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CONDITIONAL QUANTILE ESTIMATION FOR HYSTERETIC AUTOREGRESSIVE MODELS

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Abstract: The phenomenon of hysteresis has been observed in many economic time series, especially in unemployment rates. To study their hysteretic patterns at different quantiles, this paper considers the conditional quantile estimation for hysteretic autoregressive models, and its asymptotic properties are also derived. Simulation experiments are conducted to evaluate the finite-sample performance of our method, and its usefulness is further demonstrated by the analysis for the growth rates of unemployment rates.

Key words and phrases: Autoregression, conditional quantile estimation, hysteretic model, threshold model.

1. Introduction

The threshold model (Tong and Lim, 1980) has been shown highly successful, since its appearance, in interpreting time-irreversibility, limit cycles, asymmetric dynamics etc.; see Tong (1990) for the comprehensive exposition. However, this model is well known not to work well around the boundaries between different regimes (Wu and Chen, 2007), possibly due to a sudden change in the probability structure when a threshold process switches regimes. This problem has been reduced to some extent by other regime-switching models, such as smooth-transition threshold models in Chan and Tong (1986), discrete-state Markov switching models in Hamilton (1989) and McCulloch and Tsay (1994), and a threshold variable-driven switching model in Wu and Chen (2007). While they grant the threshold model more flexibility by changing the piecewise linear structure or introducing latent random variables to the regime-switching mechanism, in general they lack a physical interpretation.

The phenomenon of hysteresis has been observed in many economic time series, especially in unemployment rates (Brunello, 1990; Roed, 2002; Song and Wu, 1997, 1998). Economic theory decomposes the unemployment rate into two components: the short-term cyclical and the long-term natural rate. When there is a negative shock to the economy, the cyclical rate will rise, but the natural rate may also rise, due to the propagation effect of the shock. There are also some microeconomic interpretations. First, the unemployed may loss skills, and thus have more difficulty in returning to work. Secondly, due to the wage bargaining institution and labor turnover costs, the incumbent workers have incentives to bargain for higher wages when the economy start to recover, which renders the wage level higher than the market equilibrium level (Blanchard and Summers, 1986). As a result, the observed unemployment rate will be pushed up, and remains high for a longer-than-expected period; see Amable et al. (1995) and Perez-Alonso and Sanzo (2011).

Li et al. (2015a) proposed a hysteretic autoregressive (HAR) time series model by combining threshold models and the phenomenon of hysteresis,

$$y_t = \begin{cases} \theta_{01} + \theta_{11} y_{t-1} + \ldots + \theta_{p1} y_{t-p} + \sigma_1 \varepsilon_t, & R_t = 1, \\ \theta_{02} + \theta_{12} y_{t-1} + \ldots + \theta_{p2} y_{t-p} + \sigma_2 \varepsilon_t, & R_t = 0, \end{cases} \tag{1.1}$$

with the regime indicator

$$R_t = \begin{cases} 1 & y_{t-d} \leq r_L, \\ 0 & y_{t-d} > r_U, \\ R_{t-1} & r_L < y_{t-d} \leq r_U. \end{cases} \tag{1.2}$$

As the hysteresis zone $(r_L, r_U]$ acts as a buffer for the regime switching, a better model fit is expected. Moreover, when $r_L = r_U$, the hysteretic model will reduce to a threshold model, which corresponds to the case without hysteresis; see Section 2 for discussions. Due to the phenomenon of hysteresis in unemployment rates, it is of interest to apply the hysteretic model in (1.1) and (1.2) to the corresponding sequence. Moreover, the movement of the natural unemployment rate can be approximately separated into two phases: the one of rising up and the other of returning back, which may correspond to the upper and lower regimes at (1.2).

In the meanwhile, since Koenker and Bassett (1978), quantile regression has become a valuable tool for analyzing the conditional quantile functions of a response variable. Comparing with the conditional mean regression, quantile regression provides a more comprehensive analysis on how predictors may influence different aspects of the conditional distributions of the response; see Koenker (2005) for a comprehensive introduction. Koenker and Xiao (2006) proposed the quantile autoregressive (AR) model, which is the first quantile model in the literature of time series. Kato (2009) extended the convexity arguments to the scenario under which the estimators are derived as stochastic process, and it hence provides a very useful tool to theoretically study quantile regression and threshold models. Kuan et al. (2017) studied time series models with possible threshold structures at some quantile levels, and Cai and Stander (2008) considered a Bayesian approach to estimate the conditional quantiles of threshold AR processes; see also Cai (2010). Galvao et al. (2014) developed a uniform test for linearity against the threshold effect of quantile regression, and Zhang et al. (2014) suggested a CUSUM-type test for the threshold at some quantile levels. Especially Galvao et al. (2011) considered the conditional quantile estimation for the threshold AR model, and found that the structures differ significantly for different quantiles in the US monthly unemployment rate. Together with the phenomenon of hysteresis in economic time series, this motivates us to consider the conditional quantile estimation for HAR models.

The remainder of this paper is organized as follows. Section 2 describes the model settings and the estimating procedure. The asymptotic properties are derived in Section 3. Section 4 conducts simulation experiments to evaluate the finite-sample performance of the conditional quantile estimation. A sequence of growth rates of the unemployment rate is analyzed in Section 5. Section 6 gives a short conclusion and discussion. All technical proofs are provided in the supplementary material.

2. Conditional quantile estimation for HAR models

For a fixed $\tau \in (0,1)$, the τth conditional quantile of the HAR process, generated by (1.1) and (1.2), has the form of

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = \begin{cases} \theta_{01,\tau} + \theta_{11,\tau} y_{t-1} + \ldots + \theta_{p1,\tau} y_{t-p}, & R_{t,\tau} = 1, \\ \theta_{02,\tau} + \theta_{12,\tau} y_{t-1} + \ldots + \theta_{p2,\tau} y_{t-p}, & R_{t,\tau} = 0, \end{cases} \tag{2.1}$$

with the regime indicator

$$R_{t,\tau} = \begin{cases} 1 & y_{t-d_\tau} \leq r_{L,\tau}, \\ 0 & y_{t-d_\tau} > r_{U,\tau}, \\ R_{t-1,\tau} & r_{L,\tau} < y_{t-d_\tau} \leq r_{U,\tau}, \end{cases}$$

where $\mathcal{F}_t$ is the σ-field generated by $\{y_t, y_{t-1}, ...\}$, d_τ is the delay parameter, y_{t-d_τ} is the hysteresis variable, $(r_{L,\tau}, r_{U,\tau}]$ is the hysteresis zone. Here we use the subscripts to emphasize the dependence on τ. After some algebras, it can be verified that, in the almost surely sense,

$$\begin{aligned} R_{t,\tau} &= I\{y_{t-d_\tau} \leq r_{L,\tau}\} + I\{r_{L,\tau} < y_{t-d_\tau} \leq r_{U,\tau}\} R_{t-1,\tau} \\ &= I\{y_{t-d_\tau} \leq r_{L,\tau}\} + \sum_{j=0}^{\infty} \prod_{i=0}^{j} I\{r_{L,\tau} < y_{t-d_\tau - i} \leq r_{U,\tau}\} I\{y_{t-d_\tau - j-1} \leq r_{L,\tau}\}, \end{aligned} \tag{2.2}$$

where $I(\cdot)$ is the indicator function. When $r_{L,\tau} = r_{U,\tau}$, it holds that $R_{t,\tau} = I\{y_{t-d_\tau} \leq r_{L,\tau}\}$, and this corresponds to a threshold AR model without hysteresis. Let $n_0 = \max(p, d_{\max})$. For an observed sequence $\{y_t, -n_0+1 \leq t \leq n\}$, we next consider the conditional quantile estimation for model (2.1).

Denote the parameter vector by $\lambda_\tau = (\theta_\tau^T, d_\tau, r_{L,\tau}, r_{U,\tau})^T$, where $\theta_{1,\tau} = (\theta_{01,\tau}, \ldots, \theta_{p1,\tau})^T$, $\theta_{2,\tau} = (\theta_{02,\tau}, \ldots, \theta_{p2,\tau})^T$ and $\theta_\tau = (\theta_{1,\tau}^T, \theta_{2,\tau}^T)^T$. Let Θ be a compact set of $\mathbb{R}^{2p+2}$, $[a, b]$ be a predetermined interval and $d_{\max}$ be a predetermined positive integer. For the true value of parameter vector λ_τ, we assume that θ_τ^0 is an interior point of Θ, $a < r_{L,\tau}^0 < r_{U,\tau}^0 < b$ and $d_\tau^0 \in D = \{1, \ldots, d_{\max}\}$.

Let $x_t = (1, y_{t-1}, \ldots, y_{t-p})^T$, and model (2.1) can be rewritten into a compact form,

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = x_t^T \theta_{1,\tau} R_t(r_{L,\tau}, r_{U,\tau}, d_\tau) + x_t^T \theta_{2,\tau}[1 - R_t(r_{L,\tau}, r_{U,\tau}, d_\tau)], \quad (2.3)$$

where, from (2.2), the regime indicator function $R_t(r_{L,\tau}, r_{U,\tau}, d_\tau)$ depends on the past observations infinitely far away as $r_{L,\tau} < r_{U,\tau}$. For fixed $r_{L,\tau}, r_{U,\tau}$ and d_τ, the first few observations of the hysteresis variable, say $\{y_{1-d_\tau}, \ldots, y_{t_0-d_\tau}\}$, may fall into the hysteresis zone $(r_{L,\tau}, r_{U,\tau}]$ such that we cannot identify the corresponding regimes. For simplicity, we can assign them to the lower regime, and denote the resulting regime indicator function by $\widetilde{R}_t(r_{L,\tau}, r_{U,\tau}, d_\tau)$. Note that the exact value of $R_{t_0+1}(r_{L,\tau}, r_{U,\tau}, d_\tau)$ is known, and it holds that $\widetilde{R}_t(r_L, r_{U,\tau}, d_\tau) = R_t(r_{L,\tau}, r_{U,\tau}, d_\tau)$ for $t_0 < t \leq n$.

Let $L_n(\lambda_\tau) = \sum_{t=1}^n \rho_\tau[y_t - M_t(\lambda_\tau)]$ be the loss function, where

$$M_t(\lambda_\tau) = x_t^T \theta_{1,\tau} R_t(r_{L,\tau}, r_{U,\tau}, d_\tau) + x_t^T \theta_{2,\tau}[1 - R_t(r_{L,\tau}, r_{U,\tau}, d_\tau)],$$

and $\rho_\tau(u) = u[\tau - I(u < 0)]$ is the check function. When the regime indicator function $R_t(r_{L,\tau}, r_{U,\tau}, d_\tau)$ in $M_t(\lambda_\tau)$ and $L_n(\lambda_\tau)$ is replaced by $\widetilde{R}_t(r_{L,\tau}, r_{U,\tau}, d_\tau)$, we denote them by $\widetilde{M}_t(\lambda_\tau)$ and $\widetilde{L}_n(\lambda_\tau)$. The conditional quantile estimator of model (2.1) can then be defined as

$$\widehat{\lambda}_{n,\tau} = \operatorname{argmin} \widetilde{L}_n(\lambda_\tau),$$

where $\widehat{\lambda}_{n,\tau} = (\widehat{\theta}_{n,\tau}^T, \widehat{d}_\tau, \widehat{r}_{L,\tau}, \widehat{r}_{U,\tau})^T$.

From (2.3), numerically minimizing $\widetilde{L}_n(\lambda_\tau)$ for each fixed $r_{L,\tau}$, $r_{U,\tau}$ and d_τ is equivalent to perform a linear quantile regression. We denote the resulting minimizer by $\widetilde{\theta}_{n,\tau}(r_{L,\tau}, r_{U,\tau}, d_\tau)$. It is noteworthy to point out that $\widetilde{L}_n[\widetilde{\theta}_{n,\tau}(r_{L,\tau}, r_{U,\tau}, d_\tau), r_{L,\tau}, r_{U,\tau}, d_\tau]$ is a stepwise function with possible jumps at $d_\tau \in D$ and

$$(r_{L,\tau}, r_{U,\tau}) \in \{(y_{t-d_\tau}, y_{s-d_\tau}) : 1 \le t, s \le n; y_{t-d_\tau} \le y_{s-d_\tau}\};$$

see Li and Li (2008, 2011). As a result, it can be minimized by searching over all jumps, and the corresponding minimizer is our conditional quantile estimator $(\widehat{d}_\tau, \widehat{r}_{L,\tau}, \widehat{r}_{U,\tau})$. We can verify that $\widehat{\theta}_{n,\tau} = \widetilde{\theta}_{n,\tau}(\widehat{r}_{L,\tau}, \widehat{r}_{U,\tau}, \widehat{d}_\tau)$.

For the initial value of regime indicator function $R_t(r_{L,\tau}, r_{U,\tau}, d_\tau)$, we may alternatively assign first t_0 observations to the upper regime, and denote by $\widetilde{R}_t^*(r_{L,\tau}, r_{U,\tau}, d_\tau)$ the resulting regime indicator function. Let $\widetilde{L}_n^*(\lambda_\tau)$ be the corresponding loss function, and $\widehat{\lambda}_{n,\tau}^* = \operatorname{argmin} \widetilde{L}_n^*(\lambda_\tau)$. We can choose $\widehat{\lambda}_{n,\tau}^*$ to be the estimator if $\widetilde{L}_n^*(\widehat{\lambda}_{n,\tau}^*) < \widetilde{L}_n(\widehat{\lambda}_{n,\tau})$. In practice, the values of a and b can be set to some percentiles of the observed data.

We next adopt the Bayesian information criterion (BIC) in Lee et al. (2014) to select the order p. Denote $\widetilde{R}_{t,\tau} = \widetilde{R}_t(\widehat{r}_{L,\tau}, \widehat{r}_{U,\tau}, \widehat{d}_\tau)$ for simplicity. By temporarily assuming that ε_t at (1.1) follows the asymmetric Laplace distribution with the density of

$$f(x) = \tau(1-\tau)\exp\{-\rho_\tau(x)\},$$

we can define

$$\text{BIC}(p) = 2n_1 \log \widehat{\sigma}_{1n} + (p+1)\log n_1 + 2n_2 \log \widehat{\sigma}_{2n} + (p+1)\log n_2, \quad (2.4)$$

where $\widehat{\sigma}_{1n} = n_1^{-1}\sum_{t=1}^n \rho_\tau(y_t - x_t^T\widehat{\theta}_{1n,\tau})\widetilde{R}_{t,\tau}$, $\widehat{\sigma}_{2n} = n_2^{-1}\sum_{t=1}^n \rho_\tau(y_t - x_t^T\widehat{\theta}_{2n,\tau})(1-\widetilde{R}_{t,\tau})$, $n_1 = \sum_{t=1}^n \widetilde{R}_{t,\tau}$, $n_2 = n - n_1$, and $\widehat{\theta}_{n,\tau} = (\widehat{\theta}_{1n,\tau}^T, \widehat{\theta}_{2n,\tau}^T)^T$. Similarly, we can also define the Akaike information criterion (AIC). Moreover, it is possible to consider different orders, say p_1 and p_2, for the two regimes of model (2.1) in the information criteria proposed above.

3. Asymptotic Results

Assumption 1. *It holds that* $\theta^0_{1,\tau} \neq \theta^0_{2,\tau}$, $P(y_t \in [a,b]) < 1$, *and that time series* $\{y_t\}$ *is strictly stationary with* $E(|y_t|^{2+\varsigma}) < \infty$ *for some* $\varsigma > 0$.

Theorem 1. *If Assumption 1 holds, then* $\widehat{\lambda}_{n,\tau} \to \lambda^0_\tau$ *almost surely as* $n \to \infty$, *where* $\lambda^0_\tau = (\theta^{0T}_\tau, r^0_{L,\tau}, r^0_{U,\tau}, d^0_\tau)^T$ *and* $\theta^0_\tau = (\theta^{0T}_{1,\tau}, \theta^{0T}_{2,\tau})^T$.

Note that the delay parameter d_τ only takes discrete integer values. It then holds that $\widehat{d}_\tau = d^0_\tau$ when the sample size n is sufficiently large. Without loss of generality, the true delay parameter d^0_τ is assumed to be known for the reminder of this section, and hence is deleted from parameter vector λ_τ and corresponding functions.

Let $Y_t = (y_t, \ldots, y_{t-p+1}, R_t)^T$. From (1.1) and (1.2), it can be verified that $\{Y_t\}$ is a Markov chain, and we denote its k-step transition probability by $P^k(x,z)$. Let

$$\Omega_0 = \text{diag}\left\{E\{x_t x_t^T R_{t,\tau}\}, E\{x_t x_t^T(1-R_{t,\tau})\}\right\},$$
$$\Omega_1 = \text{diag}\left\{E\{f_t[F_t^{-1}(\tau)]x_t x_t^T R_{t,\tau}\}, E\{f_t[F_t^{-1}(\tau)]x_t x_t^T(1-R_{t,\tau})\}\right\}.$$

where $f_t(\cdot)$ and $F_t(\cdot)$ are the density and distribution functions of y_t conditional on $\mathcal{F}_{t-1}$, respectively.

Assumption 2. *Markov chain* $\{Y_t\}$ *has a unique invariant measure* $\pi(\cdot)$, *such that* $\exists K > 0$ *and* $\exists \kappa \in [0,1)$, $\forall x \in \mathbb{R}^p \times \{0,1\}$ *and* $\forall k \in \mathbb{N}$, $\|P^k(x,\cdot) -$

$\pi(\cdot)\|_v \leq K(1+\|x\|)\kappa^k$, *where* $\|\cdot\|_v$ *and* $\|\cdot\|$ *denote the total variation norm and the Euclidean norm, respectively.*

Assumption 3. *There exist* $p-1$ *constants* $z_{p-1}, \ldots, z_{p-d_\tau+1}, z_{p-d_\tau-1}, \ldots, z_0$ *such that* $Z^T(\theta^0_{1,\tau}-\theta^0_{2,\tau}) \neq 0$ *for all* $z_{p-d_\tau} \in [r^0_{L,\tau}, r^0_{U,\tau}]$, *where* $Z = (1, z_{p-1}, \ldots, z_0)^T$. *Furthermore, it is assumed that* $d_\tau \leq p$ *without loss of generality.*

Assumption 4. $F_t(\cdot)$ *is absolutely continuous,* $0 < f_t(u) < \infty$ *on* $\mathcal{U} = \{u : 0 < F_t(u) < 1\}$, *and* $f_t[F_t^{-1}(\tau)] > 0$.

Assumption 1-3 are regularity conditions used in Li et al. (2015a), and Assumption 4 is necessary to derive for the conditional quantile estimation (Koenker and Xiao, 2006).

Theorem 2. *Suppose that* $E(|y_t|^{4+\varsigma}) < \infty$ *for some* $\varsigma > 0$, *and matrix* Ω_1 *is positive definite. If Assumptions 1-4 hold, then*

(a) $n(\widehat{r}_{L,\tau} - r^0_{L,\tau}) = O_p(1)$ *and* $n(\widehat{r}_{U,\tau} - r^0_{U,\tau}) = O_p(1)$;

(b) $\sqrt{n}\sup_{n(|r_{L,\tau}-r^0_{L,\tau}|+|r_{U,\tau}-r^0_{U,\tau}|)\leq B}\left\|\widetilde{\theta}_{n,\tau}(r_{L,\tau}, r_{U,\tau}) - \widetilde{\theta}_{n,\tau}(r^0_{L,\tau}, r^0_{U,\tau})\right\| = o_p(1)$ *for any fixed* $0 < B < \infty$, *where* $\widetilde{\theta}_{n,\tau}(r_{L,\tau}, r_{U,\tau})$ *is defined in the previous section.*

(c) $\sqrt{n}(\widehat{\theta}_{n,\tau} - \theta^0_\tau) \to_d N(0, \Sigma)$, *where* $\Sigma = \tau(1-\tau)\Omega_1^{-1}\Omega_0\Omega_1^{-1}$.

In real applications, we may be interested in the quantities of $\Xi\theta_\tau$, where Ξ is a known $k \times (2p+2)$ matrix with a full rank; see, for example, the generalized linear hypotheses in regression models. From Theorem 2, it holds that $\sqrt{n}\Xi(\widehat{\theta}_{n,\tau} - \theta^0_\tau) \to_d N(0, \Xi\Sigma\Xi^T)$, and we hence can design its inference tools accordingly.

The matrix Ω_1 in the asymptotic variance of $\widehat{\theta}_{n,\tau}$ involves the conditional density $f_t(\cdot)$. As in Koenker (2005) and Li et al. (2015b), we first consider a nonparametric method to estimate $f_t[F_t^{-1}(\tau)]$,

$$\widehat{f}_t[F_t^{-1}(\tau)] = \frac{2h}{\widehat{Q}_{y_t}(\tau + h|\mathcal{F}_{t-1}) - \widehat{Q}_{y_t}(\tau - h|\mathcal{F}_{t-1})},$$

where $\widehat{Q}_{y_t}(\tau|\mathcal{F}_{t-1}) = M_t(\widehat{\lambda}_{n,\tau})$. Matrices Ω_0 and Ω_1 can then be estimated by the sample averages, and hence the asymptotic variance Σ.

We may alternatively consider a bootstrap method to approximate the variance of $\widehat{\theta}_{n,\tau}$. From Theorems 1 and 2, without loss of generality, the parameters of $r_{L,\tau}$, $r_{U,\tau}$ and d_τ can be assumed to be known. By adopting the random weighting method in Rao and Zhao (1992) and Li et al. (2015b), we suggest the bootstrapping procedure below.

(a) Generate non-negative *i.i.d.* random weights $\{\omega_t\}$ with both mean and variance one.

(b) Calculate

$$\widehat{\theta}^*_{n,\tau} = \operatorname{argmin} \sum_{t=1}^{n} \omega_t \rho_\tau[y_t - \widetilde{M}_t(\theta_\tau, \widehat{r}_{L,\tau}, \widehat{r}_{U,\tau}, \widehat{d}_\tau)],$$

where $(\widehat{r}_{L,\tau}, \widehat{r}_{U,\tau}, \widehat{d}_\tau)$ are the conditional quantile estimator of $(r_{L,\tau}, r_{U,\tau}, d_\tau)$.

(c) Repeat Steps (a) and (b) for B times, and denote the resulting quantities by $\{\widehat{\theta}^{*(1)}_{n,\tau}, ..., \widehat{\theta}^{*(B)}_{n,\tau}\}$. The sample variance of $\{\widehat{\theta}^{*(k)}_{n,\tau} - \widehat{\theta}_{n,\tau}, 1 \le k \le B\}$ can then be used to approximate the variance of $\widehat{\theta}_{n,\tau}$.

Theorem 3. *Under the conditions of Theorem 2, it holds that, conditional on $y_1, \ldots, y_n$,*

$$\sqrt{n}(\widehat{\theta}^*_{n,\tau} - \widehat{\theta}_{n,\tau}) \to_d N(0, \Sigma)$$

in probability as $n \to \infty$*.*

Let $\widehat{p}_n = \operatorname{argmin}_{0 \le p \le p_{\max}} \mathrm{BIC}(p)$. We give the theoretical justification of the proposed BIC in the previous section.

Theorem 4. *Under the conditions of Theorem 2, if* $p_{\max} \ge p_0$*, then* $P\{\widehat{p}_n = p_0\} \to 1$ *as* $n \to \infty$*, where* p_0 *is the true order, i.e.* $|\theta^0_{1,p_0}| + |\theta^0_{2,p_0}| \ne 0$*.*

By a method similar to the above theorem, we can show that the minimization of the AIC tends to select an order that is greater than or equal to p_0.

4. Simulation experiments

This section conducts three simulation experiments to evaluate the finite-sample performance of the conditional quantile estimation. In all experiments, we consider four quantiles, $\tau = 0.2$, 0.4, 0.6 and 0.8, and three sample sizes, $n = 100$, 200 and 500, and there are 100 replications for each combination of quantile and sample size. The number of bootstrapped samples B is set to 1000.

The data generating process in the first experiment is

$$y_t = \begin{cases} \theta_{01}(U_t) + \theta_{11}(U_t) y_{t-1}, & R_t = 1, \\ \theta_{02}(U_t) + \theta_{12}(U_t) y_{t-1}, & R_t = 0, \end{cases} \tag{4.1}$$

with

$$R_t = \begin{cases} 1, & y_{t-2} \le 1.12, \\ 0, & y_{t-2} > 1.85, \\ R_{t-1}, & \text{otherwise}, \end{cases}$$

where $\{U_t\}$ are *i.i.d.* standard uniform random variables over $[0, 1]$, $\theta_{01}(x) = 0.85 + 0.15x$, $\theta_{11}(x) = 1/(e^{-x} + 1)$, $\theta_{02}(x) = 0.5$ and $\theta_{12}(x) = 1/(e^{-x} + e^{0.5})$. The conditional quantile estimation in Section 2 is applied with a (or b) being the 10th (or 90th) percentiles of each sample. The asymptotic variances of $\widehat{\theta}_{n,\tau}$ are estimated by both the nonparametric method and bootstrapping approximation. As in Koenker and Xiao (2006) and Li et al. (2015b), we choose the bandwidth of $3h_{HS}$ in the nonparametric method, where

$$h_{HS} = n^{-1/3}\left[\Phi^{-1}\left(1-\alpha/2\right)\right]^{2/3}\left[\frac{1.5\phi^2\Phi^{-1}(\tau)}{2\left[\Phi^{-1}(\tau)\right]^2+1}\right]^{1/3},$$

and $\Phi(\cdot)$ is the distribution of the standard normal distribution; see Hall and Sheather (1988). Tables 1 and 2 list the estimation results for $\tau = \{0.2, 0.4\}$ and $\{0.6, 0.8\}$, respectively. They include the bias (BIAS), the calculated asymptotic variances by the nonparametric method (ASD) and by the bootstrap method (BSD). It can be seen that both bias and ESDs get smaller as the sample size increases. Despite that in most cases the BSDs are slightly above the ASDs, they are both close to each other and close to the ESDs when the sample size is as small as $n = 200$.

The second experiment employs the same data generating process as in (4.1) with the regime indicator function

$$R_t = \begin{cases} 1, & y_{t-2} \leq 2.55, \\ 0, & y_{t-2} > 3.57, \\ R_{t-1}, & \text{otherwise}, \end{cases}$$

where $\theta_{01}(x) = F^{-1}_{\chi^2_1}[1/(e^{-x}+1)]$, $\theta_{11}(x) = 1$, $\theta_{02}(x) = 0.5$, $\theta_{12}(x) = 1/(e^{-x}+1)$, and $F^{-1}_{\chi^2_1}(\cdot)$ is the quantile function of χ^2_1 distribution with degree of freedom one. The generated sequences are all nonnegative. There is a unit root in the structure at the lower regime, while the whole model is still stationary (Koenker and Xiao, 2004). All other settings are the same as in the first experiment. Estimation results are presented in Tables 3 and 4 for $\tau = \{0.2, 0.4\}$ and $\{0.6, 0.8\}$, respectively. Similar findings can be observed.

In the third experiment, we perform the conditional least square estimation (Li et al., 2015a) to the samples generated in the first two experiments. Estimation results are given in Table 5, and both bias and ESDs decrease as the sample size increases. Note that the data generating process at (4.1) can be rewritten into a mean regression form,

$$y_t = \begin{cases} \theta_{01} + \theta_{11} y_{t-1} + \varepsilon_{1t}, & R_t = 1, \\ \theta_{02} + \theta_{12} y_{t-1} + \varepsilon_{2t}, & R_t = 0, \end{cases}$$

where $\theta_{ij} = E[\theta_{ij}(U_t)]$ for $0 \le i \le 1$ and $1 \le j \le 2$, $\varepsilon_{jt} = \theta_{0j}(U_t) - \theta_{0j} + [\theta_{1j}(U_t) - \theta_{1j}]y_{t-1}$ for $1 \le j \le 2$, and $\{(\varepsilon_{1t}, \varepsilon_{2t}), \mathcal{F}_t\}$ is a martingale difference sequence. As a result, both the consistency and asymptotic normality can be obtained possibly.

5. An empirical example

We study the unemployment rates, as it provides important implications for economic policymaking. Many researchers have studied the asymmetric dynamics in the response of unemployment to economic expansion and contraction. Koenker and Xiao (2006) used the quantile AR model to analyze the US quarterly and annual unemployment rates, and the estimated AR roots vary over different quantiles. Galvao et al. (2011) carried out a thorough study of the asymmetric dynamics of the conditional distribution of the US monthly unemployment growth after World War II, based on the threshold AR model, and a stronger asymmetric persistence is suggested in the higher quantiles. Meanwhile, hysteresis has been extensively studied in the literature, and confirmed for the unemployment rate (Blanchard and Summers, 1986; Jaeger and Parkinson, 1994; Perez-Alonso and Sanzo, 2011). As a result, the HAR model may be more suitable in modeling such asymmetry.

This section considers the growth rates of US monthly unemployment rates, instead of themselves, for the clear mean reverting behavior (Galvao et al., 2011). The study period spans from January, 1948 to December, 2007, and the time plot is presented in Figure 1. We consider eight different quantiles $\tau = \{0.05, 0.10, 0.25, 0.40, 0.60, 0.75, 0.90, 0.95\}$. The BIC at (2.4) is employed to select the values of p and d with $p_{\max} = d_{\max} = 5$, and the selection results are given in Table 6. It can be seen that $p = 1$ is selected for all τ's except 0.05 and 0.90, while $d = 1$ is chosen except three quantiles. For easy comparison, we fix $p = d = 1$, and the estimating results are listed in Table 7 as well as in Figure 2.

The fitted intercept for the upper regime crosses zero at $\tau = 0.40$, whereas at $\tau = 0.60$ for the lower regime; both of them are monotonically increasing with τ. The slopes of the lower regime are all significantly greater than zero, which suggests strong serially correlated behavior of the unemployment rates for this regime. However, the slopes of the upper regime are all insignificant except for lower quantiles $\tau = 0.05$ and 0.10, and thus the return series of the unemployment rate display memoryless behavior for this regime. This observation is in line with the economic intuition that economic growths are generally considered as the 'normal' state of the economy, while economic recessions are considered to be anomalies and should not last for too long. Therefore, for the majority of the time, the return series of the unemployment rate tend to exhibit strong serially correlated behavior at relatively low levels (lower regime); the series would be pushed up by occasional large shocks into the upper regime, but will quickly exits it due to the lack of memory. However, with the presence of the hysteresis zone, the series will not immediately fall back into the lower regime, but will instead encounter some delays or even switch back and forth, leading to a longer-than-expected period of high unemployment. Hence, by explicitly incorporating a hysteresis zone, our model leads to an interpretation that dramatically differs from that of the quantile threshold autoregressive (TAR) model of Galvao et al. (2011), and is more consistent with the economic intuition.

We also compare the HAR and TAR models with two and three regimes in terms of BIC, and the results are given in Table 8. The evidence of hysteresis is further reinforced by the observation that the BIC of the HAR model is the lowest at all quantiles. We may conclude that the HAR model is more suitable than the TAR model in interpreting unemployment rates. Note that the BIC for the HAR is supposed to be smaller than that for the TAR model with two regimes since the former includes the latter as a special case.

For the sake of comparison, we further fit the quantile AR model in Koenker and Xiao (2006) to the data. The order is chosen to be one, in line with the choice of p in the HAR models, and the fitted coefficients are presented in Figure 2. It can be seen that the slope parameters of the fitted model do not significantly differ from zero for lower quantiles, but are significantly positive for upper quantiles, $\tau = 0.75, 0.90$ and 0.95. This implies the presence of asymmetric dynamics, and hence the necessity of a regime-switching model; see also Figure 4 in Koenker and Xiao (2006). Moreover, we also calculate the conditional least squares estimation for the AR and HAR models,

$$y_t = 0.14_{0.17} \ + \ 0.13_{0.04} y_{t-1} + \varepsilon_t,$$

and

$$y_t = \begin{cases} 0.15_{0.19} \ + \ 0.44_{0.07} y_{t-1} + \varepsilon_{1t}, & R_t = 1, \\ -0.02_{0.19} \ + \ 0.05_{0.06} y_{t-1} + \varepsilon_{2t}, & R_t = 0, \end{cases}$$

with $d = 2$, $r_L = -1.80$ and $r_U = 0.00$, where the standard errors are given in the subscripts. For the fitted HAR model, the slope parameter of the lower regime is significantly positive, while that of the upper regime, similar to its quantile counterparts, is not significantly different from zero. Actually they can be considered as the averaged values over all quantiles; see also the third experiment in the previous section. Finally, the fitted AR model seems a compromise of these two structures in the HAR model.

6. Conclusion and discussion

This paper develops a conditional quantile estimation for HAR models, which is useful in modeling some economic time series with hysteresis, e.g.

unemployment rates. It gives us much more flexibility in understanding the hysteresis patterns at different quantiles. The asymptotic behaviors of the estimators are established.

In the meanwhile, there are still some open problems related to the conditional quantile estimation for HAR models, and our future research will be along these directions. First, it is an important task in the literature to test for the existence of the threshold. Galvao et al. (2014) and Zhang et al. (2014) proposed tests for the threshold effect at some quantile levels, and Zhu et al. (2014) conducted a quasi-likelihood ratio test for the linearity against hysteresis AR processes. It should be feasible to construct a test for $\theta^0_{1,\tau} \neq \theta^0_{2,\tau}$ in Assumption 1 by following Kato (2009), Galvao et al. (2014), Zhang et al. (2014), and Zhu et al. (2014).

Secondly, we only provide theoretical justifications for the super-consistency of estimated boundary parameters, $\widehat{r}_{L,\tau}$ and $\widehat{r}_{U,\tau}$, and it is of interest to derive their asymptotic distributions as in Li et al. (2015a) and Kuan et al. (2017). Thirdly, it should be challenging to extend our theoretical results from a fixed τ to a close set $\mathcal{I} \in (0,1)$, while these theoretical tools in Kato (2009) may do some help here. Finally, it is important to construct a diagnostic tool for checking the adequacy of the fitted HAR model by the conditional quantile estimation.

Supplementary Materials

The supplementary file contains the proofs of Theorem 1-4.

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Table 1: Estimation results for HAR models in the first experiment with $\tau = 0.2$ and 0.4.

n		$\widehat{\theta}_{01,\tau}$	$\widehat{\theta}_{11,\tau}$	$\widehat{\theta}_{02,\tau}$	$\widehat{\theta}_{12,\tau}$	$\widehat{r}_{L,\tau}$	$\widehat{r}_{U,\tau}$
				$\tau = 0.2$			
100	BIAS	0.0035	0.0029	0.0012	-0.0002	-0.0085	-0.0204
	ESD	0.0965	0.0693	0.0358	0.0269	0.0126	0.0265
	ASD	0.0970	0.0680	0.0392	0.0295		
	BSD	0.1083	0.0744	0.0413	0.0307		
200	BIAS	0.0095	-0.0031	0.0012	-0.0008	-0.0051	-0.0068
	ESD	0.0652	0.0471	0.0285	0.0216	0.0080	0.0146
	ASD	0.0687	0.0481	0.0278	0.0209		
	BSD	0.0731	0.0512	0.0291	0.0216		
500	BIAS	-0.0006	0.0010	0.0024	-0.0009	-0.0062	-0.0060
	ESD	0.0476	0.0317	0.0165	0.0124	0.0044	0.0116
	ASD	0.0436	0.0305	0.0176	0.0133		
	BSD	0.0474	0.0328	0.0176	0.0132		
				$\tau = 0.4$			
100	BIAS	0.0033	-0.0016	-0.0015	0.0001	-0.0071	-0.0136
	ESD	0.1271	0.0878	0.0382	0.0290	0.0125	0.0247
	ASD	0.1167	0.0814	0.0445	0.0334		
	BSD	0.1192	0.0828	0.0444	0.0329		
200	BIAS	-0.0072	0.0050	-0.0031	0.0013	-0.0046	-0.0079
	ESD	0.0854	0.0590	0.0322	0.0244	0.0094	0.0185
	ASD	0.0828	0.0578	0.0317	0.0239		
	BSD	0.0888	0.0615	0.0327	0.0247		
500	BIAS	-0.0117	0.0084	0.0055	-0.0035	-0.0026	-0.0007
	ESD	0.0512	0.0360	0.0202	0.0153	0.0063	0.0100
	ASD	0.0524	0.0367	0.0200	0.0151		
	BSD	0.0533	0.0369	0.0207	0.0156		

Table 2: Estimation results for HAR models in the first experiment with $\tau = 0.6$ and 0.8.

n		$\widehat{\theta}_{01,\tau}$	$\widehat{\theta}_{11,\tau}$	$\widehat{\theta}_{02,\tau}$	$\widehat{\theta}_{12,\tau}$	$\widehat{r}_{L,\tau}$	$\widehat{r}_{U,\tau}$
				$\tau = 0.6$			
100	BIAS	-0.0026	-0.0033	-0.0009	0.0005	-0.0079	-0.0142
	ESD	0.1105	0.0753	0.0375	0.0266	0.0133	0.0259
	ASD	0.1132	0.0792	0.0406	0.0306		
	BSD	0.1141	0.0786	0.0424	0.0313		
200	BIAS	-0.0047	0.0016	0.0042	-0.0026	-0.0008	-0.0049
	ESD	0.0845	0.0577	0.0302	0.0225	0.0081	0.0170
	ASD	0.0800	0.0558	0.0288	0.0217		
	BSD	0.0822	0.0567	0.0302	0.0227		
500	BIAS	-0.0001	-0.0005	-0.0005	0.0005	0.0021	0.0031
	ESD	0.0539	0.0385	0.0202	0.0153	0.0059	0.0057
	ASD	0.0508	0.0355	0.0182	0.0137		
	BSD	0.0524	0.0363	0.0185	0.0138		
				$\tau = 0.8$			
100	BIAS	-0.0039	-0.0015	0.0095	-0.0088	-0.0049	-0.0134
	ESD	0.0923	0.0637	0.0320	0.0255	0.0126	0.0264
	ASD	0.0885	0.0616	0.0302	0.0227		
	BSD	0.0943	0.0661	0.0330	0.0253		
200	BIAS	-0.0007	-0.0014	0.0004	-0.0010	0.0006	-0.0044
	ESD	0.0667	0.0440	0.0206	0.0151	0.0082	0.0165
	ASD	0.0627	0.0437	0.0211	0.0158		
	BSD	0.0635	0.0445	0.0224	0.0168		
500	BIAS	-0.0042	0.0013	0.0011	-0.0010	0.0042	0.0047
	ESD	0.0402	0.0272	0.0124	0.0094	0.0043	0.0040
	ASD	0.0397	0.0277	0.0134	0.0101		
	BSD	0.0416	0.0290	0.0141	0.0107		

Table 3: Estimation results for HAR models in the second experiment with $\tau = 0.2$ and 0.4.

n		$\widehat{\theta}_{01,\tau}$	$\widehat{\theta}_{11,\tau}$	$\widehat{\theta}_{02,\tau}$	$\widehat{\theta}_{12,\tau}$	$\widehat{r}_{L,\tau}$	$\widehat{r}_{U,\tau}$
				$\tau = 0.2$			
100	BIAS	0.0225	-0.0039	0.0209	-0.0045	-0.0349	-0.0457
	ESD	0.1294	0.0424	0.1441	0.0460	0.0397	0.0582
	ASD	0.1298	0.0425	0.1518	0.0509		
	BSD	0.1425	0.0466	0.1595	0.0524		
200	BIAS	0.0034	-0.0001	0.0035	-0.0023	-0.0139	-0.0183
	ESD	0.0918	0.0298	0.0993	0.0334	0.0228	0.0313
	ASD	0.0918	0.0301	0.1080	0.0361		
	BSD	0.0948	0.0319	0.1118	0.0370		
500	BIAS	-0.0080	0.0037	-0.0080	0.0041	-0.0058	-0.0023
	ESD	0.0547	0.0186	0.0692	0.0234	0.0142	0.0097
	ASD	0.0576	0.0189	0.0678	0.0227		
	BSD	0.0608	0.0203	0.0702	0.0236		
				$\tau = 0.4$			
100	BIAS	0.0281	-0.0059	-0.0249	0.0070	-0.0273	-0.0385
	ESD	0.1954	0.0648	0.1564	0.0552	0.0392	0.0576
	ASD	0.1841	0.0605	0.1757	0.0589		
	BSD	0.1965	0.0640	0.1853	0.0610		
200	BIAS	-0.0085	0.0029	0.0005	0.0001	-0.0106	-0.0210
	ESD	0.1384	0.0446	0.1286	0.0445	0.0216	0.0336
	ASD	0.1292	0.0424	0.1257	0.0421		
	BSD	0.1339	0.0442	0.1313	0.0440		
500	BIAS	-0.0049	0.0017	0.0022	-0.0003	-0.0019	-0.0019
	ESD	0.0752	0.0257	0.0849	0.0274	0.0131	0.0104
	ASD	0.0816	0.0268	0.0804	0.0269		
	BSD	0.0841	0.0276	0.0821	0.0274		

Table 4: Estimation results for HAR models in the second experiment with $\tau = 0.6$ and 0.8.

n		$\widehat{\theta}_{01,\tau}$	$\widehat{\theta}_{11,\tau}$	$\widehat{\theta}_{02,\tau}$	$\widehat{\theta}_{12,\tau}$	$\widehat{r}_{L,\tau}$	$\widehat{r}_{U,\tau}$
				$\tau = 0.6$			
100	BIAS	-0.0010	0.0002	-0.0017	-0.0017	-0.0316	-0.0383
	ESD	0.2095	0.0658	0.1680	0.0549	0.0436	0.0487
	ASD	0.2061	0.0678	0.1688	0.0566		
	BSD	0.2097	0.0688	0.1784	0.0606		
200	BIAS	-0.0292	0.0087	-0.0336	0.0096	-0.0071	-0.0093
	ESD	0.1373	0.0456	0.1214	0.0429	0.0216	0.0278
	ASD	0.1456	0.0477	0.1199	0.0402		
	BSD	0.1439	0.0473	0.1293	0.0428		
500	BIAS	0.0099	-0.0041	0.0025	-0.0005	0.0014	-0.0009
	ESD	0.0997	0.0344	0.0796	0.0276	0.0104	0.0074
	ASD	0.0920	0.0302	0.0766	0.0257		
	BSD	0.0958	0.0316	0.0798	0.0262		
				$\tau = 0.8$			
100	BIAS	-0.0042	0.0006	-0.0121	0.0019	-0.0204	-0.0384
	ESD	0.2004	0.0684	0.1438	0.0458	0.0357	0.0549
	ASD	0.1881	0.0615	0.1323	0.0443		
	BSD	0.1947	0.0639	0.1375	0.0454		
200	BIAS	0.0023	-0.0027	0.0023	-0.0020	-0.0084	-0.0153
	ESD	0.1436	0.0452	0.1057	0.0357	0.0188	0.0337
	ASD	0.1328	0.0435	0.0928	0.0311		
	BSD	0.1349	0.0440	0.1006	0.0335		
500	BIAS	-0.0077	0.0004	-0.0117	0.0037	0.0036	-0.0016
	ESD	0.0838	0.0274	0.0631	0.0209	0.0079	0.0087
	ASD	0.0841	0.0276	0.0589	0.0198		
	BSD	0.0877	0.0288	0.0617	0.0205		

Table 5: Estimation results for HAR models under mean regression in the first and second experiments.

n		$\widehat{\theta}_{01,\tau}$	$\widehat{\theta}_{11,\tau}$	$\widehat{\theta}_{02,\tau}$	$\widehat{\theta}_{12,\tau}$	$\widehat{r}_{L,\tau}$	$\widehat{r}_{U,\tau}$
				First Experiment			
100	BIAS	-0.0078	0.0059	-0.0045	0.0021	-0.0088	-0.0189
	ESD	0.0733	0.0509	0.0300	0.0230	0.0106	0.0238
200	BIAS	-0.0032	0.0034	-0.0019	0.0017	-0.0036	-0.0045
	ESD	0.0516	0.0387	0.0192	0.0137	0.0068	0.0084
500	BIAS	0.0039	-0.0035	-0.0023	0.0023	-0.0004	-0.0020
	ESD	0.0334	0.0229	0.0149	0.0116	0.0018	0.0053
				Second Experiment			
100	BIAS	-0.0268	0.0079	-0.0081	0.0016	-0.0382	-0.0544
	ESD	0.1220	0.0408	0.1236	0.0392	0.0453	0.0560
200	BIAS	0.0012	0.0003	-0.0015	0.0012	-0.0153	-0.0182
	ESD	0.0848	0.0278	0.0848	0.0283	0.0202	0.0251
500	BIAS	0.0076	-0.0025	0.0037	-0.0009	-0.0017	-0.0048
	ESD	0.0586	0.0185	0.0527	0.0178	0.0095	0.0094

Table 6: Values of p and d selected by the BIC.

τ	0.05	0.1	0.25	0.4	0.6	0.75	0.9	0.95
p	4	1	1	1	1	1	2	1
d	2	1	1	2	1	1	1	2

Table 7: Estimated coefficients, together with standard errors in parentheses, of the HAR model by the conditional quantile estimation, and the percentages of observations (PO) falling into the hysteresis zone.

τ	$\widehat{\theta}_{01,\tau}$	$\widehat{\theta}_{11,\tau}$	$\widehat{\theta}_{02,\tau}$	$\widehat{\theta}_{12,\tau}$	$\widehat{r}_{L,\tau}$	$\widehat{r}_{U,\tau}$	PO
0.05	-5.66	0.48	-5.24	0.27	-1.30	3.65	0.47
	(0.99)	(0.13)	(0.36)	(0.08)			
0.1	-3.99	0.37	-4.10	0.18	-1.30	3.65	0.47
	(0.61)	(0.08)	(0.19)	(0.06)			
0.25	-2.10	0.31	-2.00	0.08	-1.70	0.00	0.06
	(0.21)	(0.06)	(0.13)	(0.06)			
0.4	-1.02	0.35	-0.76	0.14	-1.70	0.00	0.06
	(0.20)	(0.05)	(0.51)	(0.08)			
0.6	1.53	0.32	0.00	0.00	-1.30	3.10	0.44
	(0.31)	(0.08)	(0.00)	(0.00)			
0.75	2.87	0.39	1.70	-0.03	-1.30	2.30	0.38
	(0.33)	(0.09)	(0.23)	(0.06)			
0.9	7.45	0.56	3.76	0.01	1.30	3.65	0.21
	(0.84)	(0.18)	(0.31)	(0.08)			
0.95	8.70	0.62	5.70	0.08	1.30	3.65	0.21
	(0.91)	(0.22)	(0.40)	(0.11)			

Note: “0.00” refers to a value smaller than 0.005.

Table 8: Values of the BIC for the fitted HAR and TAR models with two (TAR2) and three regimes (TAR3).

τ	0.05	0.1	0.25	0.4	0.6	0.75	0.9	0.95
HAR	-694	-344	57	204	198	114	-243	-584
TAR2	-662	-322	65	213	221	127	-224	-565
TAR3	-685	-333	65	210	221	128	-232	-577

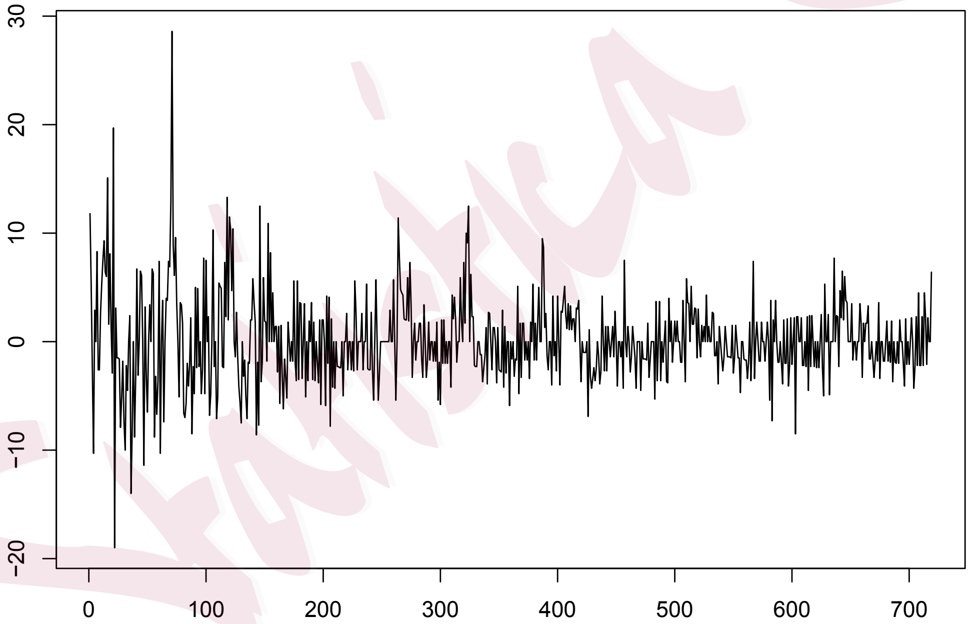

Figure 1: Growth rates, in percentages, of US monthly unemployment rates from January, 1948 to December, 2007.

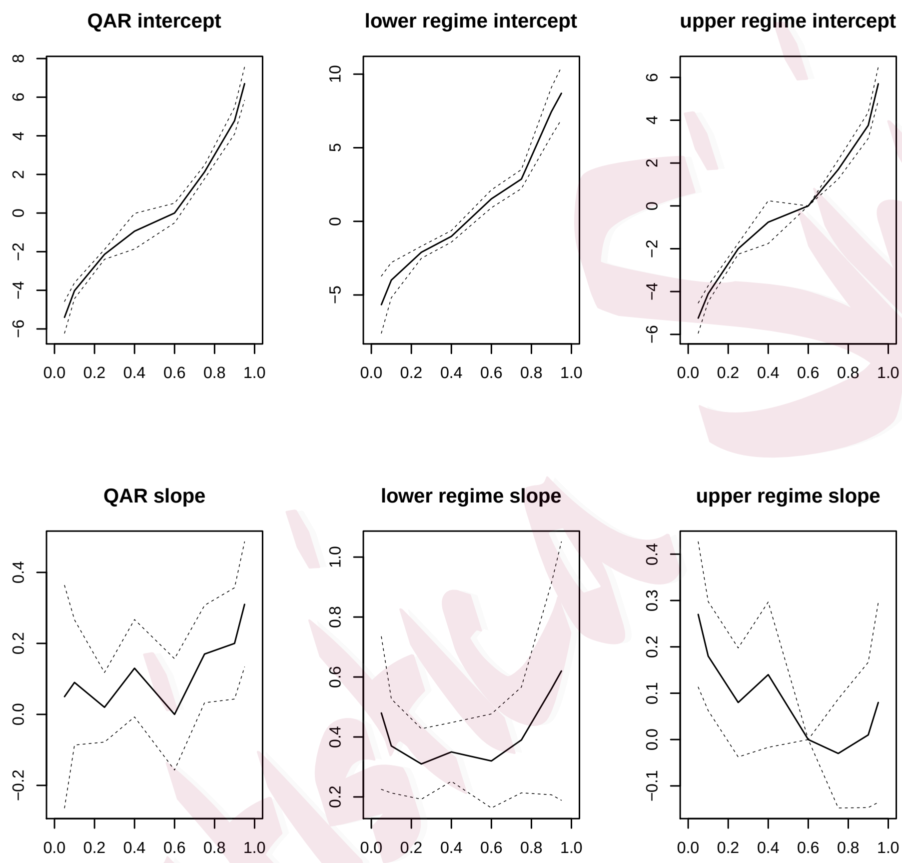

Figure 2: Estimated coefficients (solid lines) and their 95% confidence intervals (dotted lines) of the AR (left panel) and HAR (middle and right panels) models by the conditional quantile estimation.