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A Simple and Efficient Estimation Method for Models with Nonignorable Missing Data

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Abstract: This paper proposes a simple and efficient GMM estimation for the model with non-ignorable missing data studied by Wang, Shao and Kim (2014) and Morikawa and Kim (2016). Unlike the GMM estimation with fixed number of moments proposed by Wang, Shao and Kim (2014), we allow the number of moments to grow with the sample size and use optimal weighting. Hence our estimator is efficient, attaining the semiparametric efficiency bound derived in Morikawa and Kim (2016). Morikawa and Kim (2016) also proposes two semiparametric estimators by estimating the efficient score, but their approach either is locally efficient or suffers from the curse of dimensionality and the bandwidth selection problem. In contrast, our estimator does not suffer from those problems. Moreover, our proposed estimator and its consistent covariance matrix are easily

All authors contribute to the paper equally.

computed with the widely available GMM package. We propose two data-driven methods for selection of the number of moments. A small scale simulation study reveals that the proposed estimation indeed out-performs the existing alternatives in finite samples.

Key words and phrases: Generalized method of moments, nonignorable nonresponse, semiparametric efficiency.

1. Introduction

Missing data is common in many fields of applications. One way to deal with the missing data problem is to delete observations containing missing data. In doing so we may produce biased estimates and erroneous conclusions, depending on the data missing mechanism. If data are missing completely at random, standard estimation and inference procedures are still consistent when the missing data observations are ignored, see Heitjan and Basu (1996), Little (1988) among others. If data are missing at random (MAR) in the sense that the propensity of missingness depends only on the observed covariates, consistent estimation can still be obtained through covariate balancing, see Rubin (1976a,b), Little and Rubin (1989), Robins and Rotnitzky (1995), Robins, Rotnitzky and Zhao (1995), Bang and Robins (2005), Qin and Zhang (2007), Chen, Hong and Tarozzi (2008), Tan (2010), Rotnitzky, Lei, Sued and Robins (2012), Little and Rubin (2014) among others. In many applications, data are missing not at random (MNAR). For example, the income question in sample surveys is often not answered by people at the top end of the distribution, that is, their response frequency depends on an outcome variable that is often the key focus. An investigator is examining the effect of sleep on pain by calling subjects daily to ask them about last night's sleep and their pain today. Patients who are experiencing severe pain are more likely to not come to the phone leaving the data missing for that particular day; again this would violate the MAR assumption. From political science, roll-call votes, which measure legislatures ideological positions, are subject to non-ignorable nonresponse because, unsurprisingly, politicians behave strategically. In the MNAR case, the parameter of interest may not even be identified (e.g., Robins and Ritov (1997)), let alone be consistently estimated. To be more specific, let $T \in \{0, 1\}$ denote the binary random variable indicating the missing status of the outcome variable Y: Y is observed if T takes the value one and Y is not observed if T takes the value zero. Let $\boldsymbol{X}$ denote a vector of explanatory variables, let $\pi(\boldsymbol{x}, y) = P(T = 1|\boldsymbol{X} = \boldsymbol{x}, Y = y)$ denote the propensity score function and let $f_{Y|\boldsymbol{X}}(y|\boldsymbol{x})$ denote the conditional density function of Y given $\boldsymbol{X}$. Robins and Ritov (1997) shows that if both the propensity score function and the conditional density function are completely unknown, the joint distribution of (T, Y) given $\boldsymbol{X}$ is not identifiable. In this case, a necessary identification condition is the parameterization of either the propensity score function or the conditional density function. Molenberghs and Kenward (2007) proposes the parameterization of both the propensity score function and the conditional density function as an identification strategy, while Sverchkov (2008) and Riddles, Kim and Im (2016) parameterize the propensity score function and only a component of the conditional density function: $f_{Y|\boldsymbol{X},T}(y|\boldsymbol{x}, T = 1)$.

If the joint distribution is not the parameter of interest, the identification strategy above can be modified. For example, if the parameter of interest is the conditional density of Y given $\boldsymbol{X}$ (i.e., $f_{Y|\boldsymbol{X}}(y|\boldsymbol{x})$), parameterization of the propensity score function is not needed. However, parameterization of $f_{Y|\boldsymbol{X}}(y|\boldsymbol{x})$ in this case is not sufficient for identification due to missing data. Tang, Little and Raghunathan (2003) suggests parameterization of the marginal density $f_{\boldsymbol{X}}(\boldsymbol{x})$ as well, while Zhao and Shao (2015) imposes an exclusion retriction. In both studies, $f_{Y|\boldsymbol{X}}(y|\boldsymbol{x})$ is identified and consistently estimated.

We consider estimation of the parameter $\theta_0 = E[U(\boldsymbol{X}, Y)]$, where $U(\cdot)$ is any known function. We suppose that the propensity score π is parameterized but do not restrict the conditional density function of the outcome variable. In earlier work in this framework, either the coefficients in the propensity score function are known or consistently estimated from an external sample (Kim and Yu (2011)) or an exclusion restriction is imposed (Wang, Shao and Kim (2014) and Shao and Wang (2016)). Wang, Shao and Kim (2014) propose a Generalized Method of Moments (hereafter GMM) estimation for θ_0, but their estimator is not efficient because their moments are not optimal. Morikawa and Kim (2016) study estimation of θ_0. They derive the efficient score function (and hence the semiparametric efficiency bound) for θ_0. They propose to estimate the efficient score function by estimating $f_{Y|\boldsymbol{X},T}(y|,\boldsymbol{x},1)$ by a working parametric model (MK1) or by kernel nonparametric estimation (MK2). Their approach MK1 is not efficient unless the working parametric model is correct, although it is consistent. Their method MK2 suffers from the curse of dimensionality (their smoothness conditions depend on the dimensionality of the covariates through their conditions C14) and the bandwidth selection problem (about which they give no guidance).

We study the same estimation problem as in Wang, Shao and Kim (2014) and Morikawa and Kim (2016) but propose a simpler yet equally efficient estimation procedure. Our proposed method does not require explicit nonparametric estimation and hence does not suffer from the curse of dimensionality. The proposed estimator is motivated by the key insight that the model parameter satisfies a parametric conditional moment restriction, of which the semiparametric efficiency bound is identical to the bound derived in Morikawa and Kim (2016). The conditional moment restriction is then turned into an expanding set of unconditional moment restrictions and the parameter of interest is estimated by applying the widely available and easy to compute GMM estimation (see Hansen (1982)). Under some sufficient conditions, we establish that the proposed estimator is consistent and asymptotically normally distributed even if the set of unconditional moment restrictions does not expand, thereby freeing us from the curse of dimensionality and the bandwidth selection problem; when the set does expand, the proposed estimator attains the semiparametric efficiency bound.

The paper is organized as follows. Section 2 describes the estimation, Section 3 derives the large sample properties of the estimator, Section 4 provides a consistent asymptotic variance estimator, Section 5 suggests two data driven approaches to determine the number of unconditional moment restrictions, Section 6 reports on a small scale simulation study, followed by some concluding remarks in Section 7. All technical proofs are relegated to the supplementary materials Ai, Linton and Zhang (2018).

2. Basic Framework and Estimation

We begin by setting up the basic framework. Denote $\boldsymbol{Z} = (\boldsymbol{X}^\top, Y)^\top$. The following assumption shall be maintained throughout the paper:

Assumption 2.1. (i) Parameterization of data missing mechanism: $P(T = 1|Y, \boldsymbol{X}) = \pi(Y, \boldsymbol{X}; \gamma_0) = \pi(\boldsymbol{Z}; \gamma_0)$ holds for some known function $\pi(.;.)$, where $\gamma_0 \in \mathbb{R}^p$ for some known $p \in \mathbb{N}$ is the true (unknown) value; (ii) exclusion restriction: there exists some nonresponse instrument variables $\boldsymbol{X}_1$ in $\boldsymbol{X} = (\boldsymbol{X}_1^\top, \boldsymbol{X}_2^\top)^\top$ so that $\boldsymbol{X}_2$ is independent of T given both $\boldsymbol{X}_1$ and Y; and (iii) the parameter of interest is $\theta_0 = E[U(\boldsymbol{Z})]$ for some known function $U(\cdot)$.

Under Assumption 2.1 and by applying the law of iterated expectations, we obtain the following conditional moment restrictions:

$$E\left[1 - \frac{T}{\pi(\boldsymbol{Z}; \gamma_0)}\middle|\boldsymbol{X}\right] = 0, \tag{2.1}$$

$$E\left[\theta_0 - \frac{T}{\pi(\boldsymbol{Z}; \gamma_0)}U(\boldsymbol{Z})\right] = 0, \tag{2.2}$$

which will form the basis for the proposed estimation. We notice that the parameters of interest in (2.1)-(2.2) are finite dimensional (and there is no explicit infinite dimensional nuisance parameter) and can be easily estimated with GMM estimation.

The (nuisance) parameter γ_0 is identified by (2.1) and the parameter of interest θ_0 is identified by (2.2). The following condition shall also be maintained throughout the paper:

Assumption 2.2. The parameter space Γ is a compact subset of $\mathbb{R}^p$. The true value γ_0 lies in the interior of Γ and is the only solution to (2.1). The parameter space Θ is a compact subset of $\mathbb{R}$ and the true value θ_0 lies in the interior of Θ.

To estimate model (2.1)-(2.2), we first turn it into a set of unconditional moment restrictions. We work with a set of known basis functions: for each integer $K \in \mathbb{N}$ with $K \geq p$, let $u_K(\boldsymbol{X}) = (u_{1K}(\boldsymbol{X}), \ldots, u_{KK}(\boldsymbol{X}))^\top$. Discussion on the choice of $u_K(\boldsymbol{X})$ and its properties can be found in Chen (2007) and Section 2.2 of the supplemental material. Model (2.1)-(2.2) implies the unconditional moment restrictions:

$$E\left[\left(1 - \frac{T}{\pi(\boldsymbol{Z};\gamma_0)}\right) u_K(\boldsymbol{X})\right] = 0, \tag{2.3}$$

$$E\left[\theta_0 - \frac{T}{\pi(\boldsymbol{Z};\gamma_0)} U(\boldsymbol{Z})\right] = 0. \tag{2.4}$$

To avoid redundant moment restrictions, we require $E\left[u_K(\boldsymbol{X}) u_K(\boldsymbol{X})^\top\right]$ to be nonsingular for every K. The following somewhat stronger identification condition shall be maintained throughout the paper:

Assumption 2.2'. The parameter space Γ is a compact subset of $\mathbb{R}^p$. The true value γ_0 lies in the interior of Γ and is the only solution to (2.3).

The parameter space Θ is a compact subset of $\mathbb{R}$ and the true value θ_0 lies in the interior of Θ.

We can estimate the parameter of interest by the GMM method. Let $\{T_i, \boldsymbol{Z}_i\}_{i=1}^N$ denote an *i.i.d.* sample drawn from the joint distribution of $(T, \boldsymbol{Z})$. Denote $\boldsymbol{G}_K(\gamma, \theta) := \sum_{i=1}^N g_K(T_i, \boldsymbol{Z}_i; \gamma, \theta)$, where $g_K(T, \boldsymbol{Z}; \gamma, \theta) := \big([1 - T\pi(\boldsymbol{Z}; \gamma)^{-1}]\, u_K(\boldsymbol{X})^\top, \theta - T\pi(\boldsymbol{Z}; \gamma)^{-1}U(\boldsymbol{Z})\big)^\top$. The GMM estimator of γ_0 and θ_0 is defined as

$$(\check{\gamma}, \check{\theta}) = \arg \min_{\gamma \in \Gamma, \theta \in \Theta} \boldsymbol{G}_K(\gamma, \theta)^\top \cdot \mathbf{W} \cdot \boldsymbol{G}_K(\gamma, \theta)$$

where $\mathbf{W}$ is a $(K+1) \times (K+1)$ symmetric weighting matrix. For every fixed $K \geq p$, Hansen (1982) shows that, under some regularity conditions, the estimator

$$(\check{\gamma} - \gamma_0, \check{\theta} - \theta_0) = O_p(N^{-1/2}) \tag{2.5}$$

is asymptotically normally distributed, but generally not the best unless the best weighting matrix is used. The best weighting matrix is the inverse of $\boldsymbol{D}_{(K+1)\times(K+1)} := E\left[g_K(T, \boldsymbol{Z}; \gamma_0, \theta_0) g_K(T, \boldsymbol{Z}; \gamma_0, \theta_0)^\top\right]$. The best estimator (within the class defined by the specific unconditional moments) is defined as

$$(\overline{\gamma}, \overline{\theta}) = \arg \min_{\gamma \in \Gamma, \theta \in \Theta} \boldsymbol{G}_K(\gamma, \theta)^\top \cdot \boldsymbol{D}_{(K+1)\times(K+1)}^{-1} \cdot \boldsymbol{G}_K(\gamma, \theta).$$

Suppose that the propensity score function is differentiable with respect to

γ. Denote

$$\boldsymbol{B}_{(K+1)\times(p+1)} = \nabla_{\gamma,\theta} E\left[\frac{1}{N}\boldsymbol{G}_K(\gamma_0,\theta_0)\right] = \begin{pmatrix} E\left[u_K(\boldsymbol{X})\frac{\nabla_\gamma \pi(\boldsymbol{Z};\gamma_0)^\top}{\pi(\boldsymbol{Z};\gamma_0)}\right], & \mathbf{0}_{K\times 1} \\ E\left[U(\boldsymbol{Z})\frac{\nabla_\gamma \pi(\boldsymbol{Z};\gamma_0)^\top}{\pi(\boldsymbol{Z};\gamma_0)}\right], & 1 \end{pmatrix}$$

and

$$\boldsymbol{V}_K = \left\{\left(\boldsymbol{B}_{(K+1)\times(p+1)}\right)^\top \boldsymbol{D}^{-1}_{(K+1)\times(K+1)}\left(\boldsymbol{B}_{(K+1)\times(p+1)}\right)\right\}^{-1}.$$

Hansen (1982) shows that, for every fixed $K \geq p$,

$$\boldsymbol{V}_K^{-1/2}\begin{pmatrix}\sqrt{N}(\overline{\gamma}-\gamma_0) \\ \sqrt{N}(\overline{\theta}-\theta_0)\end{pmatrix} \to N\left(0, I_{(p+1)\times(p+1)}\right) \text{ in distribution.} \tag{2.6}$$

Since the best weighting matrix depends on the unknown parameter value, the best estimator $(\overline{\gamma},\overline{\theta})$ is infeasible. Hansen (1982) suggests the following two-step procedure:

Step I. Compute the initial $\sqrt{N}$-consistent estimator

$$\widehat{\boldsymbol{W}}_0 := \begin{pmatrix} \frac{1}{N}\sum_{i=1}^N u_K(\mathbf{X}_i)u_K(\mathbf{X}_i)^\top & \mathbf{0}_{K\times 1} \\ \mathbf{0}_{K\times 1}^\top & 1 \end{pmatrix},$$

$$(\check{\gamma},\check{\theta}) = \arg\min_{(\gamma,\theta)\in\Gamma\times\Theta} \boldsymbol{G}_K(\gamma,\theta)^T \cdot \widehat{\boldsymbol{W}}_0^{-1} \cdot \boldsymbol{G}_K(\gamma,\theta).$$

Step II. Compute the best weighting matrix and the best estimator

$$\hat{\boldsymbol{D}}_{(K+1)\times(K+1)} := \frac{1}{N}\sum_{i=1}^N g_K(T_i,\boldsymbol{Z}_i;\check{\gamma},\check{\theta})g_K(T_i,\boldsymbol{Z}_i;\check{\gamma},\check{\theta})^\top,$$

$$(\widehat{\gamma},\widehat{\theta}) = \arg\min_{\gamma\in\Gamma,\theta\in\Theta} \boldsymbol{G}_K(\gamma,\theta)^T \cdot \widehat{\boldsymbol{D}}^{-1}_{(K+1)\times(K+1)} \cdot \boldsymbol{G}_K(\gamma,\theta).$$

Hansen (1982) establishes that, for every fixed $K \geq p$,

$$\boldsymbol{V}_K^{-1/2}\begin{pmatrix}\sqrt{N}(\widehat{\gamma}-\gamma_0)\\ \sqrt{N}(\widehat{\theta}-\theta_0)\end{pmatrix} \to N\left(0, I_{(p+1)\times(p+1)}\right) \text{ in distribution.} \tag{2.7}$$

Moreover, denote

$$\widehat{\boldsymbol{B}}_{(K+1)\times(p+1)} := \begin{pmatrix} N^{-1}\sum_{i=1}^N u_K(\boldsymbol{X}_i)\frac{\nabla_\gamma \pi(\boldsymbol{Z}_i;\widehat{\gamma})^\top}{\pi(\boldsymbol{Z}_i;\widehat{\gamma})}, & \boldsymbol{0}_{K\times 1} \\ \\ N^{-1}\sum_{i=1}^N U(\boldsymbol{Z}_i)\frac{\nabla_\gamma \pi(\boldsymbol{Z}_i;\widehat{\gamma})^\top}{\pi(\boldsymbol{Z}_i;\widehat{\gamma})}, & 1 \end{pmatrix}$$

and

$$\widehat{\boldsymbol{V}}_K := \left\{\left(\widehat{\boldsymbol{B}}_{(K+1)\times(p+1)}\right)^\top \widehat{\boldsymbol{D}}^{-1}_{(K+1)\times(K+1)}\left(\widehat{\boldsymbol{B}}_{(K+1)\times(p+1)}\right)\right\}^{-1}.$$

Hansen (1982) proves that, for every fixed $K \geq p$,

$$\widehat{\boldsymbol{V}}_K \to \boldsymbol{V}_K \text{ in probability.} \tag{2.8}$$

Remark 1: *Despite the popularity and theoretical appeal of the use of inverse propensity score weighting, a major practical weakness of this method is the parameterization of the propensity score. It is well established that slight misspecification of the propensity score function can lead to substantial bias (Kang and Schafer, 2007). In practice, applied researchers can apply diagnostic procedures, such as "the propensity score tautology" proposed in Imai, King and Stuart (2008), to select a particular parametric* *function from a set of prespecified family of functions. The "propensity score tautology" procedure selects a particular propensity score if it balances the covariates. Specifically, for a fixed $K \in \mathbb{N}$ larger than p, Hansen's J-statistic:*

$$J = N\left\{\frac{1}{N}\boldsymbol{G}_K(\hat{\gamma},\hat{\theta})^\top \hat{\boldsymbol{D}}^{-1}_{(K+1)\times(K+1)}\frac{1}{N}\boldsymbol{G}_K(\hat{\gamma},\hat{\theta})\right\} \xrightarrow{d} \chi^2_{K-p},$$

can be employed to test the null hypothesis of the propensity score being correctly specified. If the propensity score is correctly specified, the deviation of this statistic from 0 should be within the range of sampling error. For details, see Kosuke and Marc (2015). For other model assessment in missing data analysis, see Ibrahim and Molenberghs (2009).

Remark 2: *There are two approaches in the semiparametric estimation for nonignorable missing data: the moment-based approach and the empirical likelihood approach. Morikawa and Kim (2018) establish the equivalence between the empirical likelihood estimator (Owen, 2004) and the moment-based estimator. To describe the empirical likelihood estimation for our model, suppose that the first n Y_i are observed and the remaining $(N-n)$ Y_i are missing, namely, $T_i = 1$ for $i = 1, ..., n$ and $T_i = 0$ for $i = n+1, ..., N$. Qin, Leung and Shao (2002) construct the likelihood using the data with*

$T_i = 1$ *by*

$$\prod_{i=1}^{n} \pi(\boldsymbol{Z}_i;\gamma)dF(\boldsymbol{Z}_i) \prod_{i=n+1}^{N} \int \{1-\pi(\boldsymbol{z};\gamma)\}dF(\boldsymbol{z}), \tag{2.9}$$

and discretize the distribution F *by* w_i $(i = 1,\ldots,n)$. *The discretized distribution* w_i *can be estimated by maximizing* $\prod_{i=1}^{n} w_i$ *subject to the following constraints:*

$$\begin{cases} w_i \geq 0,\ \sum_{i=1}^{n} w_i = 1,\ \sum_{i=1}^{n} w_i\{\pi(\boldsymbol{Z}_i;\gamma)-\mu_T\} = 0, \\ \sum_{i=1}^{n} w_i\{u_K(\boldsymbol{X}_i)-\bar{u}_K\} = 0,\ \sum_{i=1}^{n} w_i\{U(\boldsymbol{Z}_i)-\theta\} = 0, \end{cases}$$

where $\mu_T := E[T] = \int \pi(\boldsymbol{z};\gamma)dF(\boldsymbol{z})$ *and* $\bar{u}_K := N^{-1}\sum_{i=1}^{N} u_K(\boldsymbol{X}_i)$. *The approximating basis functions* $u_K(\boldsymbol{X})$ *shall be allowed to increase the estimation efficiency. With* λ_1, λ_2 *and* λ_3 *as Lagrange multipliers, the solution to above optimization problem is* $\hat{w}_i^{-1} = n[1+\lambda_1^\top\{u_K(\boldsymbol{X}_i)-\bar{u}_K\} + \lambda_2\{U(\boldsymbol{Z}_i)-\theta\} + \lambda_3\{\pi(\boldsymbol{Z}_i;\gamma)-\mu_T\}]$. *Profiling out the unknown* F *with the estimates* $\hat{w}_i$ $(i = 1,\ldots,n)$ *in* (2.9) *and taking logarithm, we obtain the profile pseudo-loglikelihood:*

$$\begin{aligned} \ell(\gamma,\theta,\mu_T,\lambda_1,\lambda_2) &= \sum_{i=1}^{n} \log \pi(\boldsymbol{Z}_i;\gamma) \\ &\quad - \sum_{i=1}^{n} \log[1+\lambda_1^\top\{u_K(\boldsymbol{X}_i)-\bar{u}_K\}+\lambda_2\{U(\boldsymbol{Z}_i)-\theta\}+\lambda_3\{\pi(\boldsymbol{Z}_i;\gamma)-\mu_T\}] \\ &\quad + (N-n)\log(1-\mu_T) \end{aligned} \tag{2.10}$$

where $\lambda_3 = (N/n-1)/(1-\mu_T)$. *Morikawa and Kim (2018) show that the* *empirical likelihood estimator is exactly the same as the GMM estimator. We choose the GMM method based on its computational simplicity.*

The best GMM estimator (within the class defined by the specific unconditional moments) is generally not semiparametrically efficient. To obtain the efficient estimator, we shall allow K to increase with the sample size at the rate $o(N^{1/3})$ so that $\{u_K(\boldsymbol{X})\}$ span the space of measureable functions (see also Geman and Hwang (1982) and Newey (1997)). In the next two sections, we shall establish that results in (2.5)-(2.8) still hold with expanding $K = o(N^{1/3})$.

One attraction of the proposed estimator over the existing estimators is that it does not require the estimation of $f_{Y|\boldsymbol{X},T}(y|\boldsymbol{x},1)$ and instead relies on the moment conditions. Since the number of the unknown parameters is fixed and finite and is independent of the number of covariates, the proposed estimator is always consistent as long as the number of moment conditions exceeds the number of the unknown parameters (i.e., $K \geq p$). Further increasing the moment conditions only improves efficiency. Therefore, the classical tradeoff between bias and variance phenomenon in the nonparametric estimation does not apply here. This is in constrast with the estimators proposed by Riddles et al. (2016) and Morikawa and Kim (2016) which require estimation of $f_{Y|\boldsymbol{X},T}(y|\boldsymbol{x},1)$. The estimator of Morikawa and Kim

(2016) is consistent even if the parametric specification of $f_{Y|\boldsymbol{X},T}(y|\boldsymbol{x},1)$ is incorrect, but it is inefficient. If $f_{Y|\boldsymbol{X},T}(y|\boldsymbol{x},1)$ is nonparametrically estimated (e.g., kernel estimation), the resulting estimator suffers from the curse of dimensionality and bandwidth selection problem.

3. Asymptotic Theory

In this section, we show that results in (2.5)- (2.7) still hold with expanding K, all technical proof can be found in the supplemental material Ai, Linton and Zhang (2018). First, we establish the convergence rate of the first step estimator $(\check{\gamma}, \check{\theta})$.

Theorem 1. *Under Assumptions 2.1-2.2' and Assumptions 1, 2, 4, 6, and 7 listed in Appendix, with $K = o(N^{1/3})$, the first step estimator satisfies $(\check{\gamma} - \gamma_0, \check{\theta} - \theta_0) = O_p\left(N^{-1/2}\right)$.*

Next, we establish the large sample properties of the infeasible best estimator $(\overline{\gamma}, \overline{\theta})$ without imposing the smoothness Assumptions 3 and 5 listed in Appendix.

Theorem 2. *Under Assumptions 2.1-2.2' and Assumptions 1, 2, 4, 6, and 7 listed in Appendix, with $K = o(N^{1/3})$, the infeasible best estimator* *satisfies*

$$\boldsymbol{V}_K^{-1/2}\begin{pmatrix}\sqrt{N}(\overline{\gamma}-\gamma_0)\\ \sqrt{N}(\overline{\theta}-\theta_0)\end{pmatrix} \to N\left(0, I_{(p+1)\times(p+1)}\right) \text{ in distribution.}$$

If in addition the smoothness Assumptions 3 and 5 are satisfied, the next result shows that $\boldsymbol{V}_K \to \boldsymbol{V}_{eff}$, where $\boldsymbol{V}_{eff} := E[\boldsymbol{S}_{eff}\boldsymbol{S}_{eff}^\top]^{-1}$ is the semiparametric efficiency bound of (γ_0, θ_0) derived in Morikawa and Kim (2016), $\boldsymbol{S}_{eff} = (\boldsymbol{S}_1^\top, S_2)^\top$, and $\boldsymbol{S}_1$, S_2 are defined in (10.12) and (10.13).

Theorem 3. *Under Assumption 2.1-2.2' and Assumption 1-7 listed in Appendix, with $K = o(N^{1/3})$, we obtain $\boldsymbol{V}_K \to \boldsymbol{V}_{eff}$.*

By Theorem 1-3, the infeasible best estimator attains the semiparametric efficiency bound. The next result establishes the equivalence between the best estimator $(\widehat{\gamma}, \widehat{\theta})$ and the infeasible best estimator $(\overline{\gamma}, \overline{\theta})$, implying that the best estimator also attains the semiparametric efficiency bound.

Theorem 4. *Under Assumption 2.1-2.2' and Assumption 1-7 listed in Appendix, with $K = o(N^{1/3})$, we obtain $(\sqrt{N}(\overline{\gamma}-\widehat{\gamma}), \sqrt{N}(\overline{\theta}-\widehat{\theta})) = o_p(1)$.*

4. Variance Estimation

In order to conduct statistical inference, we need a consistent covariance estimator. Notice that (2.5) implies that $\widehat{\boldsymbol{V}}_K$ is a consistent estimator of

$\boldsymbol{V}_K$ for every fixed $K \geq p$. We now show that this result still holds true with expanding K, thereby providing a consistent covariance estimator.

Theorem 5. *Under Assumption 2.1-2.2' and Assumption 1-7 listed in Appendix, with $K = o(N^{1/3})$, we obtain $\widehat{\boldsymbol{V}}_K \to \boldsymbol{V}_K$ in probability.*

We notice that our covariance estimator is much simpler and more natural than the one suggested in Morikawa and Kim (2016), which requires nonprametric estimation of $f_{Y|\boldsymbol{X},T}(y|x,1)$ and tends to have poor performance in finite samples. Our covariance estimator is the GMM covariance estimator and is easily computed by existing statistical packages.

5. Selection of K

The large sample properties of the proposed estimator established in the previous sections allow for a wide range of values for K, and theoretically the sensitivity of the estimator to the choice of K is not so pronounced, it affects higher order terms in a way that does not affect consistency and asymptotic normality. Nevertheless, there may be some higher order effect of the choice of K on perfomance. In this section, we present two data-driven approaches to select K.

Covariate balancing approach. The first approach attempts to balance the distribution of the covariates between the whole population and the non-missing population through weighting. Notice that

$$E\left[\frac{T}{\pi(\boldsymbol{Z};\gamma_0)}I(X_j \le x_j)\right] = E[I(X_j \le x_j)],\ j \in \{1,...,r\},$$

where X_j is the j^{th} component of $\boldsymbol{X}$ and $I(X_j \le x_j)$ is the indicator function. Obviously the propensity score function $\pi(\boldsymbol{Z};\gamma_0)$ plays the role of balancing. Notice that the estimator $\hat{\gamma}$ depends on K. For a given K, we compute

$$\hat{F}^j_{N,K}(x_j) := \frac{1}{N}\sum_{i=1}^{N}\frac{T_i}{\pi(\boldsymbol{X}_i;\hat{\gamma})}I(X_{ij} \le x_j),\ j \in \{1,\dots,r\}.$$

We compute the empirical distributions of the covariates

$$\tilde{F}^j_{N}(x_j) := \frac{1}{N}\sum_{i=1}^{N}I(X_{ij} \le x_j),\ j \in \{1,\dots,r\}.$$

We choose the lowest K so that the difference between $\{\hat{F}^j_{N,K}\}_{j=1}^r$ and $\{\hat{F}^j_{N}\}_{j=1}^r$ is small. Denote the upper bound of K by $\bar{K}$ (e.g. $\bar{K}=7$ in our simulation studies). We choose $K \in \{1,...,\bar{K}\}$ to minimize the aggregate Kolmogorov-Smirnov distance between $\{\hat{F}^j_{N,K}\}_{j=1}^r$ and $\{\hat{F}^j_{N}\}_{j=1}^r$:

$$\hat{K} = \arg\min_{K\in\{1,\dots,\bar{K}\}} D_N(K) := \sum_{j=1}^{r}\sup_{x_j\in\mathbb{R}}\left|\tilde{F}^j_{N}(x_j) - \hat{F}^j_{N,K}(x_j)\right|.$$

Higher order MSE approach. The second approach chooses K to minimize the mean-squared error of the estimator. Donald, Imbens and Newey (2009) derives the higher-order asymptotic mean-square error (MSE) of a linear combination $\mathbf{t}^\top\hat{\gamma}$ for some fixed $\mathbf{t} \in \mathbb{R}^p$. Let $\check{\gamma}$ be some preliminary estimator. Define:

$$\widehat{\Pi}(K;\boldsymbol{t}) = \sum_{i=1}^{N} \hat{\xi}_{ii}\rho(T_i, \boldsymbol{X}_i, Y_i; \check{\gamma}) \cdot (\mathbf{t}^\top \hat{\Omega}_{p\times p}^{-1}\tilde{\eta}_i),$$

$$\hat{\Phi}(K;\boldsymbol{t}) = \sum_{i=1}^{N} \hat{\xi}_{ii} \left\{ \mathbf{t}^\top \hat{\Omega}_{p\times p}^{-1} \left[\widehat{\mathbf{D}}_i^* \rho(T_i, \boldsymbol{X}_i, Y_i; \check{\gamma})^2 - \nabla_\gamma \rho(T_i, \boldsymbol{X}_i, Y_i; \check{\gamma}) \right] \right\}^2 - \mathbf{t}^\top \hat{\Omega}_{p\times p}^{-1} (\hat{\Gamma}_{K\times p})^\top \hat{\Upsilon}_{K\times K}^{-1} \hat{\Gamma}_{K\times p} \hat{\Omega}_{p\times p}^{-1} \boldsymbol{t} \, .$$

where $\rho(T_i, \boldsymbol{X}_i, Y_i; \check{\gamma})$, $\hat{\Omega}_{p\times p}$, $\tilde{\eta}_i$, $\hat{\xi}_{ii}$, $\hat{\boldsymbol{D}}_i^*$, $\hat{\Gamma}_{K\times p}$, and $\hat{\Upsilon}_{K\times K}$ are defined in Section 10.1 of Appendix. Notice that $\widehat{\Pi}(K;\boldsymbol{t})^2/N$ is an estimate of the squared bias term derived in Newey and Smith (2004) and $\hat{\Phi}(K;\boldsymbol{t})$ is the asymptotic variance. The second approach chooses K to minimize the following higher-order MSEs of $\hat{\gamma}_j, j = 1, \ldots, p$:

$$S_{GMM}(K) = \sum_{j=1}^{p} \left\{ \frac{1}{N} \widehat{\Pi}(K; e_j)^2 + \hat{\Phi}(K; e_j) \right\} , \tag{5.11}$$

where e_j is the j^{th} column of the p-dimensional identity matrix. In practice, we set the upper bound $\bar{K}$ and then choose $K \in \{1, 2, \ldots, \bar{K}\}$ to minimize the criteria (5.11) .

6. Simulations

After establishing the large sample properties of the proposed estimator, we now evaluate its finite sample performance through a small scale simulation study. We consider four scenarios. In all scenarios, the parameter of interest is $\theta_0 = E[Y]$ and the sample size is set respectively at $N = 200, 500$ and 1000.

- **Scenario I**: X is generated from the normal distribution $N(0, 1)$, and the outcome Y is generated from the normal distribution with mean $X + 1$ and unit variance, i.e. $Y \sim N(X + 1, 1)$. The relationship between the outcome variable and the covariate is linear, and the distribution of outcome is normal. The missing mechanism is modeled by $P(T = 1|Y, X) = [1 + \exp(\alpha_0 + \beta_0 Y)]^{-1}$ with the true value $(\alpha_0, \beta_0) = (0, -1.2)$. The true value of the parameter of interest is $\theta_0 = E[Y] = 1$.

- **Scenario II**: X is generated from the normal distribution $N(0, 1)$, and the outcome Y is generated from the normal distribution with mean $X^2 + 1$ and unit variance, i.e. $Y \sim N(X^2 + 1, 1)$. Thus the relationship between the outcome variable and the covariate is nonlinear, and the distribution of outcome is non-normal. The missing mechanism is modeled as $P(T = 1|Y, X) = [1 + \exp(\alpha_0 + \beta_0 Y)]^{-1}$ with the true value $(\alpha_0, \beta_0) = (1.25, -1.2)$. The true value of the parameter of interest is $\theta_0 = E[Y] = 2$.

- **Scenario III**. The design follows Qin et al. (2002). We generate the outcome from $Y = 0.1X^2 + ZX^{1/2}/5$, where Z and X are independent, Z is standard normal random variable, and X follows the $\chi^2_{(6)}/2$ distribution. The missing mechanism is modeled as $P(T = 1|Y, X) = [1 + \exp(\alpha_0 + \beta_0 Y)]^{-1}$ with the true value $(\alpha_0, \beta_0) = (3, -1)$. The true value of the target parameter is $\theta_0 = E[Y] = 1.2$.

- **Scenario IV**. The design is similar to that in Kang and Schafer (2007). $\boldsymbol{Z} = (Z_1, Z_2)$ is generated from the standard bivariate normal distribution, and Y is generated from the normal distribution with mean $2 + Z_1$ and unit variance. The missing mechanism is modeled as $P(T = 1|Y, X_1, X_2) = [1 + \exp(\alpha_0 Z_1 + \beta_0 Y)]^{-1}$ with $(\alpha_0, \beta_0) = (1, -1)$. The true value of the parameter of interest is $\theta_0 = E[Y] = 2$. Instead of directly observing covariates $\boldsymbol{Z}$, we observe a non-linear transformation of $\boldsymbol{Z}$: $X_1 = \exp(Z_1/2)$ and $X_2 = Z_2/(1 + \exp(Z_1))$.

In all scenarios, we generate $J = 500$ random samples, and for each sample, we compute the following three estimators:

1. Naive estimator. We compute the missing at random estimator $(\tilde{\alpha}_{MAR}, \tilde{\beta}_{MAR}, \tilde{\theta}_{MAR})$ as $\tilde{\theta}_{MAR} = N^{-1} \sum_{i=1}^{N} T_i Y_i / \pi(\boldsymbol{X}_i; \tilde{\alpha}_{MAR}, \tilde{\beta}_{MAR})$, where $\pi(\boldsymbol{X}_i; \tilde{\alpha}_{MAR}, \tilde{\beta}_{MAR})$ is an estimated response model. In Scenarios I, II & III, $\pi(\boldsymbol{X}_i; \tilde{\alpha}_{MAR}, \tilde{\beta}_{MAR}) =$

$\left[1+\exp(\tilde{\alpha}_{MAR}+\tilde{\beta}_{MAR}X_i)\right]^{-1}$ and in Scenario IV $\pi(\boldsymbol{X}_i;\tilde{\alpha}_{MAR},\tilde{\beta}_{MAR}) = \left[1+\exp(\tilde{\alpha}_{MAR}Z_{1i}+\tilde{\beta}_{MAR}X_{2i})\right]^{-1}$, where $(\tilde{\alpha}_{MAR},\tilde{\beta}_{MAR})$ are estimated by GMM.

2. MK2 estimator. We compute $(\hat{\alpha}_{MK},\hat{\beta}_{MK},\hat{\theta}_{MK})$ using the approach of Morikawa and Kim (2016), i.e. $(\hat{\alpha}_{MK},\hat{\beta}_{MK},\hat{\theta}_{MK})$ is the solution of

$$\sum_{i=1}^{N}\left(\hat{\boldsymbol{S}}_1(T_i,\boldsymbol{Z}_i;\alpha,\beta)^\top,\hat{S}_2(T_i,\boldsymbol{Z}_i;\alpha,\beta,\theta)\right)^\top = 0,$$

where

$$\hat{\boldsymbol{S}}_1(T,\boldsymbol{Z};\alpha,\beta) = -\left(1-\frac{T}{\pi(\boldsymbol{Z};\alpha,\beta)}\right)E^\star\left[\frac{\nabla_\gamma\pi(\boldsymbol{Z};\alpha,\beta)}{1-\pi(\boldsymbol{Z};\alpha,\beta)}\middle|\boldsymbol{X}\right],$$

$$\hat{S}_2(T,\boldsymbol{Z};\alpha,\beta,\theta) = -\frac{T}{\pi(\boldsymbol{Z};\alpha,\beta)}U(\boldsymbol{Z})+\theta-\left(1-\frac{T}{\pi(\boldsymbol{Z};\alpha,\beta)}\right)E^\star\left[U(\boldsymbol{Z})|\boldsymbol{X}\right],$$

and for any function $g(\boldsymbol{Z})$ the quantity $E^\star[g(\boldsymbol{Z})|\boldsymbol{X}]$ is defined by

$$E^\star[g(\boldsymbol{Z})|\boldsymbol{X}=\boldsymbol{x}] := \frac{\sum_{j=1}^{N}T_jK_h(\boldsymbol{x}-\boldsymbol{X}_j)T_j\pi(\boldsymbol{Z}_j;\alpha,\beta)^{-1}O(\boldsymbol{x},Y_j;\alpha,\beta)g(\boldsymbol{x},Y_j)}{\sum_{j=1}^{N}K_h(\boldsymbol{x}-\boldsymbol{X}_j)T_j\pi(\boldsymbol{Z}_j;\alpha,\beta)^{-1}O(\boldsymbol{x},Y_j;\alpha,\beta)},$$

$$O(\boldsymbol{z};\alpha,\beta) = \frac{1-\pi(\boldsymbol{z};\alpha,\beta)}{\pi(\boldsymbol{z};\alpha,\beta)},\ K_h(\boldsymbol{x}-\boldsymbol{w}) = K\left((\boldsymbol{x}-\boldsymbol{w}/h)\right),$$

$K(\cdot)$ is Gaussian kernel function and h is the bandwidth. Because Morikawa and Kim (2016) does not describe how to select the bandwidth, we choose $h=0.1$ in Scenarios I, II and III, and $h=0.2$ in Scenario IV (the numeric computation fails if $h=0.1$ in Scenario IV, perhaps due to overfitting in the multivariate case).

3. WSK Eestimator. We compute $(\hat{\alpha}_{WSK}, \hat{\beta}_{WSK}, \hat{\theta}_{WSK})$ using the approach of Wang, Shao and Kim (2014), i.e. $(\hat{\alpha}_{WSK}, \hat{\beta}_{WSK}, \hat{\theta}_{WSK})$ is a GMM estimator from the following moments

$$E\begin{bmatrix} \frac{T}{\pi(\boldsymbol{X},Y;\gamma)} - 1 \\ \left\{\frac{T}{\pi(\boldsymbol{X},Y;\gamma)} - 1\right\}\boldsymbol{X} \\ \frac{T}{\pi(\boldsymbol{X},Y;\gamma)}Y - \theta \end{bmatrix} = 0.$$

4. Our GMM estimator. We compute $(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ using the proposed approach and the covariate balancing approach to select K, with $\bar{K} = 7$ in Scenarios I, II, III, and with $\bar{K} = 10$ in Scenario IV. Here $\bar{K}$ is the maximal number of candidate moments to be considered.

The simulation results (the bias, the standard deviation (Stdev), the mean squared error (MSE), and the coverage probability (CP) (for significance level $\alpha = 0.05$) of the point estimates) for all scenarios are reported in Tables 1, 2, 3, and 4 respectively. The histogram of selected K's (based on 500 Monte Carlo samples) in all scenarios is reported in Figure 1. Glancing at these tables, we find:

1. In all scenarios, the naive estimator using the missing at random assumption has a large bias, because this assumption does not hold.

2. In all scenarios, our proposed estimator of $E[Y]$ out-performs the MK

estimator.

3. In all scenarios, our proposed estimator of $E[Y]$ is always consistent in the sense that its bias declines to zero as sample size increases. However, in Scenario II and IV, the WSK estimator has large bias and the bias does not shrink to zero as sample size increases.

4. In all scenarios, our proposed estimators of the nuisance parameter α_0 and β_0 in the response model are always consistent. MK estimators of the nuisance parameters have reasonable performance in Scenarios I, III and IV, but have a large bias in Scenario II. WSK estimators of the nuisance parameters have relative poor performance in Scenarios II and IV.

5. In all scenarios, our proposed variance estimator has coverage probability close to 95%, even the sample size is small. The MK's variance estimator performs well in Scenario IV, but badly in other scenarios: in Scenario I, the coverage probability using MK's approach converges to 90% rather than 95%; in Scenario II, the CP values are far from 95% in Scenario 2 no matter the sample size is small or large; in Scenario III, the MK's variance estimator is consistent only when the sample size is large.

6. When the sample size is small the optimal K tends to be 2 with large probability. When the sample size is large, the optimal K tends to be 3 with large probability. The growing rate of K is extremely slow comparing to that of the sample size n, which is consistent with our theoretical Assumption 7.

These results clearly show that the proposed approach has better finite sample performance.

7. Discussion

The data missing not at random problem is common in applications. Morikawa and Kim (2016) studies the efficient estimation of a class of missing not at random problems. They propose two estimators by parametrically or non-parametrically estimate $f_{Y|\boldsymbol{X},T}(y|\boldsymbol{x},1)$. If the working model of $f_{Y|\boldsymbol{X},T}(y|\boldsymbol{x},1)$ is misspecified, the parametric estimator is consistent but not efficient. In this paper, we study the same class of missing not at random problems but present a much simpler and more natural efficient estimator. Our approach is based on a parametric moment restriction model that does not require nonparametric estimation and hence does not suffer from the curse of dimensionality problem nor the bandwidth selection problem. Indeed the simulation results confirm that the proposed approach out-performs their

Table 1: Simulation results under Scenorio I

$n = 200$												
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}_{MK}$	$\hat{\beta}_{MK}$	$\hat{\theta}_{MK}$	$\hat{\alpha}_{WSK}$	$\hat{\beta}_{WSK}$	$\hat{\theta}_{WSK}$	$\tilde{\alpha}_{MAR}$	$\tilde{\beta}_{MAR}$	$\tilde{\theta}_{MAR}$
Bias	0.028	-0.125	0.039	0.040	0.202	0.134	-0.140	-0.448	-0.031	-0.997	0.167	0.301
Stdev	0.254	0.413	0.129	0.229	0.256	0.118	0.910	1.183	0.150	0.197	0.266	0.101
MSE	0.065	0.186	0.018	0.054	0.106	0.032	0.849	1.601	0.023	1.033	0.099	0.101
CP	—	—	0.906	—	—	0.860	—	—	0.946	—	—	0.220
$n = 500$												
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}_{MK}$	$\hat{\beta}_{MK}$	$\hat{\theta}_{MK}$	$\hat{\alpha}_{WSK}$	$\hat{\beta}_{WSK}$	$\hat{\theta}_{WSK}$	$\tilde{\alpha}_{MAR}$	$\tilde{\beta}_{MAR}$	$\tilde{\theta}_{MAR}$
Bias	0.011	-0.067	0.016	0.050	0.097	0.083	-0.029	-0.147	-0.014	-0.966	0.220	0.299
Stdev	0.161	0.282	0.090	0.152	0.173	0.075	0.207	0.403	0.108	0.126	0.160	0.063
MSE	0.026	0.084	0.008	0.025	0.039	0.013	0.044	0.184	0.012	0.949	0.074	0.093
CP	—	—	0.928	—	—	0.844	—	—	0.948	—	—	0.034

$n = 1000$												
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}_{MK}$	$\hat{\beta}_{MK}$	$\hat{\theta}_{MK}$	$\hat{\alpha}_{WSK}$	$\hat{\beta}_{WSK}$	$\hat{\theta}_{WSK}$	$\tilde{\alpha}_{MAR}$	$\tilde{\beta}_{MAR}$	$\tilde{\theta}_{MAR}$
Bias	0.005	-0.040	0.008	0.041	0.048	0.053	-0.002	-0.057	-0.002	-0.962	0.235	0.298
Stdev	0.103	0.187	0.065	0.103	0.128	0.055	0.1118	0.218	0.073	0.078	0.099	0.045
MSE	0.010	0.036	0.004	0.012	0.018	0.006	0.012	0.051	0.005	0.932	0.065	0.091
CP	—	—	0.934	—	—	0.864	—	—	0.956	—	—	0.012

Stdev: standard deviation ; MSE: mean squared error; CP: coverage probability. The bandwidth used in computing the nonparametric kernel estimators $(\hat{\alpha}_{MK}, \hat{\beta}_{MK}, \hat{\theta}_{MK})$ is $h = 0.1$.

Table 2: Simulation results under Scenorio II

$n=200$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}_{MK}$	$\hat{\beta}_{MK}$	$\hat{\theta}_{MK}$	$\hat{\alpha}_{WSK}$	$\hat{\beta}_{WSK}$	$\hat{\theta}_{WSK}$	$\tilde{\alpha}_{MAR}$	$\tilde{\beta}_{MAR}$	$\tilde{\theta}_{MAR}$
Bias	-0.208	0.096	0.084	-0.254	0.254	0.114	-0.849	-2.158	0.003	-2.053	1.215	0.530
Stdev	0.646	0.555	0.201	0.381	0.252	0.134	8.939	16.509	0.367	0.809	0.148	0.205
MSE	0.462	0.318	0.047	0.210	0.128	0.031	80.632	277.217	0.134	4.873	1.498	0.323
CP	—	—	0.950	—	—	0.910	—	—	0.890	—	—	0.138

$n=500$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}_{MK}$	$\hat{\beta}_{MK}$	$\hat{\theta}_{MK}$	$\hat{\alpha}_{WSK}$	$\hat{\beta}_{WSK}$	$\hat{\theta}_{WSK}$	$\tilde{\alpha}_{MAR}$	$\tilde{\beta}_{MAR}$	$\tilde{\theta}_{MAR}$
Bias	-0.081	0.040	0.044	-0.119	0.140	0.073	-0.648	-1.532	-0.077	-1.924	1.203	0.583
Stdev	0.406	0.363	0.131	0.262	0.188	0.096	8.289	10.832	0.349	0.175	0.064	0.132
MSE	0.171	0.134	0.019	0.083	0.055	0.014	69.132	119.697	0.128	3.732	1.451	0.357
CP	—	—	0.932	—	—	0.894	—	—	0.910	—	—	0.06

$n=1000$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}_{MK}$	$\hat{\beta}_{MK}$	$\hat{\theta}_{MK}$	$\hat{\alpha}_{WSK}$	$\hat{\beta}_{WSK}$	$\hat{\theta}_{WSK}$	$\tilde{\alpha}_{MAR}$	$\tilde{\beta}_{MAR}$	$\tilde{\theta}_{MAR}$
Bias	-0.036	0.019	0.019	-0.078	0.093	0.047	-0.111	-0.856	-0.131	-1.900	1.201	0.590
Stdev	0.260	0.225	0.086	0.184	0.142	0.068	1.092	1.547	0.311	0.086	0.044	0.078
MSE	0.069	0.051	0.007	0.040	0.029	0.007	1.206	3.127	0.114	3.618	1.445	0.354
CP	—	—	0.932	—	—	0.900	—	—	0.902	—	—	0.018

Stdev: standard deviation ; MSE: mean squared error; CP: coverage probability. The bandwidth used in computing the nonparametric kernel estimators $(\hat{\alpha}_{MK}, \hat{\beta}_{MK}, \hat{\theta}_{MK})$ is $h = 0.1$.

Table 3: Simulation results under Scenorio III

$n = 200$												
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}_{MK}$	$\hat{\beta}_{MK}$	$\hat{\theta}_{MK}$	$\hat{\alpha}_{WSK}$	$\hat{\beta}_{WSK}$	$\hat{\theta}_{WSK}$	$\tilde{\alpha}_{MAR}$	$\tilde{\beta}_{MAR}$	$\tilde{\theta}_{MAR}$
Bias	0.155	-0.171	0.003	0.047	0.015	0.071	0.184	-0.205	-0.005	-2.794	0.954	-1.146
Stdev	0.584	0.585	0.155	0.376	0.190	0.131	0.753	0.922	0.149	1.395	0.396	0.263
MSE	0.365	0.372	0.024	0.144	0.036	0.022	0.602	0.892	0.022	9.758	1.069	1.384
CP	—	—	0.934	—	—	0.884	—	—	0.942	—	—	0.032
$n = 500$												
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}_{MK}$	$\hat{\beta}_{MK}$	$\hat{\theta}_{MK}$	$\hat{\alpha}_{WSK}$	$\hat{\beta}_{WSK}$	$\hat{\theta}_{WSK}$	$\tilde{\alpha}_{MAR}$	$\tilde{\beta}_{MAR}$	$\tilde{\theta}_{MAR}$
Bias	0.034	-0.036	0.000	0.012	0.012	0.034	0.037	-0.036	-0.002	0.782	0.355	0.123
Stdev	0.305	0.224	0.103	0.250	0.128	0.085	0.304	0.216	0.100	0.433	0.113	0.101
MSE	0.094	0.051	0.010	0.062	0.016	0.008	0.094	0.048	0.010	0.799	0.139	0.025
CP	—	—	0.902	—	—	0.894	—	—	0.912	—	—	0.698

$n = 1000$												
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}_{MK}$	$\hat{\beta}_{MK}$	$\hat{\theta}_{MK}$	$\hat{\alpha}_{WSK}$	$\hat{\beta}_{WSK}$	$\hat{\theta}_{WSK}$	$\tilde{\alpha}_{MAR}$	$\tilde{\beta}_{MAR}$	$\tilde{\theta}_{MAR}$
Bias	0.009	-0.010	0.002	0.002	0.009	0.017	0.007	-0.008	0.001	0.728	0.372	0.126
Stdev	0.215	0.157	0.069	0.167	0.083	0.056	0.213	0.156	0.069	0.302	0.078	0.067
MSE	0.046	0.024	0.004	0.028	0.007	0.003	0.045	0.024	0.004	0.621	0.144	0.020
CP	—	—	0.932	—	—	0.934	—	—	0.93	—	—	0.454

Stdev: standard deviation ; MSE: mean squared error; CP: coverage probability. The bandwidth used in computing the nonparametric kernel estimators $(\hat{\alpha}_{MK}, \hat{\beta}_{MK}, \hat{\theta}_{MK})$ is $h = 0.1$.

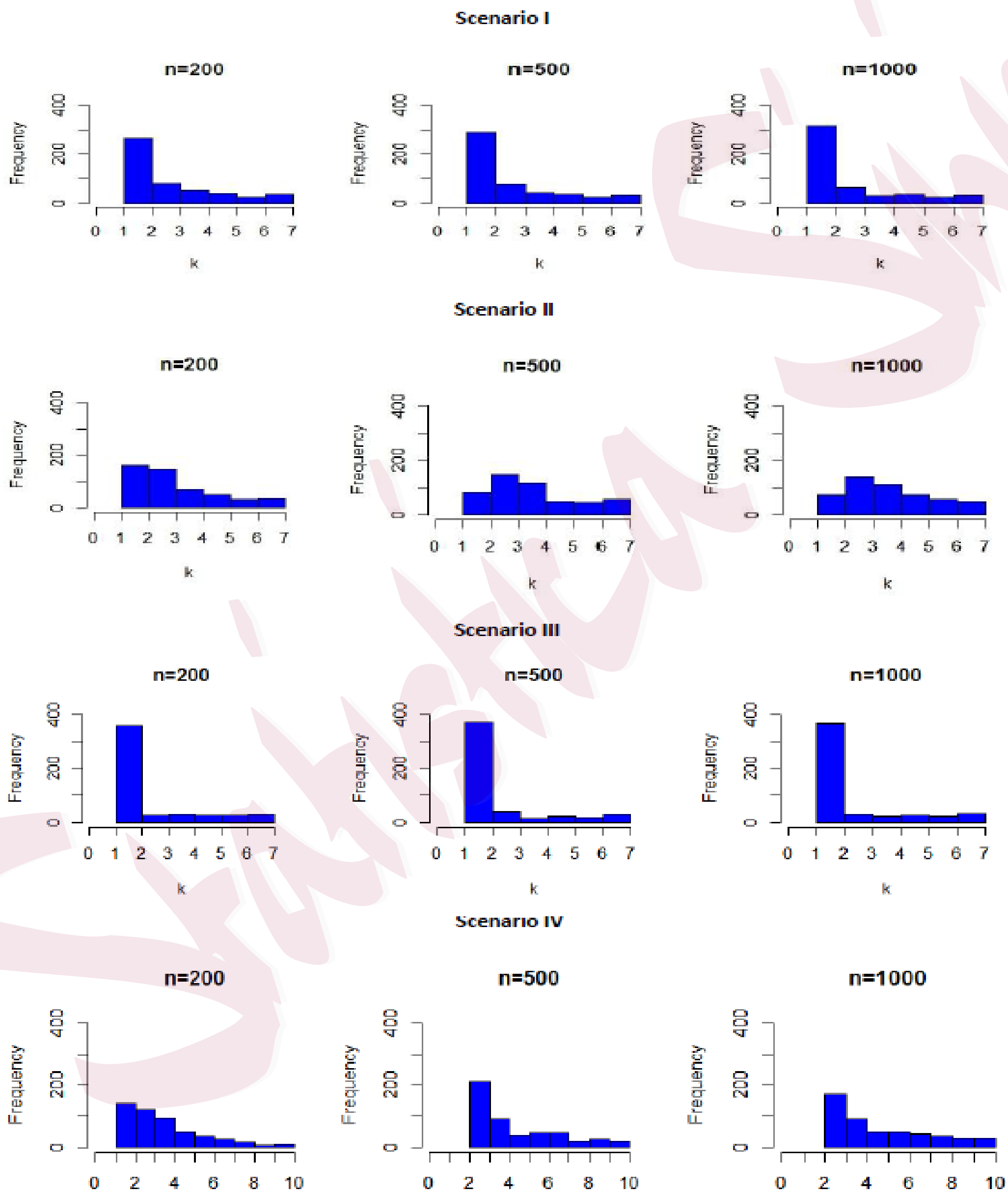
Table 4: Simulation results under Scenorio IV

$n = 200$												
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}_{MK}$	$\hat{\beta}_{MK}$	$\hat{\theta}_{MK}$	$\hat{\alpha}_{WSK}$	$\hat{\beta}_{WSK}$	$\hat{\theta}_{WSK}$	$\tilde{\alpha}_{MAR}$	$\tilde{\beta}_{MAR}$	$\tilde{\theta}_{MAR}$
Bias	0.097	-0.114	0.005	-0.018	0.027	0.043	0.738	-0.748	-0.011	-1.002	1.003	0.136
Stdev	1.140	0.721	0.118	0.308	0.185	0.103	3.241	3.079	0.134	0.081	0.139	0.348
MSE	1.310	0.533	0.014	0.095	0.035	0.013	11.055	10.046	0.0183	1.011	1.026	0.139
CP	—	—	0.914	—	—	0.920	—	—	0.882	—	—	0.998
$n = 500$												
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}_{MK}$	$\hat{\beta}_{MK}$	$\hat{\theta}_{MK}$	$\hat{\alpha}_{WSK}$	$\hat{\beta}_{WSK}$	$\hat{\theta}_{WSK}$	$\tilde{\alpha}_{MAR}$	$\tilde{\beta}_{MAR}$	$\tilde{\theta}_{MAR}$
Bias	-0.001	-0.026	0.003	-0.042	0.041	0.022	0.537	-0.564	-0.024	-1.003	1.000	0.146
Stdev	0.203	0.139	0.071	0.172	0.100	0.067	1.759	1.773	0.111	0.048	0.088	0.199
MSE	0.041	0.020	0.005	0.031	0.011	0.005	3.384	3.464	0.0131	1.010	1.009	0.061
CP	—	—	0.944	—	—	0.946	—	—	0.842	—	—	1.000

$n = 1000$												
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}_{MK}$	$\hat{\beta}_{MK}$	$\hat{\theta}_{MK}$	$\hat{\alpha}_{WSK}$	$\hat{\beta}_{WSK}$	$\hat{\theta}_{WSK}$	$\tilde{\alpha}_{MAR}$	$\tilde{\beta}_{MAR}$	$\tilde{\theta}_{MAR}$
Bias	0.010	-0.034	-0.001	-0.027	0.024	0.011	0.580	-0.586	-0.040	-1.000	0.997	0.134
Stdev	0.262	0.264	0.052	0.122	0.070	0.048	1.565	1.509	0.107	0.035	0.065	0.148
MSE	0.068	0.070	0.002	0.015	0.005	0.002	2.787	2.623	0.013	1.003	1.000	0.039
CP	—	—	0.936	—	—	0.932	—	—	0.786	—	—	1.000

Stdev: standard deviation ; MSE: mean squared error; CP: coverage probability. The bandwidth used in computing the nonparametric kernel estimators $(\hat{\alpha}_{MK}, \hat{\beta}_{MK}, \hat{\theta}_{MK})$ is $h = 0.2$.

Figure 1: Histogram of K



The Monte Carlo sample size used to plot the histogram of K is $J = 500$.

approach in finite samples. The GMM approach is also easy to adapt to stratified sampling and other sampling schemes common in survey data.

Both approaches require correct parameterization of the propensity score function. If the propensity score function is misspecified, then both approaches yield inconsistent estimates. There is some attempt in the literature to mitigate this problem. For instance, Zhao and Shao (2015) introduce a partial linear index to model missing mechanism. The proposed approach can be extended in this direction. Such extension shall be pursued in a future study.

8. Supplementary Materials

The supplementary materials contain the technical proofs for Theorems 1, 2, 3, 4, and 5.

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10. Appendix

10.1 Notation

The following notation are needed for presenting the efficiency bounds:

$$O(\boldsymbol{Z}) := \frac{1-\pi(\boldsymbol{Z};\gamma_0)}{\pi(\boldsymbol{Z};\gamma_0)}, \quad \boldsymbol{S}_0(\boldsymbol{Z}) := -\frac{\nabla_\gamma \pi(\boldsymbol{Z};\gamma_0)}{1-\pi(\boldsymbol{Z};\gamma_0)},$$

$$m(\boldsymbol{X}) := \frac{E[O(\boldsymbol{Z})\boldsymbol{S}_0(\boldsymbol{Z})|\boldsymbol{X}]}{E[O(\boldsymbol{Z})|\boldsymbol{X}]}, \quad R(\boldsymbol{X}) := \frac{E[O(\boldsymbol{Z})U(\boldsymbol{Z})|\boldsymbol{X}]}{E[O(\boldsymbol{Z})|\boldsymbol{X}]},$$

$$\boldsymbol{S}_1(T,\boldsymbol{Z};\gamma_0) := \left(1-\frac{T}{\pi(\boldsymbol{Z};\gamma_0)}\right)m(\boldsymbol{X}), \tag{10.12}$$

$$S_2(T,\boldsymbol{Z};\gamma_0,\theta_0) := -\frac{T}{\pi(\boldsymbol{Z};\gamma_0)}U(\boldsymbol{Z}) + \theta_0 - \left(1-\frac{T}{\pi(\boldsymbol{Z};\gamma_0)}\right)R(\boldsymbol{X}). \tag{10.13}$$

The following notation are needed to describe the higher order MSE criteria proposed by Donald, Imbens and Newey (2009):

$$\rho(T_i, \boldsymbol{X}_i, Y_i; \check{\gamma}) = 1 - \frac{T_i}{\pi(\boldsymbol{X}_i, Y_i; \check{\gamma})}, \quad \hat{\Upsilon}_{K\times K} = \frac{1}{N}\sum_{i=1}^{N} \rho(T_i, \boldsymbol{X}_i, Y_i; \check{\gamma})^2 u_K(\boldsymbol{X}_i)^{\otimes 2},$$

$$\widehat{\Gamma}_{K\times p} = \frac{1}{N}\sum_{i=1}^{N} u_K(\boldsymbol{X}_i)\nabla_{\gamma}\rho(T_i, \boldsymbol{X}_i, Y_i; \check{\gamma})^{\top}, \quad \hat{\Omega}_{p\times p} = (\hat{\Gamma}_{K\times p})^{\top}\hat{\Upsilon}^{-1}_{K\times K}\hat{\Gamma}_{K\times p},$$

$$\tilde{\mathbf{d}}_i = (\hat{\Gamma}_{K\times p})^{\top}\left(\frac{1}{N}\sum_{j=1}^{N} u_K(\boldsymbol{X}_j)^{\otimes 2}\right)^{-1} u_K(\boldsymbol{X}_i), \quad \tilde{\eta}_i = \nabla_{\gamma}\rho(T_i, \boldsymbol{X}_i, Y_i; \check{\gamma}) - \tilde{\mathbf{d}}_i,$$

$$\hat{\xi}_{ij} = \frac{1}{N} u_K(\boldsymbol{X}_i)^{\top}\hat{\Upsilon}^{-1}_{K\times K} u_K(\boldsymbol{X}_j), \quad \widehat{\mathbf{D}}^*_i = (\hat{\Gamma}_{K\times p})^{\top}\hat{\Upsilon}^{-1}_{K\times K} u_K(\boldsymbol{X}_i)\,.$$

10.2 Assumptions

The following assumptions are maintained in this paper:

Assumption 1. *There exists a nonresponse instrumental variable* $\mathbf{X_2}$, *i.e.,* $\mathbf{X} = (\mathbf{X}_1^{\intercal}, \mathbf{X}_2^{\intercal})^{\intercal}$, *such that* $\boldsymbol{X}_2$ *is independent of* T *given* $\boldsymbol{X}_1$ *and* Y; *furthermore,* $\boldsymbol{X}_2$ *is correlated with* Y.

Assumption 2. *The support of* $\mathbf{X}$, *which is denoted by* $\mathcal{X}$, *is a Cartesian product of* r*-compact intervals, and we denote* $\boldsymbol{X} = (X_1, ..., X_r)^{\top}$.

Assumption 3. $E[O(\boldsymbol{Z})S_0(\boldsymbol{Z})|\mathbf{X} = \boldsymbol{x}]$, $E[O(\boldsymbol{Z})U(\boldsymbol{Z})|\mathbf{X} = \boldsymbol{x}]$ *and* $E[O(\boldsymbol{Z})|\boldsymbol{X} = \boldsymbol{x}]$ *are* s*-smooth in* $\boldsymbol{x}$ *(the definition is given in page 5569 of Chen (2007) and Definition 1.1 of the supplemental material), where* $s > 0$.

Assumption 4. *There exists two finite positive constants $\underline{a}$ and $\overline{a}$ such that the smallest (resp. largest) eigenvalue of $E[u_K(\boldsymbol{X})u_K^\top(\boldsymbol{X})]$ is bounded away from $\underline{a}$ (resp. $\overline{a}$) uniformly in K, i.e.,*

$$0 < \underline{a} \leq \lambda_{\min}(E[u_K(\boldsymbol{X})u_K(\boldsymbol{X})^\top]) \leq \lambda_{\max}(E[u_K(\boldsymbol{X})u_K(\boldsymbol{X})^\top]) \leq \overline{a} < \infty \ .$$

Assumption 5. *(i) The parameter space Γ and Θ are compact; (ii) The efficient score function $\boldsymbol{S}_{eff}(T,\boldsymbol{Z};\gamma,\theta) := (\boldsymbol{S}_1^\intercal(T,\boldsymbol{Z};\gamma), S_2(T,\boldsymbol{Z};\gamma,\theta))^\intercal$ is continuously differentiable at each $(\gamma,\theta) \in \Gamma\times\Theta$, and $\mathbb{E}\left[\partial\boldsymbol{S}_{eff}(\gamma,\theta)/\partial(\gamma^\top,\theta)\right]$ is nonsingular at (γ_0,θ_0).*

Assumption 6. *(i) There exists two positive constants $\bar{c}$ and $\underline{c}$ such that $0 < \underline{c} \leq \pi(\boldsymbol{x},y;\gamma) \leq \bar{c} < 1$ for all $\gamma \in \Gamma$ and $(\boldsymbol{x},y) \in \mathcal{X}\times\mathbb{R}$; (ii) $\pi(\boldsymbol{x},y;\gamma)$ is twice continuously differentiable in $\gamma \in \Gamma$, and the derivatives are uniformly bounded.*

Assumption 7. *Suppose $K \to \infty$ and $K^3/N \to 0$.*

The Assumption 1 is used for the identification of the model, which was discussed in Wang et al. (2014). Assumptions 2 and 3 are required for L^2 approximations. Assumption 4 is a standard assumption for sieve basis, see also Newey (1997). Assumptions 5-6 are required for the convergence of our estimator as well as the boundness of the asymptotic variance. Assumption 7 is required for controlling the aymptotic variance, which is desirable in practice because K grows very slowly with N so a relatively small number of moment conditions is sufficient for the method proposed to perform well.

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