<table>
<thead>
<tr>
<th><strong>Statistica Sinica Preprint No:</strong> SS-2017-0106</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Title</strong></td>
</tr>
<tr>
<td><strong>Manuscript ID</strong></td>
</tr>
<tr>
<td><strong>URL</strong></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
</tr>
</tbody>
</table>
| **Complete List of Authors** | Pingyan Chen  
Tao Zhang and  
Soo Hak Sung |
| **Corresponding Author** | Soo Hak Sung |
| **E-mail** | sungsh@pcu.ac.kr |

Notice: Accepted version subject to English editing.
Strong laws for randomly weighted sums of random variables and applications in the bootstrap and random design regression

Pingyan Chen¹, Tao Zhang², Soo Hak Sung³,*
¹,² Jinan University and ³ Pai Chai University

Abstract: In this paper, we establish the Marcinkiewicz-Zygmund strong law of large numbers for randomly weighted sums of negatively orthant dependent random variables. A single law of logarithm for randomly weighted sums of negatively orthant dependent random variables is also established. These results are applied to the bootstrap sample means and the least squares estimators in the simple linear regression with random design.

Keywords and phrases: Bootstrap sample mean, law of single logarithm, Marcinkiewicz-Zygmund strong law, randomly weighted sum, regression with random design.

Running title: Strong laws for randomly weighted sums

Mathematics Subject Classification: 60F15, 62G08.

*Corresponding author.  E-mail: sungsh@pcu.ac.kr  Tel:+82-42-520-5369  Fax:+82-70-4362-6304
1. Introduction

Let \( \{X_n, n \geq 1\} \) be a sequence of random variables, \( \{w_{nk}, n \geq 1, 1 \leq k \leq n\} \) an array of random variables independent of \( \{X_n, n \geq 1\} \). A randomly weighted sum is defined by

\[
\sum_{k=1}^{n} w_{nk} X_k.
\]

(1.1)

The \( w_{nk}, n \geq 1, 1 \leq k \leq n \), are called random weights. In the special case of constant random variables \( w_{nk}, n \geq 1, 1 \leq k \leq n \), (1.1) is referred to as a non-randomly weighted sum or weighted sum. Many useful linear statistics, e.g., least squares estimators, nonparametric regression function estimators and jackknife estimates, are weighted sums. The randomly weighted sums play an important role in various applied and theoretical problems. For examples, in the field of queueing theory, the \( \sum_{k=1}^{n} w_{nk} X_k \) represents the total output for a customer being served by \( n \) machines, where \( w_{nk} \) is the service time in the \( k \)th machine, and \( X_k \) is the output from the \( k \)th machine. In statistics, the bootstrap sample means and the least squares estimators in the simple linear regression with random design are randomly weighted sums (see Section 3).

We first consider strong laws for weighted sums \( \sum_{k=1}^{n} a_{nk} X_k \) of independent and identically distributed (i.i.d.) random variables \( \{X, X_n, n \geq 1\} \), where \( \{a_{nk}, n \geq 1, 1 \leq k \leq n\} \) is an array of constants, i.e., each weight random variable \( w_{nk} \) has the constant value \( a_{nk} \) with probability 1. Chow (1966) proved the Kolmogorov strong law

\[
n^{-1} \sum_{k=1}^{n} a_{nk} X_k \to 0 \text{ almost surely (a.s.) when } EX = 0, E|X|^2 < \infty, \text{ and } \{a_{nk}\} \text{ is an array of the second Cesàro uniformly bounded, that is, } \sup_{n \geq 1} n^{-1} \sum_{k=1}^{n} |a_{nk}|^2 < \infty.
\]

Choi and Sung (1987) showed the Kolmogorov strong law under the conditions that \( EX = 0 \) and the weights \( a_{nk} \) are uniformly bounded, that is, \( \sup_{n \geq 1} \max_{1 \leq k \leq n} |a_{nk}| < \infty \). Let us compare these two results. If \( a_{nk} \) are uniformly bounded, then they are the second Cesàro uniformly bounded. Hence the condition on weights of Chow (1966) is weaker than that of Choi and Sung (1987), but the moment conditions of Chow (1966) are stronger. Cuzick (1995) proved the result of Chow (1966) when
the weights $a_{nk}$ are $\alpha$-th Cesàro uniformly bounded for some $1 < \alpha < \infty$ (i.e., $\sup_{n \geq 1} n^{-1} \sum_{k=1}^{n} |a_{nk}|^\alpha < \infty$), $EX = 0$, and $E|X|^\beta < \infty$, where $1/\alpha + 1/\beta = 1$. If $\alpha = \beta = 2$, the result of Cuzick (1995) reduces to that of Chow (1966). Bai and Cheng (2000) extended the result of Cuzick (1995) and obtained the Marcinkiewicz-Zygmund strong law:

$$n^{-1/p} \sum_{k=1}^{n} a_{nk} X_k \rightarrow 0 \ a.s.,$$  \hfill (1.2)

where $1 \leq p < 2$, $\{a_{nk}\}$ is an array of $\alpha$-th Cesàro uniformly bounded for some $p < \alpha < \infty$, $EX = 0$, and $E|X|^\beta < \infty$, where $1/\alpha + 1/\beta = 1/p$.

On the other hand, Bai et al. (1997) obtained the law of single logarithm:

$$\limsup_{n \rightarrow \infty} \left| \frac{\sum_{k=1}^{n} a_{nk} X_k}{\sqrt{2n \log n}} \right| \leq \limsup_{n \rightarrow \infty} \left( n^{-1} \sum_{k=1}^{n} |a_{nk}|^2 \right)^{1/2} \ a.s.,$$  \hfill (1.3)

where $\{a_{nk}\}$ is an array of $\alpha$-th Cesàro uniformly bounded for some $2 < \alpha < \infty$, $EX = 0$, $EX^2 = 1$, and $E|X|^\beta < \infty$, where $1/\alpha + 1/\beta = 1/2$. The result of Bai et al. (1997) was improved and extended by many authors. Chen and Gan (2007) weakened the moment condition to $E|X|^\beta/(\log |X|)^{3/2} < \infty$, Sung (2009) and Chen and Chen (2010) extended the result of Chen and Gan (2007) from real valued random variables to Banach valued random elements. Chen et al. (2014) and Chen et al. (2017) obtained more generalized results which include the result of Chen and Gan (2007).

We next consider strong laws for randomly weighted sums $\sum_{k=1}^{n} w_{nk} X_k$ of i.i.d. random variables $\{X, X_n, n \geq 1\}$. The strong laws for randomly weighted sums are less studied than those for non-randomly weighted sums. Cuzick (1995) proved the Kolmogorov strong law for randomly weighted sums:

$$n^{-1} \sum_{k=1}^{n} (w_{nk} X_k - EW_{nk}EX) \rightarrow 0 \ a.s.$$  \hfill (1.4)

if $\{w_{nk}, n \geq 1, 1 \leq k \leq n\}$ is an array of i.i.d. random variables independent of $\{X_n, n \geq 1\}$, $E|w_{11}|^\alpha < \infty$ for some $2 \leq \alpha < \infty$, and $E|X|^\beta < \infty$, where $1/\alpha + 1/\beta = 1$. 

3
Cuzick (1995) also proved the law of single logarithm for randomly weighted sums:

$$\limsup_{n \to \infty} \frac{\sum_{k=1}^{n} w_{nk} X_k}{\sqrt{2n \log n}} = 1 \text{ a.s.} \tag{1.5}$$

providing that \( \{w_{nk}, n \geq 1, 1 \leq k \leq n\} \) is an array of independent Rademacher random variables and \( EX^2 = 1 \). Li et al. (1995b) proved (1.5) under general conditions that \( \{w_{nk}\} \) is an array of bounded i.i.d. random variables independent of \( \{X_n, n \geq 1\} \), \( E(w_{11}X_1) = 0 \), \( E(w_{11} - Ew_{11})^2 = 1 \), and \( EX^2 = 1 \).

The Marcinkiewicz-Zygmund strong law is between the Kolmogorov strong law (1.4) and the law of single logarithm (1.5). However, the Marcinkiewicz-Zygmund strong law for randomly weighted sums of i.i.d. random variables is not proved.

Bootstrap samples were first investigated by Efron (1979) for a sequence of i.i.d. random variables. But, in general, bootstrap samples are defined for a sequence of not necessarily i.i.d. random variables. Moreover, in the bootstrap sample mean, the weights \( w_{nk} \) are not independent but negatively associated (see Section 3). Therefore, it is more interesting to study the Marcinkiewicz-Zygmund strong law for randomly weighted sums of dependent random variables under dependent random weights.

For randomly weighted sums of dependent random variables, Rosalsky and Sreehari (1998), Thanh et al. (2011), and Csörgő and Nasari (2013) provided sufficient conditions for some kind of strong laws. But, their work does not imply the Marcinkiewicz-Zygmund strong law. In this paper, we prove the Marcinkiewicz-Zygmund strong law for randomly weighted sums of negatively orthant dependent random variables. The weights are also negatively orthant dependent random variables.

Let us first recall the concept of negatively orthant dependent random variables. This concept was introduced by Lehmann (1966) as follows.

**Definition 1.** A finite family of random variables \( \{X_1, \ldots, X_n\} \) is said to be negatively orthant dependent if the following two inequalities hold:

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i)$$
and
\[ P(X_1 > x_1, \cdots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i) \]
for all real numbers \( x_1, \cdots, x_n \). An infinite family of random variables is negatively orthant dependent if every finite subfamily is negatively orthant dependent.

It is well known that if \( \{X_n, n \geq 1\} \) is a sequence of negatively orthant dependent random variables and \( \{f_n, n \geq 1\} \) is a sequence of Borel functions all of which are monotone increasing (or all monotone decreasing), then \( \{f_n(X_n), n \geq 1\} \) is still a sequence of negatively orthant dependent random variables. It is also well known that if \( X_1, \cdots, X_n \) are nonnegative negatively orthant dependent random variables, then
\[ E[X_1 \cdots X_n] \leq EX_1 \cdots EX_n. \]

The next dependence notion is negative association. This concept was introduced by Alam and Saxena (1981) as follows.

**Definition 2.** A finite family of random variables \( \{X_i, 1 \leq i \leq n\} \) is said to be negatively associated if for every pair of disjoint subsets \( A \) and \( B \) of \( \{1, 2, \cdots, n\} \),
\[ \text{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0 \]
whenever \( f_1 \) and \( f_2 \) are coordinatewise increasing (or coordinatewise decreasing) and the covariance exists. An infinite family of random variables is negatively associated if every finite subfamily is negatively associated.

As pointed out and proved by Joag-Dev and Proschan (1983), a number of well known multivariate distributions possess the negative association property, such as multinomial, convolution of unlike multinomial, multivariate hypergeometric, Dirichlet, permutation distribution, negatively correlated normal distribution, random sampling without replacement, and joint distribution of ranks.

Since negative association implies negative orthant dependence, the above multivariate distributions also possess negative orthant dependence property.

The following example shows that the random variables \( X_1, X_2, X_3 \) are negatively orthant dependent, but not negatively associated.
Example 1. Let $(X_1, X_2, X_3)$ have the joint distribution as in Table 1.

<table>
<thead>
<tr>
<th>$(x_1, x_2, x_3)$</th>
<th>(0,0,0)</th>
<th>(0,0,1)</th>
<th>(0,1,0)</th>
<th>(0,1,1)</th>
<th>(1,0,0)</th>
<th>(1,0,1)</th>
<th>(1,1,0)</th>
<th>(1,1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{3}{16}$</td>
<td>$\frac{3}{16}$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{1}{8}$</td>
</tr>
</tbody>
</table>

For $x_1, x_2, x_3 = 0$ or 1, it is easy to show that

$$P\{X_1 \leq x_1, X_2 \leq x_2, X_3 \leq x_3\} \leq P\{X_1 \leq x_1\}P\{X_2 \leq x_2\}P\{X_3 \leq x_3\},$$

$$P\{X_1 \geq x_1, X_2 \geq x_2, X_3 \geq x_3\} \leq P\{X_1 \geq x_1\}P\{X_2 \geq x_2\}P\{X_3 \geq x_3\}.$$ 

For example, if $(x_1, x_2, x_3) = (1, 0, 0)$, then

$$P\{X_1 \leq x_1, X_2 \leq x_2, X_3 \leq x_3\} = \frac{3}{16}, \quad P\{X_1 \leq x_1\}P\{X_2 \leq x_2\}P\{X_3 \leq x_3\} = \frac{1}{4},$$

$$P\{X_1 \geq x_1, X_2 \geq x_2, X_3 \geq x_3\} = \frac{1}{2}, \quad P\{X_1 \geq x_1\}P\{X_2 \geq x_2\}P\{X_3 \geq x_3\} = \frac{1}{2}.$$ 

Therefore, $X_1, X_2, X_3$ are negatively orthant dependent. But, $P\{X_1 \leq 0, X_2 + X_3 \leq 1\} > P\{X_1 \leq 0\}P\{X_2 + X_3 \leq 1\}$, since $P\{X_1 \leq 0, X_2 + X_3 \leq 1\} = \frac{7}{16}, \quad P\{X_1 \leq 0\}P\{X_2 + X_3 \leq 1\} = \frac{13}{32}.$ Hence $X_1, X_2, X_3$ are not negatively associated.

The rest of this paper is organized as follows. In Section 2, we present the main results and remarks. The applications of our main results to the bootstrap sample means and the least squares estimators are given in Section 3. The technical details are provided in the online supplementary material.

Throughout this paper, $I(A)$ denotes the indicator function of the event $A$. It proves convenient to define that $\log x = \max\{1, \ln x\}$ for $x > 0$, where $\ln x$ denotes the natural logarithm.

2. Main results and remarks

In this section, we present our main results and remarks. We first provide a lemma which plays an important role in the proofs of our main results.

Lemma 1. Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative negatively orthant dependent (NOD) random variables, $\{Y_n, n \geq 1\}$ a sequence of negatively orthant dependent
random variables. Assume that \( \{X_n, n \geq 1\} \) and \( \{Y_n, n \geq 1\} \) are independent. Then \( \{X_n Y_n, n \geq 1\} \) is a sequence of negatively orthant dependent random variables.

The following theorem is the Marcinkiewicz-Zygmund strong law for randomly weighted sums of negatively orthant dependent random variables.

**Theorem 1.** Let \( 1 \leq p < 2 \). Let \( \{X, X_n, n \geq 1\} \) be a sequence of negatively orthant dependent and identically distributed random variables, \( \{w_{nk}, n \geq 1, 1 \leq k \leq n\} \) an array of rowwise negatively orthant dependent random variables with

\[
\sup_{n \geq 1} n^{-1} \sum_{k=1}^{n} E|w_{nk}|^{\alpha} < \infty \tag{2.1}
\]

for some \( \alpha > 2p \). Assume that \( \{X_n\} \) and \( \{w_{nk}\} \) are independent. If \( E|X|^{\beta} < \infty \), where \( 1/\alpha + 1/\beta = 1/p \), then

\[
n^{-1/p} \sum_{k=1}^{n} (w_{nk}X_k - Ew_{nk}E X) \to 0 \text{ a.s.} \tag{2.2}
\]

If we further assume that all weights \( w_{nk} \) have the same distribution, then the case \( \alpha = 2p \) is possible (see the following corollary).

**Corollary 1.** Let \( 1 \leq p < 2 \). Let \( \{X, X_n, n \geq 1\} \) be a sequence of negatively orthant dependent and identically distributed random variables, \( \{w, w_{nk}, n \geq 1, 1 \leq k \leq n\} \) an array of rowwise negatively orthant dependent random variables identically distributed as \( w \). Assume that \( \{X_n\} \) and \( \{w_{nk}\} \) are independent. If \( E|w|^{\alpha} < \infty \) for some \( \alpha \geq 2p \) and \( E|X|^{\beta} < \infty \), where \( 1/\alpha + 1/\beta = 1/p \), then

\[
n^{-1/p} \sum_{k=1}^{n} (w_{nk}X_k - Ew_{nk}E X) \to 0 \text{ a.s.} \tag{2.3}
\]

**Remark 1.** The assumption \( \alpha \geq 2p \) is needed. Let \( P(X = 1) = 1 \). Then the sufficient moment condition for (2.3) is \( E|w|^{2p} < \infty \) (see Taylor et al., 2002). Further assume that \( \{w, w_{nk}, n \geq 1, 1 \leq k \leq n\} \) is an array of i.i.d. random variables. Then, by the
Borel-Cantelli lemma, (2.3) is equivalent to \( \sum_{n=1}^{\infty} P(|\sum_{k=1}^{n}(w_{nk} - Ew)| > n^{1/p}\varepsilon) < \infty, \forall \varepsilon > 0 \). The latter is also equivalent to \( E|w|^{2p} < \infty \) (see Katz, 1963). Hence, the necessary and sufficient moment condition for (2.3) is \( E|w|^{2p} < \infty \).

**Remark 2.** For any \( \alpha' \in (0, \alpha) \), by Hölder’s inequality and Jensen’s inequality,

\[
\sup_{n \geq 1} n^{-1} \sum_{k=1}^{n} E|w_{nk}|^{\alpha'} \leq \sup_{n \geq 1} \left( n^{-1} \sum_{k=1}^{n} E|w_{nk}|^{\alpha} \right)^{\alpha'/\alpha} < \infty.
\]

Hence, condition (2.1) is increasingly stronger as \( \alpha \) increases.

**Remark 3.** Let \( 1/\alpha + 1/\beta = 1/p \). If \( \alpha > 2p \), then \( \beta < 2p \), and hence \( \alpha > \beta \). Conversely, if \( \alpha > \beta \), then \( \alpha > 2p \) (if \( \alpha \leq 2p \), then \( \beta \geq 2p \), and so \( \alpha \leq \beta \)). Hence, under the condition \( 1/\alpha + 1/\beta = 1/p \), \( \alpha > 2p \) and \( \alpha > \beta \) are equivalent.

**Remark 4.** Thanh et al. (2011) obtained a randomly weighted version of Theorem 3.1 in Li et al. (1995a). But the result of Thanh et al. (2011) does not include Theorem 1.

If \( p = 2 \), we have a single law of logarithm for randomly weighted sums of negatively orthant dependent random variables.

**Theorem 2.** Let \( \{X, X_n, n \geq 1\} \) be a sequence of negatively orthant dependent and identically distributed random variables, \( \{w_{nk}, n \geq 1, 1 \leq k \leq n\} \) an array of rowwise negatively orthant dependent random variables satisfying (2.1) for some \( \alpha > 4 \). Assume that \( \{X_n\} \) and \( \{w_{nk}\} \) are independent, \( Ew_{nk}X_k = 0 \) for all \( n \geq 1 \) and \( 1 \leq k \leq n \), \( E X^2 = 1 \), and \( E|X|^\beta/(\log |X|)^{\beta/2} < \infty \), where \( 1/\alpha + 1/\beta = 1/2 \). If \( X \geq 0 \) a.s., or \( w_{nk} \geq 0 \) a.s. for all \( n \geq 1 \) and \( 1 \leq k \leq n \), then

\[
\limsup_{n \to \infty} \frac{\left|\sum_{k=1}^{n} w_{nk}X_k\right|}{\sqrt{2n\log n}} \leq \rho \quad \text{a.s.},
\]

where \( \rho = \inf\{u > 0 : \sum_{n=1}^{\infty} \exp(-u^2n\log n/\sum_{k=1}^{n} Ew_{nk}^2) < \infty\} \).

**Remark 5.** The following inequalities hold (the proof is provided in the online sup-
\[
\liminf_{n \to \infty} \left( n^{-1} \sum_{k=1}^{n} Ew_{nk}^2 \right)^{1/2} \leq \rho \leq \limsup_{n \to \infty} \left( n^{-1} \sum_{k=1}^{n} Ew_{nk}^2 \right)^{1/2}.
\]

Hence \( \rho = \lim_{n \to \infty} \left( n^{-1} \sum_{k=1}^{n} Ew_{nk}^2 \right)^{1/2} \) whenever the limit exists.

**Remark 6.** The nonnegative condition on \( X \) or on \( w_{nk} \) ensures that \( \{w_{nk}X_k, 1 \leq k \leq n\} \) is also a sequence of negatively orthant dependent random variables by Lemma 1. If the nonnegative condition is deleted, we can apply Theorem 2 to \( \{X^+, X_n^+, n \geq 1\} \) and \( \{X^-, X_n^-, n \geq 1\} \) (where and in the following \( x^+ = \max\{x, 0\} \) and \( x^- = \max\{-x, 0\} \)), or to \( \{w_{nk}^+, n \geq 1, 1 \leq k \leq n\} \) and \( \{w_{nk}^-, n \geq 1, 1 \leq k \leq n\} \), respectively, and hence (2.4) can be replaced by

\[
\limsup_{n \to \infty} \frac{\left| \sum_{k=1}^{n} w_{nk}X_k \right|}{\sqrt{2n \log n}} \leq 2\rho \text{ a.s.}
\]

Then upper bound has increased by a factor 2.

**Remark 7.** If \( \{X, X_n, n \geq 1\} \) and \( \{w_{nk}, n \geq 1, 1 \leq k \leq n\} \) are all independent, then (2.4) also holds without the nonnegative condition on \( X \) or on \( w_{nk} \). since \( \{w_{nk}X_k, 1 \leq k \leq n\} \) is a sequence of independent, and hence negatively orthant dependent.

If we further assume that all weights \( w_{nk} \) are i.i.d. and \( \{X_n, n \geq 1\} \) are independent, then the reverse inequality of (2.4) holds (see the following corollary). From this fact, we see that the upper bound of (2.4) is optimal.

**Corollary 2.** Let \( \{X, X_n, n \geq 1\} \) be a sequence of i.i.d. random variables, \( \{w, w_{nk}, n \geq 1, 1 \leq k \leq n\} \) an array of i.i.d. random variables independent of \( \{X, X_n, n \geq 1\} \). If \( E(wX) = 0 \), \( E|w|^\alpha < \infty \) for some \( \alpha > 4 \), \( EX^2 = 1 \), and \( E|X|^\beta/(\log |X|)^{\beta/2} < \infty \), where \( 1/\alpha + 1/\beta = 1/2 \), then

\[
\limsup_{n \to \infty} \frac{\left| \sum_{k=1}^{n} w_{nk}X_k \right|}{\sqrt{2n \log n}} = \sqrt{E(w - Ew)^2} \text{ a.s.} \tag{2.5}
\]
3. Applications

In this section, we apply our main results to the bootstrap sample means and the least squares estimators.

We first consider the bootstrap samples. The bootstrap samples were introduced by Efron (1979). Let $Y_{n1}, Y_{n2}, \ldots, Y_{nm(n)}$ be a sample selected randomly, with replacement, from the set of random variables $\{X_i, 1 \leq i \leq n\}$. The $\{Y_{n1}, Y_{n2}, \ldots, Y_{nm(n)}\}$ is called a bootstrap sample from $\{X_i, 1 \leq i \leq n\}$ with bootstrap sample size $m(n)$. We can write $Y_{ni} = X_{Z_{ni}}$, where $Z_{n1}, Z_{n2}, \ldots, Z_{nm(n)}$ are independent and uniformly distributed on $\{1, 2, \cdots, n\}$ and independent of $\{X_i, 1 \leq i \leq n\}$. Setting the weights $w_{nk}$ by

$$w_{nk} = \frac{1}{m(n)} \sum_{i=1}^{m(n)} I(Z_{ni} = k), \quad 1 \leq k \leq n,$$

the bootstrap sample mean is

$$\bar{X}_n^* = \frac{1}{m(n)} \sum_{i=1}^{m(n)} Y_{ni} = \sum_{k=1}^{n} w_{nk} X_k.$$

Note that $m(n)(w_{n1}, \cdots, w_{nn})$ has the multinomial distribution with parameters $(m(n), 1/n, \cdots, 1/n)$.

As an application of Theorems 1 and 2, we can obtain the convergence rate of the bootstrap strong law of large numbers for negatively orthant dependent random variables. To prove it, the following lemma is needed. The proofs of the results in this section are also provided in the online supplementary material.

**Lemma 2.** Let $X_n$ have the binomial distribution with parameters $n$ and $p_n$. Then the following statements hold.

(i) If $np_n \leq c$ for some positive constant $c$, then $EX_n^k \leq C_k np_n$ for all $k \geq 1$, where $C_k$ is a positive constant depending only on $k$.

(ii) If $np_n \geq d$ for some positive constant $d$, then $EX_n^k \leq D_k (np_n)^k$ for all $k \geq 1$, where $D_k$ is a positive constant depending only on $k$. 

10
Theorem 3. Let $1 \leq p \leq 2$, $\{X, X_n, n \geq 1\}$ be a sequence of negatively orthant dependent and identically distributed random variables with $E|X|^{\beta} < \infty$ for some $\beta > p$. If $n = O(m(n))$, then we have the followings.

(i) If $1 \leq p < 2$, then

$$n^{1-1/p} (\bar{X}_n^* - EX) \rightarrow 0 \text{ a.s.} \tag{3.1}$$

(ii) If $p = 2$, then

$$\limsup_{n \to \infty} \sqrt{\frac{n}{2\log n}} |\bar{X}_n^* - EX| \leq \sqrt{(r+1)EX^2} \text{ a.s.,} \tag{3.2}$$

where $r = \limsup_{n \to \infty} n/m(n)$.

Remark 8. By our knowledge, no literature deals with the converge rate of the unconditional strong law of large numbers for the bootstrap mean. So Theorem 3 is a new result.

Remark 9. Arenal-Gutiérrez et al. (1996) proved the bootstrap strong law of large numbers, i.e. $\bar{X}_n^* \to EX$ a.s., under the conditions that $\{X, X_n, n \geq 1\}$ is a sequence of pairwise i.i.d. random variables with $E|X| < \infty$ and $n \log n = o(m(n))$. The moment condition is weaker than that of Theorem 3, but the condition on the bootstrap sample size is stronger.

We next consider the simple linear regression model with random design:

$$Y_{nk} = a + bX_{nk} + \epsilon_k, \quad 1 \leq k \leq n, \tag{3.3}$$

where $a$ and $b$ are unknown parameters, the randomly design points $\{X_{nk}, n \geq 1, 1 \leq k \leq n\}$ is an array of rowwise negatively orthant dependent random variables identically distributed as a random variable $X$, $\{Y_{nk}, n \geq 1, 1 \leq k \leq n\}$ is an array of observable variables, the errors $\{\epsilon_n, n \geq 1\}$ is a sequence of negatively orthant dependent and identically distributed random variables independent of $\{X_{nk}, n \geq 1, 1 \leq k \leq n\}$ and with the same distribution as a random variable $\epsilon$. Then the least square (LS)
estimators of $b$ and $a$ are
\[ \hat{b}_n = \frac{\sum_{k=1}^{n}(Y_{nk} - \bar{Y}_n)(X_{nk} - \bar{X}_n)}{S_n^2}, \quad \hat{a}_n = \bar{Y}_n - \hat{b}_n \bar{X}_n, \] (3.4)

where $\bar{X}_n = n^{-1} \sum_{k=1}^{n} X_{nk}$, $\bar{Y}_n = n^{-1} \sum_{k=1}^{n} Y_{nk}$, and $S_n^2 = \sum_{k=1}^{n} (X_{nk} - \bar{X}_n)^2$, $n \geq 1$.

In the following, as applications of Corollaries 1 and 2, we can obtain the convergence rates for the strong consistency of LS estimators of the unknown parameters.

**Theorem 4.** Let $1 \leq p \leq 2$. Under the model (3.3), we have the followings.

(i) When $1 \leq p < 2$, we assume that $E|X|^\alpha < \infty$ for some $\alpha \geq 2p$, $E\epsilon = 0$, and $E|\epsilon|^\beta < \infty$, where $1/\alpha + 1/\beta = 1/p$. Then
\[ n^{1-1/p}|\hat{b}_n - b| \to 0 \text{ a.s.} \] (3.5)

and
\[ n^{1-1/p}|\hat{a}_n - a| \to 0 \text{ a.s.} \] (3.6)

(ii) When $p = 2$, we assume that $E|X|^\alpha < \infty$ for some $\alpha > 4$, $E\epsilon = 0$, and $E|\epsilon|^\beta/(\log |\epsilon|)^{\beta/2} < \infty$, where $1/\alpha + 1/\beta = 1/2$, and further assume that $\{X_{nk}, n \geq 1, 1 \leq k \leq n\}$ is an array of i.i.d. random variables, and $\{\epsilon_n, n \geq 1\}$ is a sequence of i.i.d. random variables. Then
\[ \limsup_{n \to \infty} \sqrt{n \log n} |\hat{b}_n - b| = \sqrt{\frac{E\epsilon^2}{E(X - EX)^2}} \text{ a.s.} \] (3.7)

and
\[ \limsup_{n \to \infty} \sqrt{n \log n} |\hat{a}_n - a| = |EX| \cdot \sqrt{\frac{E\epsilon^2}{E(X - EX)^2}} \text{ a.s.} \] (3.8)

**Acknowledgments**

The authors would like to thank the Co-Editor Zhiliang Ying, Associate Editor and two anonymous reviewers for careful reading of the manuscript and valuable suggestions which helped in improving an earlier version of this paper. The research of
Pingyan Chen is supported by the National Natural Science Foundation of China (No. 11271161). The research of Soo Hak Sung is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2017R1D1A1B03029898).

References


Department of Mathematics, Jinan University, Guangzhou, 510630, P.R. China.

E-mail: tchenpy@jnu.edu.cn

Department of Statistics, Jinan University, Guangzhou, 510630, P.R. China

E-mail: zhangt.jnu@foxmail.com

Department of Applied Mathematics, Pai Chai University, Daejeon, 35345, South Korea.

E-mail: sungsh@pcu.ac.kr