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High-dimensional Linear Regression for Dependent Data with Applications to Nowcasting

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Abstract

In recent years, extensive research has focused on the ℓ_1 penalized least squares (Lasso) estimators of high-dimensional linear regression when the number of covariates p is considerably larger than the sample size n. However, there is limited attention paid to the properties of the estimators when the errors and/or the covariates are serially dependent. In this paper, we investigate the theoretical properties of the Lasso estimators for linear regression with random design and weak sparsity under serially dependent and/or non-sub-Gaussian errors and covariates. In contrast to the traditional case in which the errors are i.i.d and have finite exponential moments, we show that p can be at most a power of n if the errors have only finite polynomial moments. In addition, the rate of convergence becomes slower due to the serial dependence in errors and the covariates. We also consider sign consistency for model selection via Lasso when there are serial correlations in the errors or the covariates or both. Adopting the framework of functional dependence measure, we provide a detailed description on how the rates of convergence and the selection consistency of the estimators depend on the dependence measures and moment conditions of the errors and the covariates. Simulation results show that Lasso regression can be substantially more powerful than the mixed-frequency data sampling regression (MIDAS) and Dantzig Selector in the presence of irrelevant variables. We apply the results obtained for the Lasso method to now-casting with mixed-frequency data for which serially correlated errors and a large number of covariates are common. The empirical results show that the Lasso procedure outperforms the MIDAS regression and autoregressive model with exogenous variables (ARX) in both forecasting and now-casting.

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1 Introduction

During the past two decades, there have been significant developments in high-dimensional linear regression analysis. Consider the linear regression for the response variable y_i and the covariate vector $\mathbf{x}_i$,

$$y_i = \mathbf{x}_i^T \beta + e_i, \quad 1 \le i \le n, \tag{1}$$

where $\beta \in \mathbb{R}^p$ consists of unknown coefficients, e_i is the error term, and $\mathbf{x}_i^T$ denotes the transpose of the covariate vector $\mathbf{x}_i$. Denote the dimension of $\mathbf{x}_i$ by p. In matrix form, we can write the model as $Y = X\beta + e$, where Y is the $n \times 1$ response vector, X is the $n \times p$ design matrix, and e is the $n \times 1$ error vector. Under certain sparsity conditions on β, a great deal of attention has been focused on the ℓ_1 penalized least squares (Lasso) estimator of β when the number of variables p can be much larger than the sample size n; see Efron et al. (2004), Zhao and Yu (2006), and Meinshausen and Yu (2009), among others. Other related approaches include the Dantzig-selector of Candes and Tao (2007), the adaptive Lasso of Zou (2006), the Group Lasso by Yuan and Lin (2006) and the SCAD estimator of Fan and Li (2001), among others. Theoretical properties of those estimators have been established in the literature under the independence assumption; see, for example, Bickel et al. (2009) and Bühlmann and Van De Geer (2011). Here we focus on the Lasso estimator defined as

$$\hat{\beta} = \arg\min_{\beta} \left(\frac{1}{2} |Y - X\beta|_2^2 + \lambda |\beta|_1 \right), \tag{2}$$

where $\lambda \ge 0$ is a tuning parameter, controlling the level of sparsity in $\hat{\beta}$.

Much of the available research dedicated to the Lasso problem deals with the case of large p and small n when the design matrix is static and the errors are independent and identically distributed (i.i.d.) random variables. On the other hand, in many real applications, $\mathbf{x}_i$ consists of stochastic random variables that might be dynamically dependent or e_i is serially dependent or both. Despite a considerable amount of recent work on Lasso estimators, there has been limited research on theoretical properties of the estimates when the observations are dependent. Wang et al. (2007) proposed a Lasso estimator for the regression model with autoregressive errors. Gupta (2012) investigated Lasso estimator for weakly dependent errors. Both papers concentrate on the case when n is greater than p. More recently, Basu and Michailidis (2015) investigated theoretical properties of Lasso estimators with a random design for high-dimensional Gaussian processes. Kock and Callot (2015) established oracle inequalities of the Lasso for Gaussian errors in stationary vector autoregressive models. Wu and Wu (2016) analyzed Lasso estimator with a fixed design matrix and assumed that a restricted eigenvalue condition is satisfied. Medeiros and Mendes (2016) studied the asymptotic properties of the adaptive Lasso when the errors are non-Gaussian and may be conditionally heteroskedastic. The goal of this paper is to investigate the limiting properties of Lasso estimators of Model (1) in the presence of serial dependence in both the covariate vector $\mathbf{x}_i$ and the errors. We establish rate of convergence of the lasso estimator under weak sparsity condition, and provide sign consistency of lasso regression. Our results extend beyond the fixed design and exact sparsity time series, and we do not assume the restricted eigenvalue condition on either the sample or the population covariance matrix.

In practice, many important macroeconomic variables are not sampled at the same frequency. For example, gross domestic product (GDP) data are available quarterly, industrial production data are monthly, and most interest rate data are available daily. Analyzing such data jointly is referred to as the mixed-frequency data analysis. In the econometrics literature, Ghysels et al. (2004) proposed a mixed-data sampling (MIDAS) approach to analyze mixed-frequency data. In particular, they use newly available high-frequency data to improve the prediction of a lower-frequency macroeconomic variable of interest and refer to such predictions as *now-casting*. Consider, for example, the problem of predicting quarterly GDP growth rate y_{n+1} at the forecast origin $i = n$. Here the time interval is a quarter. Traditional forecasting methods employ quarterly data available at $i = n$ to build a model, then use the fitted model to perform prediction. In practice, some monthly and daily data become available during the quarter $i = n + 1$. Now-casting is to make use of such newly available monthly and daily data to update the prediction of y_{n+1}. Therefore, the term now-casting means taking advantages of high-frequency data within a given quarter to update the prediction of GDP growth rate of that quarter. In short, the basic principle of now-casting is the exploitation of the information which is published at higher frequencies than the target variable of interest in order to obtain an improved prediction before the official lower-frequency data becomes available. Since many high-frequency data are available, a large number of covariates are common in now-casting. Therefore, Model (1) with dependent covariates and errors is applicable to now-casting, and the Lasso method is highly relevant. The mixed-data sampling approach of Ghysels et al. (2004) has proven useful for various forecasting and now-casting purposes. We compare the performance of Lasso regression with MIDAS regression and autoregressive model with exogenous variables (ARX) in this paper. To the best of our knowledge, this is the first paper to apply lasso regression to nowcasting. Both simulation studies and empirical studies show that Lasso estimator outperforms the existing MIDAS regression and ARX model.

The rest of the paper is organized as follows. Section 2 defines the high-dimensional dependence measure, adopting the concept of Wu (2005). Section 3 deals with rates of convergence of Lasso estimators. Model selection consistency of Lasso estimators is given in Section 4, and simulation studies are carried out in Section 5. Section 6 considers some real data examples, including forecasting and now-casting applications.

We begin with some basic definitions. Throughout the paper, for a matrix $A = (a_{ij}) \in \mathbb{R}^{p\times p}$, define the spectral norm $\rho(A) = \sup_{|x|\le 1} |Ax|_2$, the Frobenius norm $|A|_F = (\sum_{ij} a_{ij}^2)^{1/2}$, and infinity norm $|A|_\infty = \max_{1\le i,j\le p} |a_{ij}|$. For a vector $a = (a_1, ..., a_p)^T \in \mathbb{R}^p$, define the vector q norm $|a|_q = (\sum_{i=1}^p |a_i|^q)^{1/q}$ for $1 \le q < \infty$. Let $|a|_\infty = \max_{1\le i\le p} |a_i|$ and $|a|_0 = \#\{i : a_i \ne 0\}$. For a random variable $\xi \in \mathcal{L}^k$, denote the q-norm by $\|\xi\|_q = (\mathsf{E}|\xi|^q)^{1/q}$ for $1 \le q \le k$. For two sequences of real numbers $\{a_n\}$ and $\{b_n\}$, write $a_n = O(b_n)$ if there exists a constant C such that $|a_n| \le C|b_n|$ holds for all sufficiently large n, write $a_n = o(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$, and write $a_n \asymp b_n$ if there are positive constants c and C such that $c \le a_n/b_n \le C$ for all sufficiently large n. Denote $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

2 High Dimensional Time Series

Let $\varepsilon_i, i \in \mathbb{Z}$, be i.i.d. random vectors and σ-field $\mathcal{F}_i = (\cdots, \varepsilon_{i-1}, \varepsilon_i)$. In our random-design setting, we assume that in Model (1) the covariate process $(\mathbf{x}_i, i = 1, ..., n)$ is high-dimensional and weakly stationary in the form

$$\mathbf{x}_i = (g_1(\mathcal{F}_i), ..., g_p(\mathcal{F}_i))^T, \tag{3}$$

and the error e_i satisfies

$$e_i = g_e(\mathcal{F}_i), \tag{4}$$

where $g_1(\cdot), \ldots, g_p(\cdot)$ and $g_e(\cdot)$ are measurable functions in $\mathbb{R}$ such that $\mathbf{x}_i$ is well-defined. In the scalar case with $p = 1$, (3) and (4) include a very general class of stationary process (c.f. Wiener (1958), Rosenblatt (1971), Priestley (1988), Tong (1990), Tsay (2005), Wu (2005)). They also allow models with homogeneous or heteroscedastic errors; see Example 1 of Section 3. In the homogeneous case, the covariate process $(\mathbf{x}_i)$ and the errors (e_i) can be independent of each other.

Following Wu (2005), we define the functional dependence measure

$$\delta_{i,q,j} = \|x_{ij} - x_{ij}^*\|_q = \|g_j(\mathcal{F}_i) - g_j(\mathcal{F}_i^*)\|_q, \quad (5)$$

$$\delta_{i,q,e} = \|e_i - e_i^*\|_q = \|g_e(\mathcal{F}_i) - g_e(\mathcal{F}_i^*)\|_q, \quad (6)$$

where the coupled process $x_{ij}^* = g_j(\mathcal{F}_i^*)$ and $e_i^* = g_e(\mathcal{F}_i^*)$ with $\mathcal{F}_i^* = (..., \varepsilon_{-1}, \varepsilon_0', \varepsilon_1, ..., \varepsilon_{i-1}, \varepsilon_i)$ and $\varepsilon_0', \varepsilon_l, l \in \mathbb{Z}$, being i.i.d. random variables. We assume short-range dependence so that

$$\Delta_{m,q,j} := \sum_{i=m}^{\infty} \delta_{i,q,j} < \infty, \quad (7)$$

$$\Delta_{m,q,e} := \sum_{i=m}^{\infty} \delta_{i,q,e} < \infty. \quad (8)$$

Then for fixed m, $\Delta_{m,q,j}$ and $\Delta_{m,q,e}$ measure the cumulative effect of ε_0 on $(x_{ij})_{i\geq m}$ and $(e_i)_{i\geq m}$. We introduce the following dependence adjusted norm (DAN)

$$\|x_{\cdot j}\|_{q,\alpha} = \sup_{m\geq 0}(m+1)^{\alpha}\Delta_{m,q,j}, \quad \alpha \geq 0. \quad (9)$$

$$\|e_{\cdot}\|_{q,\alpha} = \sup_{m\geq 0}(m+1)^{\alpha}\Delta_{m,q,e}, \quad \alpha \geq 0. \quad (10)$$

It can happen that, due to dependence, $\|e_{\cdot}\|_{q,\alpha} = \infty$ while $\|e_i\|_q < \infty$. Since $e_0 = \sum_{l=-\infty}^{0}(\mathsf{E}(e_0|\mathcal{F}_l) - \mathsf{E}(e_0|\mathcal{F}_{l-1}))$, we have

$$\|e_0\|_q \leq \sum_{l=0}^{\infty} \|\mathsf{E}(e_0|\mathcal{F}_{-l}) - \mathsf{E}(e_0|\mathcal{F}_{-l-1})\|_q = \sum_{l=0}^{\infty} \|\mathsf{E}(e_l - e_l^*|\mathcal{F}_0)\|_q \leq \sum_{l=0}^{\infty} \|e_l - e_l^*\|_q = \|e_{\cdot}\|_{q,0}, \quad (11)$$

by stationarity. If e_i, $i \in \mathbb{Z}$, are i.i.d., the dependence adjusted norm $\|e_{\cdot}\|_{q,\alpha}$ and the $\mathcal{L}^q$ norm $\|e_0\|_q$ are equivalent in the sense that $\|e_0\|_q \leq \|e_{\cdot}\|_{q,\alpha} \leq 2\|e_0\|_q$.

To account for the cross-sectional dependence of the p-dimensional stationary process $(\mathbf{x}_i)$, we define the $\mathcal{L}^\infty$ functional dependence measure and its corresponding dependence adjusted norm (c.f. Chen et al. (2013), Zhang and Wu (2017))

$$\omega_{i,q} = \| \max_{1\le j\le p} |x_{ij} - x^*_{ij}| \|_q,$$

$$\| |\mathbf{x}_{\cdot}|_\infty \|_{q,\alpha} = \sup_{m\ge 0} (m+1)^\alpha \Omega_{m,q}, \quad \alpha \ge 0, \quad \text{and } \Omega_{m,q} = \sum_{i=m}^{\infty} \omega_{i,q}.$$

Additionally, we define

$$\Psi_{q,\alpha} = \max_{1\le j\le p} \|x_{\cdot j}\|_{q,\alpha} \quad \text{and} \quad \Upsilon_{q,\alpha} = \left(\sum_{j=1}^{p} \|x_{\cdot j}\|^q_{q,\alpha} \right)^{1/q},$$

where $\Psi_{q,\alpha}$ and $\Upsilon_{q,\alpha}$ can be viewed as the uniform and the overall dependence adjusted norms of $(\mathbf{x}_i)$. Clearly, $\Psi_{q,\alpha} \le \| |\mathbf{x}_{\cdot}|_\infty \|_{q,\alpha} \le \Upsilon_{q,\alpha}$.

We give an example of high-dimensional time series to illustrate how the univariate and multivariate dependence adjusted norms scale.

Example 1. Let $\varepsilon_{ij}, i, j \in \mathbb{Z}$, be i.i.d. random variables with mean 0, variance 1, and having finite qth moments, $q > 2$, and let $A_i,\ i \ge 0$, be $p \times d$ coefficient matrices with real entries such that $\sum_{i=0}^{\infty} \mathrm{tr}(A_i A_i^T) < \infty$. Write $\varepsilon_i = (\varepsilon_{i1}, ..., \varepsilon_{id})^T$. Then by Kolmogorov's three-series theorem the linear process

$$\mathbf{x}_i = \sum_{l=0}^{\infty} A_l \varepsilon_{i-l} \tag{12}$$

exists. Denote $A_l = (a_{l;jk})_{1\le j\le p, 1\le k\le d}$, $A_{l,j\cdot}$ the jth row of A_l. By Burkholder's inequality, $\|A_{l,j\cdot}\varepsilon_0\|_q \le \sqrt{q-1}|A_{l,j\cdot}|_2 \|\varepsilon_{00}\|_q$. We assume that the linear process satisfies the decay condition

$$\max_{j\le p} |A_{l,j\cdot}|_2 \le K_1 (1 \vee l)^{-\theta} \tag{13}$$

for all $l \ge 0$, where $\theta > 1/2$ and $K_1 > 0$. If $\theta > 1$, (13) implies short-range dependence (SRD) since the auto-covariance matrices $\Sigma_k = \sum_{l=0}^{\infty} A_l A_{l+k}^T$ are absolutely summable. On the other hand, if $1 > \theta > 1/2$, then $(\mathbf{x}_i)$ in (12) may not have summable auto-covariance matrices, thus allowing long-range dependence (LRD). The classical literature on LRD primarily focuses on the univariate case $p = 1$. Then under the SRD case, the dependence adjusted norms have the following bounds

$$\Psi_{q,\alpha} = \max_{1 \le j \le p} \|x_{\cdot j}\|_{q,\alpha} = \max_j \sup_{m \ge 0} (m+1)^{\alpha} \sum_{i=m}^{\infty} \|A_{i,j\cdot}\varepsilon_0\|_q \le K_1 K_2 \|\varepsilon_{00}\|_q, \tag{14}$$

$$\||\mathbf{x}_{\cdot}|_{\infty}\|_{q,\alpha} = \sup_{m \ge 0} (m+1)^{\alpha} \sum_{i=m}^{\infty} \|\max_j |A_{i,j\cdot}\varepsilon_0|\|_q \le K_1 K_2 p^{1/q} \|\varepsilon_{00}\|_q, \tag{15}$$

where $\alpha = \theta - 1$ and the constant K_2 only depends on θ and q.

In this paper, we use dependence adjusted norms $\||\mathbf{x}_{\cdot}|_{\infty}\|_{q,\alpha}$, $\Psi_{q,\alpha}$, and $\Upsilon_{q,\alpha}$ to study the limiting properties of Lasso estimators in the presence of serial dependence. These adjusted norms are more convenient than the commonly used mixing conditions for handling serial dependence in high-dimensional time series.

3 Convergence Rate of the Lasso estimator

In this section, we present the main results on convergence rate of the Lasso estimator for dependent data. In the low-dimensional case, the consistency of $\hat{\beta}$ relies on the assumption that the sample covariance matrix converges to the population covariance matrix. In the high-dimensional case ($n \ll p$), it requires that $|X(\hat{\beta} - \beta)|_2$ is small only when $|\hat{\beta} - \beta|_2$ is small. Let $\hat{\Sigma} = (\hat{\sigma}_{jk})_{1 \le j,k \le p} = n^{-1} \sum_{i=1}^{n} x_i x_i^T$ be the sample covariance. Typically, researchers assume with high probability, the following Restricted Strong Convexity condition holds,

$$u' \hat{\Sigma} u \ge \kappa_1 |u|_2^2 - \kappa_2 g(n,p) |u|_1^2, \tag{16}$$

for all $u \in \mathbb{R}^p$, where κ_1, κ_2 are positive constants and $g(n,p)$ is a function of the sample size n and ambient dimension p. It can be viewed as an analogous sufficient condition in the high-dimensional case. As shown in the proof of Theorem 1, the Restricted Strong Convexity condition for the sample covariance matrix holds with high probability under certain conditions.

To establish our theoretical results, we first impose a weak sparsity condition:

Assumption 1. There exits some $0 \leq \theta < 1$, with a uniform radius K_θ such that

$$\sum_{j=1}^{p} |\beta_j|^\theta \leq K_\theta. \tag{17}$$

The following theorem shows the L_2 and L_1 convergence rates of $\hat{\beta}$ to β depend on the moment condition and the temporal and cross-sectional dependence conditions.

Theorem 1. *Denote the population covariance matrix by $\Sigma = (\sigma_{jk}) = [Cov(x_{ij}, x_{ik})]$.Suppose the minimum eigenvalue of Σ satisfies $\lambda_{\min}(\Sigma) \geq \kappa > 0$. Assume that $\Psi_{\gamma,\alpha_X} = \max_j \|x_{\cdot j}\|_{\gamma,\alpha_X} = M_X < \infty$ and $\|e_\cdot\|_{q,\alpha_e} = M_e < \infty$, where $q > 2, \gamma > 4$ and $\alpha_X, \alpha_e > 0$. Define*

$$\nu = \begin{cases} 1 & \text{if } \alpha_X \geq 1/2 - 2/\gamma, \\ \gamma/4 - \alpha_X\gamma/2 & \text{if } \alpha_X < 1/2 - 2/\gamma. \end{cases}$$

Assume $\tau = q\gamma/(q+\gamma) > 2$ and let $\alpha = \min(\alpha_X, \alpha_e)$. Define

$$\rho = \begin{cases} 1 & \text{if } \alpha \geq 1/2 - 1/\tau, \\ \tau/2 - \alpha\tau & \text{if } \alpha < 1/2 - 1/\tau. \end{cases}$$

Denote $\omega = \sqrt{\log p/n} M_X^2 + n^{2\nu/\gamma - 1}(\log p)^{3/2} \| |\mathbf{x}_\cdot|_\infty \|_{\gamma,\alpha_X}^2$. Suppose Assumption 1 holds. Then for any λ such that

$$\lambda \gtrsim \sqrt{\log p/n} M_e M_X + n^{\rho/\tau - 1}(\log p)^{3/2} M_e \| |\mathbf{x}_\cdot|_\infty \|_{\gamma,\alpha_X},$$

and $K_\theta \omega \lambda^{-\theta} \leq C$ for some positive constant C, any Lasso solution $\hat{\beta}$ satisfies,

$$|\hat{\beta} - \beta|_2 \lesssim \sqrt{K_\theta} \left(\frac{\lambda}{\kappa}\right)^{1-\theta/2}, \tag{18}$$

$$|\hat{\beta} - \beta|_1 \lesssim K_\theta \left(\frac{\lambda}{\kappa}\right)^{1-\theta}. \tag{19}$$

with probability at least $1 - C_1(\log p)^{-\gamma/2} - C_2 p^{-C_3} - C_4(\log p)^{-\tau}$*, where* $C_1, ..., C_4$ *are positive constants.*

In the special case $\theta = 0$, the quantity of weak sparsity corresponds to an exact sparsity constraint–that is, β has at most $s := K_0$ nonzero entries. The following theorem shows the convergence rate of $\hat{\beta}$ and the prediction error $|X(\hat{\beta} - \beta)|_2^2$ for the exact sparsity case.

Theorem 2. *Suppose the same conditions of Theorem 1 hold. If* $|\beta|_0 = s$ *and* $\kappa \asymp 1$*,*

$$n \gtrsim M_X^4 s^2 \log p + s^{1/(1-2\nu/\gamma)}(\log p)^{3/(2-4\nu/\gamma)} \| |\mathbf{x}_{\cdot}|_\infty \|_{\gamma,\alpha_X}^{2/(1-2\nu/\gamma)},$$

then for any λ *such that*

$$\lambda \gtrsim \sqrt{\log p/n} M_e M_X + n^{\rho/\tau - 1}(\log p)^{3/2} M_e \| |\mathbf{x}_{\cdot}|_\infty \|_{\gamma,\alpha_X},$$

any Lasso solution $\hat{\beta}$ *satisfies,*

$$|\hat{\beta} - \beta|_2 \lesssim \lambda\sqrt{s}/\kappa, \tag{20}$$

$$|\hat{\beta} - \beta|_1 \lesssim \lambda s/\kappa, \tag{21}$$

$$|X(\hat{\beta} - \beta)|_2^2/n \lesssim \lambda^2 s/\kappa, \tag{22}$$

with probability at least $1 - C_1(\log p)^{-\gamma/2} - C_2 p^{-C_3} - C_4(\log p)^{-\tau}$*.*

Remark 1. In the exact sparsity case, instead of the condition $\lambda_{\min}(\Sigma) \geq \kappa > 0$, we may require that the restricted eigenvalue assumption RE(s,3) of Bickel et al. (2009) holds for the population covariance matrix Σ, namely

$$\kappa := \min_{J \subseteq \{1,...,p\}, |J|_0 \leq s} \; \min_{u \neq 0, |u_{J^c}|_1 \leq 3|u_J|_1} u'\Sigma u/|u|_2^2 > 0, \tag{23}$$

where J^c is the complement of the set J, *i.e.*, $J^c = \{1, 2, ..., p\} \backslash J$, u_J is defined as a modification of u by setting its elements outside J to zero. All the bounds (20), (21) and (22) still hold with high probability.

Remark 2. The best known convergence rate of Lasso estimators for i.i.d sub-Gaussian data requires that $K_\theta(\log p/n)^{1-\theta/2} \le C$ for some positive constant C. Our theorems require that $K_\theta \omega \lambda^{-\theta} \le C$, where

$$\omega = \sqrt{\log p/n} M_X^2 + n^{2\nu/\gamma - 1}(\log p)^{3/2} \| |\mathbf{x}_\cdot|_\infty \|^2_{\gamma,\alpha_X},$$

and

$$\lambda \gtrsim \sqrt{\log p/n} M_e M_X + n^{\rho/\tau - 1}(\log p)^{3/2} M_e \| |\mathbf{x}_\cdot|_\infty \|_{\gamma,\alpha_X}.$$

The second terms in ω and λ are introduced by the heavy tails, and thus are unavoidable. In other words, under heavy tailed distributions, sometimes, the allowed dimension p for Lasso methods can be at most a power of the sample size n.

In the exact sparsity case, we require $n \gtrsim M_X^4 s^2 \log p + s^{1/(1-2\nu/\gamma)}(\log p)^{3/(2-4\nu/\gamma)} \| |\mathbf{x}_\cdot|_\infty \|_{\gamma,\alpha_X}^{2/(1-2\nu/\gamma)}$. One may argue the first term $M_X^4 s^2 \log p$ can be further improved to $M_X^4 s \log p$ for short range temporal dependence data, in agreement with i.i.d. sub-Gaussian data. However, we cannot achieve it because even the optimal Bernstein type inequality for nonlinear weakly dependent data is still an open problem. The best known result is proposed by Merlevède et al. (2009).

Remark 3. Based on Theorem 2, we have the following cases: Assume $M_X \asymp 1$ and $M_e \asymp 1$. Under the weak cross-sectional dependence $\| |\mathbf{x}_\cdot|_\infty \|_{\gamma,\alpha_X} \asymp p^{1/\gamma}$, which holds if the p components x_{ij} $(1 \le j \le p)$ are nearly independent, then the required sample size for exact sparsity is $n \gtrsim s^2 \log p + s^{1/(1-2\nu/\gamma)}(\log p)^{3/(2-4\nu/\gamma)} p^{2/(\gamma-2\nu)}$ and regularization parameter satisfies $\lambda \gtrsim \sqrt{\log p/n} + n^{\rho/\tau-1}(\log p)^{3/2} p^{1/\gamma}$. In comparison, Bonferroni Inequality and Lemma 1 in the Appendix would result in $n \gtrsim s^2 \log p + s^{1/(1-2\nu/\gamma)} p^{4/(\gamma-2\nu)}$ and $\lambda \gtrsim \sqrt{\log p/n} + n^{\rho/\tau-1} p^{1/\tau}$.

In addition, under the strong cross-sectional dependence $\| |\mathbf{x}_\cdot|_\infty \|_{\gamma,\alpha_X} \asymp 1$, which holds if the p components x_{ij} $(1 \le j \le p)$ are linear combinations of fixed random variables, the required sample size for exact sparsity is $n \gtrsim s^2 \log p + s^{1/(1-2\nu/\gamma)}(\log p)^{3/(2-4\nu/\gamma)}$ and the regularization parameter satisfies $\lambda \gtrsim \sqrt{\log p/n} + n^{\rho/\tau-1}(\log p)^{3/2}$.

Next, we give an example for which the results of Theorem 1 apply.

Example 2. Consider the autoregressive model with exogenous variables, that is, the ARX(a, b) model:

$$y_i = \sum_{l=1}^{a} \phi_l y_{i-l} + \sum_{l=0}^{b} \psi_l' \mathbf{z}_{i-l} + e_i = \beta' \mathbf{x}_i + e_i, \tag{24}$$

where a and b are nonnegative integers, e_i follows a GARCH(1,1) model defined below, and $\mathbf{z}_i$ is a linear process defined by

$$\mathbf{z}_i = \sum_{l=0}^{\infty} A_l \varepsilon_{i-l}, \tag{25}$$

where the random variables ε_{ij} and coefficient matrices A_l are given in Example 1 with $E|\varepsilon_{ij}|^\gamma < \infty$ and $\gamma > 2$. Assume the roots of the polynomial $1 - \sum_{l=1}^{a} \phi_l B^l$ are outside the unit circle, which ensures stationarity of the autoregressive part of the model. Also assume the population covariance matrix $\Sigma = \mathsf{E}\mathbf{x}_i\mathbf{x}_i'$ is positive definite.

Let

$$e_i = \sqrt{h_i}\eta_i, \qquad h_i = \pi_0 + \pi_1 e_{i-1}^2 + \pi_2 h_{i-1}, \tag{26}$$

with $\pi_0 > 0$, $\pi_1 \geq 0$, $\pi_2 \geq 0$ and $\mathsf{E}(\pi_1 + \pi_2 \eta_{i-1}^2)^{q/2} < \infty$, $q > 4$. Then it is easy to show $\|e_.\|_{q,\alpha_e} < \infty$.

Again, by Burkholder's inequality, $\|A_{l,j.}\varepsilon_0\|_\gamma \leq \sqrt{\gamma-1}|A_{l,j.}|_2 \|\varepsilon_{00}\|_\gamma$. If there exist constants $K_1 > 1$ and $\alpha_Z > 0$ such that $\max_{j \leq p} |A_{l,j.}|_2 \leq K_1(l+1)^{-1-\alpha_Z}$ holds for all $l \geq 0$, then we have $\max_j \|z_{.j}\|_{\gamma,\alpha_Z} \leq K_1 K_2 \|\varepsilon_{00}\|_\gamma$, where the constant K_2 only depends on α_Z and γ. Together with the assumption that the roots of the polynomial $1 - \sum_{l=1}^{a} \phi_l B^l$ are outside the unit circle, we ensure $\max_j \|x_{.j}\|_{\gamma,\alpha_Z} < \infty$.

4 Model Selection Consistency

In this section, we extend the asymptotic properties of sign consistency for model selection via the Lasso to the dependent setting. The sign consistency of Lasso was first introduced by Zhao and Yu (2006). Without loss of generality, write $\beta = (\beta_1, ..., \beta_s, ..., \beta_p)'$, where $\beta_j \neq 0$ if $j \leq s$ and $\beta_j = 0$ if $j > s$. That is, the first s predictors are relevant variables. Denote $\beta = (\beta'_{(1)}, \beta'_{(2)})'$, where $\beta_{(1)}$ is a $s \times 1$ vector. Correspondingly, for any i, denote $\mathbf{x}_i = (\mathbf{x}'_{i(1)}, \mathbf{x}'_{i(2)})'$ and $X = (\mathbf{x}_1, ..., \mathbf{x}_n)' = (X_{(1)}, X_{(2)})$, where $X_{(1)}$ is the $n \times s$ sub-matrix consisting of the relevant variables, and $X_{(2)}$ is the $n \times (p - s)$ sub-matrix with the irrelevant ones. Similarly, consider the partition of the covariance matrix as

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where $\Sigma_{11} = \mathsf{E}\mathbf{x}_{i(1)}\mathbf{x}'_{i(1)}$ is a $s \times s$ sub-matrix associated with the relevant variables.

We impose the following assumptions.

Assumption 2. For any $1 \leq i \leq n$, $\mathsf{E}(x_{ik}|X_{(1)}, e) = \Sigma_{2k,1}\Sigma_{11}^{-1}x_{i(1)}$, where $\Sigma_{2k,1}$ is the k-th row of Σ_{21}.

Define $z_{ik} = x_{ik} - \mathsf{E}(x_{ik}|X_{(1)}, e)$ for $s + 1 \leq k \leq p$, and $\mathbf{z}_i = (z_{i,s+1}, ..., z_{i,p})'$.

Assumption 3. There exists $L > 0$ such that $\min_{1 \leq j \leq s} |\beta_j| \geq L$.

Assumption 4. There exists a constant $N_1 > 0$ such that

$$\inf_{|\zeta|_2=1} \zeta'\Sigma_{11}\zeta = N_1.$$

Assumption 5. There exists a positive constant $\eta \in (0, 1)$ such that

$$|\Sigma_{21}\Sigma_{11}^{-1}\text{sign}(\beta_{(1)})|_\infty \leq 1 - \eta. \tag{27}$$

Assumption 2 explicitly defines how the irrelevant variables depend on the relevant variables and the errors. Note that $\mathsf{Cov}(\Sigma_{2k,1}\Sigma_{11}^{-1}x_{i(1)}, x_{ik} - \Sigma_{2k,1}\Sigma_{11}^{-1}x_{i(1)}) = 0$ always holds, for all $s+1 \leq k \leq p$. That is, $\Sigma_{2k,1}\Sigma_{11}^{-1}x_{i(1)}$ and $x_{ik} - \Sigma_{2k,1}\Sigma_{11}^{-1}x_{i(1)}$ are mutually uncorrelated. We further assume they are independent. Intuitively, $\mathbf{z}_i$ can be viewed as the unique part of irrelevant variables that cannot be explained by the relevant variables. Thus, for irrelevant variables, $\mathbf{z}_i$ is more representative than $\mathbf{x}_{i(2)}$. Assumption 3 controls the lower bound of the non-zero parameters; see, for example, Bühlmann and Van De Geer (2011). Assumption 4 imposes a lower bound, N_1, on the minimal eigenvalue of the covariance matrix of relevant variables. In practice, quantifying the rate under which N_1 decreases is difficult and problem specific, and it is frequently assumed constant, e.g., Medeiros and Mendes (2016) and Kock and Callot (2015). Assumption 5 employs the strong irrepresentable condition of population covariance, which is similar to the condition in Zhao and Yu (2006).

To account for the cross-sectional dependence of the stationary process $(\mathbf{x}_{i(1)})$ and $(\mathbf{z}_i)$, we also define the $\mathcal{L}^\infty$ functional dependence measure and its corresponding dependence adjusted norm

$$\omega_{i,q,1} = \| \max_{1\leq j\leq s} |x_{ij} - x_{ij}^*| \|_q,$$

$$\||\mathbf{x}_{.(1)}|_\infty\|_{q,\alpha} = \sup_{m\geq 0} (m+1)^\alpha \Omega_{m,q,1}, \quad \alpha \geq 0, \quad \text{and } \Omega_{m,q,1} = \sum_{i=m}^{\infty} \omega_{i,q,1}.$$

Additionally, we define

$$\Psi_{q,\alpha,1} = \max_{1\leq j\leq s} \|x_{.j}\|_{q,\alpha} \quad \text{and} \quad \Upsilon_{q,\alpha,1} = \left(\sum_{j=1}^{s} \|x_{.j}\|_{q,\alpha}^q \right)^{1/q}.$$

For $(\mathbf{z}_i)$, the quantities $\||\mathbf{z}_.|_\infty\|_{q,\alpha}$, $\Psi_{q,\alpha,2}$ and $\Upsilon_{q,\alpha,2}$ can be similarly defined. Clearly, $\Psi_{q,\alpha,1} \leq \||\mathbf{x}_{.(1)}|_\infty\|_{q,\alpha} \leq \Upsilon_{q,\alpha,1}$ and $\Psi_{q,\alpha,2} \leq \||\mathbf{z}_.|_\infty\|_{q,\alpha} \leq \Upsilon_{q,\alpha,2}$.

Let $\sigma = \mathsf{E}e_i^2$. Define

$$\begin{aligned}
\delta_*(\lambda, N_1, \sigma) &= \frac{\lambda^2 s}{2nN_1} + \frac{2\sigma}{n}, \\
M(\delta_*, \eta, \iota, \gamma) &= \eta^{-1}\sqrt{\delta_* \log p} + \eta^{-1} n^{(\iota-1)/\gamma} \delta_*^{1/2} (\log p)^{3/2} \| |\mathbf{z}_\cdot|_\infty \|_{\gamma,\alpha_X}, \\
Q(\rho, \tau) &= \sqrt{n \log s} + n^{\rho/\tau} (\log s)^{3/2} \| |\mathbf{x}_{\cdot(1)}|_\infty \|_{\gamma,\alpha_X}, \\
V_1(N_1) &= \frac{s^2 \log s}{N_1}, \\
V_2(N_1) &= \frac{1}{N_1} s (\log s)^{3/2} \| |\mathbf{x}_{\cdot(1)}|_\infty \|^2_{\gamma,\alpha_X}.
\end{aligned}$$

These quantities are used in the following theorem.

Theorem 3 below extends the results of Zhao and Yu (2006) to random design linear model with dependent errors. In comparison, Medeiros and Mendes (2016) derived asymptotic properties of sign consistency for the adaptive Lasso, while our results apply to the original Lasso and do not need any assumptions on weights. Note that even for heavy-tail variables, our results show that if the dependence among $\mathbf{z}_i$ is strong, the allowed dimension p can be as large as some exponential of the sample size n; see Remark 2 for more details.

Theorem 3. *Suppose Assumptions 2, 3, 4 and 5 hold. Assume that* $\max_{1\le j\le p} \|x_{\cdot j}\|_{\gamma,\alpha_X} < C_\gamma < \infty$*, and* $\|e_\cdot\|_{q,\alpha_e} < C_q < \infty$*, where* $q, \gamma > 4$*,* $\alpha_X, \alpha_e > 0$*, constants* C_γ, C_q *only depend on* γ, q*. Define*

$$\nu = \begin{cases} 1 & \text{if } \alpha_X > 1/2 - 2/\gamma, \\ \gamma/4 - \alpha_X\gamma/2 & \text{if } \alpha_X < 1/2 - 2/\gamma, \end{cases}$$

and

$$\iota = \begin{cases} 1 & \text{if } \alpha_X > 1/2 - 1/\gamma, \\ \gamma/2 - \alpha_X\gamma & \text{if } \alpha_X < 1/2 - 1/\gamma. \end{cases}$$

Let $\alpha = \min(\alpha_X, \alpha_e)$*. Assume* $\tau = q\gamma/(q+\gamma) > 2$ *and define*

$$\rho = \begin{cases} 1 & \text{if } \alpha > 1/2 - 1/\tau, \\ \tau/2 - \alpha\tau & \text{if } \alpha < 1/2 - 1/\tau. \end{cases}$$

Furthermore, suppose $s = o(n)$*. Then, for any* λ *and the sample size* n *such that*

$$n \gtrsim V_1(N_1), \tag{28}$$

$$n^{1-2\nu/\gamma} \gtrsim V_2(N_2), \tag{29}$$

$$M(\delta_*, \eta, \iota, \gamma) + Q(\rho, \tau) \lesssim \lambda \leq \frac{nN_1 L}{4\sqrt{s}}, \tag{30}$$

the consistency probability $\mathsf{P}(\hat{\beta} =_s \beta)$ *is at least*

$$1 - C_1(\log p)^{-\gamma} - C_2(\log s)^{-\gamma/2} - C_3(\log s)^{-\tau} - C_4 p^{-C_5} - C_6 s^{-C_7} - \frac{\|e_{\cdot}\|_{q,\alpha_e}^q}{n^{q-1}\sigma^q} - \exp\left(-\frac{n\sigma^2}{\|e_{\cdot}\|_{2,\alpha_e}^2}\right). \tag{31}$$

Remark 4. In particular, assume $N_1 \asymp 1$, $\eta \asymp 1$. Also assume the weak temporal dependence case $\alpha_X > 1/2 - 1/\gamma$ and $\alpha > 1/2 - 1/\tau$. If the dependence measure $\||\mathbf{x}_{\cdot(1)}|_\infty\|_{\gamma,\alpha_X} \asymp s^{1/\gamma}$ and $\||\mathbf{z}_{\cdot}|_\infty\|_{\gamma,\alpha_X} \asymp p^{1/\gamma}$, which would hold if all the components x_{ij} $(1 \leq j \leq s)$ and z_{ik} $(s+1 \leq k \leq p)$ are nearly independent, then (28), (29) and (30) reduce to

$$n \gtrsim s^2 \log s + s^{\frac{1+2/\gamma}{1-2/\gamma}} (\log s)^{\frac{3}{2-4/\gamma}} + sp^{2/\gamma}(\log p)^3$$

and

$$\sqrt{n \log s} + n^{1/\tau} s^{1/\tau} (\log s)^{3/2} \lesssim \lambda \lesssim \frac{nL}{\sqrt{s}}.$$

Additionally, if $s = O(n^{c_1})$ for some $c_1 < \min\{1/2, (\gamma-2)/(\gamma+2)\}$, then the valid regularization parameter λ has the range $n^{1/2} + n^{1/\tau + c_1/\gamma} \ll \lambda \ll n^{1-c_1/2} L$. The dimension p satisfies that $p \ll n^{\gamma(1-c_1)/2}$.

On the other hand, assume $\||\mathbf{x}_{\cdot(1)}|_\infty\|_{\gamma,\alpha_X} \asymp s^{1/\gamma}$ and $\||\mathbf{z}_{\cdot}|_\infty\|_{\gamma,\alpha_X} \asymp 1$, that is, all the components z_{ik} $(s+1 \le k \le p)$ are strongly dependent. Let $s = O(n^{c_1})$ for some $c_1 < \min\{1/2, (\gamma-2)/(\gamma+2)\}$, then the existence of regularization parameter λ requires $n^{1/2} + n^{1/\tau + c_1/\gamma} \ll \lambda \ll n^{1-c_1/2}L$. The dimension p satisfies $p \ll \exp\{n^{(1-c_1)/3}\}$.

Furthermore, if $\||\mathbf{x}_{\cdot(1)}|_\infty\|_{\gamma,\alpha_X} \asymp 1$ and $\||\mathbf{z}_{\cdot}|_\infty\|_{\gamma,\alpha_X} \asymp 1$, $s = O(n^{c_1})$ for some $c_1 < 1/2$, then the existence of regularization parameter λ requires $n^{1/2} \ll \lambda \ll n^{1-c_1/2}L$, and the dimension p satisfies $p \ll \exp\{n^{(1-c_1)/3}\}$.

In summary, the allowed dimension p varies from $n^{\gamma(1-c_1)/2}$ to $\exp\{n^{(1-c_1)/3}\}$ depending on the cross-sectional dependence of $z_{ik}, s+1 \le k \le p$.

Note that if the assumptions in Example 2 hold, together with the Strong Irrepresentable Condition, the results of Theorem 3 continue to apply. In general, the Strong Irrepresentable Condition is non-trivial, particularly since we do not know $\mathrm{sign}(\beta)$ a priori. Then, we need the Strong Irrepresentable Condition to hold for every possible combination of different signs and placement of zeros. We give a simple example below in which the Strong Irrepresentable Condition is guaranteed. All diagonal elements of Σ are assumed to be 1 which is equivalent to normalizing all covariates in the model to the same scale since Strong Irrepresentable Condition is invariant under any common scaling of Σ.

Example 3. Consider the following autoregressive model with exogenous variables:

$$y_i = \sum_{l=1}^{a} \phi_l y_{i-l} + \psi \mathbf{z}_i + e_i = \beta' \mathbf{x}_i + e_i, \tag{32}$$

where a is nonnegative finite integer, $\mathbf{z}_i$ is independent of e_i, and the errors e_i are homogeneous. Assume the roots of the polynomial $1 - \sum_{l=1}^{a} \phi_l B^l$ are outside the unity circle, which ensures stationarity of the autoregressive part of the model. Also assume $\Sigma = \mathsf{E}\mathbf{x}_i\mathbf{x}_i'$ is positive definite.

Furthermore, suppose β has s nonzero entries. Similar to Corollary 2 in Zhao and Yu (2006), Σ has 1's on the diagonal and bounded correlation $|\sigma_{jk}| \leq c/(2s-1)$ for a constant $0 < c < 1$ then Strong Irrepresentable Condition holds. In this case, we need autocorrelation of y_i to be weak, and all the covariates $\mathbf{z}_i$ are slightly correlated.

Remark 5. Lasso may fail in the presence of strong serial dependence. Consider two scalar Gaussian autoregressive, AR(3), models:

$$y_i = 1.9y_{i-1} - 0.8y_{i-2} - 0.1y_{i-3} + e_i, \tag{33}$$

and

$$y_i = y_{i-1} - 0.8y_{i-2} - 0.1y_{i-3} + e_i, \tag{34}$$

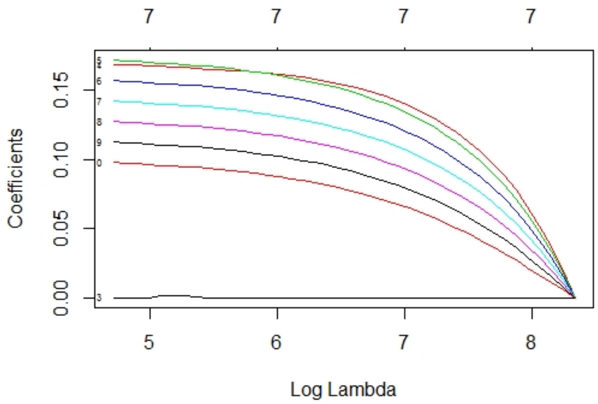
where e_i follows the standard normal distribution. Then AR(3) model (33) is unit-root nonstationary, but model (34) is stationary. We generate 2000 observations from each of the two models. We choose $y_{i-10}, y_{i-9}, ..., y_{i-1}$, and $x_{1i}, ..., x_{10,i}$ as regressors, where x_{li} are i.i.d. standard normal. Figure 1 shows the model selection results for scaling vs. not scaling the predictors.

The default Lasso procedure standardizes each variable in y_i. For unit-root non-stationary time series, standardization might wash out the dependence of the stationary part; see Parts (a) and (b) of Figure 1. In this paper, we only consider stationary time series for which scaling the predictors does not affect the estimation consistency of the Lasso estimates; see Parts (c) and (d) of Figure 1.

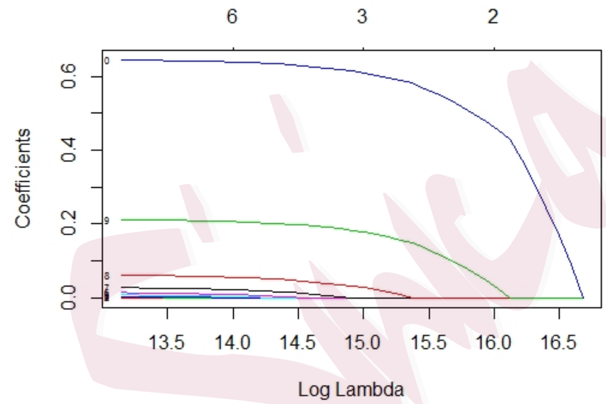
The following proposition shows a necessary and sufficient condition for a stationary AR(2) model under which the Strong Irrepresentable Condition (Assumption 5) holds. Similar results also hold for the general stationary AR(d) model.

Proposition 1. *Consider the stationary AR(2) model,*

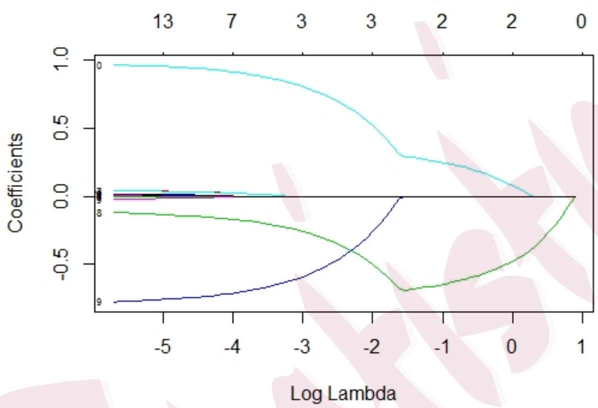
$$y_i = \phi_1 y_{i-1} + \phi_2 y_{i-2} + e_i,$$



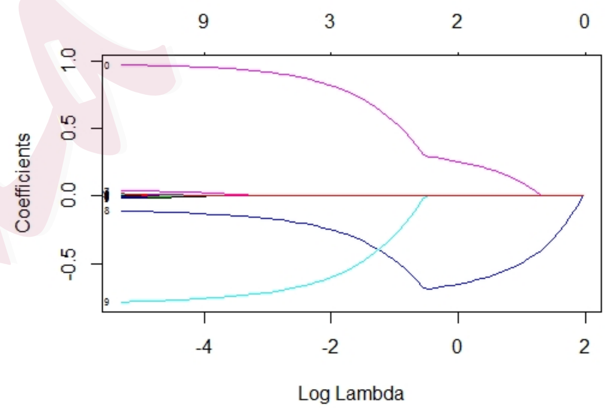
(a) scaling for AR model (33)



(b) not scaling for AR model (33)



(c) scaling for AR model (34)



(d) not scaling for AR model (34)

Figure 1: Results of Lasso regression for the two AR(3) series in (33) and (34) via the glmnet package of R.

where e_i are i.i.d. random variates with mean zero and finite variance. We also normalize y_i such that the variance of y_i is 1. Then, the Strong Irrepresentable Condition (Assumption 5) holds if and only if

$$|\phi_1| + |\phi_2| < 1. \tag{35}$$

5 Simulation Study

In this section, we use simulation to demonstrate the performance of Lasso regression for dependent data in finite samples and to compare its efficacy with the mixed-frequency data sampling regression (MIDAS) commonly used in the econometric literature; see Ghysels et al. (2004). In addition, we also compare the model selection consistency and parameter estimation of Lasso estimator and Dantzig estimator for dependent data in finite samples.

We first consider the following data generating process,

$$\begin{aligned} y_i &= \phi y_{i-1} + \boldsymbol{x}_{i-1,1}^T \beta_s + e_i, \\ \boldsymbol{x}_i &= \begin{bmatrix} \boldsymbol{x}_{i,1} \\ \boldsymbol{x}_{i,2} \end{bmatrix} = \sum_{j=1}^{m} A_j \begin{bmatrix} \boldsymbol{x}_{i-j,1} \\ \boldsymbol{x}_{i-j,2} \end{bmatrix} + \boldsymbol{\eta}_i, \end{aligned} \tag{36}$$

where $\phi = 0.6$ and each element of β_s is given by $\beta_{s,j} = \frac{1}{\sqrt{s}}(-1)^j$, $\boldsymbol{x}_{i,1}$ is a $s \times 1$ vector of relevant variables. Let $\beta = (\beta_s, \beta_{s^c})$, where $\beta_{s^c} = \mathbf{0}$ is a $(p-s) \times 1$ vector. The errors e_i and $\boldsymbol{\eta}_{ij}$ are i.i.d random variables of Student-t distribution with 5 degrees of freedom, and e_i and $\boldsymbol{\eta}_i$ are all mutually uncorrelated. The explanatory variable process $\boldsymbol{x}_i$, which has $p-s$ irrelevant variables, follows a vector autoregressive, VAR(m), model. The following two choices of $\boldsymbol{x}_i$ are considered, denoted as Model 1 and Model 2, respectively.

(1). **Model 1**: The explanatory process $\boldsymbol{x}_i$ is a VAR(4) process, where A_1 and A_4 assume a block-diagonal structure and $A_2 = A_3 = 0$. In particular, the first two and the last two blocks are 5×5 matrices with all entries of the blocks of A_1 equal to 0.15 and all entries of the blocks of A_4 equal to -0.1. The other blocks are 10×10 matrices with all elements of the blocks of A_1 equal to 0.075 and all elements of the blocks of A_4 equal to -0.05. This structure could be motivated by a model built for mixed-frequency data with some quarterly time series often encountered in macroeconomic analysis.

(2). **Model 2**: The explanatory process $\boldsymbol{x}_i$ follows a VAR(1) model, where A_1 is block-diagonal with the same block structure given by Model 1. The (j, k)th entry of the block is $(-1)^{|j-k|}\rho^{|j-k|+1}$ with $\rho = 0.4$. Hence, the entries decrease exponentially fast with their distances from the diagonal.

We employ sample sizes $n = 50, 100, 200$ with different choices of p and s. We set $p = 100, 200, 400$ and $s = 5, 10, 20$. For comparison, we also simulate a response series from a MIDAS model. In Model (36), for $s = 5, 10, 20$, let $\beta_s = \beta(1)$, $(\beta(1)^T, \beta(2)^T)^T$ or $(\beta(1)^T, \beta(2)^T, \beta(3)^T)^T$ respectively, with

$$\beta_j(l) = \frac{\exp(\delta_1 j + \delta_2 j^2)}{\sum_{k=1}^{|\beta(l)|_0} \exp(2\delta_1 k + 2\delta_2 k^2)} \tag{37}$$

where $\beta(1)$ and $\beta(2)$ have 5 variables, $\beta(3)$ has 10 variables, and $\delta = (\delta_1, \delta_2)' = (0.5, -1)'$. All the other settings are the same as before. The two choices of $\boldsymbol{x}_i$ as in Models 1 and 2 are used, and we denote the resulting MIDAS models as Models 3 and 4, respectively. The models estimated by Lasso are with λ selected by the BIC; see Bühlmann and Van De Geer (2011). The consistency of Lasso estimator selected by BIC was first proved by Zou et al. (2007) under the case $p < n$. Then, Tibshirani and Taylor (2012) studied the effective degrees of freedom of the Lasso when $p > n$. It is interesting to investigate the theoretical justification of the consistency of the BIC criterion for

Lasso under the time series setting. We leave this to future work. We also employed models with λ selected by cross validation but found that cross-validation does not improve the results while being considerably much slower in computation. For the models estimated by MIDAS, we only consider Exponential Almon lag polynomial weighting scheme (see (37)) for the first 100 variables and impute the true values as initial values.

Table 1 shows the average of absolute error (AE), the average of root mean squared error (RMSE) for the Lasso estimators and MIDAS estimators over the 10,000 Monte Carlo simulations for the data generating processes used. The AE and the RMSE are defined as,

$$\begin{aligned} \mathrm{AE} &= \frac{1}{MC}\sum_{l=1}^{MC}|(\hat{\phi};\hat{\beta})-(\phi;\beta)|_1, \\ \mathrm{RMSE} &= \sqrt{\frac{1}{MC}\sum_{l=1}^{MC}|(\hat{\phi};\hat{\beta})-(\phi;\beta)|_2^2}, \end{aligned}$$

where MC denotes the number of Monte Carlo repetitions. From the table, it is clear that both the AE and RMSE measures show that the Lasso regression provides substantially more accurate parameter estimation than the mixed-frequency data sampling regression (MIDAS) in the presence of irrelevant variables. Also, as expected, the AE and the RMSE of the estimators decrease with n, but increase with s and p.

To evaluate the performance of out-of-sample forecasts, we use the estimated parameters to compute one-step-ahead forecasts and consider a total of 10 out-of-sample predictions, denoted by $y_{n+1},\dots,y_{n+10}$. Table 2 shows the average absolute forecast error (AFE) and the average root mean squared forecast error (RMSFE) over the 10,000 Monte Carlo simulations, which are calculated as

$$\begin{aligned} \mathrm{AFE} &= \frac{1}{10MC}\sum_{l=1}^{MC}\sum_{k=1}^{10}|\hat{y}_{n+k}-y_{n+k}|, \\ \mathrm{RMSFE} &= \sqrt{\frac{1}{10MC}\sum_{l=1}^{MC}\sum_{k=1}^{10}|\hat{y}_{n+k}-y_{n+k}|^2}. \end{aligned}$$

The forecasting results in Table 2 show that the Lasso regression has smaller AE and RMSFE than the MIDAS in all settings. Furthermore, the results show clearly that the performance of the Lasso regression and the MIDAS improves with the sample size, but deteriorates as the number of relevant variables s increases. Finally, both AE and RMSFE of the Lasso regression decrease faster than those of MIDAS as the sample size n increases. As a matter of fact, the AE and the RMSFE of the MIDAS remain high even when $n = 200$. Since we only fit MIDAS through the first 100 variables, the performance of the MIDAS does not change as p increases. Overall, in the presence of irrelevant variables, the Lasso regression significantly outperforms the MIDAS regression.

Next, we compare the model selection and parameter estimation of Lasso estimator and Dantzig estimator for dependent data. We use the same data generating process (36), where $\phi = 0.6$ and $\boldsymbol{x}_{i,1}$ is a $s \times 1$ vector of relevant variables. Here, we set each element of β_s by $\beta_{s,j} = 3(-1)^j$. Model 1 and Model 2 defined before are chosen for $\boldsymbol{x}_i$. Table 3 shows the number of noise covariates that are selected (False Positive), the number of signal covariates that are not selected (False Negative), the average of root mean squared error (RMSE) for the Lasso estimators and the Dantzig estimators over the 10,000 Monte Carlo simulations for the data generating processes used. As expected, False Positive and RMSE decrease with n, but increase with s and p. False Negative for both methods are almost the same. In terms of False Negative and RMSE, Lasso estimator substantially outperforms the Dantzig selector. Dantzig selector might be more sensitive to heavy tails and outliers, since Dantzig selector uses L_∞ norm. The rate of convergence for Lasso estimator in our paper is faster than that for Dantzig selector in Wu and Wu (2016). They built L_∞ type rate of convergence for Dantzig estimator, which is related to the unknown L_1 norm of true coefficients and matrix L_1 norm of population matrix. In this paper, we overcome this weakness and achieve the same bounds for Lasso regression under i.i.d. data, but with different requirements for regularization parameter

Table 1: Accuracy in Parameter Estimation of Lasso Regression and Mixed-Frequency Data Sampling Regression. The results are based on 10,000 repetitions, where AE and RMSE denote the average of mean absolute errors and average of root mean square errors over Monte Carlo repetitions and parameters. In the table, s, p, and n denote the number of non-zero parameters, the dimension of regressors, and sample size, respectively.

s	n	Absolute Error (AE) $\times 10^2$						Root Mean Square Error (RMSE) $\times 10^2$					
		Lasso			MIDAS			Lasso			MIDAS		
		p											
		100	200	400	100	200	400	100	200	400	100	200	400
Model 1													
	50	2.44	2.64	2.84	6.63	6.64	6.67	3.08	3.75	4.55	3.73	4.44	5.29
5	100	1.89	2.07	2.22	6.24	6.26	6.28	2.79	3.44	4.21	3.64	4.33	5.15
	200	1.27	1.49	1.70	5.91	5.91	5.94	2.28	2.92	3.71	3.56	4.23	5.04
	50	4.60	4.99	5.30	8.26	8.31	8.32	3.66	4.45	5.38	4.09	4.88	5.80
10	100	3.69	4.11	4.39	7.86	7.88	7.90	3.36	4.17	5.10	4.02	4.78	5.69
	200	2.28	2.74	3.29	7.50	7.55	7.56	2.65	3.39	4.42	3.96	4.71	5.60
	50	7.83	8.81	8.93	10.76	10.82	10.83	4.08	5.00	6.00	4.42	5.26	6.26
20	100	6.56	7.33	7.70	10.38	10.43	10.44	3.84	4.75	5.77	4.35	5.18	6.16
	200	4.69	5.55	6.56	10.08	10.12	10.15	3.31	4.21	5.40	4.30	5.12	6.09
Model 2													
	50	0.95	1.14	1.38	4.95	4.97	4.99	2.02	2.53	3.19	3.31	3.94	4.70
5	100	0.54	0.60	0.67	4.55	4.58	4.58	1.56	1.92	2.36	3.18	3.79	4.50
	200	0.34	0.36	0.38	4.20	4.21	4.22	1.26	1.53	1.87	3.06	3.64	4.33
	50	1.91	2.40	2.92	5.46	5.46	5.46	2.54	3.26	4.18	3.53	4.20	4.99
10	100	1.06	1.24	1.46	5.03	5.07	5.08	1.92	2.41	3.04	3.39	4.04	4.81
	200	0.65	0.71	0.79	4.60	4.63	4.65	1.52	1.87	2.31	3.25	3.87	4.61
	50	3.19	4.21	4.94	6.12	6.15	6.18	2.95	3.85	4.96	3.76	4.48	5.34
20	100	1.75	2.14	2.59	5.68	5.69	5.70	2.23	2.85	3.64	3.63	4.32	5.15
	200	1.07	1.21	1.38	5.26	5.27	5.29	1.77	2.20	2.74	3.51	4.18	4.98
Model 3													
	50	1.71	2.05	2.43	6.57	6.60	6.62	2.62	3.29	4.11	3.74	4.46	5.30
5	100	0.93	1.06	1.21	6.27	6.31	6.33	2.03	2.54	3.18	3.65	4.35	5.18
	200	0.57	0.63	0.69	6.17	6.20	6.21	1.62	2.02	2.50	3.61	4.30	5.11
	50	3.74	4.47	5.07	8.41	8.44	8.46	3.34	4.17	5.16	4.13	4.92	5.86
10	100	2.06	2.52	3.00	8.20	8.24	8.25	2.59	3.32	4.25	4.08	4.85	5.77
	200	1.20	1.38	1.58	8.10	8.14	8.16	2.00	2.52	3.18	4.05	4.82	5.73
	50	7.23	8.77	9.38	11.02	11.07	11.09	3.90	4.88	5.95	4.47	5.32	6.32
20	100	4.45	5.81	7.01	10.92	10.97	11.00	3.22	4.16	5.32	4.43	5.28	6.28
	200	2.53	2.93	3.50	10.87	10.93	10.95	2.49	3.11	3.97	4.42	5.26	6.26
Model 4													
	50	1.39	1.58	1.78	5.14	5.16	5.16	2.49	3.10	3.83	3.47	4.13	4.90
5	100	0.96	1.05	1.12	4.58	4.59	4.59	2.12	2.63	3.22	3.31	3.95	4.69
	200	0.71	0.77	0.83	4.22	4.23	4.25	1.83	2.28	2.81	3.23	3.85	4.58
	50	2.40	2.79	3.14	6.03	6.07	6.10	2.90	3.64	4.54	3.80	4.53	5.39
10	100	1.67	1.86	2.02	5.50	5.53	5.55	2.47	3.08	3.80	3.69	4.39	5.23
	200	1.23	1.38	1.50	5.13	5.15	5.16	2.11	2.65	3.30	3.62	4.31	5.13
	50	3.68	4.58	4.97	6.88	6.93	6.93	3.22	4.09	5.14	4.06	4.84	5.75
20	100	2.43	2.77	3.06	6.38	6.42	6.45	2.71	3.41	4.25	3.96	4.72	5.62
	200	1.78	2.00	2.22	6.04	6.08	6.08	2.32	2.91	3.66	3.91	4.66	5.55

Table 2: Performance of Out-of-sample predictions of Lasso regression and mixed frequency data sampling regression (MIDAS). The results are based on 10 one-step ahead predictions and 10,000 iterations, where AFE and RMSFE denote the average absolute forecast errors and root mean squared forecast errors, respectively, and s, p, and n are the number of non-zero parameters, the dimension of regressors, and sample size. For MIDAS, the maximum p is fixed at 100.

s	n	Absolute Error (AE) $\times 10^2$						Root Mean Square Forecast Error (RMSFE) $\times 10^2$					
		Lasso			MIDAS			Lasso			MIDAS		
		p											
		100	200	400	100	200	400	100	200	400	100	200	400
Model 1													
5	50	120.0	125.8	130.3	169.2	161.5	162.3	147.2	153.8	158.8	206.0	197.4	198.0
	100	102.7	106.6	110.7	162.2	156.7	156.4	127.7	132.3	136.9	197.7	191.7	191.3
	200	86.9	90.6	95.4	156.8	152.4	153.2	109.7	114.1	119.4	191.4	186.7	187.4
10	50	151.6	159.6	166.4	185.0	178.8	179.9	185.2	194.3	202.0	225.9	218.6	219.9
	100	125.6	133.9	141.7	177.6	171.6	172.3	155.1	164.9	173.8	216.9	210.1	211.4
	200	96.0	101.9	112.2	171.9	167.7	168.2	120.3	127.2	139.3	210.2	206.1	206.3
20	50	177.7	188.8	195.0	205.0	200.0	199.9	216.1	229.2	236.2	250.2	244.5	244.1
	100	150.2	162.2	170.0	195.8	191.7	191.1	184.0	198.5	207.7	239.5	235.0	234.4
	200	118.6	128.7	145.1	190.1	185.9	188.2	146.7	159.2	178.4	232.4	228.4	230.5
Model 2													
5	50	96.4	101.8	107.2	147.3	148.8	148.5	119.7	125.5	131.7	179.9	181.5	180.9
	100	84.1	85.7	88.1	142.1	142.7	142.9	106.2	108.1	110.4	173.3	174.0	174.0
	200	78.4	79.6	80.6	138.6	137.6	138.8	99.9	101.5	102.3	169.0	168.1	169.2
10	50	114.1	125.7	140.0	171.5	164.2	163.9	139.9	153.7	169.8	208.0	199.6	199.5
	100	90.9	95.3	100.8	156.7	157.9	158.0	114.1	118.7	124.9	190.6	191.9	191.9
	200	81.7	83.1	85.3	151.4	151.1	151.9	103.7	105.1	107.6	184.1	183.7	184.5
20	50	126.9	144.5	167.8	178.2	173.1	173.5	155.4	175.9	202.9	216.5	211.1	211.6
	100	97.7	105.1	113.7	169.7	164.3	164.7	121.6	130.1	139.9	206.5	200.4	200.9
	200	85.3	87.9	91.9	161.7	157.1	158.0	107.8	110.5	115.1	196.9	191.9	192.9
Model 3													
5	50	117.4	128.7	140.5	152.9	153.1	153.3	143.0	155.5	168.3	187.5	187.6	187.9
	100	89.4	92.8	97.1	144.7	145.0	145.0	112.2	116.0	120.4	144.7	178.2	178.1
	200	80.6	81.2	82.8	142.3	141.0	141.1	102.3	103.0	105.0	174.7	173.6	173.5
10	50	154.4	172.5	188.9	178.9	179.1	179.8	185.9	206.0	224.6	218.4	218.8	219.7
	100	103.1	112.9	124.3	171.2	171.3	170.6	127.9	138.7	152.2	209.7	209.5	209.1
	200	84.7	88.2	91.0	166.9	168.2	167.4	107.1	111.1	114.3	204.7	205.9	205.3
20	50	197.3	224.5	244.3	206.9	205.0	205.3	236.1	266.5	288.4	251.6	249.7	249.8
	100	130.9	150.8	172.2	196.6	197.6	196.4	160.2	182.9	207.6	240.2	240.9	239.3
	200	97.0	101.2	109.0	193.1	193.8	193.7	121.2	125.9	134.8	236.5	237.4	237.1
Model 4													
5	50	103.0	108.7	113.2	131.7	131.9	130.9	126.8	133.3	138.2	162.6	162.9	161.7
	100	88.4	90.4	92.9	121.6	122.0	121.5	110.9	113.0	115.8	150.9	151.5	150.8
	200	81.3	82.6	83.4	118.0	117.1	116.8	103.3	104.4	105.4	147.0	145.7	145.2
10	50	117.6	126.6	136.2	148.6	148.5	148.5	144.1	154.4	165.7	183.7	183.0	183.0
	100	95.8	99.8	103.4	139.4	139.5	139.3	119.8	124.2	128.2	172.5	172.6	172.3
	200	84.9	87.5	89.7	134.2	135.0	134.7	107.3	110.1	112.7	166.5	167.3	167.0
20	50	132.2	148.5	162.3	163.8	164.7	163.7	161.3	180.2	196.3	201.7	202.7	201.6
	100	102.4	108.9	115.4	154.2	154.0	154.7	127.2	134.8	142.3	190.6	190.7	190.9
	200	88.6	92.1	96.2	150.0	149.8	150.2	111.7	115.7	120.2	185.8	185.4	185.7

λ and sample size n.

6 Empirical Analysis

6.1 Predicting GDP growth

We consider the problem of predicting the growth rate of U.S. quarterly gross domestic product (GDP). In addition, nine (9) macroeconomic variables with different sampling frequencies are also available. The data are obtained from the St. Louis Federal Reserve Economic Data website. The predictive regression used is

$$y_i = \phi_0 + \phi_1 y_{i-1} + \cdots + \phi_a y_{i-a} + \sum_{l=1}^{9} \sum_{b=0}^{B_l} \beta_{l,b} z_{l,i\times m_l - b} + e_i \tag{38}$$

where a and B_l are nonnegative integers, y_i is the growth rate (first difference of natural logarithm) of U.S. quarterly seasonally adjusted real GDP and $z_{l,\cdot}$'s are the high-frequency covariates with frequency m_l, e.g., $m_l = 3$ for monthly data. The nine covariates considered in this study are: $z_{1,\cdot}$ is the change of monthly civilian unemployment rates, $z_{2,\cdot}$ is the growth rate of monthly all employees total payrolls, $z_{3,\cdot}$ is the growth rate of monthly industrial production total index, $z_{4,\cdot}$ is the growth rate of monthly consumer price index, $z_{5,\cdot}$ is the growth rate of monthly Moody's Seasoned Baa Corporate Bond Yields, $z_{6,\cdot}$ is the change of daily 3-Month Treasury Bill Secondary Market Rate, $z_{7,\cdot}$ is the change of daily 10-Year Treasury Constant Maturity Rate, $z_{8,\cdot}$ is the change of daily NASDAQ Composite Index, and $z_{9,\cdot}$ is the change of daily Wilshire 5000 Total Market Full Cap Index. The transformations of all variables are based on those of Stock and Watson (2002). Note that all data are seasonally adjusted if necessary, and the explanatory variables are monthly or daily data. For daily variables $z_{6,\cdot}$ and $z_{7,\cdot}$, we only use data of the first 16 trading days in a month. For daily variables $z_{8,\cdot}$ and $z_{9,\cdot}$, we only use data of the first 15 trading days. The sampling

Table 3: Accuracy in Model Selection and Parameter Estimation of Lasso Estimator and Dantzig Estimator for Linear Regression. The results are based on 10,000 repetitions, where RMSE denote the average of root mean square errors over Monte Carlo repetitions and parameters. In the table, s, p, and n denote the number of non-zero parameters, the dimension of regressors, and sample size, respectively.

s	n	Model 1						Model 2					
		Lasso			Dantzig			Lasso			Dantzig		
		p											
		100	200	400	100	200	400	100	200	400	100	200	400
False Negative													
5	50	0.077	0.20	0.67	0.072	0.28	0.73	0	0	0.003	0	0	0.002
	100	0	0	0.01	0	0	0.04	0	0	0	0	0	0
	200	0	0	0	0	0	0	0	0	0	0	0	0
10	50	0.67	2.07	4.28	0.81	2.50	4.04	0.011	0.14	0.79	0.045	0.17	0.95
	100	0	0.004	0.11	0	0.006	0.15	0	0	0	0	0	0
	200	0	0	0	0	0	0	0	0	0	0	0	0
20	50	4.65	7.45	9.61	4.83	7.29	10.1	1.94	5.70	7.18	2.48	5.45	8.34
	100	0	0.21	2.27	0.03	0.28	2.14	0	0	0.029	0	0.002	0.052
	200	0	0	0	0	0	0	0	0	0	0	0	0
False Positive													
5	50	10.9	15.0	22.5	15.85	24.20	32.0	5.25	8.01	11.5	6.45	13.3	21.7
	100	5.30	8.97	13.6	8.03	14.11	19.8	2.05	3.02	4.00	4.14	7.43	9.91
	200	1.63	3.01	4.95	3.66	6.25	9.52	0.49	0.79	1.31	2.88	3.56	5.03
10	50	13.7	23.2	29.1	19.2	32.5	37.5	10.8	18.2	23.5	13.8	23.4	32.3
	100	8.69	15.2	23.8	12.4	23.6	30.9	4.60	7.46	9.52	9.01	13.5	18.9
	200	2.12	4.96	7.24	4.02	8.17	10.6	1.08	2.05	3.68	3.37	6.09	10.75
20	50	17.6	26.9	31.8	28.9	37.4	39.8	16.5	25.3	30.0	21.3	28.8	37.3
	100	12.0	23.1	25.0	16.6	30.2	30.5	8.21	16.2	23.9	14.0	24.7	31.1
	200	3.99	7.01	10.6	5.84	10.2	15.1	2.49	4.05	8.46	8.22	11.3	16.2
RMSE													
5	50	1.78	2.63	3.88	2.06	2.76	4.07	0.80	0.98	1.17	0.88	1.04	1.21
	100	0.87	1.04	1.19	0.95	1.02	1.27	0.44	0.48	0.54	0.50	0.52	0.57
	200	0.69	0.64	0.70	0.61	0.80	0.83	0.33	0.33	0.34	0.42	0.39	0.41
10	50	4.53	7.49	9.22	5.50	7.79	9.29	1.83	3.02	5.67	2.43	3.74	6.39
	100	1.52	1.76	2.78	1.59	2.00	2.41	0.76	0.84	0.96	0.90	0.96	1.09
	200	0.97	1.01	1.09	0.94	1.21	1.24	0.55	0.56	0.58	0.69	0.70	0.68
20	50	10.6	13.3	14.4	11.1	13.3	14.5	8.48	12.8	15.1	9.58	13.0	15.3
	100	2.61	4.06	8.25	3.53	5.55	8.95	1.46	1.81	2.71	2.21	2.67	3.98
	200	1.46	1.65	1.78	1.69	1.76	1.91	0.91	0.94	1.00	1.13	1.22	1.31

period was from January 1980 to February 2017, but the prediction origin started with the second quarter of 2013 and ended with the first quarter of 2017. There was no trading activities during weekends and holidays, and there exist some missing data in the trading activities. Trading days for each month varies. We choose the first 15 or 16 trading days simply because they are the minimum number of trading days available for each month (mainly February).

Two types of empirical analysis are entertained. First, we consider a linear model with all explanatory variables and estimated by the Lasso procedure. For comparison, we include a model with all explanatory variables except the NASDAQ Composite Index and Wilshire 5000 Total Market Full Cap Index, estimated by the MIDAS regression (denoted by MIDAS-B model), a model with monthly all-employees total payrolls as the only explanatory variable, also estimated by MIDAS (denoted by MIDAS-A model), and a simple ARMA model of the GDP growth rates (denoted by ARMA model). We use BIC to select the number of autoregressive lags (a) and the lags (B_l) of explanatory variables. The Lasso tuning parameter λ is also chosen by the BIC; see Bühlmann and Van De Geer (2011). Here we aggregate daily explanatory variables z_6 and z_7 to weekly frequency for the MIDAS regression.

Table 4 shows the median absolute deviation (MAD), the mean absolute error (MAE), and the root mean squared error (RMSE) for the prediction period. From the table, it is clear that the Lasso based model outperforms all the other models in this particular instance. The poor performance of MIDAS-B is likely due to using too many explanatory variables with multiple sampling frequencies.

Figure 2 displays the cumulative absolute errors and the cumulative squared errors for different models in predicting the GDP growth rate. It shows clearly that the Lasso model is the best one. The MIDAS-A model also improves the prediction errors over the simple ARMA model. However, the MIDAS-B model fares poorly. Consequently, unlike the Lasso model, the MIDAS regression is not robust to the presence of irrelevant regressors. In fact, the MIDAS regression is also sensitive to the weighting schemes and the starting points of its optimization program.

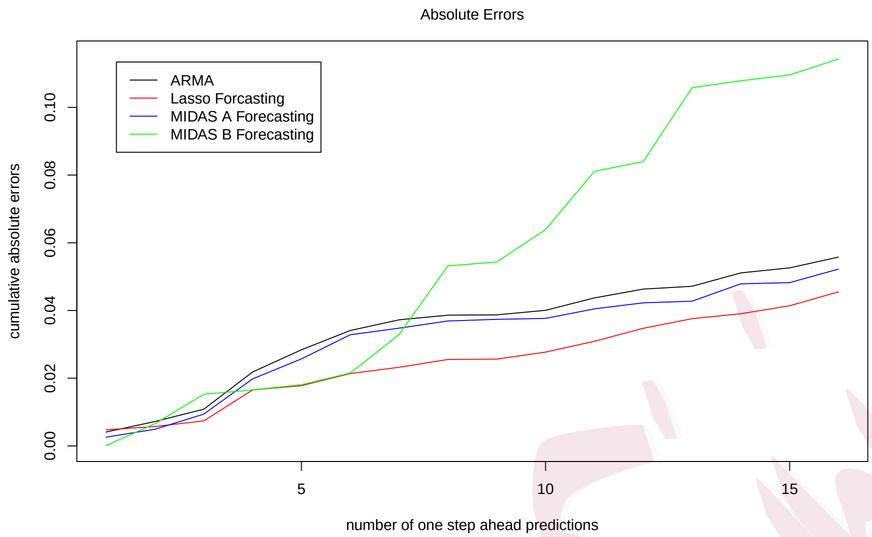
Table 4: Results of out-of-sampling prediction of U.S. quarterly real GDP growth rate. The data span is from 1980 to February 2017, but the forecast origins start from the second quarter of 2013 to the first quarter of 2017. All measurements are multiplied by 10^3. In the table, MAD, MAE, and RMSE are the median absolute error, mean absolute error, and root mean squared error, respectively.

Model	MAD	MAE	RMSE
ARMA	3.175	3.486	4.319
Lasso	2.328	2.845	3.491
MIDAS-A	2.463	3.264	4.245
MIDAS-B	4.089	7.143	9.920

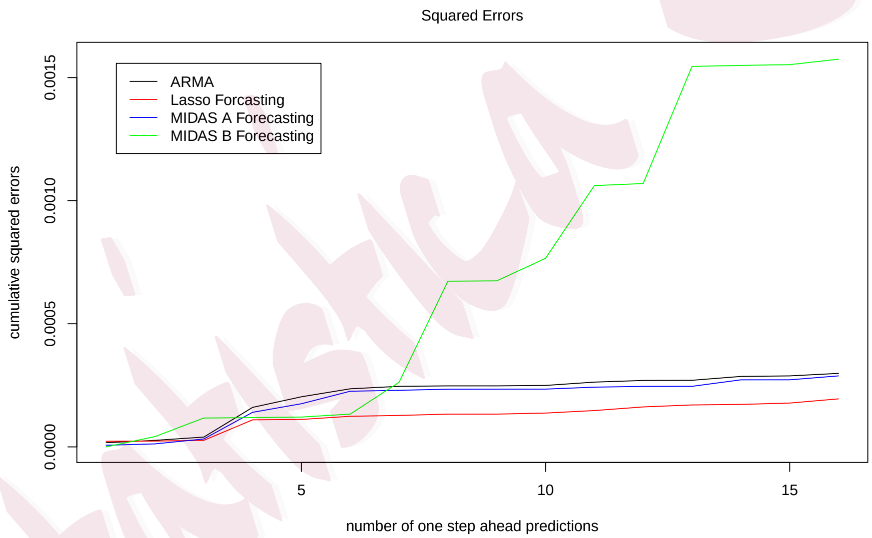
Turn to comparison between forecasting and now-casting. Recall that the goal of now-casting is to take advantages of available high-frequency data to improve the prediction of lower-frequency variables of interest. For the quarterly GDP growth rate, during the quarter of interest, some monthly macroeconomic variables and even some daily economic variables become available, now-casting attempts to update the GDP prediction by incorporating those newly available high-frequency explanatory variables. In this exercise, we consider now-casting with the first month data within the quarter available and the first two months data available.

For comparison purpose, we employ an autoregressive (AR) model

$$y_i = \phi_0 + \phi_1 y_{i-1} + \cdots + \phi_a y_{i-a} + \epsilon_i, \tag{39}$$



(a)



(b)

Figure 2: Panel (a): Cumulative absolute errors. Panel (b): Cumulative squared errors. MIDAS-A represents the MIDAS regression model using only monthly all-employees total payrolls as the explanatory variable. MIDAS-B represents the MIDAS regression model with seven regressors $z_{1,\cdot}, \cdots, z_{7,\cdot}$, where $z_{6,\cdot}$ and $z_{7,\cdot}$ are aggregated into weekly data.

as a benchmark for prediction. The AR order is selected by the BIC in the modeling subsample and is assumed to be fixed in the forecasting subsample. The AR model in Equation (39) is estimated by two ways. First, it is estimated by the ordinary least squares method and we denote the model by AR-OLS. Second, assuming sparsity, we estimate the AR model via Lasso method with the tuning parameter λ also selected by BIC. The forecasting result of this model is denoted by AR-Lasso. These two models represent the performance of forecasting.

For now-casting, we augment the AR model in Equation (39) with all explanatory variables available in the first month of the quarter and denote the results by Now-casting 1. Similarly, if we augment the AR model with all explanatory variables available in the first two months of the quarter, then the results are denoted by Now-casting 2. Specifically, for now-casting, we employ the model

$$y_i = \phi_0 + \phi_1 y_{i-1} + \cdots + \phi_a y_{i-a} + \beta^T \mathbf{x}_i + \epsilon_i,$$

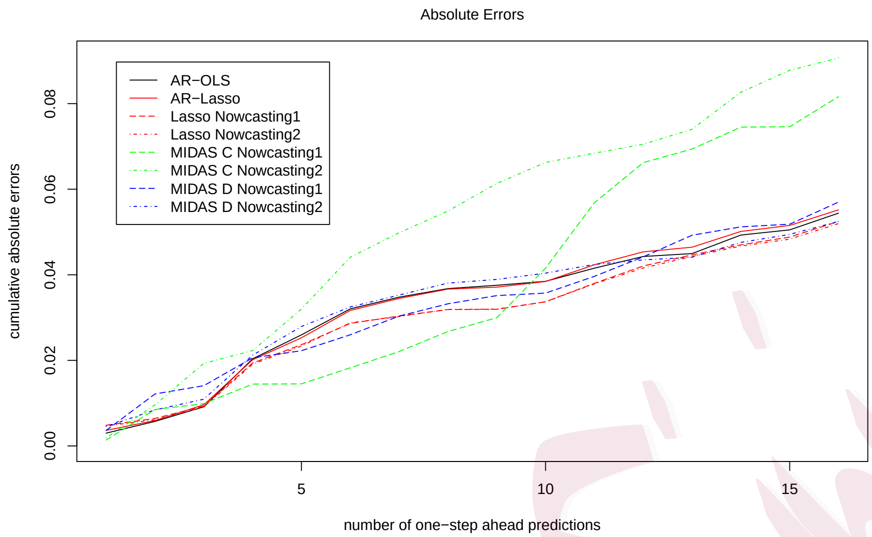
where $\mathbf{x}_i$ denotes the available high-frequency explanatory variables. For Now-casting 1, $\mathbf{x}_i$ consists of data of the first month into a given quarter whereas for Now-casting 2, it consists of data of the first two months into a given quarter. In this exercise, we use all monthly and daily high-frequency variables $z_{1,\cdot}, \cdots, z_{9,\cdot}$. We denote the results for MIDAS regression as MIDAS-C Now-casting 1 and MIDAS-C Now-casting 2, respectively. Finally, we also employ a MIDAS regression that only uses explanatory variables $z_{1,\cdot}, \cdots, z_{7,\cdot}$ in the now-casting and denote the results as MIDAS-D.

Table 5 summarizes the performance of now-casting in predicting U.S. quarterly GDP growth rates in the forecasting period. From the table, we make the following observations. First, as expected, now-casting fares better than forecasting. The only exception is MIDAS-D now-casting. Second, also as expected, Now-casting 2 shows some improvement over Now-casting 1 for a given model. Keep in mind, however, Now-casting 1 is available one month into a quarter whereas Now-casting 2 needs to wait for an additional month. Third, from the performance of MIDAS-C and MIDAS-D, the stock market indexes do not seem to be helpful in predicting the GDP growth rate. In real applications, there exist many high-frequency explanatory variables, but their contributions to predicting the low-frequency variable of interest in unknown a priori. In this situation, the results obtained in this paper suggest that the Lasso regression could be helpful.

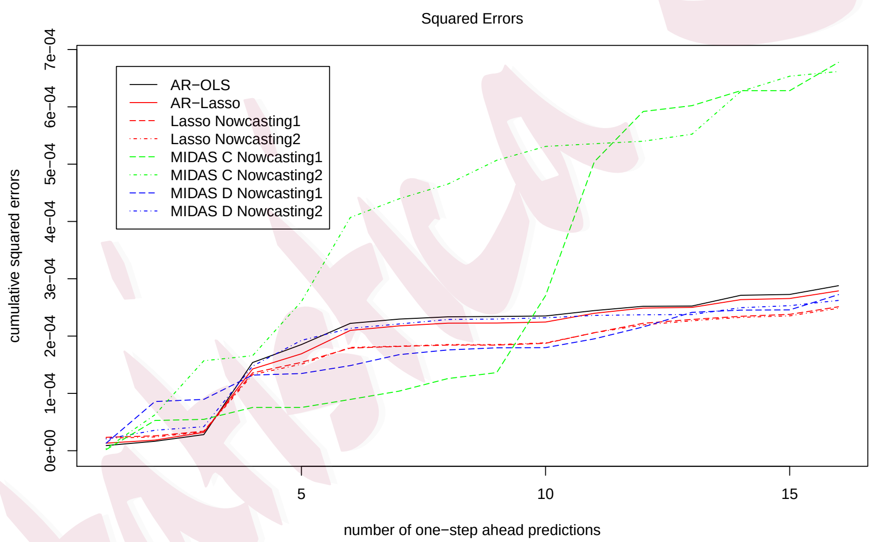
Figure 3 shows both the Lasso model and the MIDAS-B model improve the prediction via now-casting. But when irrelevant variables exist, MIDAS regression might encounter some difficulties.

Table 5: Comparison between forecasting and now-casting in predicting the U.S. quarterly real GDP growth rate. The data span is from 1980 to February 2017, but the forecast origins are from the second quarter of 2013 to the first quarter of 2017. All measurements are multiplied by 10^3. In the table, MAD, MAE, RMSE are the median absolute deviation, mean absolute error, and root mean squared error, respectively.

Model	MAD	MAE	RMSE
AR-OLS	2.865	3.400	4.242
AR-Lasso	3.327	3.448	4.174
Lasso Now-casting 1	2.731	3.278	3.962
Lasso Now-casting 2	2.834	3.247	3.941
MIDAS-C Now-casting 1	4.181	5.102	6.507
MIDAS-C Now-casting 2	5.108	5.666	6.430
MIDAS-D Now-casting 1	3.670	3.561	4.125
MIDAS-D Now-casting 2	2.784	3.279	4.048



(a)



(b)

Figure 3: Panel (a): Cumulative absolute errors. Panel (b) Cumulative squared errors. MIDAS-D represents the MIDAS regression model with seven regressors $z_{1,\cdot}, ..., z_{7,\cdot}$. MIDAS-C represents the MIDAS regression model with nine regressors $z_{1,\cdot}, ..., z_{9,\cdot}$. Now-casting 1 and Now-casting 2 represent predicting quarterly GDP growth rate when the first month and the first two months data are available, respectively.

6.2 Nowcasting $PM_{2.5}$

Consider next the prediction of $PM_{2.5}$. The response y is the square-root transformed daily maximum of $PM_{2.5}$. Hourly data of a monitoring station in the southern part of Taiwan are used. To see the nowcasting effects, we consider adding 6 covariates, which are the first 6 hourly $PM_{2.5}$ readings of the same day, starting from midnight. The time period is from 2006 to 2015 so that there are 3650 observations. (Feb 29 was dropped.) We reserve the last 730 data points (2 years) for one-step ahead out-of-sample forecasts.

For comparison purpose, we first consider the square-root $PM_{2.5}$ (i.e. response y) as a pure time series. An AR(22) model is selected. Thus, the baseline model is a univariate AR(22). We denote the model by AR-OLS. For now-casting, we augment the AR model by the first 6 hourly readings. If we augment the AR model with the first hourly $PM_{2.5}$ reading, then the results are denoted by Now-casting 1. Similarly, if we augment the AR model with the first two hourly $PM_{2.5}$ readings, then the results are denoted by Now-casting 2, so on so forth. We denote the results for autoregressive model with exogenous variables as ARX Now-casting 1, ARX Now-casting 2, etc. We use BIC to select the number of autoregressive lags. The Lasso tuning parameter λ is also chosen by the BIC.

Table 6 summarizes the performance of now-casting in predicting daily maximum of $PM_{2.5}$. From the table, we make the following observations. First, as expected, now-casting outperforms forecasting. Second, also as expected, for a given model, Now-casting 2 shows some improvement over Now-casting 1, Now-casting 3 shows some improvement over Now-casting 2, so on so forth. Third, of most interest, Lasso estimator significantly outperforms the ARX model and the benchmark model. In short, Lasso regression seems to be helpful in applying now-casting to $PM_{2.5}$.

Table 6: Comparison between forecasting and now-casting in predicting the daily maximum of $PM_{2.5}$. The data span is from 2006 to 2015, and the forecast origins are from 2013 to the end of 2015.(Feb 29 was dropped). In the table, MAE, RMSE are the mean absolute error, and root mean squared error for one-step ahead predictions, respectively.

Model	MAE	RMSE
AR-OLS	1619.6	73.71
ARX Now-casting 1	975.9	46.54
ARX Now-casting 2	940.9	44.92
ARX Now-casting 3	904.2	43.40
ARX Now-casting 4	879.7	42.31
ARX Now-casting 5	850.6	41.24
ARX Now-casting 6	835.3	40.31
Lasso Now-casting 1	659.3	31.74
Lasso Now-casting 2	628.4	30.58
Lasso Now-casting 3	623.2	30.72
Lasso Now-casting 4	600.7	29.63
Lasso Now-casting 5	595.0	29.49
Lasso Now-casting 6	576.3	28.47

Supplementary Materials

In the supplemental materials, we provide the proofs.

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