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The Broken Adaptive Ridge Procedure 
and Its Applications

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Abstract: In this paper, we study the broken adaptive ridge method to estimate lower-dimensional patterns of coefficients in regression models. Based on the re-weighted $\ell_2$-penalization, the new method can simultaneously recover both the true sparsity and the inherent structures of the features, which makes it theoretically and practically appealing. The resulting estimate is shown to enjoy the oracle property. The proposed method also contains a set of variable selection or pattern estimation methods. As a special case, the fused broken adaptive ridge, which penalizes the differences between adjacent coefficients, is thoroughly discussed with applications in signal approximation and image processing. The associated algorithms are numerically easy to implement. Various simulation studies and real data analyses illustrate its advantages over the fused lasso method.

Key words and phrases: Oracle estimator, Linear regression, Re-weighted $\ell_2$-
1. Introduction

Identifying the underlying dynamics of the dataset of interest is an important task in many applications, including, for instance, denoising, forecasting, filtering and even more sophisticated analysis in machine learning research. In a high dimensional setting, the underlying patterns of regression coefficients usually have a lower-dimensional structure. In particular, when the candidate variables can be treated individually, the true coefficients are assumed to contain many zeros. Many state-of-the-art variable selection methods have been developed such as lasso (Tibshirani, 2011), bridge penalty (Fu, 1998; Huang et al., 2008a, 2009), SCAD (Fan and Li, 2001), elastic net (Zou and Hastie, 2005), and MCP (Zhang, 2010), among many others. These methods have gained much attention in recent years and are widely used to find a parsimonious model. In this paper, we mainly focus on variables that naturally have some local structures, such as piece-wise constancy, linear trend or being grouped. An overarching goal is to effectively reduce the dimension of covariates as well as estimate their underlying structures.

The $\ell_0$-penalized regression is one of the most natural methods for variable selection, which directly penalizes the cardinality of a model. Rinaldo
(2009) considered the application of the $\ell_0$-penalization in finding a piecewise constant function to approximate a signal. Due to the lack of convexity, the $\ell_0$-penalization procedure is computationally difficult to implement, especially for high dimensional datasets. A body of effort has been devoted to approaches based on different penalties including $\ell_1$-norm penalties and $\ell_2$-norm penalties. The fused lasso method (Tibshirani et al., 2005) based on the $\ell_1$-penalization simultaneously captures the sudden jumps and infers those nonzero segments in a noisy signal or gene sequence. This novel approach uses the $\ell_1$ as well as the fusion (or total variation) penalties, which favors solutions that are both sparse and piecewise constant. As an extension, the two-dimensional (2D) fused lasso (Tibshirani and Taylor, 2011) is introduced in image denoising. From the algorithmic viewpoint, the idea of fused lasso penalization has its roots in the well-known total variation method (Rudin et al., 1992) that impacted enormously on the modern imaging science. For more recent developments of fused lasso and its variants in network inference, see Shen and Huang (2010); Zhu et al. (2013); Wang et al. (2016); Shin et al. (2016); Tang and Song (2016). On the other hand, for grouped data, such as assayed genes or proteins in biological applications, Yuan and Lin (2006) invented the group lasso methods by imposing the $\ell_2$-penalty on the coefficients within each group. Simon et al. (2013)
studied a sparse group lasso method, which yields solutions that are sparse at both the group and individual feature levels. Other advancements using the \( \ell_p \)-penalization to capture local structures of coefficients can be found in Eilers (2003), Rippe et al. (2012), Price et al. (2015) and Lam et al. (2016). Despite their impressive performance in empirical studies, theoretical justification of the oracle property (Fan and Li, 2001) of many of them remains quite challenging.

In this paper, instead of finding a desirable solution in one single step, we propose an iterative re-weighted \( \ell_2 \)-penalization procedure, referred to as \textit{broken adaptive ridge} (BAR) method. It has several distinctive features compared with other existing methods in the literature. First, it is in a general form in the sense that it can be used to estimate any local linear structure of regression coefficients. Some special cases of the BAR method, such as the fused BAR method, are introduced with applications in signal processing, gene detection, trend filtering or image denoising. Second, it can simultaneously produce a sparse solution and estimate the underlying pattern of covariates. Moreover, under certain conditions, it is shown that the BAR procedure converges to a fixed point and that the resulting estimate possesses the oracle property, i.e., it performs as well as if the correct underlying model were given in advance. Since the adaptive objective func-
tion is strictly convex and differentiable, the iterative procedure is easy to implement with a closed form iterative function. To avoid computational overflows in each iterative step, we establish efficient algorithms by using the Lagrange multiplier technique. The results of numerical studies demonstrate that, compared with the fused lasso, the fused BAR method has a good performance on variable selection and structure estimation.

The rest of the paper is organized as follows. The BAR method is detailed in Section 2, along with its special cases. Section 3 presents the oracle property of the proposed method. We establish a general algorithm for the BAR method in Section 4. Numerical studies on signal approximation and image processing are conducted in Sections 5 and 6. Technical proofs are provided in the supplementary material.

2. Broken adaptive ridge procedure

Consider the linear model

\[ y = \sum_{j=1}^{p_n} \beta_j x_j + \varepsilon \]

where \( y \in \mathbb{R}^n \) is a response variable, \( x_j \in \mathbb{R}^n \) are feature vectors and \( \varepsilon \) is a vector of independent and identically distributed random variables with mean zero and finite variance \( \sigma^2 \). Suppose that the response variable \( y = (y_1, \ldots, y_n) \) is centered and the covariates matrix \( X = (x_1, \ldots, x_{p_n}) \) is standardized by column vectors. We are concerned with the problem of
the recovery of the sparsity and the underlying patterns of feature vectors. Throughout the paper, \( \| \cdot \| \) represents the Euclidean norm for a vector and the spectral norm for a matrix.

Let \( d_k \in \mathbb{R}^{p_n}, k = 1, \ldots, K_n \) be nonzero column vectors, implying a prior knowledge of data structure. Define

\[
g(\tilde{\beta}) \equiv \arg \min_{\beta} \|y - X\beta\|^2 + \lambda_n \sum_{k=1}^{K_n} \frac{(d_k'\beta)^2}{c_k^2(\beta)} = \arg \min_{\beta} \|y - X\beta\|^2 + \lambda_n \beta' D(\tilde{\beta}) \beta,
\]

where

\[
D(\tilde{\beta}) = \sum_{k=1}^{K_n} \frac{d_k d_k'}{c_k^2(\tilde{\beta})}, \quad c_k(\tilde{\beta}) = d_k' \tilde{\beta}.
\]

It results from the convexity and differentiability of the objective function in (2.1) that

\[
g(\tilde{\beta}) = \{X'X + \lambda_n D(\tilde{\beta})\}^{-1}X'y. \tag{2.2}
\]

In principle, the ridge estimator

\[
\hat{\beta}(\text{ridge}) = (X'X + \xi I)^{-1}X'y
\]

is chosen to be the initial value, where \( \xi > 0 \) is a tuning parameter. The proposed estimator is, thereby, referred to as the \textit{broken adaptive ridge} (BAR) estimator, which is defined as the limit of the iterative algorithm

\[
\hat{\beta}^{(i)} = g(\hat{\beta}^{(i-1)}), \quad \text{namely,}
\]

\[
\hat{\beta}(\text{BAR}) = \lim_{j \to \infty} \hat{\beta}^{(j)}. \tag{2.3}
\]
Since the subsequent updates $d_k'\tilde{\beta}^{(j)}$ usually do not yield any zeros, the weights $\{c_k(\tilde{\beta})\}^{-2}$ in each iteration are well defined. It is worth emphasizing that the data-dependent weight $\{c_k(\tilde{\beta})\}^{-2}$ is more clever than a constant weight $c^{-2}$. As the sample size grows, the weights for zero $d_k'\beta$ get to infinity, whereas those for nonzero $d_k'\beta$ converge to finite constants. In this sense, the proposed BAR procedure and the adaptive lasso (Zou, 2006) are similar in spirit. As pointed out by a reviewer, the BAR method provides us with new insights into the ridge penalty: it is able to produce a sparse solution as well as estimate local structures of predictors through an iterative procedure.

Noticing that the term $d_k'\beta$ represents any linear combination of $\beta$, which allows us to design the vector $d_k$ in line with some believed structure or geometry in the feature vectors, such as sparsity, piecewise constancy, and grouping effect. We present below a set of illustrative examples that motivate our work on the BAR procedure.

**Example 2.1** (*Broken adaptive ridge estimator for variable selection*). Let $K_n = p_n$ and $d_j = e_j$, where $e_j$ is the standard basis vector with the $j$th component being 1. The design of $d_k$ only encourages the sparsity of the coefficients and virtually ignores any other underlying patterns of the feature vectors. As a result, the BAR method in this instance is appropriate
to do variable selection for those variables that can be treated individually.

**Example 2.2** (*Fused broken adaptive ridge estimator*). Setting $X = I$ yields an interesting but highly nontrivial class of problems including signal approximation, gene detection and image denoising. In signal approximation, a noisy signal is usually approximated by a piecewise constant function. A variety of denoising methods have been developed including lowess (Cleveland, 1979), kernel estimators (Gasser *et al.*, 1985; Müller and Stadtmüller, 1987), penalized smoothing splines (Ruppert *et al.*, 2009), Markov random field (Geman and Geman, 1984), and wavelets (Donoho and Johnstone, 1994; Chang *et al.*, 2000). To encourage the underlying sparse or blocky structure of $y$, we set $d_j = e_j$ for $j = 1, 2, \ldots, p_n$ and the remaining $d_j = (0, \ldots, -1, 1, \ldots, 0)'$ with the $(j - p_n)$th element being -1 and $(j - p_n + 1)$th element being 1. In such a way, the expression (2.1) can be written as

$$g(\tilde{\beta}) = \arg\min_\beta \|y - \beta\|^2 + \lambda_1 \sum_{i=1}^{n} \frac{\beta_i^2}{\tilde{\beta}_i^2} + \lambda_2 \sum_{i=2}^{n} \frac{(\beta_i - \beta_{i-1})^2}{(\tilde{\beta}_i - \tilde{\beta}_{i-1})^2}, \quad (2.4)$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are tuning parameters. We refer to the limit of the iterative procedure based on (2.4) as 1D fused BAR estimator, since the penalties are imposed both on the coefficients and the differences between the adjacent coefficients. A general form of the 1D fused BAR method
is induced by allowing the design matrix $X$ to be arbitrary. In a similar fashion, the 2D fused BAR estimator has the iterative function

$$
 g(\tilde{\beta}) = \arg \min_{\beta} \|y - \beta\|^2 + \lambda_1 \sum_{i=1}^{n} \frac{\beta_i^2}{\tilde{\beta}_i^2} + \lambda_2 \sum_{(i,j) \in E} \frac{(\beta_i - \beta_j)^2}{(\tilde{\beta}_i - \tilde{\beta}_j)^2} \quad (2.5)
$$

where $E$ is the edge set of the graph. It is seen that the second penalty term on the right-hand side of (2.5) favors the flatness of the proximal coefficients. Therefore, the 2D fused BAR estimator is useful to cope with the adjacent pixels in image denoising.

**Example 2.3 (Broken adaptive ridge trend filter).** Identifying the unknown underlying trend of a given noisy signal or sequence is of great importance for a wide range of applications. In many cases, the signal can be approximated by piecewise linear trends. To both select and estimate the trend’s components, we take into account the optimization rule

$$
 g(\tilde{\beta}) = \arg \min_{\beta} \|y - \beta\|^2 + \lambda_1 \sum_{i=1}^{n} \frac{\beta_i^2}{\tilde{\beta}_i^2} + \lambda_2 \sum_{i=2}^{n-1} \frac{(\beta_i - 2\beta_i + \beta_{i+1})^2}{(\tilde{\beta}_i - 2\tilde{\beta}_i + \tilde{\beta}_{i+1})^2} \quad (2.6)
$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are both tuning parameters. It is clear that (2.6) is a special case of the iterative function (2.1). The second penalty on the right-hand side of (2.6) constrains the slopes between two consecutive coefficients and thus results in a solution that has fewer linear segments.

3. **Oracle properties**
In this section, we investigate the oracle property of the BAR estimator. Assume that the true $\beta_0$ satisfies
\[
d_k'\beta_0 \neq 0 \text{ for } k = 1, \ldots, q_n,
\]
\[
d_k'\beta_0 = 0 \text{ for } k = q_n + 1, \ldots, K_n,
\]
where $\|d_k\| \neq 0$. Let $D$ denote the space spanned by vectors $d_{q_n+1}, \ldots, d_K$, and the dimensionality of $D$ is $(p_n - m_n)$, where $m_n$ is the dimensionality of subspace orthogonal to $D$. There exists an orthonormal basis of $\mathbb{R}^{p_n}$, $T = (T_1, T_2) = (u_1, \ldots, u_{m_n}; u_{m_n+1}, \ldots, u_{p_n})$, such that $u_{m_n+1}, \ldots, u_{p_n} \in D$.

Then,
\[
u_j'\beta_0 = 0 \text{ for } j = m_n + 1, \ldots, p_n,
\]
\[
u_i'd_k = 0 \text{ for } i = 1, \ldots, m_n \text{ and } k = q_n + 1, \ldots, K_n.
\]
Let $X_1 = XT_1$, $X_2 = XT_2$, $\Sigma_n = n^{-1}X'X$ and $\tilde{\Sigma} = n^{-1}X_1'X_1$. For the simplicity of notation, we write $\hat{\beta}$ (BAR) as $\hat{\beta}$ and omit the tilde on $\beta$ in (2.2). Define $b_n \equiv \min_{1 \leq k \leq q_n} |d_k'\beta_0|$. The following regularity conditions are assumed:

(A1) $0 < 1/C < \lambda_{\min}(\Sigma_n) \leq \lambda_{\max}(\Sigma_n) < C < \infty$, for some $C > 1$;

(A2) As $n \to \infty$,
\[
m_n/n \to 0, \quad \lambda_n \to \infty, \quad \lambda_n q_n \to 0, \quad p_n b^2_n \to 0;
\]

(A3) For $1 \leq k \leq q_n$, $0 < \|d_k\| \leq c_0 < \infty$ for some constant $c_0$. 

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(A4) The initial estimator satisfies $\|\hat{\beta}^{(0)} - \beta_0\| = O_p\{(p_n/n)^{1/2}\}$. Condition (A1) assumes that the $\ell_2$-norm of covariance matrix $\Sigma_n$ is bounded away from zero and infinity. Condition (A2) restricts the number of covariates, the number of nonzero linear combinations of covariates, the tuning parameter, and the smallest nonzero linear combination. It is also made to ensure that those nonzero $d_k'\beta_0$ are identifiable. Condition (A3) is made for the simplicity of proof and is satisfied for many commonly-used penalties, such as fusion penalty and trend filter penalty. For high-dimensional data, $\|d_k\|$ would be allowed to diverge to infinity at some rate as $n \to \infty$. Such relaxation would not necessarily affect the asymptotic properties of the BAR estimate, since the penalty term in the first line of (2.1) remains the same when its numerator and denominator divided simultaneously by $\|d_k\|^2$. The initial value needs to satisfy condition (A4).

Lemma 1. Suppose that regularity conditions (A1)–(A4) are satisfied. For any positive sequence $\delta_n \to \infty$ such that $\lambda_n/(\delta_np_n) \to \infty$, define $\mathcal{B} = \{\beta \in \mathbb{R}^{pn} : \|\beta - \beta_0\| \leq \delta_n \sqrt{p_n/n}\}$. Then, with probability tending to 1:

(a) $g(\beta)$ is a mapping from $\mathcal{B}$ to itself;

(b) $\sup_{\beta \in \mathcal{B}} \frac{\|T'g(\beta)\|}{\|T'\beta\|} < \frac{1}{C_0}$, for some constant $C_0 > 1$. \hspace{1cm} (2.7)

Remark 1. The statement (2.7) reveals that $\lim_{k \to \infty} T'\hat{\beta}^{(k)} \equiv T'\hat{\beta} = 0$. 
with probability tending to 1. In other words, the BAR estimator is zero-consistent in the sense that those zero linear combinations of coefficients would be exactly zero as $n \to \infty$. Additionally, the result that $g(\cdot)$ is a mapping of $\mathcal{B}$ to itself is necessary for the convergence of $\hat{\beta}^{(k)}$.

On the other hand, since $T_2'\beta_0 = 0$, the regression model is reduced to

$$y = XT_1T_1'\beta_0 + \epsilon.$$  \hfill (2.8)

Define

$$f(T_1'\hat{\beta}) \equiv \arg \min_{\beta} \|y - X_1T_1'\beta\|^2 + \lambda_n \sum_{k=1}^{q_n} \frac{(d_k' T_1 T_1' \beta)^2}{\tilde{c}_k^2(T_1' \beta)},$$

$$= \arg \min_{\beta} \|y - X_1T_1'\beta\|^2 + \lambda_n \beta' \tilde{D}(T_1' \hat{\beta}) \beta,$$

where

$$\tilde{c}_k(T_1' \beta) = d_k' T_1 T_1' \beta \quad \text{and} \quad \tilde{D}(T_1' \beta) = T_1' \left( \sum_{k=1}^{q_n} \frac{d_k d_k'}{\tilde{c}_k^2(T_1' \beta)} \right) T_1.$$  

Similarly, by a gradient rule, we can obtain

$$f(T_1' \beta) = \{X_1' X_1 + \lambda_n \tilde{D}(T_1' \beta)\}^{-1}X_1' y.$$  \hfill (2.9)

The asymptotic normality of those nonzero linear combinations $d_k' \hat{\beta}$, $k = 1, \ldots, q_n$ are shown in Lemma 2 below.

**Lemma 2.** Suppose that regularity conditions (A1)–(A4) are satisfied. For any $q_n$-vector $a_n$ with $\|a_n\| \leq 1$, let $s_n^2 = \sigma^2 a_n' \tilde{\Sigma}_n^{-1} a_n$. Define $\mathcal{B}_1 = \{T_1' \beta \in \mathcal{B} : a_n' \tilde{\Sigma}_n^{-1} a_n \leq s_n^2 \}$.
\[ \mathbb{R}^{m_n} : \| T'_i \beta - T'_1 \beta_0 \| \leq \delta_n \sqrt{p_n/n} \] and assume that \( \inf_{\beta \in \mathbb{B}_1} (d'_k T'_1 T'_1 \beta)^2 \geq c_1 (d'_k T'_1 \theta_0)^2 \) for \( 1 \leq k \leq q_n \), where \( \theta_0 = T'_1 \beta_0 \). Then, in region \( \mathbb{B}_1 \), with probability tending to 1, there exists a unique fixed point of \( f(\cdot) \) denoted by \( \hat{\theta}^{\circ} \). Furthermore, as \( n \to \infty \),

\[ \sqrt{n s_n^{-1} a'_n (\hat{\theta}^{\circ} - T'_1 \beta_0)} \to N(0, 1) \]

with probability tending to 1.

**Remark 2.** Lemma 2 shows the existence and uniqueness of the fixed point of \( f(\cdot) \) defined as (2.9). The asymptotic properties of the fixed point \( \hat{\theta}^{\circ} \) implies that \( \hat{\theta}^{\circ} \) is consistent with the true \( T'_1 \beta_0 \). To show that the BAR estimator is asymptotically normal, it suffices to show that \( P(T'_1 \hat{\beta} = \hat{\theta}^{\circ}) \to 1 \) as \( n \to \infty \).

**Theorem 1 (Oracle property).** Suppose conditions (A1)–(A4) hold and that \( \inf_{\beta \in \mathbb{B}_1} (d'_k T'_1 T'_1 \beta)^2 \geq c_1 (d'_k T'_1 \beta_0)^2 \) for \( 1 \leq k \leq q_n \), where \( \mathbb{B}_1 \) is defined in Lemma 2. For any \( q_n \)-vector \( a_n \) with \( \| a_n \| \leq 1 \), set \( s_n^2 = \sigma^2 a'_n \Sigma_n^{-1} a_n \). Then, with probability tending to 1:

(i) The BAR estimator \( T'_1 \hat{\beta} \) exists and is the unique fixed point of \( T'_1 g(\cdot) \) in the region \( \mathbb{B} \) defined as Lemma 1;

(ii) \( T'_2 \hat{\beta} = 0 \);

(iii) \( \sqrt{n s_n^{-1} a'_n (T'_1 \hat{\beta} - T'_1 \beta_0)} \to N(0, 1) \).
Remark 3. For more insight into the BAR procedure and its oracle properties, we recall that the initial value $\hat{\beta}^{(0)}$ is asymptotically consistent with $\beta_0$ and $d'_k T_1 = 0$ for $(q_n + 1) \leq k \leq K_n$. The oracle properties of the BAR estimator is essentially due to the iterative weight $(d'_k \hat{\beta})^{-2}$. Specifically, when the true $d'_k \beta_0$ is zero or alternatively $d'_k T_2 T'_2 \beta_0 = 0$ for $(q_n + 1) \leq k \leq K_n$, the weight $\{d'_k T_2 T'_2 \hat{\beta}^{(0)}\}^{-2}$ would be large, resulting in a smaller estimate of $d'_k \hat{\beta}^{(j)}$ per iteration. This leads to the zero-consistency of the BAR estimator. On the other hand, for those nonzero $d'_k \beta_0$ for $1 \leq k \leq q_n$, we have the nonzero weight function

$$\{d'_k \hat{\beta}^{(j)}\}^{-2} = \{d'_k (T_1 T'_1 + T_2 T'_2) \hat{\beta}^{(j)}\}^{-2} \approx \{d'_k T_1 T'_1 \hat{\beta}^{(j)}\}^{-2},$$

where $\{d'_k T_1 T'_1 \hat{\beta}^{(j)}\}^{-2}$ is the weight function when the true model (2.8) known in advance. Hence, the asymptotic normality of $d'_k \beta_0$ is very closed to the unique fixed point of $f(\cdot)$.

In real applications, when $d'_k \beta_0 = 0$, the denominator $c_k^2 (\hat{\beta}^{(k)})$ will inevitably run into a small value close to zero, causing an arithmetic overflow. In the next section, we attempt to use the Lagrange multiplier to overcome this computational difficulty. As a result, Algorithm 1 is proposed for the BAR procedure and can be used to do signal approximation, image processing, and gene detection.
4. Algorithm

4.1. 1D fused BAR implementation

We consider the 1D fused BAR method with an arbitrary $X$. Let

$$M = \begin{pmatrix}
-1 & 1 & 0 & \cdots & 0 \\
0 & -1 & 1 & \cdots & 0 \\
& & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & -1 & 1
\end{pmatrix}$$

of size $(p_n - 1) \times p_n$. The iterative function (2.4) can be written as

$$g(\tilde{\beta}) = \arg\min_\beta \|y - X\beta\|^2 + \lambda_1 \beta' H_1(\tilde{\beta}) \beta + \lambda_2 \beta' M' H_2(\tilde{\beta}) M \beta, \quad (4.1)$$

where $H_1(\tilde{\beta}) = \text{diag}(\tilde{\beta}^{-2})$, $H_2(\tilde{\beta}) = \text{diag}\{(M\tilde{\beta})^{-2}\}$ with $(M\tilde{\beta})_i$ being the $i$th component of $M\tilde{\beta}$. Since the objective function in (4.1) is differentiable and strictly convex, there exists a unique global minimum. After a few iterative steps, however, some elements of $\tilde{\beta}$ and $M\tilde{\beta}$ would be close to zero. As a result, the division in the diagonal entries of $H_1(\tilde{\beta})$ and $H_2(\tilde{\beta})$ will run into an overflow and the iterative procedure would stop at a suboptimal value. To avoid those divisions in $H_1(\tilde{\beta})$ and $H_2(\tilde{\beta})$, we set $\tilde{z} = M\tilde{\beta}$ and

$$\min_{\beta, z} \|y - X\beta\|^2 + \lambda_1 \beta' H_1(\tilde{\beta}) \beta + \lambda_2 z' H_1(\tilde{z}) z \text{ subject to } z = M\beta,$$

where $H_1(\tilde{z}) = \text{diag}(\tilde{z}^{-2})$ by the preceding definition.
The Lagrange function is

\[ L(\beta, z, u) = \|y - X\beta\|^2 + \lambda_1 \beta' H_1(\tilde{\beta}) \beta + \lambda_2 z' H_1(\tilde{z}) z + u'(M\beta - z), \]

where \( u \) is the Lagrange multiplier. The Lagrange dual of (4.1) is

\[ \max_u \min_{\beta, z} L(\beta, z, u). \] (4.2)

To further solve the problem, we first minimize \( L(\beta, z, u) \) over \( \beta \) and \( z \).

The term of \( L(\beta, z, u) \) that involves \( \beta \) is

\[ \|y - X\beta\|^2 + \lambda_1 \beta' H_1(\tilde{\beta}) \beta + u'M\beta. \]

It follows that

\[ \min_{\beta} \|y - X\beta\|^2 + \lambda_1 \beta' H_1(\tilde{\beta}) \beta + u'M\beta \]

\[ = y'y - (X'y - M'u/2)' \{X'X + \lambda_1 H_1(\tilde{\beta})\}^{-1}(X'y - M'u/2) \]

and the optimal \( \beta \) is \( \beta^* = \{X'X + \lambda_1 H_1(\tilde{\beta})\}^{-1}(X'y - M'u/2) \). In a similar way, minimizing \( L(\beta, z, u) \) over \( z \), we have

\[ \min_z \lambda_2 z' H_1(\tilde{z}) z - u'z = -\frac{1}{4\lambda_2} u'H_1^{-1}(\tilde{z}) u. \]

Therefore, the dual problem (4.2) is equivalent to

\[ \min_u \{X'y - M'u/2\}' \{X'X + \lambda_1 H_1(\tilde{\beta})\}^{-1}(X'y - M'u/2) + \frac{1}{4\lambda_2} u'H_1^{-1}(\tilde{z}) u - y'y. \]

It is straightforward to get the solution denoted as

\[ \hat{u}^* = 2\{MB(\tilde{\beta})M' + H_1^{-1}(\tilde{z})/\lambda_2\}^{-1}MB(\tilde{\beta})X'y. \]
where $B(\tilde{\beta}) = \{X'X + \lambda_1H_1(\tilde{\beta})\}^{-1}$. In practice, if the inverse of a matrix does not exist, we suggest using the Moore–Penrose pseudo-inverse denoted as $\text{Pinv}()$. One the other hand, to avoid the division in $B(\tilde{\beta})$, we instead calculate $B(\tilde{\beta}) = H_1^{-1}(\tilde{\beta})\{X'H_1^{-1}(\tilde{\beta}) + \lambda_1I_n\}^{-1}$. To implement the 1D fused BAR procedure, we take $M_0 = M$ in Algorithm 1. In particular, set $X = I$, when we do signal approximation.

4.2. 2D fused BAR implementation

We now investigate the implementation of the 2D fused BAR method with application in image denoising. Distinguishing from signals, the adjacent pixels of one image include both the horizontal level neighbors and vertical level neighbors. The objective function in (2.5) can be written as

$$
\min_{\beta} \|y - \beta\|^2 + \lambda_1\beta'H_1(\tilde{\beta})\beta + \lambda_2\beta'(M'_1H_3(\tilde{\beta})M_1 + M'_2H_4(\tilde{\beta})M_2)\beta,
$$

where $M_1$ and $M_2$ respectively capture the vertical and horizontal neighbors in a graph, $H_3(\tilde{\beta}) = \text{diag}\{(M_1\tilde{\beta})_j^{-2}\}$, and $H_4(\tilde{\beta}) = \text{diag}\{(M_2\tilde{\beta})_j^{-2}\}$. To overcome the numerically difficulty in $H_3(\tilde{\beta})$ and $H_4(\tilde{\beta})$, we derive the refined iterative procedure based on the Lagrange multiplier. In a similar vein, the Lagrange function is

$$
\tilde{L}(\beta, z, u) = \|y - \beta\|^2 + \lambda_1\beta'H_1(\tilde{\beta})\beta + \lambda_2z'H^*(\tilde{z})z + u'(M^*\beta - z),
$$

where $\tilde{z}_1 = M_1\tilde{\beta}$, $\tilde{z}_2 = M_2\tilde{\beta}$, $z = (z'_1, z'_2)'$, $u = (u'_1, u'_2)'$, $M^* = (M'_1, M'_2)'$, $X = I$, when we do signal approximation.
and

\[
H^*(\tilde{z}) = \begin{pmatrix}
H_1(\tilde{z}_1) & 0 \\
0 & H_1(\tilde{z}_2)
\end{pmatrix}.
\]

It is observed that the above Lagrange function \( \tilde{L}(\beta, \mathbf{z}, \mathbf{u}) \) is in the same form of \( L(\beta, \mathbf{z}, \mathbf{u}) \) in the 1D fused BAR method. As a consequence, we set \( \mathbf{M}_0 = \mathbf{M}^* \) in the Algorithm 1 for image denoising.

It is worth mentioning that the Algorithm 1 is a general algorithm for the BAR method and its application is not restricted to signal approximation and image processing. Generally speaking, any design of \( d_k \) can be incorporated in \( \mathbf{M} \). The Algorithm 1 is flexible in the sense that it allows different penalties imposed on different types of structures of \( d_k \).

### 4.3. Choice of tuning parameters

To implement the BAR procedure, the initial value \( \hat{\beta}^{(0)} \) and parameters \( \lambda_1 \) and \( \lambda_2 \) need to be chosen carefully. The BAR method recommends the ridge estimator as the initial \( \hat{\beta}^{(0)} \) with the tuning parameter \( \xi \) being carefully chosen by the 5-fold cross-validation (CV). However, when sample size \( n \) is small, the value of \( \xi \) chosen by CV may vary due to the different partitions of the data. To avoid this problem, we instead use the univariate regression estimator as the initial estimator \( \hat{\beta}^{(0)} \) whenever \( p \gg n \), namely

\[
\hat{\beta}_j^{(0)} = \frac{\sum_{i=1}^{n} x_{ij} y_i}{\sum_{i=1}^{n} x_{ij}^2}, \quad j = 1, ..., p_n,
\] (4.3)
Algorithm 1: Fused BAR Algorithm

Result: Fused BAR estimator $\hat{\beta}^*.$

1. Input $y, X, M_0, \hat{\beta}^{(0)}, \lambda_1, \lambda_2$ and $\epsilon$;
2. $k \leftarrow 0$;
3. $\tilde{\beta} \leftarrow \hat{\beta}^{(0)}$;
4. while ($\|\tilde{\beta} - \hat{\beta}^{(k)}\| > \epsilon$ or $k = 0$) do
5. $\tilde{\beta} \leftarrow \hat{\beta}^{(k)}$;
6. $\tilde{z} \leftarrow M_0 \tilde{\beta}$;
7. $H_1^{-1}(\tilde{\beta}) \leftarrow \text{diag}(\tilde{\beta}_j^2)$;
8. $H_0^{-1}(\tilde{z}) \leftarrow \text{diag}(\tilde{z}_j^2)$;
9. $B(\tilde{\beta}) \leftarrow H_1^{-1}(\tilde{\beta})\{X'XH_1^{-1}(\tilde{\beta}) + \lambda_1 I_n\}^{-1}$;
10. $\hat{u}^* \leftarrow \text{Pinv}\{M_0 B(\hat{\beta})M_0' + H_0^{-1}(\tilde{z})/\lambda_2\}M_0 B(\tilde{\beta}) X' y$;
11. $k \leftarrow k + 1$;
12. $\hat{\beta}^{(k)} \leftarrow B(\tilde{\beta})(X' y - M_0' \hat{u}^*)$;
13. end

which is adopted as the initial value in the adaptive lasso (Huang et al., 2008b) to handle the high-dimensional problem. Huang et al. (2008b) also showed that under certain conditions, the adaptive lasso estimator is consistent in variable selection and estimation if the initial estimator is the marginal regression estimator. This is due to the fact that the univariate regression estimator is zero-consistent in the sense that the estimators of zero coefficients converge to zero, while the estimators of non-zero coeffi-
coefficients do not converge to zero. Our simulation results evidenced that the univariate estimator is a good initial value.

For the selections of $\lambda_1$ and $\lambda_2$, we adopt $k$-fold CV method. Specifically, in signal approximation, we pick all of the odd coefficients as the training set and the even coefficients as the validation set. And we search a grid of $\lambda_1$ and $\lambda_2$ by the two-fold CV method. For example, 10 grids evenly distributed on the interval $[0.1, 10]$ for $\lambda_1$ and 10 grids evenly distributed on $[1, 20]$ for $\lambda_2$. Then we select the optimal $(\lambda_1, \lambda_2)$ with the minimum CV error by searching over values in the 2D grid. Further, if $X$ is a general matrix, we recommend to use five-fold CV to find the optimal tuning parameters $\lambda_1$ and $\lambda_2$.

5. Simulation study

In this section, we carry out simulations on the fused BAR method, the BAR fusion, the fused lasso and the $\ell_1$ fusion. Notice that the difference between “fused” and “fusion” constraints is that the former encourages sparsity both in the coefficients and their differences, while the latter only penalizes the flatness of coefficients. For instance, the BAR fusion method is a special case of the BAR method with penalties imposed only on the differences between adjacent coefficients. The comparisons of their performance in terms of variable selection, estimation and prediction are presented. We
use the R package *genlasso* for $\ell_1$ fusion and fused lasso. The response variable is generated from the regression model

$$y = X\beta_0 + \sigma \epsilon, \quad \epsilon \sim \mathcal{N}(0, I).$$

The following Examples 4.4–4.8 are considered. Specifically, Example 4.4 is designed to check the ability of producing a piecewise constant estimate with fewer number of jumps, while Example 4.5 focuses on checking that whether these methods can successfully detect the single nonzero coefficients. In Examples 4.6–4.7, we assess the performance of Algorithm 1 for a general design of $X$ when $p > n$ and $p < n$, respectively. To examine the performance of these methods on detecting true smaller jumps, we conduct Examples 4.8–4.9. In Example 4.10, we simulate a toy image and illustrate the performance of fused BAR method compared with the fused lasso method in image denoising.

**Example 4.4.** (Signal approximation.) Set $X = I$. Let $\sigma = 0.8$, $n = 200$ and the true signal be

$$\beta_0 = \left(0, \ldots, 0, 5, \ldots, 5, 0, \ldots, 0, 3.5, \ldots, 3.5, 0, \ldots, 0, 4.5, \ldots, 4.5, 0, \ldots, 0\right)'.$$

The number of nonzero coefficients is 37.

**Example 4.5.** (Singular nonzero value.) Set $X = I$. Let $\sigma = 0.8$, $n = 250$
and the true signal be

\[ \beta_0 = (0, \ldots, 0, 5, 0, \ldots, 0, 4.5, \ldots, 4.5, 0, \ldots, 0)' . \]

The number of nonzero coefficients is 50 with a single nonzero coefficient \( \beta_{25} = 5 \).

**Example 4.6.** (A general matrix \( X \) with \( p_n > n \).) Let \( \sigma = 10, \ p = 250, \ n = 200 \) and the true coefficients be

\[ \beta_0 = (0, \ldots, 0, 5, 0, \ldots, 0, 3, \ldots, 3, 0, \ldots, 0, -4, \ldots, -4, 0, \ldots, 0)' . \]

The number of nonzero coefficients is 83. We generate \( x_{ij} \sim \mathcal{N}(0,1) \), for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq p_n \).

**Example 4.7.** (A general matrix \( X \) with \( p_n < n \).) Let \( \sigma = 5, \ p = 100, \ n = 200 \) and the true coefficients be

\[ \beta_0 = (0, \ldots, 0, -2, \ldots, -2, 0, \ldots, 0, 4, \ldots, 4, 0, \ldots, 0)' . \]

The number of nonzero coefficients is 34. We generate \( x_{ij} \sim \mathcal{N}(0,1) \), for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq p_n \).

**Example 4.8.** We use the same model as in example 4.4 but with \( \beta_j = 0.85 \) for all \( \beta_j \neq 0 \).
Example 4.9. We use the same model as in example 4.5 but with $\beta_j = 0.85$ for all $\beta_j \neq 0$.

Example 4.10. (Image denoising.) We design a $20 \times 20$ pixel toy image. The noise following normal distribution with mean zero and variance $(14/51)^2$ is added to the original image. We calculate the reconstruction errors by the fused BAR and the fused lasso denoising, respectively.

Tables 1–2 summarize the results of Examples 4.4–4.7 and Examples 4.8–4.9 each with 20 replications, respectively. The two tables present the number of selected features (NOS), the number of falsely selected variables (NOFS), the percent of true nonzero coefficients model selected (TM), the number of jumps (NOJ), the mean absolute bias (MAB), the fitted mean squared error (FMSE), the single value selection (SVS) and the minimum CV error. Figures 1–5 depict the estimated coefficients using the fused BAR, the BAR fusion, the fused lasso and the $\ell_1$ fusion for Examples 4.4–4.9. The image processing results of Example 4.10 are shown in Figure 6.

It can be seen from Tables 1-2 that the NOS, NOFS and MAB of the fused BAR estimator are relatively smaller than that of fused Lasso. This implies that the resulting fused BAR estimator has a better performance on variable selection. Moreover, Figures 1–3 show that the fused BAR
Table 1: Mean and the standard deviation (in parentheses) of results using the fused BAR and the fused Lasso for Examples 4.4–4.7.

<table>
<thead>
<tr>
<th></th>
<th>Example 4.4</th>
<th></th>
<th>Example 4.5</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>fused BAR</td>
<td>fused Lasso</td>
<td>fused BAR</td>
<td>fused Lasso</td>
<td></td>
</tr>
<tr>
<td>NOS</td>
<td>37.350 (0.933)</td>
<td>96.850 (34.973)</td>
<td>49.950 (0.394)</td>
<td>114.550 (45.542)</td>
<td></td>
</tr>
<tr>
<td>NOFS</td>
<td>0.350 (0.933)</td>
<td>59.850 (34.973)</td>
<td>0.050 (0.224)</td>
<td>64.700 (45.542)</td>
<td></td>
</tr>
<tr>
<td>TM</td>
<td>100% (0.000)</td>
<td>100% (0.000)</td>
<td>99.8% (0.006)</td>
<td>99.7% (0.007)</td>
<td></td>
</tr>
<tr>
<td>NOJ</td>
<td>6.850 (1.496)</td>
<td>37.050 (13.839)</td>
<td>3.250 (1.070)</td>
<td>16.450 (7.749)</td>
<td></td>
</tr>
<tr>
<td>MAB</td>
<td>0.056 (0.018)</td>
<td>0.148 (0.039)</td>
<td>0.050 (0.022)</td>
<td>0.112 (0.027)</td>
<td></td>
</tr>
<tr>
<td>FMSE</td>
<td>0.620 (0.058)</td>
<td>0.545 (0.097)</td>
<td>0.679 (0.076)</td>
<td>0.688 (0.072)</td>
<td></td>
</tr>
<tr>
<td>CV error</td>
<td>1.200 (0.132)</td>
<td>1.211 (0.138)</td>
<td>0.922 (0.096)</td>
<td>0.892 (0.083)</td>
<td></td>
</tr>
<tr>
<td>SVS</td>
<td>–</td>
<td>–</td>
<td>0.950 (0.224)</td>
<td>0.850 (0.366)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Example 4.6</th>
<th></th>
<th>Example 4.7</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>fused BAR</td>
<td>fused Lasso</td>
<td>fused BAR</td>
<td>fused Lasso</td>
<td></td>
</tr>
<tr>
<td>NOS</td>
<td>96.150 (16.000)</td>
<td>202.800 (42.010)</td>
<td>35.250 (2.197)</td>
<td>56.400 (13.697)</td>
<td></td>
</tr>
<tr>
<td>NOFS</td>
<td>13.300 (15.885)</td>
<td>119.800 (42.010)</td>
<td>1.250 (2.197)</td>
<td>22.400 (13.697)</td>
<td></td>
</tr>
<tr>
<td>TM</td>
<td>99.8% (0.004)</td>
<td>100% (0.000)</td>
<td>100% (0.000)</td>
<td>100% (0.000)</td>
<td></td>
</tr>
<tr>
<td>NOJ</td>
<td>7.250 (2.124)</td>
<td>23.450 (5.094)</td>
<td>4.250 (0.639)</td>
<td>13.500 (1.318)</td>
<td></td>
</tr>
<tr>
<td>MAB</td>
<td>0.060 (0.033)</td>
<td>0.162 (0.041)</td>
<td>0.025 (0.005)</td>
<td>0.058 (0.006)</td>
<td></td>
</tr>
<tr>
<td>FMSE</td>
<td>102.699 (9.217)</td>
<td>91.191 (11.323)</td>
<td>8.218 (0.263)</td>
<td>8.124 (0.175)</td>
<td></td>
</tr>
<tr>
<td>CV error</td>
<td>116.314 (13.951)</td>
<td>125.785 (12.500)</td>
<td>8.371 (0.161)</td>
<td>9.548 (0.227)</td>
<td></td>
</tr>
<tr>
<td>Test error</td>
<td>113.799 (27.965)</td>
<td>113.629 (22.332)</td>
<td>8.796 (0.945)</td>
<td>10.699 (1.314)</td>
<td></td>
</tr>
</tbody>
</table>
Table 2: Mean and the standard deviation (in parentheses) of results using the fused BAR and the fused Lasso for Examples 4.8–4.9.

<table>
<thead>
<tr>
<th></th>
<th>Example 4.8</th>
<th>Example 4.9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>fused BAR</td>
<td>fused Lasso</td>
</tr>
<tr>
<td>NOS</td>
<td>45.050 (17.760)</td>
<td>94.200 (31.629)</td>
</tr>
<tr>
<td>NOFS</td>
<td>18.350 (14.449)</td>
<td>60.550 (29.366)</td>
</tr>
<tr>
<td>TM</td>
<td>72.2% (0.170)</td>
<td>90.9 % (0.121)</td>
</tr>
<tr>
<td>MAB</td>
<td>0.122 (0.040)</td>
<td>0.126 (0.030)</td>
</tr>
<tr>
<td>FMSE</td>
<td>0.584 (0.075)</td>
<td>0.564 (0.085)</td>
</tr>
<tr>
<td>CV error</td>
<td>0.732 (0.081)</td>
<td>0.712 (0.074)</td>
</tr>
<tr>
<td>SVS</td>
<td>0.000 (0.000)</td>
<td>0.500 (0.513)</td>
</tr>
</tbody>
</table>

can obtain coefficients that are piecewise constant with fewer jumps. This fact is also evidenced in Table 1, where the NOJ is much smaller than the
Figure 1: Estimated coefficients using $\ell_1$ fusion, BAR fusion, fused lasso and fused BAR for Example 4.4.

fused Lasso. In addition, Figure 2 indicates that the fused BAR method is sensitive to the single value coefficient and its SVS in Table 1 is larger than the fused lasso method. On the other hand, we see from Table 2 and
Figure 2: Estimated coefficients using $\ell_1$ fusion, BAR fusion, fused lasso and fused BAR for Example 4.5.

Figures 4–5 that when the true jumps are relatively smaller, the fused BAR and BAR fusion can still detect these smaller jumps with a flatter estimate, compared with the fused lasso and the $\ell_1$ fusion. In Example 4.9, the
Figure 3: Estimated coefficients using fused lasso and fused BAR for Example 4.6 (the first row) and Example 4.7 (the second row).

fused lasso seems to successfully detect a single nonzero value since it has a larger SVS value seeing from Table 2. However, we observed by our limited experiments that the fused lasso estimate does not essentially capture an
Figure 4: Estimated coefficients using $\ell_1$ fusion, BAR fusion, fused lasso and fused BAR for Example 4.8.

up-and-down jumping structure, similar to Figure 5. This phenomenon is reasonable because the single true nonzero is merged with larger noise and thus hard to detect. At last, it is seen from Figure 4 that the fused
Figure 5: Estimated coefficients using $\ell_1$ fusion, BAR fusion, fused lasso and fused BAR for Example 4.9.

BAR method is comparable to the fused lasso on image processing. More precisely, the 2D fused BAR reduces the reconstruction error of the 2D fused lasso from 5.525 to 1.051. Overall, the simulation results contain
Figure 6: Results of Example 4.8 for image processing using the fused lasso and the fused BAR.

Supportive evidence that the fused BAR method works reasonably well in variable selection, estimation and prediction compared with the fused lasso. As one reviewer pointed out, one reason is likely to be that lasso is a biased estimation for large coefficients and differences of coefficients. On the other hand, this phenomenon is possibly due to the variable selection inconsistency of lasso in some scenarios (Zou and Hastie, 2005).
6. **Real examples**

6.1. **CGH array denoising**

In cancer research, the copy number variations data (CNV) is one of the important datasets that have an adjacent relationship. CNVs is typically in the form of segments of various lengths (Rippe *et al.*, 2012). The comparative genomic hybridization (CGH) array is a powerful tool to detect genetic alterations such as deletions and copy number increases, and regions of gains or losses in DNA copy numbers (Pinkel *et al.*, 1998; Wang *et al.*, 2005). To facilitate the detection of alterations, the array of CGH data is set to be the log2 ratio of the number of DNA copies in tumor cells over that in normal or reference cells. Therefore, a positive CGH value called a gain indicates an increase in the number of DNA copies, while a loss would occur when a negative value is given. CGH signals are usually approximated by a piecewise constant sequence or function with segmented areas of zero values. In recent years, many approaches, such as the EM-based method (Myers *et al.*, 2004), hidden markov models (Fridlyand *et al.*, 2004; Liu *et al.*, 2010), and a segmentation algorithm (Venkatraman and Olshen, 2007), have been developed for the visualization of CGH signal and the inference of segmented values. The fused lasso method has been applied to identify those gains and losses in the CGH arrays (Tibshirani and Wang,
Table 3: Summary of the results of analysing CGH data.

<table>
<thead>
<tr>
<th>Method</th>
<th>Tuning parameters</th>
<th>NOS</th>
<th>NOJ</th>
<th>FMSE</th>
<th>CV error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fused BAR</td>
<td>$\lambda_1 = 2.154e - 05$, $\lambda_2 = 0.889$</td>
<td>732</td>
<td>11</td>
<td>0.166</td>
<td>0.370</td>
</tr>
<tr>
<td>BAR fusion</td>
<td>$\lambda = 0.910$</td>
<td>–</td>
<td>11</td>
<td>0.167</td>
<td>0.371</td>
</tr>
<tr>
<td>Fused Lasso</td>
<td>$\lambda_1 = 0.005$, $\lambda_2 = 2.081$</td>
<td>942</td>
<td>40</td>
<td>0.176</td>
<td>0.321</td>
</tr>
<tr>
<td>L1 Fusion</td>
<td>$\lambda = 2.081$</td>
<td>–</td>
<td>40</td>
<td>0.176</td>
<td>0.321</td>
</tr>
</tbody>
</table>

We apply the fused BAR, BAR fusion, fused Lasso and $\ell_1$ fusion methods to the CGH arrays. The CGH data is obtained from the R package cgFLasso. The results are illustrated in Table 3 and Figure 7. Table 3 indicates that the fused BAR selects a smaller number of features than fused lasso, and the mean squared errors of fused BAR fitting is smaller. Figure 7 shows that the fused BAR is sensitive to the outliers. The signals recovered using fused BAR and BAR fusion are flatter than those recovered using fused lasso and $\ell_1$ fusion.

6.2. Lena image processing

We explore the use of the two-dimensional fused lasso and fused BAR for denoising the Lena image in the R package filling. We added Gaussian noise with a standard deviation of 20 to the original image. As zero does not represent a natural baseline in this image, we tried the $\ell_1$ fusion model with $\lambda_1 = 0$ and similarly the BAR fusion model. We found the optimal value
Figure 7: The 1D fused lasso and fused BAR applied to CGH data. The red lines are the estimated CGH signals. The black solid line is $y = 0$. of $\lambda_2$ for each method using two-fold cross-validation. The reconstruction errors from the original noiseless image are 6.256 for the BAR fusion and 6.731 for the $\ell_1$ fusion, respectively. Although the BAR fusion has a smaller reconstruction error than the $\ell_1$ fusion, the two methods have their respective advantages on image processing. Specifically, the BAR fusion solution shown in the bottom right panel of Figure 8 gives a better approximation to the smoothness of the image, especially for the background, while the $\ell_1$ fusion estimate shown in the bottom left panel recovers more details in the
characterization of Lena.

7. Discussion

In this paper, we have proposed the BAR method to do variable selection and pattern estimation of regression coefficients. Its oracle properties are demonstrated under proper conditions. As one of the special cases, the fused BAR is introduced and thoroughly discussed with applications in signal approximation and image denoising. To make it easy to implement, the associated algorithms are established based on the Lagrange method. The simulation study and real data analysis show that fused BAR method is comparable to fused lasso in terms of recovering a lower-dimensional piecewise constant structure and reconstructing an image. The BAR approach can be further connected with those methods that penalize the linear combinations of coefficients and is expected to be applied in many other scientific fields.

Supplementary Materials

The technical proofs are provided in the supplementary material.

Acknowledgements

The authors are grateful to the Editor, the Associate Editor and two anonymous reviewers for their insightful comments and suggestions that
Figure 8: Top-left panel: 128 × 128 pixels gray-scale image of Lena. Top-right panel: Gaussian noise with standard deviation 20 has been added. Bottom-left panel: solution of fused lasso with $\lambda_1 = 0$ and $\lambda_2$ chosen by CV. Bottom-right panel: solution of fused BAR with $\lambda_1 = 0$ and $\lambda_2$ chosen by CV.
have substantially improved the presentation and the content of this paper. Linlin Dai’s research was supported by the Fundamental Research Funds for the Central Universities with approval numbers JBK140507 and JBK1806002. Kani Chen’s research was supported by the Hong Kong Research Grant Council grants 16309816, 16300714, 600813 and 600612. The research of Gang Li was partly supported by National Institute of Health Grants P30 CA-16042, UL1TR000124-02, and P50 CA211015.

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