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MULTI-ASSET EMPIRICAL MARTINGALE PRICE ESTIMATORS FOR FINANCIAL DERIVATIVES

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Abstract: This study proposes an empirical martingale simulation (EMS) and an empirical P-martingale simulation (EPMS) price estimators for multi-asset financial derivatives. Under mild assumptions on the payoff functions, the strong consistency and asymptotic normality of the proposed estimators are established. Several simulation scenarios are conducted to investigate the performance of the proposed price estimators under multivariate geometric Brownian motion, multivariate GARCH models, multivariate jump-diffusion models and multivariate stochastic volatility models. Numerical results indicate that the multi-asset EMS and EPMS price estimators are capable of improving the efficiency of their Monte Carlo counterparts. In addition, the asymptotic distribution serves as a persuasive approximation to the finite-sample distribution of the EPMS price estimator, which helps to reduce the computation time of finding confidence intervals for the prices of multi-asset derivatives.

Key words and phrases: Empirical martingale simulation, Esscher transform, multi-asset derivatives pricing.

1. Introduction

Due to the acceleration of cross-market integration and the globalization of financial markets, market participants have become increasingly interested in multi-asset derivatives and use them to construct diversified portfolios. On the other hand, the issuers of multi-asset derivatives are facing the task of pricing and hedging these products. In this study, the multi-asset derivative means that a derivative whose payoff depends on multiple underling assets.

For European multi-asset options, there are three broad categories: basket options, rainbow options and quanto options, which are popular and commonly traded over-the-counter (OTC). More specifically, a buyer of a currency basket option has the right, without the obligation, to receive certain currencies in exchange for a base currency, either at the spot market rate or at a predetermined rate of exchange. This kind of options is generally used by multinational corporations which have to deal with multicurrency cash flows. Meanwhile, using the basket option costs significantly less than buying an option on the individual components of the portfolio, this fact is also mentioned in Dimitroff, Lorenz and Szimayer (2011). For hedging risks arising from several events, rainbow options are useful tools and have various types (Ouwehand and West, 2006). A special type of rainbow options is the exchange options proposed by Margrabe (1978). A holder of an exchange option has the right to exchange one asset for the other at maturity. Rainbow options are most commonly used when valuing natural resources since they depend on both the price of the natural resource and how much of the resource is available in a deposit. A quanto option is a cash-settled and cross-currency derivative, whose underlying asset is measured in one currency and the payoff is quoted in another currency. The CME Nikkei 225 Dollar Futures is an example of quantos. In the contract, the underlying asset, the Nikkei 225 Stock Average Index, is settled in U.S. dollars (USD), as opposed to Japanese yen. It provides investors with an efficient way to access the opportunities of the Japanese equity market and trade using USD. On September 16, 2016, the daily trading volume of Nikkei 225 Dollar Futures with the maturity date, December 16, 2016, is 10,479, which is comparable to the daily trading volume 11,477 of S&P 500 Futures with the same maturity. Another example is the MSCI Taiwan Index Futures, which is traded on the Singapore exchange and is settled in USD. The average daily trading volume from January 7, 2014, to December 20, 2016, is around 52,500. Other similar derivatives traded on the Singapore exchange include the MSCI Hong Kong Index Futures, the MSCI Indonesia Index Futures and the FTSE China A50 Index Futures, whose underlying assets are the Indexes of different stock markets in Asia but are all settled in USD.

The above examples indicate that multi-asset derivatives play an important role in global investment and risk management. However, pricing and hedging multi-asset derivatives are more challenging than single-asset derivatives since we need to face multiple uncertainties of the underlying assets. Furthermore, there are usually no closed-form solutions for computing the prices and sensitivities (Greeks) of multi-asset derivatives if a complicated model is used to describe the dynamics of the underlying assets. Hence market participants rely on numerical or simulation procedures such as Monte Carlo (MC) method to estimate the prices and Greeks of multi-asset derivatives.

For pricing options based on single-asset, many variance reduction techniques have been proposed to improve the computational efficiency of the standard MC method. For example, Duan and Simonato (1998) proposed an empirical martingale simulation (EMS), which modifies the standard MC simulation procedure, for single-asset option pricing. The EMS method imposes the martingale property on the simulated sample paths of the underlying asset prices under a risk-neutral model and is capable of reducing the variance of the MC price estimator. In practice, a risk-neutral counterpart of a complex model may not be conveniently obtained. In this case, we can not proceed with the EMS under a risk-neutral environment. To overcome this difficulty, Huang (2014) proposed an empirical P-martingale simulation (EPMS) under the dynamic P measure. By imposing the martingale property on the simulated sample paths of both the change of measure process and the underlying asset prices under the dynamic P measure, the EPMS method has a comparable performance to the EMS method on single-asset option pricing. The EMS and EPMS methods not only can be used in pricing derivatives but also can be applied to energy investment program in the power industry. For example, Contreras and Rodriguez (2014) used the EMS and EPMS methods to evaluate investments in wind energy.

Traditionally, practitioners repeatedly generate derivative prices with independent random copies of the underlying asset prices for computing the standard deviations of the EMS and EPMS price estimators, which is time-consuming. Recently, the asymptotic distributions of the EMS and EPMS price estimators have been derived. Duan, Gauthier and Simonato (2001) showed that the EMS price estimators of derivative contracts are asymptotically normally distributed for piecewise linear and continuous payoffs. Yuan and Chen (2009) extended the asymptotic normality result of the EMS price estimator to piecewise smooth and continuous payoffs and made a conjecture for discontinuous payoffs. For the EPMS price estimator, the asymptotic normality was derived for piecewise smooth payoffs, which could be either continuous or discontinuous in Huang and Tu (2014). In addition, numerical results presented in Duan, Gauthier and Simonato (2001), Yuan and Chen (2009) and Huang and Tu (2014) indicate that the asymptotic distributions of the EMS and EPMS price estimators provide satisfactory approximations to the finite-sample distributions even when the number of sample paths is as few as 500. Consequently, market participants can quickly obtain accurate confidence interval estimates of the derivatives prices for making investment decisions by using the asymptotic distribution.

Since the EMS and EPMS price estimators are easy to implement and have satisfactory performance in pricing single-asset derivatives, we are interested in investigating whether both methods still retain the nice theoretical and numerical properties of multi-asset derivatives. This study is devoted to answering this question. The strong consistency and asymptotic normality of the proposed multi-asset EMS and EPMS price estimators are successfully derived under mild assumptions on the payoff functions. Numerical findings also indicate that the proposed methods for pricing multi-asset derivatives are capable of reducing simulation errors substantially and the asymptotic distribution provides a satisfactory approximation to the finite-sample distribution. In particular, if the change of measure process is equal to 1 in the EPMS procedure, then the EPMS price estimator coincides its EMS counterpart. This phenomenon can be observed from comparing the results of Duan and Simonato (1998) and Huang (2014) and from the results of Yuan and Chen (2009) and Huang and Tu (2014). Consequently, the EMS is a special case of the EPMS estimator and thereby we present the derivation of the large sample properties of the multi-asset EPMS estimator in this study.

The rest of this paper is organized as follows. Section 2 introduces the procedures of obtaining the multi-asset EMS and EPMS price estimators. Section 3 presents the large sample properties of the proposed price estimators. Simulation studies are conducted in Section 4 to investigate the efficiency of the proposed price estimators and the performance of the asymptotic distribution. Conclusions are given in Section 5. Detailed proofs and detailed illustrations of our simulation scenarios are presented in the online supplement (http://www3.stat.sinica.edu.tw/statistica/).

2. The proposed multi-asset empirical martingale price estimators

Let $\mathbf{S}_t = (S_{1,t}, \ldots, S_{n,t})$ denote the vector of the prices of n underlying assets at time t and satisfy a multivariate stochastic process. Let $f(\mathbf{S}_t, 0 \leq t \leq T)$ denote the payoff of a European contingent claim, whose profit depends on multiple underlying assets with expiration date T. For example, the payoff of a multi-asset and path-independent European call option is defined by $f(\mathbf{S}_t, 0 \leq t \leq T) = \max\{g(\mathbf{S}_T) - K, 0\}$ and the corresponding no-arbitrage price is

$$\begin{aligned} C_0(\mathbf{S}_0) &= e^{-rT} E^Q(\max\{g(\mathbf{S}_T) - K, 0\}) \\ &= e^{-rT} \int \cdots \int_{\mathrm{ITM}_T} \{g(\mathbf{S}_T) - K\} dF(\mathbf{S}_T), \end{aligned} \tag{2.1}$$

where the 1st equality is the so-called risk-neutral pricing formula, r is the risk-free interest rate, $g(\cdot)$ is a real-valued function with domain $\mathbb{R}^n$ and range $\mathbb{R}$, K is the strike price, ITM_T=$\{\mathbf{S}_T : g(\mathbf{S}_T) > K\}$ denotes the in-the-money (ITM) event at time T, $F(\mathbf{S}_T)$ denotes the joint distribution function of $\mathbf{S}_T$, and E^Q denotes the expectation under a risk-neutral measure Q. In particular, if the dimension of underlying assets is reduced to 1, Black and Scholes (1973) derived the famous Black-Scholes formulae for European call and put options by assuming that the prices of an underlying asset satisfy the following risk-neutral model:

$$dS_{1,t} = rS_{1,t}dt + \sigma S_{1,t} d\widetilde{W}_{1,t}, \tag{2.2}$$

where σ is the instantaneous volatility and $\widetilde{W}_{1,t}$ is a Brownian motion under the Q measure. The pricing formula for a European call option for a non-dividend-paying underlying stock with payoff function $f(S_{1,T}) =$

$\max(S_{1,T} - K, 0)$ is

$$C_0(S_{1,0}) = S_{1,0}\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where $d_1 = \log(S_{1,0}/K) + (r + 0.5\sigma^2)T/(\sigma T^{1/2})$, $d_2 = d_1 - \sigma T^{1/2}$ and $\Phi(\cdot)$ denotes the standard normal cumulative distribution function. The Black-Scholes pricing formula leads to a boom in options trading in global financial markets. Nowadays, many more complicated options than European call options are traded in the exchanges and OTC markets around the world. Many more realistic models than (2.2) are also proposed to describe the dynamics of underlying asset prices. However, a closed-form representation of the multiple integrals on the right-hand-side of (2.1) with complicated payoff functions under realistic models are usually difficult to obtain. Therefore, how to approximate derivative prices accurately and efficiently attracts the attention of traders.

Let $\Lambda_T = dQ/dP$ be a Radon-Nikodým derivative of the risk-neutral Q measure with respect to the physical P measure. Furthermore, define $\Lambda_t = E(\Lambda_T|\mathcal{F}_t) := E_t(\Lambda_T),\ 0 \leq t < T$, where $E(\cdot)$ denotes the expectation under the physical measure P and $\mathcal{F}_t$ denotes the set of information from time 0 up to time t. Consequently, $\{\Lambda_t, 0 \leq t \leq T\}$ is a change of measure process depending on $\mathbf{S}_u, 0 \leq u \leq t$, and is a martingale process under the P measure (abbreviated as P-martingale). For European-style options, due to the identity of $E^Q\{(f(\mathbf{S}_t, 0 \leq t \leq T)\} = E\{f(\mathbf{S}_t, 0 \leq t \leq T)\Lambda_T\}$, the no-arbitrage price of a financial derivative can also be computed under the physical measure P. If $E^Q\{f(\mathbf{S}_t, 0 \leq t \leq T)\}$ and $E\{f(\mathbf{S}_t, 0 \leq t \leq T)\Lambda_T\}$ do not have closed-form representations, the former can be approximated by the MC price estimator, $m^{-1}\sum_{j=1}^m f(\mathbf{S}_{j,t}, 0 \leq t \leq T)$, where $\mathbf{S}_{j,t}$, $j = 1, \ldots, m$, are independent and identically distributed (i.i.d.) random vectors generated from the risk-neutral model at time t, and the later can be approximated by $m^{-1}\sum_{j=1}^m f(\mathbf{S}_{j,t}, 0 \leq t \leq T)\Lambda_{j,T}$ with $\mathbf{S}_{j,t}$, $j = 1, \ldots, m$, being i.i.d. random vectors generated from the physical model at time t. Throughout this study, we use MCQ and MCP to denote the MC price estimators under the Q measure and P measure, respectively.

In order to improve the efficiency of the MC price estimators, we propose a multi-asset EMS price estimator under the Q measure and a multi-asset EPMS price estimator under the P measure in discrete time. In the following, we first introduce the proposed EPMS procedure for pricing multi-asset derivatives:

1. Use the standard MC method to generate m independent random paths for the prices of the ith asset under the P measure, denoted by $\hat{S}_{i,j,t}$, where $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $t = 1, \ldots, T$.

2. Let $\Lambda^*_{j,0} = \hat{\Lambda}_{j,0} = \Lambda_0 = 1$ and define the empirical martingale change of

measure process by $\Lambda^*_{j,t} = \hat{\Lambda}_{j,t}/\bar{\Lambda}_{m,t}$, for $j = 1, \ldots, m$ and $t = 1, \ldots, T$, where $\bar{\Lambda}_{m,t} = m^{-1}\sum_{j=1}^m \hat{\Lambda}_{j,t}$ and $\hat{\Lambda}_{j,t} = \Lambda_t(\hat{S}_{1,j,u}, \ldots, \hat{S}_{n,j,u}, 0 \le u \le t)$ is a function of the jth path of the MC asset prices generated in step 1.

3. Let $S^*_{i,j,0} = \hat{S}_{i,j,0} = S_{i,0}$ be the initial price of the ith asset, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, and define the empirical martingale stock prices $S^*_{i,j,t}$ by $S^*_{i,j,t} = e^{rt}S_{i,0}\hat{S}_{i,j,t}/\bar{S}^*_{i,m,t}$, for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $t \ge 1$, where $\bar{S}^*_{i,m,t} = m^{-1}\sum_{j=1}^m \hat{S}_{i,j,t}\Lambda^*_{j,t}$.

4. Define the multi-asset EPMS price estimator with a payoff function f by

$$C^{(m)}_{\text{EPMS}} = \frac{1}{m}e^{-rT}\sum_{j=1}^m f(\mathbf{S}^*_{j,1}, \ldots, \mathbf{S}^*_{j,T})\Lambda^*_{j,T}, \tag{2.3}$$

where $\mathbf{S}^*_{j,t} = (S^*_{1,j,t}, \ldots, S^*_{n,j,t})$.

Note that in steps 2 and 3 we create dependencies among $\Lambda^*_{j,t}$, $j = 1, \ldots, m$, and also among $S^*_{i,j,t}$, $j = 1, \ldots, m$, for each asset at time t. These dependencies provide an opportunity for variance reduction for the multi-asset EPMS price estimator. Moreover, the processes $\{\Lambda_t\}$ and $\{e^{-rt}S_t\Lambda_t\}$ are martingales under the P measure. After the modification in steps 2 and 3, the generated processes $\{\Lambda^*_{j,t}\}$ and $\{S^*_{i,j,t}\}$ both satisfy the following empirical P-martingale properties:

$$\frac{1}{m}\sum_{j=1}^{m}\Lambda_{j,t}^{*}=\Lambda_0=1 \quad\text{and}\quad \frac{1}{m}\sum_{j=1}^{m}e^{-rt}S_{i,j,t}^{*}\Lambda_{j,t}^{*}=S_{i,0},$$

for each $i=1,\ldots,n$, and $t=1,\ldots,T$.

Remark 1. The above multi-asset EPMS procedure can be conveniently modified to obtain a multi-asset EMS price estimator by the following scheme: (i) generate random samples from a Q measure in step 1, (ii) skip step 2, and (iii) let $\Lambda_{j,t}^{*}=1$, for all $j=1,\ldots,m$ and $t=1,\ldots,T$, in steps 3 and 4.

In view of the EPMS procedure and Remark 1, the EMS and EPMS are easy to implement since the EMS and EPMS corrections are obtained by modifying the standard MC samples. In addition, the modification is truly simple and basically does not require an expensive computational cost.

3. Main results

3.1 The strong consistency of the multi-asset EPMS method

Throughout this paper, the notation $\|\cdot\|$ stands for the Euclidean norm and the domain of the payoff function f is denoted by $D^f\subseteq\mathbb{R}^n$ for an integer n. First, we give two definitions.

Definition 1. A function $f: D^f\to\mathbb{R}$ is said to have growth rate q if there exist a constant $c>0$ and a positive integer q such that $|f(x)|\le c(1+||x||^q)$

for any $x \in D^f$.

Definition 2. A function $f : D^f \to \mathbb{R}$ is said to satisfy the Lipschitz condition if there exists $c < \infty$ such that $|f(x) - f(y)| \leq c\|x - y\|$ for any $x, y \in D^f$.

In addition, the following assumptions are needed for establishing our theoretical results.

(A1) The payoff function f is piecewise smooth.

(A2) D^f has a finite partition, denoted by $\{A_\ell, \ell = 1, \ldots, k\}$, and each partition is a connected set such that f is Lipschitz continuous on A_ℓ.

(A3) f has growth rate q on D^f.

(A4) $E^Q(|f(\mathbf{S}_1, \ldots, \mathbf{S}_T)|) < \infty$.

(A5) The multivariate distribution of $(\mathbf{S}_1, \ldots, \mathbf{S}_T)$ under Q has a bounded density function and $E^Q(||(\mathbf{S}_1, \ldots, \mathbf{S}_T)||^q) < \infty$, where q is the same as in (A3).

In financial markets, many derivatives have payoff involving multiple underlying assets. For examples, the payoff of an arithmetic basket put option is defined by $f(\mathbf{S}_T) = \max(K - n^{-1}\sum_{i=1}^n S_{i,T}, 0)$, the payoff of a maximum call option is $\max(\max(S_{1,T}, \ldots, S_{n,T}) - K, 0)$, the payoff of an exchange option is $\max(S_{1,T}-S_{2,T},0)$, and the payoff of a quanto call option is $f(S_T,X_T)=\max(X_TS_T-K,0)$, where X_T is the exchange rate at time T. It can be checked that these payoff functions satisfy the assumptions (A1)-(A3). In the following, we use a two-dimensional geometric average put option (or called geometric basket put option) as an example for illustration.

Example 1. A two-dimensional geometric average put option is defined by $f(S_{1,T},S_{2,T})=\max(K-(S_{1,T}S_{2,T})^{1/2},0)$. In particular, the domain of the payoff f here is set up to be $D^f=[\eta,\infty)\times[\eta,\infty)$ with an $0<\eta<K^{1/2}$. For practical implementation, we set η to be a very small number, say $\eta=10^{-8}$. Apparently, f is piecewise smooth on D^f. Hence, (A1) holds. Let $A_1=\{(S_{1,T},S_{2,T})\mid(S_{1,T}S_{2,T})^{1/2}\le K$ for $S_{1,T}\ge\eta$ and $S_{2,T}\ge\eta\}$ and $A_2=\{(S_{1,T},S_{2,T})\mid(S_{1,T}S_{2,T})^{1/2}>K$ for $S_{1,T}\ge\eta$ and $S_{2,T}\ge\eta\}$ be a partition of D^f. That is, A_1 and A_2 are disjoint and $D^f=A_1\cup A_2$. Since $f(S_{1,T},S_{2,T})=0$ for $(S_{1,T},S_{2,T})\in A_2$, thus f is Lipschitz continuous on A_2. In Section S2.1 of the online supplement, we prove that f is also Lipschitz continuous on A_1. Therefore, (A2) holds. In addition, f has growth rate $q=1$ on D^f since $|f(S_{1,T},S_{2,T})|\le K$, for $(S_{1,T},S_{2,T})\in D^f$, which ensures (A3). Moreover, (A4) is a natural assumption for derivative pricing. (A5) is satisfied for most payoff functions under the multivariate geometric Brownian motion, multivariate GARCH models, multivariate versions of Merton

(1976)'s jump-diffusion models and multivariate versions of Hull and White (1987)'s and Heston (1993)'s stochastic volatility models. We will discuss the above models in Section S1 of the online supplement.

Moreover, some single-asset derivatives also satisfy (A1)-(A3) with $n = 1$. For example, the payoff functions of European call, European put, digital and barrier options all satisfy (A1)-(A3) with growth rate $q = 1$. The payoff function of a self-quanto call option, defined by $f(S_T) = S_T \max(S_T - K, 0)$, for $S_T \in D^f = [0, \xi]$, where ξ is a large positive number, say $\xi = 10^8$, also satisfies (A1)-(A3) and is an example of growth rate $q = 2$. Therefore, the assumptions (A1)-(A3) on the payoff function are satisfied for many financial derivatives traded in the market. As mentioned in Example 1, (A4) and (A5) are also satisfied by popular models for describing the dynamics of the underlying assets, like the multivariate models discussed in Section S1 of the online supplement.

In the following, we prove that the derivative prices obtained from the multi-asset EPMS method converge to the theoretical values. Details of the proof are given in Section 2.2 of the online supplement.

Theorem 1. *Let $\{\Lambda_t\}$ be a change of measure process of Q with respect to P, and $\{e^{-rt}S_{i,t}\Lambda_t\}$ be a positive P-martingale process over the time index* *set* $\{t : t = 0, 1, \ldots, T\}$, $i = 1, \ldots, n$. *If* (A1)-(A5) *hold, then as* $m \to \infty$,

$$\frac{1}{m}\sum_{j=1}^{m} f(\mathbf{S}^*_{j,1}, \ldots, \mathbf{S}^*_{j,T})\Lambda^*_{j,T} \to E_0\{f(\mathbf{S}_1, \ldots, \mathbf{S}_T)\Lambda_T\},$$

almost surely, where $\mathbf{S}^*_{j,t} = (S^*_{1,j,t}, \ldots, S^*_{n,j,t})$ *and both of* $\Lambda^*_{j,t}$ *and* $S^*_{i,j,t}$ *are generated from the multi-asset EPMS method.*

Remark 2. In Theorem 1, if $n = 1$ and assume that f has a growth rate $q = 1$ on D^f, then (A1)-(A3) ensure that f satisfies the generic Lipschitz condition in Huang (2014), and f is Lipschitz continuous over each partition set of D^f. Consequently, Theorem 2.2 in Huang (2014) is a special case of Theorem 1.

In the following, we provide a counter-example to demonstrate that (A5) is crucial to the strong consistency of the proposed price estimator. Let $S_T = \lambda|X_T|$ under a risk-neutral measure Q, where X_T is t-distributed with degrees of freedom $\nu \in (1, 2)$ and λ is a positive constant. In order to keep the martingale property of discounted stock prices under the Q measure, we choose $\lambda = S_0 e^{rT}(\nu - 1)\Gamma(1/2)\Gamma(\nu/2)/\{2\nu^{1/2}\Gamma((\nu + 1)/2)\}$ such that $E^Q(e^{-rT}S^T) = S_0$. Consider a payoff function $f(S_T) = (S_T/S_0)\log S_T I_{(S_T<K)}$, which satisfies $f(S_T) < c(1 + S_T^2)$ for some positive constants c. That is, $f(S_T)$ has growth rate $q = 2 > \nu$. Note that $E^Q\{f(S_T)\} < \infty$ but $E^Q(S_T^2)$ does not exist. In other words, (A4) holds but

Table 1: The estimated option prices of the MCQ and EMS methods for $f(S_T) = (S_T/S_0)\log S_T I_{(S_T<100)}$ with 10^k sample paths, where $S_T = \lambda|X_T|$ with X_T being t-distributed with degrees of freedom $\nu = 1.01, 1.1$ or 1.3, $\lambda = S_0 e^{rT}(\nu-1)\Gamma(1/2)\Gamma(\nu/2)/\{2\nu^{1/2}\Gamma((\nu+1)/2)\}$, $S_0 = 100$, $r = 0.05$ and $T = 1$.

	k	4	5	6	7
ν=1.01	MCQ	0.10	0.10	0.10	0.10
	EMS	0.57	0.65	0.55	0.47
ν=1.1	MCQ	0.59	0.59	0.59	0.59
	EMS	0.27	0.76	0.75	0.70
ν=1.3	MCQ	0.96	0.94	0.94	0.94
	EMS	0.99	0.95	0.96	0.95

(A5) is violated. Table 1 presents the estimated option values of the MCQ and EMS methods with $S_0 = K = 100$, $r = 0.05$, $T = 1$, $\nu = 1.01, 1.1, 1.3$ and numbers of sample paths $m = 10^k$, $k = 4, 5, 6, 7$. Numerical results show that the EMS price estimator does not converge when $\nu = 1.01$ and 1.1.

3.2 Asymptotic distribution for the multi-asset EPMS price estimator

In this section, the asymptotic distribution of the multi-asset EPMS price estimator defined in (2.3) is derived, where $f(\cdot)$ is assumed to be piecewise smooth and continuous. Moreover, according to Duan, Gauthier and Simonato (2001), Yuan and Chen (2009) and Huang and Tu (2014), this study only considers establishing the asymptotic distribution of the multi-asset EPMS price estimator for path-independent derivatives. That is, the payoff function only depends on the prices of the underlying assets at time T.

Let $f : D^f \to \mathbb{R}$ be piecewise smooth and continuous and can be written as

$$f(\mathbf{x}) = \sum_{\ell=1}^{k} f_\ell(\mathbf{x}) I_{A_\ell}(\mathbf{x}), \tag{3.1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, A_ℓ's form a partition of D^f and $I_{A_\ell}(\mathbf{x})$ is an indicator function defined by $I_{A_\ell}(\mathbf{x}) = 1$, if $\mathbf{x} \in A_\ell$, and $I_{A_\ell}(\mathbf{x}) = 0$, if $\mathbf{x} \notin A_\ell$. We further denote the boundary set of D^f by

$$G = \bigcup_{\ell=1}^{k} (\overline{A}_\ell - A_\ell^*), \tag{3.2}$$

where $\overline{A}_\ell$ and A_ℓ^* denote the closure and interior of A_ℓ, respectively. To ensure the continuity of f, we also assume that $f_\ell(\mathbf{x}) = f_s(\mathbf{x})$ for $\mathbf{x} \in \overline{A}_\ell \cap \overline{A}_s$, $\ell \neq s$ and $\ell, s \in \{1, \ldots, k\}$. In addition, we write $\nabla f(\mathbf{x}) = \sum_{\ell=1}^{k} \nabla f_\ell(\mathbf{x}) I_{A_\ell}(\mathbf{x})$, where $\nabla f_\ell(\mathbf{x}) = (\partial f_\ell/\partial x_1, \ldots, \partial f_\ell/\partial x_n)^\top$. Moreover, we strengthen the conditions (A1), (A2), (A3) and (A5) in Section 3.1 by the following (A1'), (A2'), (A3') and (A5'), respectively, and let (A4') = (A4) for deriving the asymptotic distribution.

(A1') The payoff function f is piecewise smooth and continuous.

(A2') $\nabla f_\ell(\mathbf{x})$ exists and is continuous for $\mathbf{x} \in \overline{A}_\ell$. That is , f_ℓ is continuously differentiable on $\overline{A}_\ell$, $\ell = 1, \ldots, k$.

(A3') There exists a positive integer q such that each component of $\nabla f(\cdot)$ has growth rate q. Therefore, $||\nabla f(\mathbf{x})||_1 \leq c(1 + ||\mathbf{x}||^q)$ for some positive constant $c < \infty$.

(A5') The multivariate distribution of $(\mathbf{S}_1, \ldots, \mathbf{S}_T)$ under Q has a bounded density function and $E(\|\mathbf{S}_T\|^{q+1}\Lambda_T) = E^Q(\|\mathbf{S}_T\|^{q+1}) < \infty$, where q is the same as in (A3').

We continue to use the two-dimensional geometric average put option mentioned in Example 1 to demonstrate that the above assumptions are satisfied.

Example 2. It is trivial to find that (A1') is satisfied by a two-dimensional geometric average put option. By using the same notations as in Example 1 and by (3.1), we have $f_1(S_{1,T}, S_{2,T}) = K - (S_{1,T}S_{2,T})^{1/2}$ and $f_2(S_{1,T}, S_{2,T}) = 0$ for $(S_{1,T}, S_{2,T}) \in D^f$. As a result,

$$\nabla f_1 = (-0.5(S_{2,T}/S_{1,T})^{1/2}, -0.5(S_{1,T}/S_{2,T})^{1/2}) \quad \text{and} \quad \nabla f_2 = (0, 0).$$

Apparently, ∇f_1 is continuous on $\overline{A}_1 = \{(S_{1,T}, S_{2,T}) : (S_{1,T}S_{2,T})^{1/2} \leq K \text{ for } S_{1,T} \geq \eta \text{ and } S_{2,T} \geq \eta\}$ and ∇f_2 is continuous on $\overline{A}_2 = \{(S_{1,T}, S_{2,T}) :$

$(S_{1,T}S_{2,T})^{1/2} \geq K$ for $S_{1,T} \geq \eta$ and $S_{2,T} \geq \eta\}$ Therefore, (A2') holds. In addition, since $\max\{(S_{2,T}/S_{1,T})^{1/2}, (S_{1,T}/S_{2,T})^{1/2}\} \leq \eta^{-1}(1+||(S_{1,T}, S_{2,T})||)$ for all $(S_{1,T}, S_{2,T}) \in D^f$, thus (A3') holds with $q = 1$. Moreover, (A5') is also satisfied if a multivariate model discussed in Section S1 of the online supplement is used to describe the dynamics of the underlying asset prices.

Before illustrating the asymptotic result, we introduce the following notation.

Definition 3. For two matrices, A and B of the same dimension $m \times n$, the Hadamard product, denoted by $A \circ B$, is a matrix of the same dimension as the operands with elements given by $(A \circ B)_{i,j} = (A)_{i,j} \times (B)_{i,j}$, where $(X)_{i,j}$ denotes the (i, j)th element of a matrix X.

Moreover, for a random variable X and a random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ we use $\mathrm{Cov}(X, \mathbf{Y}) = (\mathrm{Cov}(X, Y_1), \ldots, \mathrm{Cov}(X, Y_n))^\top$ to denote a vector of covariances of X and Y_i, $i = 1, \ldots, n$, use $\mathrm{Cov}(\mathbf{Y}) = (\mathrm{Cov}(Y_i, Y_j))$, for $i, j = 1, \ldots, n$, to denote the covariance matrix of $\mathbf{Y}$, and let $\mathbf{Y}^{-1} = (Y_1^{-1}, \ldots, Y_n^{-1})$ to simplify the notation hereafter.

Theorem 2. *Let the underlying assets prices* $\mathbf{S}_T = (S_{1,T}, \ldots, S_{n,T})$ *be a positive random vector and* Λ_T *be a Radon-Nikodým derivative.*

(i) *If* (A1')-(A5') *hold, then*

$$C_{\text{MC}}^{(m)} - C_{\text{EPMS}}^{(m)} = e^{-rT}\{(\bar{\mathbf{S}}_{m,T} - e^{rT}\mathbf{S}_0)\Phi + (\bar{\Lambda}_{m,T} - 1)\Psi\} + o_p(m^{-1/2}),$$

where $C_{\text{MC}}^{(m)}$ *is the derivative value computed by the MC method,*

$$\Phi = e^{-rT}E[\Lambda_T \nabla f(\mathbf{S}_T) \circ (\mathbf{S}_T \circ \mathbf{S}_0^{-1})^\top]$$

is an $n \times 1$ *vector,*

$$\Psi = E[f(\mathbf{S}_T)\Lambda_T] - e^{rT}\mathbf{S}_0\Phi$$

is a scale, $\bar{\mathbf{S}}_{m,T} = (\bar{S}_{1,m,T}, ..., \bar{S}_{n,m,T})$ *with* $\bar{S}_{i,m,T} = m^{-1}\sum_{j=1}^m \hat{S}_{i,j,T}\hat{\Lambda}_{j,T}$ *for* $i = 1, \ldots, n$, $\bar{\Lambda}_{m,T} = m^{-1}\sum_{j=1}^m \hat{\Lambda}_{j,T}$, *and* $H_m = o_p(m^{-k})$ *denotes that a sequence of random variables,* H_m, $m = 1, 2, \ldots$, *satisfies* $\lim_{m\to\infty} m^k H_m = 0$ *in probability.*

(ii) *Moreover, if* (A5') *is strengthened to* $E(\|\mathbf{S}_T\|^{2(q+1)}\Lambda_T^2) < \infty$, *then*

$$m^{1/2}(C_{\text{EPMS}}^{(m)} - C) \xrightarrow{\mathcal{L}} N(0, V), \text{as } m \to \infty,$$

where C *is the true derivative price,* $\xrightarrow{\mathcal{L}}$ *denotes convergence in distribution, and*

$$\begin{aligned} V \;=\; e^{-2rT}\Big\{ & \text{Var}(f(\mathbf{S}_T)\Lambda_T) + \Phi^\top \text{Cov}(\Lambda_T \mathbf{S}_T)\Phi \\ & + \Psi^2 \text{Var}(\Lambda_T) - 2\Phi^\top \text{Cov}(f(\mathbf{S}_T)\Lambda_T, \Lambda_T \mathbf{S}_T) \\ & - 2\Psi \text{Cov}(f(\mathbf{S}_T)\Lambda_T, \Lambda_T) \\ & + 2\Psi\Phi^\top \text{Cov}(\Lambda_T, \Lambda_T \mathbf{S}_T)\Big\}. \end{aligned} \tag{3.3}$$

Remark 3. If $n = 1$, then the asymptotic results shown in Theorem 2 reduce to the results in Theorem 3.1 of Huang and Tu (2014) for single asset case.

The asymptotic properties shown in Theorems 1 and 2 for the multi-asset EPMS price estimator can be conveniently modified to the multi-asset EMS price estimator defined in Remark 1. We illustrate this fact in the following Corollary.

Corollary 1. *Under the framework of a risk-neutral Q measure, if we let* Λ_t, $\hat{\Lambda}_{j,t}$ *and* $\Lambda^*_{j,t}$ *be 1, for all* $j = 1, \ldots, m$ *and* $t = 1, \ldots, T$, *in Theorems 1 and 2, then we obtain the strong consistency and asymptotic distribution for the multi-asset EMS price estimator defined in Remark 1. Moreover, if* $n = 1$, *then the results shown in Theorems 1 and 2 also reduce to the results of Duan and Simonato (1998), Duan, Gauthier and Simonato (2001) and Yuan and Chen (2009).*

4. Simulation study

In this section, we investigate the efficiency of the multi-asset EMS and EPMS price estimators in several simulation scenarios and examine the performance of the asymptotic distribution of the multi-asset EPMS price estimator. Four types of frequently used models are considered: multivariate geometric Brownian motion, multivariate GARCH model, multivariate versions of Merton (1976)'s jump-diffusion model and multivariate versions of Hull and White (1987)'s and Heston (1993)'s stochastic volatility models.

Two multi-asset derivatives are employed in this section. The first one is the maximum call option, whose payoff function is defined by

$$f(S_{1,T},\dots,S_{n,T}) = \max\{\max(S_{1,T},\dots,S_{n,T}) - K, 0\}.$$

The second one is the geometric average put option with payoff function

$$f(S_{1,T},\dots,S_{n,T}) = \max\{K - (S_{1,T}\dots S_{n,T})^{1/n}, 0\}.$$

The payoff functions of the maximum call option and the geometric average put option are piecewise smooth and continuous. In addition, by using the results in Theorem 2, we construct the following asymptotic $(1-\alpha)$ confidence interval for the multi-asset EPMS price estimator,

$$\Big(C^{(m)}_{\text{EPMS}} - z_{\alpha/2}(\hat{V}/m)^{1/2},\ C^{(m)}_{\text{EPMS}} + z_{\alpha/2}(\hat{V}/m)^{1/2}\Big), \tag{4.1}$$

where $C^{(m)}_{\text{EPMS}}$ is defined in (2.3), $z_{\alpha/2}$ is the $(1-\alpha/2)$ quantile of a standard normal random variable and $\hat{V}$ is an estimator of V defined in (3.3) and can be obtained simply by using the MC samples.

For evaluating the performance of the proposed price estimators, we define the following ratios. If the maximum call option and the geometric average put option have closed-form solutions, we report the ratios of mean squared errors (MSE): $\mathrm{MR}^{\mathrm{Q}} = \mathrm{MSE(MCQ)}/\mathrm{MSE(EMS)}$ under the Q measure and $\mathrm{MR}^{\mathrm{P}} = \mathrm{MSE(MCP)}/\mathrm{MSE(EPMS)}$ under the P measure, where $\mathrm{MSE}(\cdot)$ denotes the MSE of the corresponding price estimator on the basis of 1,000 replications. If a closed-form solution of an option does not exist, the expected option value is replaced by using the standard MC with 10^5 simulations for computing MR^{Q} and MR^{P}.

Due to the limitation of manuscript pages, detailed illustrations and numerical results of our simulation scenarios are presented in the online supplement. In general, the proposed price estimators and the asymptotic distribution have satisfactory performance, especially for ITM options.

5. Conclusion

In this study, we propose a multi-asset EMS price estimator and a multi-asset EPMS price estimator for financial derivatives based on multiple underlying assets. The strong consistency and asymptotic normality of the proposed price estimators are derived under mild assumptions on the payoff functions. Simulation results given in the online supplement indicate that the proposed price estimators are capable of improving the efficiency of their MC counterparts under multivariate geometric Brownian motion, multivariate GARCH models, multivariate versions of Merton (1976)'s jump-diffusion models and multivariate versions of Hull and White

(1987)'s and Heston (1993)'s stochastic volatility models. Numerical results also provide strong evidence that the asymptotic distribution has a satisfactory approximation to the finite-sample distribution in our scenarios, which helps to reduce the computation time of finding confidence intervals for the prices of multi-asset derivatives.

Supplementary Materials

In the supplement, several simulation scenarios are conducted under multivariate Brownian motion, multivariate GARCH models, multivariate versions of jump-diffusion models and multivariate versions of stochastic volatility models to investigate the efficiency of the proposed price estimators. In addition, the performance of the asymptotic distribution of the multi-asset EPMS price estimator is examined under various simulation cases. Detailed proofs and numerical results are also given.

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