mathbf{0}$ and the elements of $\mathrm{var}\{\mathrm{vec}(\boldsymbol{\ell}_k(\mathbf{x}^*, Y))\}$ are bounded, for $k = 1, 2$.

(C3). Let $\boldsymbol{\phi}^* : \mathbb{R}^d \to \mathbb{C}^p$ be a complex-valued zero-mean Gaussian process with covariance function

$$\begin{aligned}\mathrm{cov}_{\boldsymbol{\phi}^*}(\mathbf{s}, \mathbf{s}_0) \stackrel{\text{def}}{=} E\Big[&\big\{\boldsymbol{\varepsilon}\exp(i\mathbf{s}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{x}^*) - \boldsymbol{\ell}_2(\mathbf{x}^*, Y)\mathbf{g}^*(\mathbf{s}) + \mathbf{h}^*(\mathbf{s})\boldsymbol{\ell}_1(\mathbf{x}^*, Y)\mathbf{s}\big\} \\ &\big\{\boldsymbol{\varepsilon}\exp(-i\mathbf{s}_0^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{x}^*) - \boldsymbol{\ell}_2(\mathbf{x}^*, Y)\mathbf{g}^*(-\mathbf{s}_0) - \mathbf{h}^*(-\mathbf{s}_0)\boldsymbol{\ell}_1(\mathbf{x}^*, Y)\mathbf{s}_0\big\}^{\mathrm{T}}\Big],\end{aligned}$$

where $\mathbf{g}^*(\mathbf{s}) \stackrel{\text{def}}{=} E\left\{\mathbf{x}^*\exp(i\mathbf{s}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{x}^*)\right\}$ and $\mathbf{h}^*(\mathbf{s}) \stackrel{\text{def}}{=} E[\boldsymbol{\varepsilon}\{i\cos(\mathbf{s}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{x}^*) - \sin(\mathbf{s}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{x}^*)\}(\mathbf{x}^*)^{\mathrm{T}}]$. Suppose $\mathrm{cov}_{\boldsymbol{\phi}^*}(\mathbf{s}, \mathbf{s}_0)$ is equal to $\mathrm{cov}_{\boldsymbol{\phi}}(\mathbf{s}, \mathbf{s}_0)$ defined in (3.2) as long as $E(\mathbf{x}^*) = E(\mathbf{x})$ and $\mathrm{var}(\mathbf{x}^*) = \mathrm{var}(\mathbf{x})$.

(C4) Assume that $\psi(\mathbf{B}) \stackrel{\text{def}}{=} E(\mathbf{Q}_{\mathbf{B}}\boldsymbol{\varepsilon}^* \mid \mathbf{P}_{\mathbf{B}}\mathbf{x}^*)$ is Lipschitz continuous.

Theorem 2. *Suppose $E(\mathbf{x}) = \mathbf{0}$, $\mathrm{var}(\mathbf{x}) = \mathbf{I}_p$, conditions (C1)-(C4) hold. Then $\widehat{\omega}_n^{(t)}$ has the same asymptotic null distribution as $\widehat{\omega}_n$. Specifically, $n\widehat{\omega}_n^{(t)} \xrightarrow{d} \|\boldsymbol{\phi}(\mathbf{s})\|^2$ as n goes to infinity.*

4. An Extension

If the predictor dimension p is very large, we assume, under the principle of sparsity that, $Y \perp\!\!\!\perp \mathbf{x} \mid \boldsymbol{\beta}_{\mathcal{A}_1}^{\mathrm{T}}\mathbf{x}_{\mathcal{A}_1}$ when the central space is considered and $Y \perp\!\!\!\perp E(Y \mid \mathbf{x}) \mid \boldsymbol{\alpha}_{\mathcal{A}_2}^{\mathrm{T}}\mathbf{x}_{\mathcal{A}_2}$ when the central mean space is considered, where

$$\begin{aligned}\mathcal{A}_1 &\stackrel{\text{def}}{=} \{k \mid F(y \mid \mathbf{x}) \text{ relies functionally on } X_k \text{ for } y \in \mathbb{R},\, k = 1, \ldots, p\}, \\ \mathcal{A}_2 &\stackrel{\text{def}}{=} \{k \mid E(y \mid \mathbf{x}) \text{ relies functionally on } X_k \text{ for } y \in \mathbb{R},\, k = 1, \ldots, p\},\end{aligned}$$

$F(y \mid \mathbf{x})$ and $E(y \mid \mathbf{x})$ are the respective conditional distribution and conditional mean functions of Y given $\mathbf{x}$. To ease subsequent presentations, we use $\mathcal{A}$ to denote either $\mathcal{A}_1$ or $\mathcal{A}_2$, and $\mathbf{B}_{\mathcal{A}}$ to denote either $\boldsymbol{\beta}_{\mathcal{A}_1}$ or $\boldsymbol{\alpha}_{\mathcal{A}_2}$.

When p is moderately large, we can first apply some sparse SDR methods, such as Li (2007), Bondell and Li (2009), Chen et al. (2010), to simultaneously select variables (i.e. estimate the active index set $\mathcal{A}$) and reduce the dimension (i.e. estimate the basis matrix $\mathbf{B}_{\mathcal{A}}$).

When p is extremely large, we recommend to use a model free screening approach, such as SIRS in Zhu et al. (2011), DC-SIS in Li et al. (2012), or MDC-SIS in Shao and Zhang (2014), to exclude as many inactive predictors as possible, before we use SDR methods to further reduce the predictor dimension. Once the number of active predictors is reduced to a moderate scale, we implement the sparse SDR methods to obtain a consistent estimator of $\mathcal{A}$ and $\mathbf{B}_{\mathcal{A}}$. Subsequent test procedure is based on $\mathbf{x}_{\widehat{\mathcal{A}}}$ and $\widehat{\mathbf{B}}_{\mathcal{A}}$. Our proposal remains valid as long as the estimate of $\mathbf{B}_{\mathcal{A}}$ is consistent.

We advocate a two-stage test procedure in the high dimensional case. In particular, we randomly split the whole sample data $\mathcal{D}$ into two equal halves, $\mathcal{D}_1$ and $\mathcal{D}_2$. First we implement DC-SIS (Li et al., 2012) on data set $\mathcal{D}_1$ and retain the top ranked covariates as the active ones. Next we implement the sparse SDR method of Li (2007) on data set $\mathcal{D}_2$ to estimate

$\mathcal{A}$ and $\mathbf{B}_{\mathcal{A}}$. We conduct our test procedure based on $\mathbf{x}_{\widehat{\mathcal{A}}}$ and $\widehat{\mathbf{B}}_{\mathcal{A}}$. We adopt a data splitting strategy to avoid inflating type I error rates in our test procedure. When some inactive covariates are retained in the screening stage, directly implementing our testing procedure without random splitting will lead to inflated type I error rates (Fan et al., 2012).

5. Numerical studies

Example 1. We conduct simulations to demonstrate the performance of our proposed test. We fix the sample size $n = 200$. We evaluate the predictor dimension $p = 8$ for low dimensional case and $p = 1000$ for high dimensional case, respectively. We consider two models.

(I): In the first model, the central space is spanned by $(1, 0, 0, \ldots, 0)^{\mathrm{T}}$ and $Y = X_1 + \delta$. Thus, $d = 1$. The predictors $\mathbf{x} = (X_1, \ldots, X_p)$ are generated as follows: $X_1, X_3, \ldots, X_p$ are drawn independently from standard normal distribution, and $X_2 = X_1 + c_1(X_1^2 - 1) + |c_2 X_1 + 1|\epsilon$.

(II): In the second model, the central space is spanned by $(1, 0, 0, \ldots, 0)^{\mathrm{T}}$ and $(0, 1, 0, \ldots, 0)^{\mathrm{T}}$, and $Y = 5X_1/\{0.5 + (X_2 + 1.5)^2\} + \delta$. Thus, $d = 2$. The predictors $\mathbf{x} = (X_1, \ldots, X_p)$ are generated as follows: $X_1, X_2, X_4, \ldots, X_p$ are independently drawn from standard normal distribution, and $X_3 = X_1 + X_2 + c_1(X_1^2 - 1) + |c_2 X_2 + 1|\epsilon$.

In both models, we generate ϵ and δ independently from standard normal.

We first evaluate the performance of testing the LCM condition. We fix $c_2 = 0$ and evaluate $c_1 = 0, 0.1, \ldots, 0.5$. To illustrate the performance of our proposal, we evaluate two test statistics: (a) based on the observed dataset $(\mathbf{x}_i, Y_i)_{i=1}^n$, we estimate $\mathbf{B}$ via SIR method and obtain $\widehat{\mathbf{B}}^{sir}$, we then construct test statistics by replacing $\widehat{\mathbf{B}}$ with $\widehat{\mathbf{B}}^{sir}$ in (3.1); (b) suppose the true $\mathbf{B}$ matrix is known as a prior, we then construct test statistics by replacing $\widehat{\mathbf{B}}$ with $\mathbf{B}$ in (3.1), which acts as a benchmark.

When evaluating the performance of the joint test (i.e., simultaneously test the LCM and the CCV conditions), we consider $c_1 = c_2 = 0, 0.1, \ldots, 0.5$. We also evaluate two test statistics: (a) test statistics based on $\widehat{\mathbf{B}}^{save}$, where $\widehat{\mathbf{B}}^{save}$ is obtained through the SAVE method; (b) test statistics based on the true $\mathbf{B}$ matrix by assuming that it is known as a prior.

To put our test procedure into practice, we may choose estimates of $\mathbf{B}$ which do not rely on the linear mean or constant variance conditions, such as Xia et al. (2002), Xia (2007), Fukumizu et al. (2009), Li and Dong (2009), Dong and Li (2010), Ma and Zhu (2012), etc.. However, these estimates are usually computationally intensive, when compared with the classical SIR and SAVE. Therefore, we simply suggest to estimate $\mathbf{B}$ via a computationally intensive method. To be precise, we implement SIR

or SAVE to obtain $\widehat{\mathbf{B}}$, then we use our proposed test based on this $\widehat{\mathbf{B}}$ to check whether the LCM condition or the joint conditions hold. If the null hypothesis is not rejected, then we are confident that $\widehat{\mathbf{B}}$ is valid. If the null hypothesis is rejected, we can choose some other methods which avoid the LCM condition or the joint conditions to re-estimate $\mathbf{B}$. In this way, however, we may lose power because $E(\boldsymbol{\varepsilon} \mid \widehat{\mathbf{B}}^{\mathrm{T}}\mathbf{x})$ may be very close to $E(\boldsymbol{\varepsilon})$ for a lousy estimate $\widehat{\mathbf{B}}$ obtained under the alternative hypothesis.

We decide whether to reject the null hypothesis through the bootstrap procedures with $T = 500$. We repeat each experiment for 500 times and study the size and the power of the our tests separately.

We first evaluate the size of the test. Note that the LCM condition holds if and only if $c_1 = 0$, and the joint condition holds if and only if $c_1 = c_2 = 0$. We thus fix $c_1 = c_2 = 0$ to study the size of all tests. We investigate different significance levels with $\alpha = 0.01, 0.02, 0.05$. The empirical sizes based on 500 repetitions are summarised in Table 1, which indicates that tests based on $\widehat{\mathbf{B}}$s behave similarly as those based on the true $\mathbf{B}$ matrix, and the empirical sizes are close to the nominal level α.

We then study the power performance of the test procedures. We fix $c_2 = 0$ and evaluate $c_1 = 0.1, 0.2, \ldots, 0.5$ when test the LCM condition and $c_1 = c_2 = 0.1, 0.2, \ldots, 0.5$ when test the joint condition LCM+CCV. We fix

Table 1: The empirical sizes of the test procedures when $c_1 = c_2 = 0$.

p	Test	Method	$d=1$			$d=2$		
			α			α		
			0.01	0.02	0.05	0.01	0.02	0.05
$p=8$	LCM	$\widehat{\mathbf{B}}^{sir}$	0.02	0.03	0.04	0.01	0.02	0.05
		$\mathbf{B}$	0.01	0.03	0.05	0.01	0.01	0.04
	LCM+CCV	$\widehat{\mathbf{B}}^{save}$	0.01	0.03	0.05	0.01	0.02	0.04
		$\mathbf{B}$	0.01	0.02	0.05	0.01	0.02	0.04
$p=1000$	LCM	$\widehat{\mathbf{B}}^{sir}$	0.01	0.02	0.05	0.01	0.02	0.05
		$\mathbf{B}$	0.01	0.02	0.05	0.01	0.02	0.06
	LCM+CCV	$\widehat{\mathbf{B}}^{save}$	0.01	0.01	0.04	0.01	0.02	0.04
		$\mathbf{B}$	0.01	0.02	0.05	0.01	0.02	0.04

the significant level $\alpha = 0.05$ and the results are summarized in Table 2.

Table 2 indicates that our proposals perform quite satisfactory. Generally, the powers of all the tests gradually rises to one when c_1 goes up form 0.1 to 0.5. In the low dimensional case $p = 8$, the powers of both the LCM test and the joint test LCM+CCV exceed 0.85 when the signal intensity parameter c_1 increases to 0.3 in Model (I) where the structure dimension $d = 1$, and finally reaches one when $c_1 = 0.5$. The results for Model (II) where $d = 2$ are quite similar, though a little inferior, to those for Model (I). This is reasonable since it is a more complicated problem in SDR when the structure dimension increases. The story is quite similar in the ultrahigh dimensional case when $p = 1000$.

Example 2. We apply our proposal to the horse mussels dataset fur-

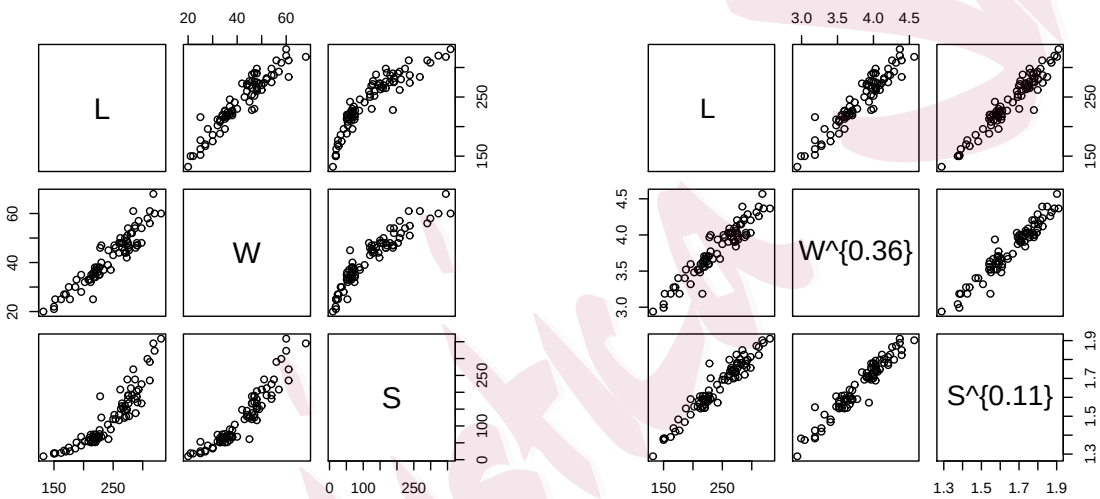
Table 2: The empirical powers of the test procedures with $\alpha = 0.05$.

p	Test		$d=1$					$d=2$				
		c_1	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
	LCM	$\widehat{\mathbf{B}}^{sir}$	0.13	0.54	0.87	0.99	1.00	0.10	0.49	0.87	0.98	1.00
		$\mathbf{B}$	0.17	0.65	0.95	1.00	1.00	0.14	0.49	0.88	0.99	1.00
		$c_1 = c_2$	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
8	LCM+CCV	$\widehat{\mathbf{B}}^{save}$	0.12	0.49	0.85	0.98	1.00	0.07	0.28	0.59	0.78	0.88
		$\mathbf{B}$	0.15	0.51	0.87	0.99	1.00	0.10	0.40	0.80	0.98	1.00
		c_1	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
	LCM	$\widehat{\mathbf{B}}^{sir}$	0.09	0.46	0.85	0.97	1.00	0.12	0.45	0.86	0.97	1.00
		$\mathbf{B}$	0.16	0.59	0.94	1.00	1.00	0.14	0.49	0.87	0.98	1.00
		$c_1 = c_2$	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
1000	LCM+CCV	$\widehat{\mathbf{B}}^{save}$	0.09	0.43	0.80	0.97	1.00	0.06	0.25	0.48	0.69	0.80
		$\mathbf{B}$	0.12	0.45	0.89	0.99	1.00	0.09	0.36	0.78	0.97	1.00

nished by Mike Camden, Wellington Polytechnic, Wellington, New Zealand. The response variable Y is mussels' muscle mass M, the edible portion of the mussel, which is measured in grams. The covariates $\mathbf{x}$ include shell length L in millimeters, shell width W in millimeters and shell mass S in grams. The sample size is 82.

We first visually evaluate whether the LCM condition holds. The scatter plot matrix of the shell length L, shell width W and shell mass S is presented in Figure 1 (A). From Figure 1, the curvature between L and S, together with that between W and S raises doubts about the LCM condition required in the sliced inverse regression method. Thus, Cook (1998) used power transformations of the covariates to make the LCM condition holds

approximately. With the application of the maximum likelihood estimates, shell width W was transformed to $W^{0.36}$, shell mass S was transformed to $S^{0.11}$, while shell length L was not transformed. The scatter plot matrix after transformations is shown in Figure 1 (B). It seems that the LCM condition holds true after transformation.



(A) $\mathbf{x} = (L, W, S)^{\mathrm{T}}$ (B) $\mathbf{x}^* = (L, W^{0.36}, S^{0.11})^{\mathrm{T}}$

Figure 1: Scatter plot matrices for covariate vectors $\mathbf{x}$ and $\mathbf{x}^*$.

Now we apply our proposal to test whether the LCM condition hold. Recall that $\mathbf{x} = (L, W, S)^{\mathrm{T}}$ is the covariate vector before transformation, we further set $\mathbf{x}^* = (L, W^{0.36}, S^{0.11})^{\mathrm{T}}$ to be the covariate vector after transformation. Then we need to test the LCM condition for datasets $(\mathbf{x}_i, Y_i)_{i=1}^{82}$

and $(\mathbf{x}_i^*, Y_i)_{i=1}^{82}$, respectively. Given the structure dimension $d = 1$, our proposal can be directly applied. To be specific, we first apply the SIR method to the original dataset $(\mathbf{x}_i, Y_i)_{i=1}^{82}$ to estimate $\mathbf{B}$ and $\widehat{\mathbf{B}}$ denotes the corresponding estimator. We then carry out the test procedure based on $\mathbf{x}_i$ and $\widehat{\mathbf{B}}$ and obtain the p-value 0.000. Then we test the LCM condition on the transformed dataset $(\mathbf{x}_i^*, Y_i)_{i=1}^{82}$ similarly. We estimate $\mathbf{B}$ via SIR method and get $\widehat{\mathbf{B}}^*$, and then we conduct the test based on $\mathbf{x}_i^*$ and $\widehat{\mathbf{B}}^*$. The p-value of the test is 0.908.

From the above tests, we can soundly reject the null hypothesis that the LCM condition holds true for dataset $(\mathbf{x}_i, Y_i)_{i=1}^{82}$, while we accept it for dataset $(\mathbf{x}_i^*, Y_i)_{i=1}^{82}$. That is to say, the LCM condition is violated on the original dataset, but the power transformation proposed by Cook (1998) acts as a remedy to this problem. Such results are in accordance with that we see in Figure (1), and thus proves the validity of our proposal.

Besides, to see whether the power transformation helps getting more accurate estimators, we conduct a simple bootstrap procedure as follows. For the original data $(\mathbf{x}_i, Y_i)_{i=1}^{82}$, we estimate $\mathbf{B}$ via the SIR method and get $\widehat{\mathbf{B}}$ and we treat $\widehat{\mathbf{B}}$ as the true $\mathbf{B}$. Then we bootstrap from the original data 500 times and obtain $\widehat{\mathbf{B}}^{(t)}$ through SIR, where $t = 1, \ldots, 500$. To assess the distance between $\widehat{\mathbf{B}}$ and $\widehat{\mathbf{B}}^{(t)}$, we adopt the trace correlation

proposed in Ferré (1998) and obtain $r(d)^{(t)}, t = 1, \ldots, 500$. Based on the 500 repetitions, the average of $r(d)^{(t)}$ is 0.74, and the standard deviation is 0.21. Similarly, for the transformed data $(\mathbf{x}_i^*, Y_i)_{i=1}^{82}$, we can get $r(d)^{*(t)}, t = 1, \ldots, 500$ with the average 0.95 and the standard deviation 0.06. According to Ferré (1998), the trace correlation $r(d) \in [0, 1]$ and larger value indicates that the two subspaces are closer. Thus the power transformation results in more accurate estimators when the LCM fails.

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