In recent years, demand for reliable small area statistics has considerably increased, but the size of samples obtained in small areas is too often small to produce accurate predictors of quantities of interest. To overcome this difficulty, a common approach is to use auxiliary data from other areas or other sources, and produce estimators that combine them with direct data. A popular model for combining direct and indirect data sources is the Fay-Herriot model, which assumes that the auxiliary variables are observed accurately. However, these variables are often subject to measurement errors, and not taking this into account can lead to estimators that are even worse than those based exclusively on the direct data. In this paper, we consider structural measurement error models and a semi-parametric approach based on the Fay-Herriot model to produce reliable prediction intervals for small area characteristics of interest. Our theoretical study reveals the surprising fact that the properties of the prediction interval are not the same for all values of the noisy covariate. Indeed, the convergence rates are slower when the contaminated covariate takes the value zero than in other cases. Our procedure is illustrated with an application and simulation studies.

Statistica Sinica: Newly accepted Paper
(accepted version subject to English editing)

Semi-parametric prediction intervals in small areas when auxiliary data are measured with error

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Foreword: *Our friend and colleague Peter Hall died in Melbourne, Australia on January 9, 2016, before this manuscript was completed. Peter worked hard on this paper before he fell ill, deriving all the theoretical results of the manuscript, whence our decision to submit this manuscript for the special issue in his honour. His theory is particularly striking since it reveals that the properties of the prediction interval depend on whether or not the contaminated covariate takes the value zero. Peter was very interested by this phenomenon but could not find an intuitive explanation to it. We checked his proofs thoroughly but could not find an intuitive explanation either, except that a similar behaviour is sometimes encountered in other problems.*

Abstract: In recent years, demand for reliable small area statistics has considerably increased, but the size of samples obtained in small areas is too often small to produce accurate predictors of quantities of interest. To overcome this difficulty, a common approach is to use auxiliary data from other areas or other sources, and produce estimators that combine them with direct data. A popular model for combining direct and indirect data sources is the Fay-Herriot model, which assumes that the auxiliary variables are observed accurately. However, these variables are often subject to measurement errors, and not taking this into account can lead to estimators that are even worse than those based exclusively on the direct data. We consider structural measurement error models and a semi-parametric approach based on the Fay-Herriot model to produce reliable prediction intervals for small area characteristics of interest. Our theoretical study reveals the surprising fact that the properties of the prediction interval are not the same for all values of the noisy covariate. Indeed, the convergence rates are slower when the contaminated covariate takes the value zero than in other cases. Our procedure is illustrated with an application and simulation studies.

Keywords: Deconvolution; density estimation; Fay-Herriot model; Fourier transform; Laplace distribution.

1 Introduction

Small area estimation methods are indispensable statistical tools to the administrators and policy makers in National Statistical Offices and many world organizations. Economic planning and welfare activities of the governments and non-government organizations rely heavily on accurate data measuring income, employment, living and health conditions for various geographic and demographic segments. While nationwide surveys such as the American Community Survey, the National Health and Nutrition Examination Survey (NHANES) and the National Health Interview Survey (NHIS), collect large samples at the national level, the subset of the data collected in local geographic and demographic domains, also known as small areas, is usually of too small size to compute accurate small area statistics.

To produce more reliable estimators at the small area level, a common approach is to use model-based methods which combine data from multiple sources, surveys, administrative records, registers and social media; see for example Rao (2003), Pfeffermann (2013), Ybarra and Lohr (2008) and Rao and Molina (2015). Suppose we are interested in predicting a random quantity T_j in small area j, where $j = 1, \ldots, n$, and that we have at our disposal a sample of independent pairs $(A_1, Y_1), \ldots, (A_n, Y_n)$, where the component A_j is a vector of auxiliary variables and Y_j is a direct estimator of T_j. The estimator Y_j is computed based on a sample only from area j, and it is referred to in small area estimation literature as a direct estimator (see Rao and Molina, 2015, p. 1). A popular small area estimation model assumes the decomposition $Y_j = T_j + \epsilon_j$, where $\epsilon_j \sim N(0, \tau_j)$ and the T_js and the ϵ_j's are completely independent. We shall follow the small area literature and assume that the τ_j's known; see González-Manteiga et al. (2010) for how these can be estimated in practice. See also Otto and Bell (1995). The widely used Fay-Herriot model (Fay and Herriot, 1979) decomposes T_j as $T_j = \beta_0 + \beta_1 A_j + V_j$, where $V_j \sim N(0, \sigma_V^2)$, with

various independence models in use. Here we assume that the A_js, the V_js and the ϵ_js are completely independent.

A difficulty in applications is that it is not always possible to measure all the components of the auxiliary vectors A_j accurately, and the techniques developed for covariates without measurement error may perform rather poorly in this case; see for example, Ghosh et al. (2006), Ybarra and Lohr (2008) and Torabi et al. (2009). In particular, if the measurement error is not taken into account, using the auxiliary A_js may result in estimators that are even less accurate than those based on the direct data from the small areas; see Ybarra and Lohr (2008), who propose a corrected small area predictor based on the empirical best linear unbiased prediction approach. While Ghosh and Sinha (2007), Ybarra and Lohr (2008) and Datta et al. (2010) used the frequentist approach to the problem, Ghosh et al. (2006), Torabi et al. (2009), Arima et al. (2012, 2015) and Datta et al. (2010), proposed a Bayesian approach.

While the aforementioned methods are useful, their focus is on point predictors, whereas in the small area estimation literature where the covariates are measured without error, there is substantial interest in the construction of prediction intervals for the T_js; see for example Datta et al. (2002), Hall and Maiti (2006), Chatterjee et al. (2008) and Diao et al. (2014). In this work, our goal is to construct prediction intervals for the small area population means $T_1, \ldots, T_n$ by taking the measurement errors of covariates into account and by relaxing some of the distributional assumptions for the random effects and sampling errors often employed in the literature. Although we focus on the construction of those intervals, we note that since our method is based on estimating the conditional distribution of T_j given the observed data, it can also be used to construct predictors under these relaxed assumptions.

In Section 2.1, we introduce a Fay-Herriot model with a covariate subject to measurement error. To construct prediction intervals, we derive the conditional dis-

tribution of the small area characteristics T_j given the values of its direct estimator Y_j and other observed data, and suggest estimators of this conditional distribution in Section 2.2. Section 3 gives the asymptotic properties of the estimators of the model parameters and the prediction intervals. Simulation studies and an illustrative example are presented in Section 4. In Section 5, we extend our approach to the case where one of the error distributions is unknown. Estimation of the parameters is relatively standard and is deferred to Appendix B. Proofs of the theoretical results are given in Section 6 and in Appendices E and F.

2 Model and estimators

2.1 Model and data

We are interested in predicting T_j, for $j = 1, \ldots, n$. We have at our disposal a sample of independent $(p+2)$-vectors $(W_j, Q_j^{\mathrm{T}}, Y_j)$, for $1 \leq j \leq n$, where the connection between (W_j, Q_j^{T}) and A_j is described in the next pargraph. We assume the following measurement error version of the Fay-Herriot model:

$$Y_j = T_j + \epsilon_j\,, \quad T_j = \beta_0 + \beta_1\, X_j + \beta_2^{\mathrm{T}}\, Q_j + V_j\,, \quad W_j = X_j + U_j\,, \tag{2.1}$$

where β_0, β_1, U_j, V_j, W_j, X_j and Y_j are scalars, β_2 and Q_j are p-vectors, β_0, β_1 and β_2 are unknown, the variables Q_j, U_j, V_j, X_j and ϵ_j, for $j \geq 1$, are completely independent, the U_js have a common, known distribution symmetric around zero, the X_js have a common, unknown distribution, the V_js have zero mean and unknown variance σ_V^2, the ϵ_js have a known distribution symmetric around zero and known variance τ_j, and the τ_js are uniformly bounded. We consider two cases: the distribution of the V_js is known except for the variance σ_V^2, and the distribution of V is totally unknown.

The data come from two sources: direct data $Y_1, \ldots, Y_n$, and indirect auxiliary observations or covariates (W_j, Q_j^{T}), which are a partially noisy version of $A_j = (X_j, Q_j^{\mathrm{T}})$;

W_j is a noisy version of X_j, and the measurement error U_j reflects the inaccuracy in the measurement process, for example due to sampling variability. Note that the model at (2.1) resembles a classical measurement error linear model, and estimating the unknown parameters can be done using standard methods (Appendix B). However, our prediction problem differs from the one in that setting because, since we are in a small area context, we have at our disposal two measurements, Y_j and (W_j, Q_j^T), of the variable T_j to be predicted. In a standard linear prediction problem with errors, we would observed only (W_j, Q_j^T).

Throughout we use $(Q^{\mathrm{T}}, T, W, Y, \epsilon, \tau)$ to denote a generic $(Q_j^{\mathrm{T}}, T_j, W_j, Y_j, \epsilon_j, \tau_j)$, where $\tau = \mathrm{var}(\epsilon)$ is known. (Here, τ denotes the variance of a generic ϵ and so we have dropped the index j in τ_j.) Our aim is to develop methodology for constructing a prediction interval for T, given the value of (Q^{T}, W, Y) using the data $(Q_k^{\mathrm{T}}, W_k, Y_k)$, $1 \le k \le n$. To summarise, we observe $n+1$ triplets: (Q^{T}, W, Y), which corresponds to the individual whose value of T we wish to predict, and $(Q_j^{\mathrm{T}}, W_j, Y_j)$, for $j = 1, \ldots, n$, which we use to construct estimators of all the unknowns in this prediction problem. Of course the procedure can be applied for all individuals in the study, using in each case the other n observations to estimate the unknown quantities.

To do this we need to construct an estimator of the density of T conditional on (Q^{T}, W, Y). In Appendix A, we prove that it is given by

$$f_{T\,|\,Q,W,Y}(t\,|\,q, w, y) = \frac{f_\epsilon(t-y) \int f_V(t - \beta_0 - \beta_1 x - \beta_2^{\mathrm{T}} q)\, f_X(x)\, f_U(w-x)\, dx}{\int f_{V+\epsilon}(y - \beta_0 - \beta_1 x - \beta_2^{\mathrm{T}} q)\, f_U(w-x)\, f_X(x)\, dx}\,, \quad (2.2)$$

where β_0, β_1, β_2, the variance σ_V^2 of V, and the density f_X are unknown. Estimating the unknown parameters is relatively standard; see Appendix B. In Section 2.2, we show how to estimate the other unknown quantities, and deduce our prediction intervals. In practice it is commonly assumed that f_V and f_ϵ are known (usually normal), and for the main part of this work we shall focus on that setting. However, we shall also see that it is possible to relax this assumption; see Section 5.

2.2 Prediction intervals

To construct a prediction interval for T, we need to estimate the conditional distribution $F_{T\,|\,Q,W,Y}$ corresponding to the density $f_{T\,|\,Q,W,Y}$ at (2.2). The latter depends on f_X, f_V and f_ϵ. In the small area literature, it is often assumed that f_ϵ is known and f_V is known up to its variance σ_V^2, which is the setting we use in this section to derive an estimator of $F_{T\,|\,Q,W,Y}$. In particular, if we let g denote the density of V/σ_V, then $f_V(\cdot) = \sigma_V^{-1} g(\cdot/\sigma_V)$, where g is assumed to be known. In this case, we can use relatively standard deconvolution methods, and the most interesting aspect of this problem is the theory, which reveals unusual and intriguing properties; see Section 3. Our simulation results in Section 4.2 suggest that our procedure seems relatively robust against misspecification. In Section 5 we derive an estimation procedure in the case where one of those two densities is unknown and estimated from the data.

When f_ϵ and g are known, the only unknowns in (2.2) are β_0, β_1, β_2 and σ_V^2, which can be replaced by standard estimators from the measurement error literature, using techniques discussed in Fuller (2009), Buonaccorsi (2010) and Delaigle and Hall (2011) (see Appendix B), and $f_X(x)$, which can be estimated by the kernel deconvolution estimator of Carroll and Hall (1988) and Stefanski and Carroll (1990), defined by

$$\hat{f}_X(x) = \frac{1}{nh}\sum_{j=1}^{n} K_U\Big(\frac{x - W_j}{h}; h\Big)\,,\ K_U(x;h) = \frac{1}{2\pi}\int \exp(-itx)\,\frac{\phi_K(t)}{\phi_U(t/h)}\,dt\,, \quad (2.3)$$

where ϕ_K is the Fourier transform of a kernel function K, $h > 0$ is a smoothing parameter called bandwidth, and, for any random variable R, $\phi_R(t) = E(e^{itR})$ denotes the characteristic function of R.

Moreover, $f_{V+\epsilon}(s) = \int f_\epsilon(v)\, f_V(s-v)\,dv = \sigma_V^{-1}\int f_\epsilon(v)\, g\{(s-v)/\sigma_V\}\,dv$, which can be estimated by $\hat{f}_{V+\epsilon}(s) = \hat{\sigma}_V^{-1}\int f_\epsilon(v)\, g\{(s-v)/\hat{\sigma}_V\}\,dv$. Using (2.2), we can estimate $F_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \int_{-\infty}^{t} f_{T\,|\,Q,W,Y}(s\,|\,q,w,y)\,ds$ by

$$\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$$

$$= \frac{\int_{-\infty}^{t} f_\epsilon(s-y) \int f_V(s - \hat\beta_0 - \hat\beta_1 x - \hat\beta_2^{\mathrm{T}} q)\, \hat f_X(x)\, f_U(w-x)\, dx\, ds}{\int \hat f_{V+\epsilon}(y - \hat\beta_0 - \hat\beta_1 x - \hat\beta_2^{\mathrm{T}} q)\, f_U(w-x)\, \hat f_X(x)\, dx} \,. \quad (2.4)$$

Next, let $\alpha \in (0,1)$ and define $\hat t_\alpha = \hat t_\alpha(q,w,y)$ to be the solution, in t, of $\widehat F_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \alpha$. Approximate one-sided prediction intervals of nominal coverage $1-\alpha$ for T given (Q^{T}, W, Y) can be defined by $\hat{\mathcal{I}}_\alpha^L = (-\infty, \hat t_{1-\alpha}]$ or $\hat{\mathcal{I}}_\alpha^R = [\hat t_\alpha, \infty)$. Approximate two-sided intervals of nominal coverage $1-\alpha$ can be defined by $\hat{\mathcal{I}}_\alpha = [\hat t_{\alpha_1}, \hat t_{1-\alpha_2}]$, where $\alpha_1 + \alpha_2 = \alpha$; a typical choice is to take $\alpha_1 = \alpha_2 = \alpha/2$, which corresponds to equal-tailed intervals.

In the small area literature it is often assumed that V and ϵ are normally distributed, with zero means and respective variances σ_V^2 and τ. In this case we have

$$F_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \Psi_1(t,y,q,w)\big/\Psi_2(t,y,q,w)\,, \quad (2.5)$$

where, for $k = 1,2$, $\Psi_k(t,y,q,w) = \int \psi_k(t,y,q,w,x)\, f_X(x)\, dx$, and

$$\psi_k(t,y,q,w,x) \;=\; \left[\Phi\left\{t - \frac{\sigma_V^2 y + \tau\left(\beta_0 + \beta_1 x + \beta_2^T q\right)}{\sigma_V^2 + \tau}; \frac{\tau\sigma_V^2}{\sigma_V^2+\tau}\right\}\right]^{2-k} \times\phi\big(y - \beta_0 - \beta_1 x - \beta_2^{\mathrm{T}} q\,;\, \sigma_V^2 + \tau\big)\, f_U(w-x)\,, \quad (2.6)$$

with $\phi(x;\sigma^2)$ and $\Phi(x;\sigma^2)$ corresponding to the univariate normal density and distribution functions when the distribution has zero mean and variance σ^2.

In that case, the estimator at (2.4) can be simplified into

$$\widehat F_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \widehat\Psi_1(t,y,q,w)\big/\widehat\Psi_2(t,y,q,w)\,, \quad (2.7)$$

where $\widehat\Psi_k(t,y,q,w) = \int \hat\psi_k(t,y,q,w,x)\, \hat f_X(x)\, dx$ and

$$\hat\psi_k(t,y,q,w,x) \;=\; \left[\Phi\left\{t - \frac{\hat\sigma_V^2 y + \tau\left(\hat\beta_0 + \hat\beta_1 x + \hat\beta_2^T q\right)}{\hat\sigma_V^2 + \tau}; \frac{\tau\hat\sigma_V^2}{\hat\sigma_V^2+\tau}\right\}\right]^{2-k} \times\phi\left(y - \hat\beta_0 - \hat\beta_1 x - \hat\beta_2^T q; \hat\sigma_V^2 + \tau\right)\, f_U(w-x)\,. \quad (2.8)$$

3 Theoretical properties

The procedure we have derived for the case where g, the density of V/σ_V, and f_ϵ are known, uses relatively standard arguments from the deconvolution literature. However, establishing theoretical properties of the prediction intervals derived in Section 2.2 is rather difficult and requires a number of steps. An interesting aspect of this problem is that the theory reveals some intriguing properties. Indeed, the accuracy of the prediction interval depends on the value taken by W. Specifically, prediction is more difficult when $W = 0$, where parametric rates are not possible, than for other values where we can reach the parametric rate.

In Appendix B we establish consistency of the parameter estimators derived there. In Section 3.2, we derive theoretical properties of the estimator of the conditional distribution $F_{T\,|\,Q,W,Y}$ from Section 2.2. In Section 3.3 we investigate theoretical properties of the quantile estimators $\hat{t}_\alpha$ defined in Section 2.2. Finally, we deduce the theoretical properties of our prediction intervals in Section 3.4.

Throughout this section we assume that the characteristic function ϕ_U is real-valued and does not vanish on the real line. Moreover, we establish our results under the assumption that V and ϵ are normally distributed. In particular, we consider prediction intervals based on the version of the estimator $\widehat{F}_{T\,|\,Q,W,Y}$ of $F_{T\,|\,Q,W,Y}$ defined at (2.7). It would be possible to extend our results to the prediction intervals under other distributional assumptions, as well as for the fully nonparametric version we shall suggest in Section 5, but the arguments would be even more tedious than those for the estimator from Section 2.2, which are already quite long and technical.

3.1 Conditions

We start by describing conditions that will be needed to prove our results. Recall the notation ψ_k from Section 2.2. Define $\chi(t) = \chi(t\,|\,s, y, q, w) = \int e^{itx}\,\psi_k(s, y, q, w, x)\,dx$

for either $k = 1$ or $k = 2$, let $\chi_1 = \Re\,\chi$, $\chi_2 = \Im\,\chi$, denoting the real and imaginary parts of χ, put $\rho_j = \chi_j/\phi_U$, let $r \geq 1$ be an integer, and set $\beta(t) = t^{2r}\,\phi_U(t)$,

$$\Psi_{kr}(x) = \left(\frac{\partial}{\partial x}\right)^{2r-1} \psi_k(s, q, w, x)\,.$$

We shall assume that, for constants $C_1, C_2, \ldots$,

(i) Ψ_{kr} has a jump discontinuity of size s_k at w, and $(1 + |x|)\,|\Psi_{kr}^{(\ell)}(x)|$ is bounded and integrable on $(-\infty, w) \cup (w, \infty)$ for $\ell = 1, 2, 3, 4$; (ii) $\phi_U(t)$ is real valued and does not vanish for any real t, ϕ_K is real valued and compactly supported, $|\phi_K'|$ is bounded, and $|\rho_j'|$ is bounded; (iii) $|\phi_W(t)| \leq C_1\,(1+|t|)^{-C_2}$, where $C_1 > 0$, $C_2 > 2$; (iv) $\beta(t) = b_1 + O(|t|^{-b_2})$ as $|t| \to \infty$, where $b_1, b_2 > 0$, and $|\beta'(t)| \leq C_3\,(1+|t|)^{-C_4}$ for all t, where $C_3 > 0$, $C_4 > 1$; (3.1)

and that for each $w \in \mathbb{R}$,

$$\max_k \sup_{s,y,q} \sup_{-\infty<t<\infty} |\chi(t\,|\,s, y, q, w)/\phi_U(t)| < \infty\,, \tag{3.2}$$

where $\max_k$ denotes the maximum over $k = 1, 2$ and the second supremum is over s, y and q in compact sets $\mathcal{S} \subset \mathbb{R}$, $\mathcal{Y} \subset \mathbb{R}$ and $\mathcal{Q} \subset \mathbb{R}^p$, respectively. See Appendix C for a discussion of these conditions. We assume too that, for an integer $\ell \geq 1$,

the functions $\lambda_k(u\,|\,s, y, q, w) \equiv \int \psi_k(s, y, q, w, x + u)\, f_X(x)\, dx$, for $k = 1, 2$ have ℓ partial derivatives with respect to u, for u in a neighbourhood of the origin, and those derivatives are bounded uniformly in $s \in \mathcal{S}$, $y \in \mathcal{Y}$ and $q \in \mathcal{Q}$. (3.3)

Defining the function $\lambda_k^{[\ell]}(u\,|\,s, y, q, w) = (\partial/\partial u)^\ell\, \lambda_k(u\,|\,s, y, q, w)$, we assume that:

$$\text{for } k = 1, 2\ \lambda_k^{[\ell]}(u\,|\,s, y, q, w) \text{ is continuous in a neighbourhood of } u = 0. \tag{3.4}$$

Of the kernel K and bandwidth h we assume additionally that:

$\int (1 + |u|)^\ell\, |K(u)|\, du < \infty$, where ℓ is as in (3.3), $\kappa_j \equiv \int u^j\, K(u)\, du = 0$ for $1 \leq j \leq \ell - 1$, and $\int K = 1$; and $h = h(n) \to 0$ and $nh \to \infty$ as $n \to \infty$. (3.5)

Finally, the following assumption will be useful to prove some of our results:

(a) the bandwidth h is chosen such that $\int E(\hat{f}_X^2) = O(n^a)$, where $a \geq 0$, and $n^{1-\eta}\, h \to \infty$ for some $\eta > 0$; (b) if a is as in part (a), then $n^{a+\varepsilon}\, h = O(1)$ as $n \to \infty$, for some $\varepsilon > 0$; and (c) the random quantities Q, U, V and X all have at least ν finite moments, where the value of $\nu \geq 4$ depends on a and ε in parts (a) and (b). (3.6)

Part (a) of (3.6) is milder than the condition usually assumed, where h would be chosen so that $\int E(\hat{f}_X - f_X)^2$ converges to zero as $n \to \infty$, and hence $\int E(\hat{f}_X^2) = O(1)$.

3.2 Consistency of conditional distribution estimator

The next theorem establishes consistency of the conditional distribution estimator $\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$ defined at (2.7). Its proof is provided in Section 6.1.

Theorem 1. *Assume the conditions imposed in Theorem 4, and that* (3.1)–(3.3) *and* (3.5) *hold. Then: (i) For each real t and y, and each $q \in \mathbb{R}^p$,*

$$\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) - F_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \begin{cases} O_p\Big\{(nh)^{-1/2} + h^\ell\Big\} & \text{if } w = 0 \\ O_p\big(n^{-1/2} + h^\ell\big) & \text{if } w \neq 0\,, \end{cases} \tag{3.7}$$

where, if (3.4) *holds, when $w = 0$, the term $O_p(h^\ell)$ can be written more explicitly as $\big(c_{01}/\tau_{02} - \tau_{01}\; c_{02}/\tau_{02}^2\big)\, h^\ell + o_P\big(h^\ell\big)$, where τ_{0k} and c_{0k}, $k = 1, 2$, are defined in* (6.1). *(ii) For each $\eta > 0$,*

$$\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) - F_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \begin{cases} O_p\Big\{\big(n^{1-\eta}h\big)^{-1/2} + h^\ell\Big\} & \text{if } w = 0 \\ O_p\big(n^{-(1-\eta)/2} + h^\ell\big) & \text{if } w \neq 0\,, \end{cases}$$

uniformly in t, q and y in any compact subsets of their respective domains, where in the case $w = 0$ we ask in addition that $n^{1-\eta}h \to \infty$.

When $w = 0$, we deduce from Theorem 1 that, by choosing $h \asymp n^{-1/(2\ell+1)}$, $\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$ converges to $F_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$ at the rate $n^{-\ell/(2\ell+1)}$, in a pointwise sense. When $w \neq 0$ we obtain root-n consistency, i.e. $\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) =$

$F_{T\,|\,Q,W,Y}(t\,|\,q,w,y) + O_p\big(n^{-1/2}\big)$, for any choice of h satisfying $h = O(n^{-1/(2\ell)})$. It follows from the proof of the theorem that the rates at (3.7) for $w = 0$ cannot be improved. In particular, the convergence rates for $w = 0$ are slower than those for $w \neq 0$. This is an intriguing property for which we do not have an intuitive explanation. However, it is encountered in related problems, also without an intuitive justification; see for example Hall and Lahiri (2008), where the authors also prove that the rates of their estimator are optimal even in the neighbourhood of the origin.

The methods used to derive Theorem 1 can be employed to show that, under the same conditions, all partial derivatives of $\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$ with respect to t converge at the same rate to the respective derivatives of $F_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$. See Theorem 5 in Appendix D. These results are of independent interest, but they are also particularly useful to derive the properties of our prediction intervals in Section 3.4.

3.3 Theoretical properties of quantiles estimators

Let $\alpha \in (0,1)$, define $t_\alpha = t_\alpha(q,w,y)$ to be the solutions, in t, of $F_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \alpha$, and recall the definition of $\hat{t}_\alpha = \hat{t}_\alpha(q,w,y)$ in Section 2.2. Strictly speaking, there is a small probability that $\hat{t}_\alpha$ is not uniquely defined, but since the probability converges to 0 at a rate faster than n^{-1} then the event of non-uniqueness can be neglected in Theorems 2 and 3 below. Our proofs are valid under the assumption that $\hat{t}_\alpha$ is the solution nearest to t_α, in cases where there is ambiguity.

The next theorem establishes asymptotic properties of $\hat{t}_\alpha - t_\alpha$. See Appendix G for a proof. It can be seen from part (i) of Theorem 2 that asymptotic properties of $\hat{t}_\alpha - t_\alpha$ are readily and directly deducible from Theorem 1. In particular, $\hat{t}_\alpha - t_\alpha$ converges to zero at the same rate as $\widehat{F}_{T\,|\,Q,W,Y}(t_\alpha\,|\,q,w,y) - F_{T\,|\,Q,W,Y}(t_\alpha\,|\,q,w,y)$, with the same distinction between the cases $w = 0$ and $w \neq 0$ as in Thoerem 1.

Theorem 2. *Assume the conditions imposed in Theorem 4, and that* (3.1)–(3.3) *and*

(3.5) *hold. Then: (i) For each* q, w *and* y,

$$\hat{t}_\alpha(q,w,y) - t_\alpha(q,w,y) \;=\; -\,\frac{\widehat{F}_{T\,|\,Q,W,Y}(t_\alpha\,|\,q,w,y) - F_{T\,|\,Q,W,Y}(t_\alpha\,|\,q,w,y)}{(\partial/\partial t)\,F_{T\,|\,Q,W,Y}(t\,|\,q,w,y)\,\big|_{t=t_\alpha}} + \begin{cases} O_p\big\{(nh)^{-1} + h^{2\ell}\big\} & \text{if } w = 0 \\ O_p\big(n^{-1} + h^{2\ell}\big) & \text{if } w \neq 0\,; \end{cases}$$

and (ii) For each $\eta > 0$,

$$\hat{t}_\alpha(q,w,y) - t_\alpha(q,w,y) = \begin{cases} O_p\Big\{\big(n^{1-\eta}h\big)^{-1/2} + h^{\ell}\Big\} & \text{if } w = 0 \\ O_p\big(n^{-(1-\eta)/2} + h^{\ell}\big) & \text{if } w \neq 0\,, \end{cases}$$

uniformly in q *and* y *in any compact subsets of their respective domains, where in the case* $w = 0$ *we ask in addition that* $n^{1-\eta}h \to \infty$.

3.4 Theoretical properties of prediction intervals

Finally, in Theorem 3 below, we explore the coverage accuracy of the prediction interval $(-\infty, \hat{t}_\alpha]$; similar results can be established for two-sided intervals. In stating the theorem we assume that the vector $(T, Q^{\mathrm{T}}, W, Y)$ is independent of the data $(Q_j^{\mathrm{T}}, W_j, Y_j)$, for $1 \le j \le n$, from which $\hat{t}_\alpha$ is computed. See Appendix H for a proof of the theorem. As for the results in the previous sections, the properties for $w = 0$ are different from those for $w \neq 0$. See the discussion under Theorem 1.

Theorem 3. *If* (3.1)–(3.3), (3.5) *and* (3.6) *hold, then*

$$P\Big(T \le \hat{t}_\alpha \,\Big|\, Q = q, W = w, Y = y\Big) = \alpha + \begin{cases} O_p\big\{(nh)^{-1/2} + h^{\ell}\big\} & \text{if } w = 0 \\ O_p\big(n^{-1/2} + h^{\ell}\big) & \text{if } w \neq 0\,, \end{cases} \tag{3.8}$$

uniformly in q *and* y *in any compact subsets of their respective domains, and in* $\alpha \in [\alpha_1, \alpha_2]$ *for any* $0 < \alpha_1 < \alpha_2 < 1$.

If we assume (3.4) in addition to the conditions in Theorem 3, and slightly strengthen (3.6)(a); and if we use a more intricate argument than that in Section 6 to determine the "remainder" term in (3.8); then the right-hand side of (3.8) can be refined to $\alpha + C_1\,(nh)^{-1/2} + C_6\,h^\ell + o\{(nh)^{-1/2} + h^\ell\}$ when $w = 0$, and to $\alpha + C_5\,n^{-1/2} + C_6\,h^\ell + o(n^{-1/2} + h^\ell)$ otherwise, where C_5 and C_6 are constants. (The more intricate argument is sketched in the last paragraph of the proof of Theorem 3.) However, the implications of this property are rather complex. For example, when $w = 0$ and in cases where C_5 and C_6 are both nonzero, the absolute value of the coverage error is minimised, at $O(n^{-\ell/(\ell+1)})$, by taking h to be of size $n^{-1/(\ell+1)}$, although it is easy to give examples where choosing h so as to produce over- or under-coverage might be advantageous. (Under-coverage, in the context of (3.8), is relevant if our real interest is in a prediction interval $[\hat{t}_\alpha, \infty)$ rather than $(-\infty, \hat{t}_\alpha]$.)

4 Numerical properties

4.1 Smoothing parameter choice

To compute our prediction interval, we need to choose the bandwidth h used by $\hat{f}_X$ at (2.3), a notoriously very difficult task for nonparametric prediction and confidence intervals. In our contaminated data case, we can exploit the error structure to suggest a selection technique inspired by ideas used in nonparametric errors-in-variable regression. There, instead of trying to consistently estimate optimal smoothing parameters, Delaigle and Hall (2008) suggest numerical approximation procedures based on mimicking the contamination process via resamples of data contaminated with additional levels of noise. We propose a method of that type tailored to our problem.

We describe our approach for the interval $\hat{\mathcal{I}}_\alpha$ (it is straightforward to adapt it to $\hat{\mathcal{I}}_\alpha^L$ and $\hat{\mathcal{I}}_\alpha^R$). If we had access to direct data $T_1, \ldots, T_n$, we would choose h so as to

minimise an estimator of coverage error, e.g. $\{n^{-1}\sum_{i=1}^n I\big(T_i \in \hat{\mathcal{I}}_{\alpha,-i}\big) - (1-\alpha)\}^2$, or, to make h less variable, $\int_{\alpha/2}^{3\alpha/2} \{n^{-1}\sum_{i=1}^n I\big(T_i \in \hat{\mathcal{I}}_{\alpha,-i}\big) - (1-\alpha)\}^2\, d\alpha$. Since we do not observe the T_is, we cannot compute this error. Instead we shall mimic it using contaminated versions of the data. Roughly speaking, our idea is to create a new sample of observations $(Q_i, W_i^\circ, Y_i^\circ)$, $1 \le i \le n$, where, compared to the original (W_i, Y_i)s, the (W_i°, Y_i°)s are contaminated with an additional level of error, so that the relationship between X_i and W_i in the original sample is mimicked by that between W_i and W_i°. In this new sample, the variable T_i° that plays the role of T_i in the original sample is observed, and thus we can compute the prediction error of a prediction interval for T_i°. Since the new sample is created in a way that mimics the model at (2.1), our h can be well approximated by minimising that version of prediction error.

To implement these ideas, first we draw a conventional bootstrap resample, $\mathcal{X}^* = \{(Q_i^{*\mathrm{T}}, W_i^*, Y_i^*) : 1 \le i \le n\}$, with replacement from the dataset $\mathcal{X} = \{(Q_i^{\mathrm{T}}, W_i, Y_i) : 1 \le i \le n\}$. Each triple $(Q_i^{*\mathrm{T}}, W_i^*, Y_i^*)$ is identical to one of the data triples in $\mathcal{X}$; let that triple be $(Q_{j_i^*}^{\mathrm{T}}, W_{j_i^*}, Y_{j_i^*})$, where, conditional on $\mathcal{X}$, the j_i^*s, for $1 \le i \le n$, are independent and identically distributed on $1, \ldots, n$. Write $\tau_{j_i^*}$ for the associated value of the variance of $\epsilon_{j_i^*}$, conditional on j_i^*, and let $(\epsilon_i^\dagger, U_i^\dagger)$ denote a pair of random variables that, conditional on both $\mathcal{X}$ and $\mathcal{X}^*$, have respectively the distribution with density $\tau_{j_i^*}^{-1/2} f_1(\,\cdot\,/\tau_{j_i^*}^{1/2})$, with f_1 the density of $\epsilon_i/\tau_i^{1/2}$, and the distribution of U. (We use the notation $\epsilon_i^\dagger$ and $U_i^\dagger$, rather than ϵ_i^* and U_i^*, to recall that $\epsilon_i^\dagger$ and $U_i^\dagger$ are drawn by sampling from known distributions rather than by resampling from a sample.) We take the pairs $(\epsilon_i^\dagger, U_i^\dagger)$, for $1 \le i \le n$, to be independent, conditional on the data. Put $Y_i^\circ = Y_i^* + \hat{\beta}_1 W_i^* + \epsilon_i^\dagger$ and $W_i^\circ = W_i^* + U_i^\dagger$, and note that $Y_i^\circ = T_i^\circ + \epsilon_i^\dagger$, where $T_i^\circ = Y_i^* + \hat{\beta}_1 W_i^* = \beta_0 + \hat{\beta}_1 W_{j_i^*} + \beta_2^{\mathrm{T}} Q_{j_i^*} + V_i^\circ$ and $V_i^\circ = \beta_1 X_{j_i^*} + V_{j_i^*} + \epsilon_{j_i^*}$.

In essence, the dataset $\mathcal{X}^\circ = \{(Q_i^{*\mathrm{T}}, W_i^\circ, Y_i^\circ) : 1 \le i \le n\}$ is generated by the model at (2.1), the difference being that V_i in (2.1) is replaced here by V_i°, which is

generally more variable than V_i. This motivates us to choose h by minimising the prediction error of a prediction interval for T_i° constructed from $\mathcal{X}^\circ$. Specifically, we omit the ith data triple from $\mathcal{X}^\circ$, obtaining $\mathcal{X}_{-i}^\circ$, say, and we use the methodology from Section 2.3 to construct, from $\mathcal{X}_{-i}^\circ$, a prediction interval $\hat{\mathcal{I}}_{\alpha,-i}^\circ$, say, for T_i°, having nominal coverage $1-\alpha$. To reduce variability, we repeat this procedure B times, generating in this way B datsets $\mathcal{X}_b^\circ = \{({Q_{i,b}^*}^{\mathrm{T}}, W_{i,b}^\circ, Y_{i,b}^\circ) : 1 \le i \le n\}$, $b = 1, \ldots, B$, and obtaining their corresponding $T_{i,b}^\circ$ and $\hat{\mathcal{I}}_{\alpha,-i,b}^\circ$. Then, recalling that for these data, the $T_{i,b}^\circ$s are known, we choose h by minimising the objective function

$$J(h) = \frac{1}{B}\sum_{b=1}^{B}\int_{\alpha/2}^{3\alpha/2}\left\{\frac{1}{n}\sum_{i=1}^{n} I\big(T_{i,b}^\circ \in \hat{\mathcal{I}}_{\alpha,-i,b}^\circ\big) - (1-\alpha)\right\}^2 d\alpha\,. \tag{4.1}$$

Based on our experience, taking $B = 10$ or 20 often suffices to find appropriate smoothing parameters. In our numerical work, we took $B = 10$. All our codes were written in MATLAB, and to reduce computational burden, h was chosen from the grid $h_{PI} \times (0.25, 0.5, 1, 1.5, 2)$, where h_{PI} is the plug-in bandwidth of Delaigle and Gijbels (2004, 2004), computed using the MATLAB code `PI_deconvUknownth4`, available at `http://www.ms.unimelb.edu.au/~aurored/links.html#Code`.

4.2 Simulations

We applied our methodology for constructing two-sided prediction intervals $\hat{\mathcal{I}}_\alpha = [\hat{t}_{\alpha/2}, \hat{t}_{1-\alpha/2}]$ on simulated examples. We generated data $(W_j, Q_j^{\mathrm{T}}, Y_j)$, $1 \le j \le n$, for $n = 30$ and 50 small areas, from the following model:

$$Y_j = T_j + \epsilon_j\,, \quad T_j = 5 + 3\,X_j + 2\,Q_j + V_j\,, \quad W_j = X_j + U_j\,,$$

where $X_j \sim N(5, 9)$, $Q_j \sim \mathrm{Uniform}(0, 5)$, $V_j \sim t(5)$ and $\epsilon_j \sim N(0, \tau_j)$. Rather than choosing the τ_js, for $j = 1, \ldots, n$, arbitrarily by hand, we generated them from a gamma distribution with mean 8 and standard deviation 4. Finally, we considered two types of measurement errors: $U_j \sim N(0, 3/4)$ and $U_j \sim \mathrm{Laplace}(\sqrt{3/8})$.

Recall that our procedure relies on knowing F_{V/σ_V} and F_ϵ. To examine robustness against misspecified distributions, in each case we compared our prediction interval $\widehat{F}_{T\,|\,Q,W,Y}$ at (2.4) constructed using the correct F_{V/σ_V} or pretending that it was equal to the standard normal distribution, where we used $\widehat{F}_{T\,|\,Q,W,Y}$ at (2.7). To demonstrate the importance of taking the noise into account, we also computed the naive prediction intervals obtained when ignoring the measurement errors U_j.

We generated 100 datasets for each combination of n and F_U, and in each case, we constructed the intervals $\hat{\mathcal{I}}_\alpha$ for three nominal levels, $\alpha = 0.99$, 0.95 and 0.90. For each generated sample, and for $i = 1, \ldots, n$, we constructed each prediction interval for T_i using the data $\mathcal{X}_{-i} = \{(Q_j^{\mathrm{T}}, W_j, Y_j) : 1 \leq j \leq n, j \neq i\}$. Let $\hat{\mathcal{I}}_\alpha^i$ denote a prediction interval for T_i obtained in this way, using either approach described above. For each approach we calculated the coverage rate of $\hat{\mathcal{I}}_\alpha$, $n^{-1} \sum_{i=1}^n I\big(T_i \in \hat{\mathcal{I}}_\alpha^i\big)$, and we averaged this number over the 100 generated samples to obtain an empirical measure of the coverage probability of the prediction intervals, which we denote below by ECP.

We report the ECP in Table 1 for each configuration. The closer ECP is to $1-\alpha$, the more accurate the prediction interval is. These results suggest that our method is relatively robust against error misspecification, the main effect of misspecification being to increase the interval length. In Section 5 we also report simulation results for the case where F_V is estimated from the data. The latter approach is much more complex to implement, and while it improves slightly the level of the interval for small sample sizes, this comes at the cost of a significant increase in interval length. Those simulation results suggest that the rough parametric approximations of F_{V/σ_V} are preferable to a complex full nonparametric procedure. We also see that ignoring the errors completely leads to shorter intervals, but with very poor coverage.

Table 1: Empirical coverage probabilities of the prediction interval (average length of the prediction intervals) computed from simulated data, using the correctly specified F_{V/σ_V} (True), erroneously pretending it is a normal distribution (Wrong), or using the naive prediction interval (Naive) which ignores the errors U_i.

U	n	F_{V/σ_V}	$1-\alpha$		
			0.99	0.95	0.90
Normal	30	True	0.987 (10.14)	0.943 (7.69)	0.889 (6.44)
		Wrong	0.985 (10.25)	0.944 (7.79)	0.890 (6.54)
		Naive	0.866 (8.67)	0.805 (6.60)	0.753 (5.54)
	50	True	0.991 (9.92)	0.949 (7.52)	0.903 (6.31)
		Wrong	0.990 (10.05)	0.953 (7.65)	0.905 (6.42)
		Naive	0.846 (7.98)	0.788 (6.07)	0.732 (5.10)
Laplace	30	True	0.988 (10.27)	0.943 (7.60)	0.888 (6.29)
		Wrong	0.987 (10.42)	0.945 (7.74)	0.893 (6.42)
		Naive	0.826 (8.03)	0.766 (6.11)	0.719 (5.12)
	50	True	0.990 (10.01)	0.947 (7.41)	0.900 (6.14)
		Wrong	0.990 (10.14)	0.949 (7.54)	0.901 (6.26)
		Naive	0.832 (7.74)	0.776 (5.89)	0.727 (4.94)

4.3 Real data examples

Example 1. We considered an application to two survey datasets: one from the 2003-2004 US NHANES, and the other from the 2004 US NHIS. The small areas in this study are 50 demographic subgroups classified by race and ethnicity (Ybarra and Lohr, 2008). Our goal was to construct prediction intervals for the population mean body mass index (BMI) for the demographic subgroups (small areas), using the NHIS as auxiliary information. The datasets were combined according to the small areas.

In the NHANES, the height and weight for each respondent were measured carefully to calculate the BMI = weight(kg)/ height(m)^2, but in the NHIS, measurements of height and weight are provided by the interviewers during the interview. Thus, the auxiliary variable (reported BMI) is prone to measurement error. For the jth demographic subgroup, let Y_j be the mean BMI from the NHANES. Let

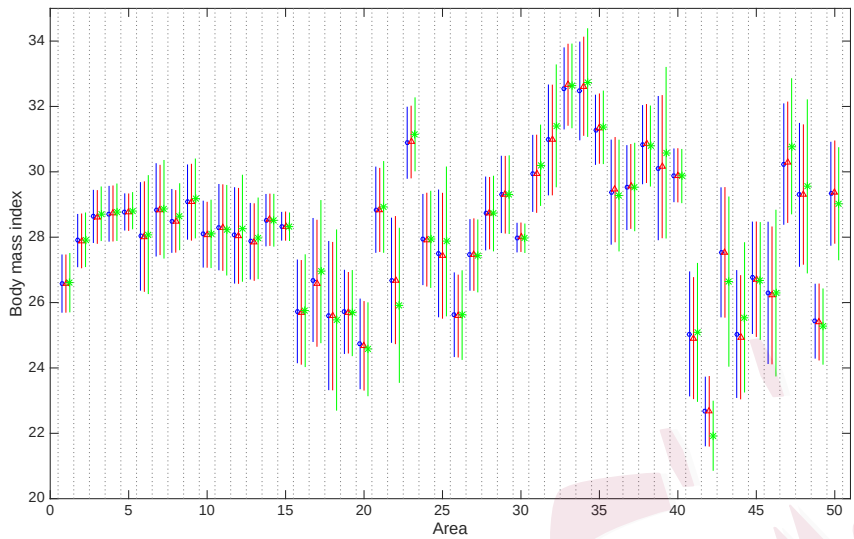

Figure 1: Each line represents a 95% prediction interval of the population mean BMI for one of 50 demographic subgroups from Example 1, assuming that f_{V/σ_V} is a standard normal density (—∘—: naive method; —△—: our method) or estimating it nonparametrically as in Section 5 (—∗—).

Table 2: Example 1: average and median length of prediction intervals for 50 areas.

f_{V/σ_V}	Average length	Median length
normal-naive	2.69	2.56
normal	2.69	2.52
estimated as in Section 5	2.95	2.69

X_j be the true value of the mean BMI from the NHIS. We considered the model $Y_j = T_j + \epsilon_j$, $T_j = \beta_0 + \beta_1 X_j + V_j$. Denoting by W_j the mean of reported BMI from the NHIS, we assumed that the measurement error was additive through $W_j = X_j + U_j$.

We constructed the 95% prediction intervals for the population mean BMI for all demographic subgroups using $\widehat{F}_{T\,|\,Q,W,Y}$ at (2.7), assuming that f_{V/σ_V} is the standard normal density. For comparison, we also computed the prediction intervals based on the approach introduced in Section 5, where f_{V/σ_V} is estimated nonparametrically. Finally, we computed the naive prediction intervals which ignore the errors U_i. For $j = 1, \ldots, 50$, Figure 1 shows the resulting 95% prediction intervals of T_j conditional

Table 3: Example 2: average and median length of prediction intervals for 51 areas.

f_{V/σ_V}	Coverage	Average length	Median length
normal-naive	78.43%	6.55e+03	6.65e+03
normal	84.31%	7.48e+03	7.41e+03
estimated as in Section 5	86.27%	1.04e+04	1.01e+04

on (W_j, Y_j). Table 2 reports the average and median length of those 50 prediction intervals. The intervals obtained using both corrected approaches are relatively similar, although as in our simulations, the interval lengths using the approach from Section 5 tend to be larger than those obtained under the normality assumption. Our simulation results suggest that the latter are preferable, since generally shorter while not much less accurate. In this example, the naive prediction intervals are close to the corrected ones, but since we do not know the truth, we do not know which method gives the most accurate prediction intervals.

Example 2. The US Department of Health and Human Services administers a program of energy assistance to low-income families. An important variable that determines the eligibility of a family for benefits from the program is an estimate of the median income of four-person families in the state. Through the years prior to 2000, the basic data, also known as the direct estimates of the state median incomes, came from the March Annual Demographic Supplement (ADS), which collects the income data of the three-, four-, and five-person households statewide. Due to smallness of sample sizes for all 51 states (50 states and the District of Columbia), the direct estimates are subject to considerable sampling variability, and the US Bureau of the Census (BOC) annually provides estimates of the state median income for four-person families by using small area estimation methods (Fay, 1987; Datta et al., 1991).

The 1989 CPS three-person households state median income estimates are strongly

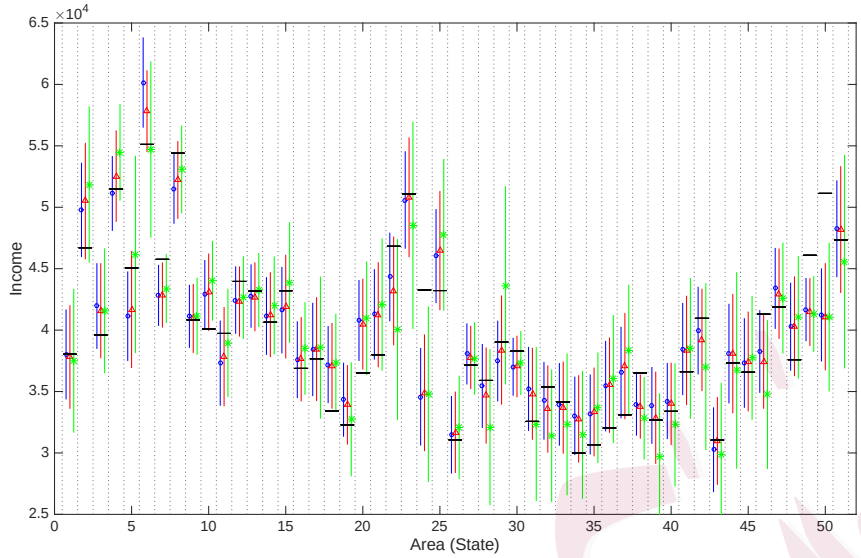

Figure 2: Each line represents a 95% prediction interval of the median income for one of the 51 states from Example 2, assuming that f_{V/σ_V} is a standard normal density (—○—: naive estimator; —△—: our method) or estimating it nonparametrically as in Section 5 (—∗—). The black flat line (−) shows year 1989 four-person family median incomes from the 1990 Census records, which can be regarded as the "true values".

correlated with the corresponding 1989 CPS four-person ones, and we use them as the W_js in our prediction model (they are subject to measurement error due to their large sampling variability). For $j = 1, \ldots, 51$, we constructed 95% prediction intervals for the four-person family median incomes for the year 1989, conditional on observed values of (W_j, Y_j); see Figure 2. Numbers for this variable are available from the 1990 Census records, and we can compute the coverage of our intervals by treating these numbers as the "true values". Table 2 reports the coverage of the prediction intervals for 51 states using the three methods used in Example 1, and the average and median length of those prediction intervals. The prediction intervals constructed using the naive method which completely ignores the measurement errors provided the worst coverage rate (78.43%). With our methods that take measurement errors into account, the coverage rates were much higher; they were 84.31% and 86.27% when

assuming that f_{V/σ_V} is a standard normal density or estimating it nonparametrically as in Section 5, respectively, but for the latter the intervals were again much longer.

5 Estimating $F_{T\,|\,Q,W,Y}$ when F_V is unknown

The known distributions assumptions used in Section 2.2 can be relaxed, although since only (Q^{T}, W, Y) are observed, the distributions of V/σ_V and ϵ are confounded in the model at (2.1), and so neither of the distributions can be estimated without knowing at least one of them. In this section, we show how to estimate $F_{T\,|\,Q,W,Y}$ from data $(Q_j^{\mathrm{T}}, W_j, Y_j)$ when the distribution of V is unknown and that of ϵ is known; similar ideas can be used if it is the distribution of ϵ that is unknown. We assume that f_ϵ is equal to $\tau^{-1/2}\, f_1(\,\cdot\,/\tau^{1/2})$, where $\tau = \mathrm{var}(\epsilon)$ and f_1 is known.

Recall that, by the Fourier inversion theorem, we have

$$f_V(v) = (2\pi)^{-1} \int e^{-itv} \phi_V(t)\, dt\,. \tag{5.1}$$

To estimate f_V, we shall estimate ϕ_V, and plug a regularised version of it in (5.1). We start by expressing ϕ_V in terms of quantities that are either known, or which can be estimated directly from the $(Q_j^{\mathrm{T}}, W_j, Y_j)$s. It follows from (2.1) that $Y_j = \beta_0 + \beta_1\, X_j + \beta_2^{\mathrm{T}}\, Q_j + V_j + \epsilon_j$. Therefore,

$$\phi_{Y_j}(t) = \exp(it\beta_0)\, \phi_X(\beta_1 t)\, \phi_{\beta_2^{\mathrm{T}} Q}(t)\, \phi_{\epsilon_j}(t)\, \phi_V(t)\,, \tag{5.2}$$

so that we can write $\phi_V(t) = \exp(-it\beta_0)\phi_{Y_j}(t)\big/\big\{\phi_X(\beta_1 t)\, \phi_{\beta_2^{\mathrm{T}} Q}(t)\, \phi_{\epsilon_j}(t)\big\}$. Here, ϕ_{ϵ_j} is known and we can estimate $\phi_{\beta_2^{\mathrm{T}} Q}(t)$ and $\phi_X(\beta_1 t) = \phi_W(\beta_1 t)/\phi_U(\beta_1 t)$ by

$$\widehat{\phi_{\beta_2^{\mathrm{T}} Q}}(t) = n^{-1} \sum_{j=1}^{n} \exp\big(it\, \hat\beta_2^{\mathrm{T}} Q_j\big)\,, \quad \widehat{\phi_X(\beta_1 t)} = \widehat{\phi_W(\beta_1 t)}\big/\phi_U(\hat\beta_1 t)\,, \tag{5.3}$$

respectively, with $\hat\beta_1$ and $\hat\beta_2$ as in Section B and $\widehat{\phi_W(\beta_1 t)} = n^{-1} \sum_j \exp(it\, \hat\beta_1 W_j)$. However, the Y_js are not identically distributed and we only have one observation,

Y_j, to estimate $\phi_{Y_j}(t)$. To overcome this difficulty, note that (5.2) also implies that $\phi_{Y'}(t) = \exp(it\,\beta_0)\,\phi_X(\beta_1 t)\,\phi_{\beta_2^{\mathrm{T}}Q}(t)\,\phi_{\epsilon'}(t)\,\phi_V(t)$, where we used the notation $\phi_{Y'}(t) = n^{-1}\sum_{j=1}^n \phi_{Y_j}(t)$ and $\phi_{\epsilon'}(t) = n^{-1}\sum_{j=1}^n \phi_{\epsilon_j}(t)$. Therefore, we can write

$$\phi_V(t) = \exp(-it\beta_0)\,\phi_{Y'}(t)\big/\big\{\phi_X(\beta_1 t)\,\phi_{\beta_2^{\mathrm{T}}Q}(t)\,\phi_{\epsilon'}(t)\big\}\,, \tag{5.4}$$

where $\phi_{Y'}(t)$ can be estimated by $\hat\phi_{Y'}(t) = n^{-1}\sum_{j=1}^n e^{itY_j}$, and $\phi_{\beta_2^{\mathrm{T}}Q}(t)$ and $\phi_X(\beta_1 t)$ can be estimated as above.

As in standard nonparametric errors-in-variable problems, substituting this estimator and (5.3) into (5.4), and plugging the resulting estimator $\hat\phi_V$ of ϕ_V directly into (5.1) needs to be done in combination with some regularisation. Several approaches can be taken, such as one based on kernel regularisation as in (2.3). However, the denominator of $\hat\phi_V$ can vanish, and a more suitable approach consists in replacing the denominator, when it gets too small, by a ridge parameter $\rho > 0$. More precisely, recalling that, for a complex number a, we have $1/a = \bar a/|a|^2$, and recalling that the ϵ_js are symmetric around zero, so that $\phi_{\epsilon'}$ is real, we suggest using

$$\hat f_V(v) = \frac{1}{2\pi}\int e^{-it\,(v+\hat\beta_0)}\hat\phi_{Y'}(t)\phi_U(t\,\hat\beta_1)\hat\phi_{W;\,Q;\,\rho}(t)\;dt,$$

where

$$\hat\phi_{W;\,Q;\,\rho}(t) = \frac{\overline{\widehat{\phi_W(\beta_1 t)}}\;\overline{\widehat{\phi_{\beta_2^{\mathrm{T}}Q}}}(t)}{\max\big\{\rho, \big|\widehat{\phi_W(\beta_1 t)}\big|^2\big\}\max\big\{\rho, \big|\widehat{\phi_{\beta_2^{\mathrm{T}}Q}}(t)\big|^2\big\}\,\phi_{\epsilon'}(t)}$$

if $\inf_t \phi_{\epsilon'}(t) \geq 0$, and

$$\hat\phi_{W;\,Q;\,\rho}(t) = \frac{\overline{\widehat{\phi_W(\beta_1 t)}}\;\overline{\widehat{\phi_{\beta_2^{\mathrm{T}}Q}}}(t)\,\phi_{\epsilon'}(t)}{\max\big\{\rho, \big|\widehat{\phi_W(\beta_1 t)}\big|^2\big\}\max\big\{\rho, \big|\widehat{\phi_{\beta_2^{\mathrm{T}}Q}}(t)\big|^2\big\}\max\big\{\rho, \phi_{\epsilon'}^2(t)\big\}}$$

otherwise. Recalling that $\mathrm{var}(\epsilon) = \tau$, we can then estimate $f_{V+\epsilon}(s)$ by $\hat f_{V+\epsilon}(s) = \int \tau^{-1/2}\,f_1\big(v/\tau^{1/2}\big)\,\hat f_V(s-v)\,dv$. Finally, using (2.2), we can estimate the distribution function of T, given that $(Q, W, Y) = (q, w, y)$, by

$$\widehat F_{T\,|\,Q,W,Y}(t\,|\,q, w, y)$$

Table 4: Empirical coverage probabilities of the prediction interval (average length of the prediction intervals) computed from simulated data when f_V is estimated nonparametrically.

U	n	$1-\alpha$		
		0.99	0.95	0.90
Normal	30	0.985 (13.59)	0.948 (10.49)	0.895 (8.80)
	50	0.986 (13.03)	0.950 (9.99)	0.904 (8.41)
Laplace	30	0.987 (14.06)	0.949 (10.9)	0.894 (9.17)
	50	0.987 (13.47)	0.946 (10.44)	0.892 (8.80)

$$= \frac{\int_{-\infty}^{t} f_1\{(s-y)/\tau^{1/2}\} \int \hat{f}_V(s-\hat{\beta}_0-\hat{\beta}_1 x-\hat{\beta}_2^{\mathrm{T}} q)\, \hat{f}_X(x)\, f_U(w-x)\, dx\, ds}{\int f_U(w-x)\, \hat{f}_X(x) \int f_1(v/\tau^{1/2})\, \hat{f}_V(y-\hat{\beta}_0-\hat{\beta}_1 x-\hat{\beta}_2^{\mathrm{T}} q-v)\, dv\, dx}\,, \tag{5.5}$$

where unqualified integrals are over the entire real line.

It can be proved that this estimator is consistent under sufficient regularity conditions, but estimating all the unknowns as above is challenging. Recall that f_V and f_ϵ only appear in an indirect way in the expression for $\widehat{F}_{T\,|\,Q,W,Y}$, and our simulation results in Section 4.2 already indicated that our method was somewhat robust to error misspecification, with the main effect being to increase the length of our prediction interval. Therefore, it is not clear that a purely nonparametric approach is worth the additional complexity it incurs, since its main effect will be to increase the variability of the predictors, which too will have the effect of increasing the interval length.

To investigate this in practice, we constructed prediction intervals based on (5.5), using the same simulation settings as in Section 4.2. To compute them, we need to choose h used in $\hat{f}_X$ at (2.3) and the ridge parameter ρ. We used the approach described in Section 4.1, replacing there the minimisation of $J(h)$ at (4.1) by a minimisation of $J(H)$, where $H=(h,\rho)$ and $J(H)$ is the version of $J(h)$ obtained when replacing $\widehat{F}_{T\,|\,Q,W,Y}$ at (2.4) by $\widehat{F}_{T\,|\,Q,W,Y}$ at (5.5). We searched for h in the grid described in Section 4.1, and for ρ on an equispaced grid of ten values between $0.2/n$

and $5/n$, which was very time consuming given the bivariate grid search.

The results are summarised in Table 4, where, as in Table 1, we report the EPC of the prediction intervals and the mean interval lengths, both computed over 100 simulated samples. Comparing with Table 1, the results confirm that making a rough guess for the unknown densities, as in Section 4.2, seems preferable to using a purely nonparametric approach. The former is also much faster to compute than the latter.

6 Technical arguments

6.1 Proof of Theorem 1

The following notation will be useful. Put

$$\tau_{0k} = \int \psi_k(t, y, q, w, x)\, f_X(x)\, dx\,, \quad c_{0k} = \frac{1}{\ell!}\, \kappa_\ell\, \lambda_k^{[\ell]}(0 \,|\, s, y, q, w)\,, \tag{6.1}$$

where the subscript 0 on τ_{0k} and c_{0k} indicates that, in this instance, the true values of the parameters β_0, β_1, β_2 and σ_V^2 are used to construct ψ_k. In this notation,

$$F_{T\,|\,Q,W,Y}(t \,|\, q, w, y) = \tau_{01}(t, y, q, w)\big/\tau_{02}(t, y, q, w)\,. \tag{6.2}$$

6.1.1 Proof of part (i) of Theorem 1

Step 1: Approximation of $\hat{\psi}_k$ by ψ_k. Let $\mathcal{S}$, $\mathcal{Y}$ and $\mathcal{Q}$ be compact sets in the respective domains of s, y and q. Using Taylor expansion it can be proved that

$$\hat{\psi}_k(s, y, q, w, x) = \psi_k(s, y, q, w, x) + \Delta(s, y, q, w, x)\,, \tag{6.3}$$

where ψ_k and $\hat{\psi}_k$ are as at (2.6) and (2.8), respectively, ψ_k is at (2.6), and, for constants $C_1, C_2 > 0$, depending on $\mathcal{S}$, $\mathcal{Y}$ and $\mathcal{Q}$, and the true values of the parameters β_0, β_1, β_2 and σ_V^2 (which we denote as here), but not on n,

$$|\Delta(s, y, q, w, x)| \;\leq\; C_1\,(1 + |x| + \|q\|)\, f_U(w - x) \Big\{ \big|\hat{\beta}_0 - \beta_0\big|$$

$$+\left|\hat{\beta}_1 - \beta_1\right| |x| + \left\|\hat{\beta}_2 - \beta_2\right\| \|q\| + \left|\hat{\sigma}_V^2 - \sigma_V^2\right| \Big\} \quad (6.4)$$

whenever

$$\max\Big\{\left|\hat{\beta}_0 - \beta_0\right|, \left|\hat{\beta}_1 - \beta_1\right|, \left\|\hat{\beta}_2 - \beta_2\right\|, \left|\hat{\sigma}_V^2 - \sigma_V^2\right|\Big\} \le C_2\,. \quad (6.5)$$

The bound at (6.4) holds uniformly in all $s \in \mathcal{S}$, all $y \in \mathcal{Y}$, all $q \in \mathcal{Q}$ and all real x, provided that (6.5) holds.

Step 2: Mean and variance of numerator and denominator on right-hand side of (2.7). Formulae for the mean square and mean are given at (6.13) and (6.14), respectively, leading to formula (6.16) for the variance.

Recall the definition of ψ_k at (2.6), let $\psi = \psi_k$ for either $k = 1$ or $k = 2$, define

$$J_1 = \int \psi(s, y, q, w, W + hu)\, K_U(u; h)\, du\,, \quad \tau_j = E\big(J_1^j\big)\,, \quad (6.6)$$

for $j = 1, 2$, and note that

$$\begin{aligned} J_2 &\equiv \int \psi(s, y, q, w, x)\, \hat{f}_X(x)\, dx \quad (6.7)\\ &= \frac{1}{nh} \sum_{j=1}^{n} \int \psi(s, y, q, w, x)\, K_U\Big(\frac{x - W_j}{h}; h\Big)\, dx \\ &= \frac{1}{n} \sum_{j=1}^{n} \int \psi(s, y, q, w, W_j + hu)\, K_U(u; h)\, du\,, \end{aligned}$$

$$n\,\mathrm{var}(J_2) = \mathrm{var}\Big\{\int \psi(s, y, q, w, W + hu)\, K_U(u; h)\, du\Big\} = \tau_2 - \tau_1^2\,, \quad (6.8)$$

$$\int e^{itu}\, \psi(s, y, q, w, W + hu)\, du = h^{-1}\, \exp(-it\, W/h)\, \chi(t/h)\,,$$

where $\chi(t) = \chi(t \,|\, s, y, q, w) = \int e^{itu}\, \psi(s, y, q, w, u)\, du$. Therefore, by Parseval's identity,

$$\begin{aligned} &\int \psi(s, y, q, w, W + hu)\, K_U(u; h)\, du \\ &= \frac{1}{2\pi h} \int \exp(-it\, W/h)\, \chi(t/h)\, \frac{\phi_K(t)}{\phi_U(t/h)}\, dt = \frac{1}{2\pi} \int \exp(-it\, W)\, \chi(t)\, \frac{\phi_K(ht)}{\phi_U(t)}\, dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi} \int \Re\big\{ \exp(-it\,W)\,\chi(t)\big\}\,\frac{\phi_K(ht)}{\phi_U(t)}\,dt \\
&= \frac{1}{2\pi} \int \big\{ \cos(t\,W)\,\chi_1(t) - \sin(t\,W)\,\chi_2(t)\big\}\,\frac{\phi_K(ht)}{\phi_U(t)}\,dt\,,
\end{aligned}
$$

where $\chi_1 = \Re\,\chi$, $\chi_2 = \Im\,\chi$, and we have used the fact that $\phi_K(ht)/\phi_U(t)$ is a symmetric function of t. Hence, defining $\rho_j = \chi_j/\phi_U$,

$$
\begin{aligned}
&(2\pi)^2\,\tau_2 \\
&\quad= \int\!\!\int E\Big\{ \cos(t_1W)\,\cos(t_2W)\,\chi_1(t_1)\,\chi_1(t_2) + \sin(t_1W)\,\sin(t_2W)\,\chi_2(t_1)\,\chi_2(t_2) \\
&\qquad - \cos(t_1W)\,\sin(t_2W)\,\chi_1(t_1)\,\chi_2(t_2) - \sin(t_1W)\,\cos(t_2W)\,\chi_2(t_1)\,\chi_1(t_2)\Big\} \\
&\qquad \times \frac{\phi_K(ht_1)\,\phi_K(ht_2)}{\phi_U(t_1)\,\phi_U(t_2)}\,dt_1\,dt_2 \\
&\quad= \tfrac{1}{2}\int\!\!\int \Big\{ E\Big[\cos\{(t_1+t_2)\,W\} + \cos\{(t_1-t_2)\,W\}\Big]\,\chi_1(t_1)\,\chi_1(t_2) \\
&\qquad + E\Big[\cos\{(t_1-t_2)\,W\} - \cos\{(t_1+t_2)\,W\}\Big]\,\chi_2(t_1)\,\chi_2(t_2) \\
&\qquad - E\Big[\sin\{(t_1+t_2)\,W\} + \sin\{(t_2-t_1)\,W\}\Big]\,\chi_1(t_1)\,\chi_2(t_2) \\
&\qquad - E\Big[\sin\{(t_1+t_2)\,W\} + \sin\{(t_1-t_2)\,W\}\Big]\,\chi_2(t_1)\,\chi_1(t_2)\Big\} \\
&\qquad \times \frac{\phi_K(ht_1)\,\phi_K(ht_2)}{\phi_U(t_1)\,\phi_U(t_2)}\,dt_1\,dt_2 \\
&\quad= \tfrac{1}{2}\int\!\!\int \Big[\Re\big\{\phi_W(t_1+t_2) + \phi_W(t_1-t_2)\big\}\,\rho_1(t_1)\,\rho_1(t_2) \\
&\qquad + \Re\big\{\phi_W(t_1-t_2) - \phi_W(t_1+t_2)\big\}\,\rho_2(t_1)\,\rho_2(t_2) \\
&\qquad - \Im\big\{\phi_W(t_1+t_2) + \phi_W(t_2-t_1)\big\}\,\rho_1(t_1)\,\rho_2(t_2) \\
&\qquad - \Im\big\{\phi_W(t_1+t_2) + \phi_W(t_1-t_2)\big\}\,\rho_2(t_1)\,\rho_1(t_2)\Big]\,\phi_K(ht_1)\,\phi_K(ht_2), dt_1\,dt_2\,. \qquad (6.9)
\end{aligned}
$$

For notational simplicity, take the support of ϕ_K to equal $[-1, 1]$. Then, defining ϕ_{W0} to be either $\Re\,\phi_W$ or $\Im\,\phi_W$; and, as in (3.1), we have, taking the $\pm$ signs respectively throughout:

$$
\int\!\!\int \phi_{W0}(t_1 \pm t_2)\,\chi_{j_1}(t_1)\,\chi_{j_2}(t_2)\,\frac{\phi_K(ht_1)\,\phi_K(ht_2)}{\phi_U(t_1)\,\phi_U(t_2)}\,dt_1\,dt_2
$$

$$
\begin{aligned}
&= \int_{-1/h}^{1/h} \int_{-1/h}^{1/h} \phi_{W0}(t_1 \pm t_2)\, \rho_{j_1}(t_1)\, \rho_{j_2}(t_2)\, \phi_K(ht_1)\, \phi_K(ht_2)\, dt_1\, dt_2 \\
&= \int_{-1/h}^{1/h} \rho_{j_1}(t_1)\, \phi_K(ht_1)\, dt_1 \int_{t_1-1/h}^{t_1+1/h} \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K\{h\,(t-t_1)\}\, dt \\
&= \frac{1}{h} \int_{-1}^{1} \rho_{j_1}(t_1/h)\, \phi_K(t_1)\, dt_1 \int_{(t_1-1)/h}^{(t_1+1)/h} \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1/h)\} \\
&\quad \times \phi_K(ht-t_1)\, dt \\
&\equiv R_1(h)\,. \qquad (6.10)
\end{aligned}
$$

We shall prove in Appendix F that, uniformly in $s \in \mathcal{S}$, $y \in \mathcal{Y}$ and $q \in \mathcal{Q}$,

$$
R_1(h) = \begin{cases} O(1) & \text{if } w \neq 0 \\ C_1\,(s_k/b_1)^2\, h^{-1} + o(h^{-1}) & \text{if } w = 0 \text{ and } j_1 = j_2 = 1 \\ o(h^{-1}) & \text{if } w = 0 \text{ and } j_1 = j_2 = 1 \text{ fails}\,, \end{cases} \qquad (6.11)
$$

where s_k and b_1 are as in (3.1), and the positive constant C_1 depends only on K and $f_W(0)$.

Combining (6.9)–(6.11) we deduce that, if $w \neq 0$,

$$
\tau_2 = O(1)\,. \qquad (6.12)
$$

If $w = 0$ then, noting from (6.11) that $R_1(h) = o(h^{-1})$ if $j_1 = j_2 = 1$ fails, we have:

$$
\begin{aligned}
\tau_2 &= \tfrac{1}{2}\,(2\pi)^{-2} \iint \Re\left\{\phi_W(t_1+t_2) + \phi_W(t_1-t_2)\right\} \chi_1(t_1)\, \chi_1(t_2) \\
&\qquad \times \frac{\phi_K(ht_1)\, \phi_K(ht_2)}{\phi_U(t_1)\, \phi_U(t_2)}\, dt_1\, dt_2 + o\big(h^{-1}\big) \\
&= C_2\,(s_k/b_1)^2\, h^{-1} + o\big(h^{-1}\big)\,, \qquad (6.13)
\end{aligned}
$$

where on this occasion $\phi_{W0} = \Re\, \phi_W$ and we define $C_2 = C_1/(2\pi)^2$.

More simply, recalling the definition of κ_ℓ immediately below (3.5), and assuming that (3.4) holds, we can write:

$$
\tau_1 = \iint \psi(s, y, q, w, w_1 + hu)\, f_W(w_1)\, K_U(u; h)\, du\, dw_1
$$

$$
\begin{aligned}
&= \int\!\!\int \psi(s,y,q,w,x+hu)\, f_X(x)\, K(u)\, du\, dx = \int K(u)\, \lambda_k(hu \,|\, s,y,q,w)\, du \\
&= \int \psi(s,y,q,w,x)\, f_X(x)\, dx + \frac{1}{\ell!}\, \kappa_\ell\, h^\ell\, \lambda_k^{[\ell]}(0 \,|\, s,y,q,w) + o\big(h^\ell\big)\,. \qquad (6.14)
\end{aligned}
$$

Combining (6.8), (6.12) and (6.14) we deduce that if $w \neq 0$,

$$
\mathrm{var}\bigg\{ \int \psi(s,y,q,w,x)\, \hat{f}_X(x)\, dx \bigg\} = O\big(n^{-1}\big)\,, \qquad (6.15)
$$

whereas if, using (6.13) and (6.14) we deduce that if $w = 0$,

$$
n\, \mathrm{var}\bigg\{ \int \psi(s,y,q,w,x)\, \hat{f}_X(x)\, dx \bigg\} = C_2\, (s_k/b_1)^2\, h^{-1} + o\big(h^{-1}\big)\,. \qquad (6.16)
$$

Step 3: Completion. For simplicity we treat only the case $w = 0$. The quantities J_2 and τ_1 each have two forms, depending on whether we take $k = 1$ or $k = 2$ in the formula $\psi = \psi_k$ used when defining J_2 at (6.7) and τ_1 at (6.6). Write J_1, J_2 and τ_1 as J_{1k}, J_{2k} and τ_{1k}, respectively, to indicate these possibilities, put

$$
\Delta_k = J_{2k} - E(J_{2k}) = J_{2k} - E(J_{1k}) = J_{2k} - \tau_{1k}\,,
$$

let τ_{0k} denote the version of τ_{1k} when we take $h = 0$ in the latter (this is equivalent to the definition at (6.1)); note that, by (6.14), $\tau_{1k} = \tau_{0k} + c_{0k}\, h^\ell + o(h^\ell)$; and recall that J_2, and hence also J_{2k} and τ_{1k}, are functions of s, q and w.

Using (6.3), Theorem 4, and the definition of $\widehat{F}_{T\,|\,Q,W,Y}$ at (2.7), we have:

$$
\begin{aligned}
\widehat{F}_{T\,|\,Q,W,Y}(t \,|\, q,w,y) &= \frac{J_{21}}{J_{22}} + O_p\big(n^{-1/2}\big) = \frac{\tau_{11} + \Delta_1}{\tau_{12} + \Delta_2} + O_p\big(n^{-1/2}\big) \\
&= \frac{\tau_{11}}{\tau_{12}} + \frac{\Delta_1}{\tau_{12}} - \frac{\tau_{11}}{\tau_{12}^2}\, \Delta_2 + o_p\big\{(nh)^{-1/2}\big\} \\
&= \frac{\tau_{01}}{\tau_{02}} + \bigg(\frac{c_{01}}{\tau_{02}} - \frac{\tau_{01}}{\tau_{02}^2}\, c_{02} \bigg)\, h^\ell + \tau_{02}^{-1}\, \Delta_1 - \tau_{01}\, \tau_{02}^{-2}\, \Delta_2 \\
&\qquad + o_p\big\{(nh)^{-1/2}\big\} + o\big(h^\ell\big)\,, \qquad (6.17)
\end{aligned}
$$

where the quantities J_{2k}, Δ_k and τ_{1k}, which earlier we defined as functions of s, q and w, are here computed for $(s,y,q,w) = (t,y,q,w)$.

Note too that

$$\mathrm{var}\big(\tau_{02}^{-1}\,\Delta_1 - \tau_{01}\,\tau_{02}^{-2}\,\Delta_2\big) = \frac{1}{\tau_{02}^2}\,\mathrm{var}(\Delta_1) + \frac{\tau_{01}^2}{\tau_{02}^4}\,\mathrm{var}(\Delta_2) - \frac{2\,\tau_{01}}{\tau_{02}^3}\,\mathrm{cov}(\Delta_1,\Delta_2)\,. \quad (6.18)$$

By (6.13), $\mathrm{var}(\Delta_k) = (nh)^{-1}\,C_2\,(s_k/b_1)^2 + o\big\{(nh)^{-1}\big\}$, and similarly it can be proved that $\mathrm{cov}(\Delta_1,\Delta_2) = (nh)^{-1}\,C_2\,\big(s_1\,s_2/b_1^2\big) + o\big\{(nh)^{-1}\big\}$. Hence, by (6.18),

$$nh\,\mathrm{var}\big(\tau_{02}^{-1}\,\Delta_1 - \tau_{01}\,\tau_{02}^{-2}\,\Delta_2\big) = \frac{C_2}{b_1^2}\left(\frac{s_1}{\tau_{02}} - \frac{s_2\,\tau_{01}}{\tau_{02}^2}\right)^2 + o(1)\,. \quad (6.19)$$

Thus, part (i) of Theorem 1 follows from (6.2), (6.17) and (6.19).

6.1.2 Proof of part (ii) of Theorem 1

We treat only the case $w = 0$. Recall the definitions of J_1 and J_2 at (6.6) and (6.7), respectively, and define $\tau_j = E(J_1^j)$ for each integer $j \geq 1$. In step 3 of the proof of part (i) of the theorem we showed that $\tau_4 = O(h^{-3})$, and more generally it can be proved that $\tau_j = O(h^{1-j})$. Hence, since $nh \to \infty$, we have for each integer $\nu \geq 2$, $\tau_{2\nu} = o(n^{\nu-1}\,h^{-\nu})$. (As before, these convergence results, and the order of magnitude bounds below, hold uniformly in s, y and q in respective compact sets $\mathcal{S}$, $\mathcal{Y}$ and $\mathcal{Q}$; recall that J_1 and J_2 are functions of (s,y,q).) Therefore, by Rosenthal's inequality,

$$n^{2\nu}\,E(J_2 - EJ_2)^{2\nu} = O\big\{(n\tau_2)^\nu + n\,\tau_{2\nu}\big\} = O\big\{(n/h)^\nu\big\}\,,$$

whence $E(J_2 - EJ_2)^{2\nu} = O\{(nh)^{-\nu}\}$. Hence, by Markov's inequality,

$$P\Big\{\big|J_2(s,y,q) - EJ_2(s,y,q)\big| > n^{\varepsilon_3}\,(nh)^{-1/2}\Big\} = O\big(n^{-B_5}\big)$$

for all $B_5, \varepsilon_3 > 0$.

As mentioned above, this bound applies uniformly in $(s,y,q) \in \mathcal{S}\times\mathcal{Y}\times\mathcal{Q}$. Therefore, if $\mathcal{S}'$, $\mathcal{Y}'$ and $\mathcal{Q}'$ are subsets of $\mathcal{S}$, $\mathcal{Y}$ and $\mathcal{Q}$, respectively, each representing a regular lattice and containing no more than $O(n^{B_6})$ points for some $B_6 > 0$, then

for all $B_5, \varepsilon_3 > 0$,

$$P\Big\{ \sup_{s \in \mathcal{S}'} \sup_{y \in \mathcal{Y}'} \sup_{q \in \mathcal{Q}'} \big| J_2(s,y,q) - EJ_2(s,y,q) \big| > n^{\varepsilon_3} \, (nh)^{-1/2} \Big\} = O\big(n^{-B_5}\big) \,, \qquad (6.20)$$

Given $(s,y,q) \in \mathcal{S} \times \mathcal{Y} \times \mathcal{Q}$, let $(s',y',q') \in \mathcal{S}' \times \mathcal{Y}' \times \mathcal{Q}'$ minimise the distance from (s',y',q') to (s,y,q). Making use of continuity properties of J_2, as a function of s, y, q and w, it can be proved that if B_6 is sufficiently large then

$$P\Big\{ \sup_{s \in \mathcal{S}} \sup_{y \in \mathcal{Y}} \sup_{q \in \mathcal{Q}} \big| J_2(s,y,q) - J_2(s',y',q') \big| > n^{-1} \Big\} = O\big(n^{-B_5}\big) \,, \qquad (6.21)$$

$$\sup_{s \in \mathcal{S}} \sup_{y \in \mathcal{Y}} \sup_{q \in \mathcal{Q}} \big| E\{J_2(s,y,q)\} - E\{J_2(s',y',q')\} \big| = O\big(n^{-1}\big) \,. \qquad (6.22)$$

Combining (6.20)–(6.22) we deduce that for all $B_5, \varepsilon_3 > 0$,

$$P\Big\{ \sup_{s \in \mathcal{S}} \sup_{y \in \mathcal{Y}} \sup_{q \in \mathcal{Q}} \big| J_2(s,y,q) - EJ_2(s,y,q) \big| > n^{\varepsilon_3} \, (nh)^{-1/2} \Big\} = O\big(n^{-B_5}\big) \,. \qquad (6.23)$$

There are of course two versions of J_2, depending on whether we take $\psi = \psi_1$ or ψ_2 in the definition of J_2 at (6.7). Result (6.23) holds for both versions, and we shall distinguish them by writing J_{k2} for J_2 when $\psi = \psi_k$. In this notation, (6.14) implies that

$$\frac{E\{J_{21}(t,y,q)\}}{E\{J_{22}(t,y,q)\}} = F_{T\,|\,Q,W,Y}(t\,|\,q,w,y) + O\big(h^{\ell}\big) \,, \qquad (6.24)$$

uniformly in $(t,y,q) \in \mathcal{S} \times \mathcal{Y} \times \mathcal{Q}$. More simply, (2.7) implies that

$$\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \frac{J_{21}(t,y,q)}{J_{22}(t,y,q)} \,,$$

and so by (6.23), for each $\varepsilon_3 > 0$,

$$\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \frac{E\{J_{21}(t,y,q)\}}{E\{J_{22}(t,y,q)\}} + O_p\big\{ n^{\varepsilon_3} \, (nh)^{-1/2} \big\} \,, \qquad (6.25)$$

uniformly in $(t,y,q) \in \mathcal{S} \times \mathcal{Y} \times \mathcal{Q}$. Part (ii) of Theorem 1 follows from (6.24) and (6.25).

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A Conditional distribution of T

We have

$$\begin{aligned} f_{T\,|\,Q,W,Y}(t\,|\,q,w,y) &= \int f_{T\,|\,Q,W,X,Y}(t\,|\,q,w,x,y)\, f_{X\,|\,Q,W,Y}(x\,|\,q,w,y)\, dx \\ &= \int f_{T\,|\,Q,X,Y}(t\,|\,q,x,y)\, f_{X\,|\,Q,W,Y}(x\,|\,q,w,y)\, dx\,. \end{aligned} \tag{A.1}$$

Then, using basic properties of conditional densities, we note that

$$\begin{aligned} &f_{T\,|\,Q,X,Y}(t\,|\,q,x,y) = f_\epsilon(y-t)\, f_V(t-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q)\big/ f_{V+\epsilon}(y-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q)\,, \\ &f_{X\,|\,Q,W,Y}(x\,|\,q,w,y) = \frac{f_{V+\epsilon}(y-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q)\, f_X(x)\, f_U(w-x)\, f_Q(q)}{f_{Q,W,Y}(q,w,y)}\,, \\ &f_{Q,W,Y}(q,w,y) = f_Q(q) \int f_{V+\epsilon}\big(y-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q\big)\, f_U(w-x)\, f_X(x)\, dx\,. \end{aligned}$$

Hence,

$$\begin{aligned} &f_{T\,|\,Q,X,Y}(t\,|\,q,x,y)\, f_{X\,|\,Q,W,Y}(x\,|\,q,w,y) \\ &\quad = \frac{f_\epsilon(y-t)\, f_V(t-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q)\, f_X(x)\, f_U(w-x)}{\int f_{V+\epsilon}(y-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q) f_U(w-x)\, f_X(x)\, dx}\,. \end{aligned} \tag{A.2}$$

Combining (A.1) and (A.2), and recalling that ϵ has a symmetric distribution, we deduce that

$$f_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \frac{f_\epsilon(t-y) \int f_V(t-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q)\, f_X(x)\, f_U(w-x)\, dx}{\int f_{V+\epsilon}(y-\beta_0-\beta_1 x-\beta_2^{\mathrm{T}} q)\, f_U(w-x)\, f_X(x)\, dx}\,. \tag{A.3}$$

B Estimating the unknown parameters in (2.2)

Let $\sigma_U^2 = \mathrm{var}(U)$, $\sigma_W^2 = \mathrm{var}(W)$ and $\sigma_X^2 = \mathrm{var}(X)$. We can estimate the unknown parameters using standard approaches employed in classical measurement error linear models (see e.g. Fuller, 2009 and Buonaccorsi, 2010). Like there, since $\sigma_W^2 = \sigma_X^2 + \sigma_U^2$ and σ_U^2 is known, we start by estimating σ_X^2 by $\hat\sigma_X^2 = \max\big(0, \hat\sigma_W^2 - \sigma_U^2\big)$, where $\hat\sigma_W^2 = n^{-1}\sum_{j=1}^n (W_j - \bar W)^2$ and $\bar W = n^{-1}\sum_j W_j$. Then, letting $Z_j = (1, W_j, Q_j^{\mathrm{T}})^{\mathrm{T}}$

and $\mathbf{Z} = (Z_1, \ldots, Z_n)^{\mathrm{T}}$, and defining the $(p+2)\times(p+2)$ matrix $\Sigma_U = (\Sigma_{U,i,j})_{i,j=1,\ldots,p+2}$ to be zero everywhere except for the (2,2)th component, which is equal to σ_U^2, we take $\widehat{M} = n^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{Z} - \Sigma_U$. Then, letting $\bar{Y} = n^{-1}\sum_j Y_j$, $T_{WY} = n^{-1}\sum_{j=1}^n W_j\, Y_j$, $T_{QY} = n^{-1}\sum_{j=1}^n Q_j\, Y_j$, and assuming that $\det \widehat{M} > 0$, we estimate β_0, β_1 and β_2 by

$$\left(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2^{\mathrm{T}}\right)^{\mathrm{T}} = \widehat{M}^{-1}\left(\bar{Y}, T_{WY}, T_{QY}^{\mathrm{T}}\right)^{\mathrm{T}}. \tag{B.1}$$

Finally, to estimate σ_V^2, let $\bar{\tau} = n^{-1}\sum_j \tau_j$ and $\hat{\sigma}_Y^2 = n^{-1}\sum_{j=1}^n (Y_j - \bar{Y})^2$. It follows from (2.1) that $\mathrm{var}(Y_j) = \beta_1^2\,\sigma_X^2 + \beta_2^{\mathrm{T}}\,\Sigma_Q\,\beta_2 + \sigma_V^2 + \tau_j$, which suggests using

$$\hat{\sigma}_V^2 = \max\left\{0, \hat{\sigma}_Y^2 - \hat{\beta}_1^2\,\hat{\sigma}_X^2 - \hat{\beta}_2^{\mathrm{T}}\,\widehat{\Sigma}_Q\,\hat{\beta}_2 - \bar{\tau}\right\}. \tag{B.2}$$

In our numerical examples in Section 4, our sample sizes are small, and in that case, Fuller (2009) and Buonaccorsi (2010) noted that, although it is a covariance matrix, the matrix $\widehat{M}$ is not always invertible. To overcome this difficulty, we apply to it the same correction as in page 121 of Buonaccorsi (2010). A similar problem arises with $\hat{\sigma}_V^2$, and we overcome it by applying the bagging technique described in Section 2.2 of Delaigle and Hall (2011).

The next theorem establishes root-n consistency of the estimators $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\sigma}_V^2$, defined at (B.1) and (B.2). The proof follows the arguments in Fuller (2009) and thus is omitted.

Theorem 4. *If the random quantities Q, U, V and X all have finite fourth moments, if $M = E\{(1, X, Q^{\mathrm{T}})^{\mathrm{T}}(1, X, Q^{\mathrm{T}})\}$ is nonsingular and $\sigma_V^2\,\sigma_X^2 \neq 0$, then $\hat{\beta}_0 - \beta_0$, $\hat{\beta}_1 - \beta_1$, $\|\hat{\beta}_2 - \beta_2\|$ and $\hat{\sigma}_V^2 - \sigma_V^2$ all equal $O_p(n^{-1/2})$ as n increases. Moreover, as $n \to \infty$ we have*

$$n^{1/2}\left\{\left(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2^{\mathrm{T}}\right)^{\mathrm{T}} - \left(\beta_0, \beta_1, \beta_2^{\mathrm{T}}\right)^{\mathrm{T}}\right\} \xrightarrow{D} N(0, \Sigma),$$

where, using the notation $\tau^* = \lim_{n\to\infty} \bar{\tau}$ *and* $\sigma^2_{\rm err} = \tau^* + \sigma^2_V + \beta_1^2 \sigma^2_U$,

$$\Sigma = \sigma^2_{\rm err} M^{-1} + \left\{ \beta_1^2 {\rm var}(U^2) + (\tau^* + \sigma^2_V)\sigma^2_U \right\} M^{-1} \begin{pmatrix} 0 & 0 & 0_{1\times p} \\ 0 & 1 & 0_{1\times p} \\ 0 & 0 & 0_{p\times p} \end{pmatrix} M^{-1} \ .$$

C Discussion of the conditions in Section 3.1

It can be proved from the definition of χ, and the first assumption in (3.1)(ii), that ρ_j and ρ_j' are both bounded on any compact interval. If $\phi_U(t)$ is asymptotic to a constant multiple of t^{-2r} as $|t| \to \infty$, as it would be if (for example) the distribution of U were that of an r-fold convolution of Laplace-distributed random variables, then (3.1)(iv) is readily proved. When (3.1) holds, integrations by parts (see Appendix E) can be used to prove that, as $|t| \to \infty$,

$$\rho_1(t) = \beta(t)^{-1} \left[\cos(tw)\, s_k + \frac{\sin t}{t} \left\{ \Psi'_{kr}(w-) - \Psi'_{kr}(w+) \right\} \right] + O\big(t^{-2}\big) \,, \quad \text{(C.1)}$$

$$\rho_2(t) = \beta(t)^{-1} \left[\sin(tw)\, s_k - \frac{\cos t}{t} \left\{ \Psi'_{kr}(w-) - \Psi'_{kr}(w+) \right\} \right] + O\big(t^{-2}\big) \,, \quad \text{(C.2)}$$

and so $|\rho_j|$ is bounded on $\mathbb{R}$. Moreover, in the Laplace case, (C.1) and (C.2) continue to hold if both sides of each equation are differentiated naively with respect to t. Therefore, in this case, $|\rho_j'|$ is bounded on $\mathbb{R}$, establishing the last part of (3.1)(ii). Also, (3.1)(i) holds if the distribution of U is an r-fold convolution of Laplace distributions.

D Theorem 5

The methods used to derive Theorem 1 can be employed to show that, under the same conditions, all partial derivatives of $\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$ with respect to t converge at the same rate to the respective derivatives of $F_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$. In particular, if for

each integer $r \geq 0$ we define

$$\begin{aligned} \widehat{F}^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) &= \left(\frac{\partial}{\partial t}\right)^r \widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q,w,y)\,, \\ F^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) &= \left(\frac{\partial}{\partial t}\right)^r F_{T\,|\,Q,W,Y}(t\,|\,q,w,y)\,, \end{aligned}$$

then the following result holds.

Theorem 5. *Assume the conditions imposed in Theorem 4, and that* (3.1)–(3.3) *and* (3.5) *hold, and let $r \geq 0$ be an integer. Then: (i) For each real t and y, and each $q \in \mathbb{R}^p$,*

$$\widehat{F}^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) - F^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \begin{cases} O_p\big\{(nh)^{-1/2} + h^\ell\big\} & \text{if } w = 0 \\ O_p\big(n^{-1/2} + h^\ell\big) & \text{if } w \neq 0\,; \end{cases} \tag{D.1}$$

and (ii) For each $\eta > 0$,

$$\widehat{F}^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) - F^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \begin{cases} O_p\Big\{\big(n^{1-\eta}h\big)^{-1/2} + h^\ell\Big\} & \text{if } w = 0 \\ O_p\big(n^{-(1-\eta)/2} + h^\ell\big) & \text{if } w \neq 0\,, \end{cases}$$

uniformly in t, q and y in any compact subsets of their respective domains, where in the case $w = 0$ we ask in addition that $n^{1-\eta}h \to \infty$.

The methods employed to establish these results are similar to those used to derive Theorem 1. The reason the convergence rates of estimators of the distribution function derivatives $F^{(r)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y)$ do not depend on r is that the derivatives have the same form as the original function estimators. For example, if we define

$$\Psi^{(r)}_k(t,y,q,w) = \left(\frac{\partial}{\partial t}\right)^r \Psi_k(t,y,q,w)\,, \quad \widehat{\Psi}^{(r)}_k(t,y,q,w) = \left(\frac{\partial}{\partial t}\right)^r \widehat{\Psi}_k(t,y,q,w)\,,$$

then it can be proved that $\widehat{\Psi}^{(r)}_k(t,y,q,w) = \Psi^{(r)}_k(t,y,q,w) + O_p\{(nh)^{-1/2} + h^\ell\}$ for each (t,y,q,w), each $r \geq 0$ and $k = 1,2$. Therefore, using standard formulae for derivatives, such as

$$\widehat{F}^{(2)}_{T\,|\,Q,W,Y}(t\,|\,q,w,y) = \frac{\widehat{\Psi}'_1(t,y,q,w)\,\widehat{\Psi}_2(t,y,q,w) - \widehat{\Psi}_1(t,y,q,w)\,\widehat{\Psi}'_2(t,y,q,w)}{\widehat{\Psi}_2(t,y,q,w)^2}$$

(compare (2.7)), it can be proved that (D.1) holds.

E Proof of (C.1) and (C.2)

Define

$$\begin{aligned}\gamma_r(t) &= \int \Psi_{kr}(x)\left(\frac{\partial}{\partial x}\, e^{itx}\right) dx = -\int e^{itx}\, d\Psi_{kr}(x) \\ &= -\left\{e^{itw}\, s_k + \left(\int_{-\infty}^{w-} + \int_{w+}^{\infty}\right) e^{itx}\, \Psi'_{kr}(x)\, dx\right\} = -\left\{e^{itw}\, s_k + \delta_r(t)\right\}\end{aligned}$$

where, in view of (3.1)(i), the function δ_r satisfies $\sup_{-\infty<t<\infty} |\delta_r(t)| < \infty$. Recall that $\chi_1 = \Re\,\chi$ and $\chi_2 = \Im\,\chi$, and put $\gamma_{r1} = \Re\,\gamma_r$, $\gamma_{r2} = \Im\,\gamma_r$, $\alpha_1(t) = \cos(tw) + \Re\,\delta_r(t)$ and $\alpha_2(t) = \sin(tw) + \Im\,\delta_r(t)$. In this notation,

$$\rho_j(t) = \frac{\chi_j(t)}{\phi_U(t)} = -\frac{\gamma_{rj}(t)}{t^{2r}\,\phi_U(t)} = \frac{\alpha_j(t)}{\beta(t)}\,. \tag{E.1}$$

Using (3.1)(i) it can be shown that

$$\begin{aligned}-\gamma_r(t) &= e^{itw}\, s_k + \frac{1}{it}\left(\int_{-\infty}^{w-} + \int_{w+}^{\infty}\right) \Psi'_{kr}(x)\left(\frac{\partial}{\partial x}\, e^{itx}\right) dx \\ &= e^{itw}\, s_k + (it)^{-1}\, e^{itw}\left\{\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right\} \\ &\qquad -\frac{1}{it}\left(\int_{-\infty}^{w-} + \int_{w+}^{\infty}\right) \Psi''_{kr}(x)\; e^{itx}\, dx \\ &= e^{itw}\, s_k + (it)^{-1}\, e^{itw}\left\{\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right\} \\ &\qquad -\frac{1}{(it)^2}\left(\int_{-\infty}^{w-} + \int_{w+}^{\infty}\right) \Psi''_{kr}(x)\left(\frac{\partial}{\partial x}\, e^{itx}\right) dx \\ &= e^{itw}\, s_k + (it)^{-1}\, e^{itw}\left\{\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right\} + O\big(t^{-2}\big)\,.\end{aligned}$$

Hence, the functions α_1 and α_2 can be written as

$$\alpha_1(t) = \cos(tw)\, s_k + \frac{\sin t}{t}\,\left\{\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right\} + O\big(t^{-2}\big)\,, \tag{E.2}$$

$$\alpha_2(t) = \sin(tw)\, s_k - \frac{\cos t}{t}\,\left\{\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right\} + O\big(t^{-2}\big)\,, \tag{E.3}$$

where the remainders are of that order as $|t| \to \infty$; and more simply, $|\rho_1|$ and $|\rho_2|$ are bounded uniformly on $\mathbb{R}$. The desired results (C.1) and (C.2) follow from (E.2) and (E.3), respectively.

F Proof of (6.11)

Recall that χ_j, and hence also $\rho_j = \phi_j/\phi_U$, depends on k, which equals 1 or 2, and that $\phi_{W0} = \Re\,\phi_W$ or $\Im\,\phi_W$. Therefore $R_1(h)$, at (6.10), depends on j_1, j_2 and k. In each step the quantities $B_1, B_2, \ldots$ denote generic constants.

Step 1: Difference between R_1 and R_2; see (F.1). Define

$$R_2(h) \;=\; \frac{1}{h}\int_{t_1\,:\,h<|t_1|<1} \rho_{j_1}(t_1/h)\,\phi_K(t_1)\,dt_1 \int \phi_{W0}(t)\,\rho_{j_2}\{\pm(t-t_1/h)\}\,\phi_K(ht-t_1)\,dt\,.$$

Then,

$$|R_1(h)-R_2(h)| \le \frac{B_1}{h}\int_{-h}^{h}|\phi_K(t_1)|\,dt_1\int_{-\infty}^{\infty}|\phi_{W0}(t)|\,dt \le \frac{B_2}{h}\int_{-h}^{h}dt_1 = 2\,B_2\,. \quad \text{(F.1)}$$

Step 2: Difference between R_2 and R_3; see (F.3). In view of (E.1) to (E.3) in Appendix E we can write

$$\rho_j(t) = \beta(t)^{-1}\Big[\mathrm{cs}_{j1}(tw)\,s_k + (-1)^{j+1}\,\frac{\mathrm{cs}_{j2}\,t}{t}\,\big\{\Psi'_{kr}(w-)-\Psi'_{kr}(w+)\big\}\Big] + O\big(t^{-2}\big)\,, \quad \text{(F.2)}$$

where $(\mathrm{cs}_{j1},\mathrm{cs}_{j2}) = (\cos,\sin)$ or $(\sin,\cos)$ according as $j=1$ or 2, respectively. In this notation, define

$$\begin{aligned} R_3(h) \;=\; \frac{1}{h}\int_{t_1\,:\,h<|t_1|<1} & \beta(t_1/h)^{-1}\Big[\mathrm{cs}_{j_1 1}(t_1 w/h)\,s_k \\ & + (-1)^{j_1+1}\,\frac{\mathrm{cs}_{j_1 2}\,(t_1/h)}{t_1/h}\,\big\{\Psi'_{kr}(w-)-\Psi'_{kr}(w+)\big\}\Big]\,\phi_K(t_1)\,dt_1 \\ & \times\int \phi_{W0}(t)\,\rho_{j_2}\{\pm(t-t_1/h)\}\,\phi_K(ht-t_1)\,dt\,. \end{aligned}$$

Then,

$$|R_2(h)-R_3(h)| \le \frac{B_3}{h}\int_h^1 (t_1/h)^{-2}\,dt_1\int_{-\infty}^{\infty}|\phi_{W0}(t)|\,dt \le B_4\,h\int_h^1 t_1^{-2}\,dt_1 \le B_4\,. \quad \text{(F.3)}$$

Step 3: Difference between R_3 and R_4; see (F.5). For b_1 as in (3.1), define

$$R_4(h) \;=\; \frac{1}{h}\int_{t_1\,:\,h<|t_1|<1}\Big[\beta(t_1/h)^{-1}\,\mathrm{cs}_{j_1 1}(t_1 w/h)\,s_k$$

$$+ (-1)^{j_1+1} b_1^{-1} \frac{\mathrm{cs}_{j_1 2}(t_1/h)}{t_1/h} \left\{\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right\}\Big] \phi_K(t_1)\, dt_1$$
$$\times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t - t_1/h)\}\, \phi_K(ht - t_1)\, dt\,.$$

Now,

$$\left|\beta(t)^{-1} - b_1^{-1}\right| \le B_5\,(1+|t|)^{-b_2} \tag{F.4}$$

for all $|t| > 1$, where $B_5 > 0$ is a constant. See (3.1)(iv). Hence,

$$\begin{aligned} |R_3(h) - R_4(h)| &\le \frac{B_5}{h} \int_{t_1\,:\,h<|t_1|<1} (1+|t_1/h|)^{-b_2}\, |t_1/h|^{-1} \left|\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right| \\ &\qquad \times |\phi_K(t_1)|\, dt_1 \int \left|\phi_{W0}(t)\, \rho_{j_2}\{\pm(t - t_1/h)\}\, \phi_K(ht - t_1)\right| dt \\ &\le \frac{B_6}{h} \int_h^1 (t_1/h)^{-(1+b_2)}\, dt_1 \le B_7\,. \end{aligned} \tag{F.5}$$

Step 4: Difference between R_4 and R_5; see (F.11). Using (3.1)(ii), (C.1), (C.2), (F.2) and (F.4) it can be proved that, for constants $B_8, B_9 > 0$, and for all $|t| > 1$,

$$\left|\rho_j(t) - b_1^{-1}\, \mathrm{cs}_{j1}(tw)\, s_k\right| \le B_8\,(1+|t|)^{-B_9}\,. \tag{F.6}$$

Let

$$\begin{aligned} R_5(h) &= \frac{s_k}{h} \int_{t_1\,:\,h<|t_1|<1} \beta(t_1/h)^{-1}\, \mathrm{cs}_{j_1 1}(t_1 w/h)\, \phi_K(t_1)\, dt_1 \\ &\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t - t_1/h)\}\, \phi_K(ht - t_1)\, dt\,. \end{aligned}$$

Then,

$$\begin{aligned} \left|b_1\,\{R_4(h) - R_5(h)\}\right| &= \frac{1}{h} \left|\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right| \Bigg| b_1 \int_{t_1\,:\,h<|t_1|<1} \frac{\mathrm{cs}_{j_1 2}(t_1/h)}{\beta(t_1/h)\, t_1/h}\, \phi_K(t_1)\, dt_1 \\ &\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t - t_1/h)\}\, \phi_K(ht - t_1)\, dt \Bigg| \\ &\le h^{-1} \left|\Psi'_{kr}(w-) - \Psi'_{kr}(w+)\right| \{S_1(h) + S_2(h)\}\,, \end{aligned} \tag{F.7}$$

where, in view of (3.1)(ii), (3.1)(iii), (3.1)(iv), (C.1), (C.2) and (F.6),

$$\begin{aligned} S_1(h) &= b_1^{-1} \left| s_k \int_{t_1\,:\,h<|t_1|<1} \frac{\mathrm{cs}_{j_1 2}\,(t_1/h)}{t_1/h}\,\phi_K(t_1)\,dt_1 \right. \\ &\qquad\qquad \left. \times \int \phi_{W0}(t)\,\mathrm{cs}_{j_2 1}\{\pm(t-t_1/h)\}\,\phi_K(ht-t_1)\,dt \right|, \qquad \text{(F.8)} \\ S_2(h) &= B_{10} \int_{t_1\,:\,h<|t_1|<1} |t_1/h|^{-1}\,|\phi_K(t_1)|\,dt_1 \int |\phi_{W0}(t)|\,(1+|t-t_1/h|)^{-B_9}\,dt \\ &\le B_{10}B_{11}\,h \int_{t_1\,:\,h<|t_1|<1} |t_1|^{-1}\,|\phi_K(t_1)|\,dt_1 \int (1+|t|)^{-B_{13}}\,(1+|t_1/h|)^{-B_{12}}\,dt \\ &\le B_{14}\,h^{1+B_{12}} \int_{t_1\,:\,h<|t_1|<1} |t_1|^{-1-B_{12}}\,dt_1 \le B_{15}\,h\,. \qquad \text{(F.9)} \end{aligned}$$

Here we have used the fact that there exist constants B_{11}, $B_{12} > 0$ and $B_{13} > 1$ so that, for all t and all t_1,

$$(1+|t|)^{-C_2}(1+|t-t_1/h|)^{-B_9} \le B_{11}(1+|t|)^{-B_{13}}(1+|t_1/h|)^{-B_{12}}.$$

We claim that

$$S_1(h) \le B_{18}\,h\,. \qquad \text{(F.10)}$$

To appreciate why, assume for the sake of definiteness that $j_1 = j_2 = 1$. Then, $\mathrm{cs}_{j_1 2} = \sin$ and $\mathrm{cs}_{j_2 1} = \cos$, and so

$$\mathrm{cs}_{j_2 1}\{\pm(t-t_1/h)\} = \cos(t)\,\cos(t_1/h) \mp \sin(t)\,\sin(t_1/h)\,,$$

whence by (F.8),

$$\begin{aligned} & b_1\,S_1(h) \\ &= \left| \int_{t_1\,:\,h<|t_1|<1} \frac{\sin(t_1/h)\,\cos(t_1/h)}{t_1/h}\,\phi_K(t_1)\,dt_1 \int \phi_{W0}(t)\,\cos(t)\,\phi_K(ht-t_1)\,dt \right. \\ &\qquad \left. - \int_{t_1\,:\,h<|t_1|<1} \frac{\sin(t_1/h)\,\sin(t_1/h)}{t_1/h}\,\phi_K(t_1)\,dt_1 \int \phi_{W0}(t)\,\sin(t)\,\phi_K(ht-t_1)\,dt \right|. \end{aligned}$$

The two terms on the right-hand side can be bounded using similar arguments. In either case the integral over $h < |t_1| < 1$ is broken up into two parts, addressing

respectively $h < t_1 < 1$ and $-1 < t_1 < -h$. We illustrate by treating the first term on the right-hand side, and the first of the two integrals, which we multiply here by $2/h$:

$$\begin{aligned}
&\frac{2}{h}\left|\int_h^1 \frac{\sin(t_1/h)\,\cos(t_1/h)}{t_1/h}\,\phi_K(t_1)\,dt_1 \int \phi_{W0}(t)\,\cos(t)\,\phi_K(ht-t_1)\,dt\right| \\
&\quad = \frac{1}{h}\left|\int_h^1 \frac{\sin(2t_1/h)}{t_1/h}\,\phi_K(t_1)\,dt_1 \int \phi_{W0}(t)\,\cos(t)\,\phi_K(ht-t_1)\,dt\right| \\
&\quad = \left|\int_h^1 \phi_K(t_1)\left\{\frac{\partial}{\partial t_1}\,\xi_1(t_1/h)\right\}dt_1 \int \phi_{W0}(t)\,\cos(t)\,\phi_K(ht-t_1)\,dt\right| \\
&\quad \le B_{19} + \int_h^1 |\phi_K'(t_1)\,\xi_1(t_1/h)|\,dt_1 \int |\phi_{W0}(t)\,\phi_K(ht-t_1)|\,dt \\
&\qquad\qquad + \int_h^1 |\phi_K(t_1)\,\xi_1(t_1/h)|\,dt_1 \int |\phi_{W0}(t)\,\phi_K'(ht-t_1)|\,dt \le B_{20}\,,
\end{aligned}$$

where we have defined

$$\xi_1(u) = \int_1^u \frac{\sin(2v)}{v}\,dv$$

and we have used the fact that $|\phi_K|$, $|\phi_K'|$ and $|\phi_{W0}|$ are integrable, and $|\phi_K|$, $|\phi_K'|$ and $|\xi_1|$ are uniformly bounded (see (3.1)(ii) and (3.1)(iii)). This proves (F.10).

Combining (F.7), (F.9) and (F.10) we deduce that

$$|R_5(h) - R_6(h)| \le B_{21}\,. \tag{F.11}$$

Step 5: Bound for R_6; see (F.13). First we treat the case where $w \neq 0$. There, defining

$$\xi_2(u) = \int_0^u \mathrm{cs}_{j_1 1}(v)\,dv\,,$$

we have:

$$\begin{aligned}
R_6(h) &= s_k \int_{t_1\,:\,1<|t_1|<1/h} \beta(t_1)^{-1}\,\mathrm{cs}_{j_1 1}(t_1 w)\,\phi_K(ht_1)\,dt_1 \\
&\qquad\qquad \times \int \phi_{W0}(t)\,\rho_{j_2}\{\pm(t-t_1)\}\,\phi_K(ht-ht_1)\,dt \\
&= \frac{s_k}{w}\int_{t_1\,:\,1<|t_1|<1/h} \beta(t_1)^{-1}\,\phi_K(ht_1)\left\{\frac{\partial}{\partial t_1}\,\xi_2(t_1 w)\right\}dt_1
\end{aligned}$$

$$\times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K(ht-ht_1)\, dt$$

$$= \; s_k\, w^{-1}\, \{R_{61}(h) + \ldots + R_{64}(h)\} + O(1)\,, \tag{F.12}$$

where

$$\begin{aligned}
R_{61}(h) &= \int_{t_1\,:\,1<|t_1|<1/h} \beta'(t_1)\, \beta(t_1)^{-2}\, \phi_K(ht_1)\, \xi_2(t_1 w)\, dt_1 \\
&\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K(ht-ht_1)\, dt\,, \\
R_{62}(h) &= -h \int_{t_1\,:\,1<|t_1|<1/h} \beta(t_1)^{-1}\, \phi_K'(ht_1)\, \xi_2(t_1 w)\, dt_1 \\
&\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K(ht-ht_1)\, dt \\
&= -\int_{t_1\,:\,h<|t_1|<1} \beta(t_1/h)^{-1}\, \phi_K'(t_1)\, \xi_2(t_1 w/h)\, dt_1 \\
&\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1/h)\}\, \phi_K(ht-t_1)\, dt\,, \\
R_{63}(h) &= h \int_{t_1\,:\,1<|t_1|<1/h} \beta(t_1)^{-1}\, \phi_K(ht_1)\, \xi_2(t_1 w)\, dt_1 \\
&\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K'(ht-ht_1)\, dt\,, \\
R_{64}(h) &= \pm \int_{t_1\,:\,1<|t_1|<1/h} \beta(t_1)^{-1}\, \phi_K(ht_1)\, \xi_2(t_1 w)\, dt_1 \\
&\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}'\{\pm(t-t_1)\}\, \phi_K(ht-ht_1)\, dt\,,
\end{aligned}$$

and the term represented by $O(1)$ is equal to

$$\frac{s_k}{w} \left[\beta(t_1)^{-1}\, \phi_K(ht_1)\, \xi_2(t_1 w)\, dt_1 \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K(ht-ht_1)\, dt \right]_1^{1/h}$$
$$+ \frac{s_k}{w} \left[\beta(t_1)^{-1}\, \phi_K(ht_1)\, \xi_2(t_1 w)\, dt_1 \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t-t_1)\}\, \phi_K(ht-ht_1)\, dt \right]_{-1/h}^{-1}.$$

It can be proved from (3.1), the fact that $|\xi_2|$ and each $|\rho_j'|$ is bounded, and the fact that $|\phi_K|$, $|\phi_K'|$ and $|\phi_{W0}|$ are bounded and integrable, that $R_{6\ell}(h) = O(1)$ for

$\ell = 1, \ldots, 4$. This result and (F.12) imply that, when $w \neq 0$,

$$R_6(h) = O(1)\,. \tag{F.13}$$

When $w = 0$, $\mathrm{cs}_{j_1 1}(t_1 w/h) \equiv 1$ or 0 according as $j_1 = 1$ or 2, respectively, and so $R_6(h) = 0$ if $j_1 = 2$, whereas if $j_1 = 1$,

$$\begin{aligned} h\, s_k^{-1}\, R_6(h) &= \int_{t_1\,:\,h<|t_1|<1} \beta(t_1/h)^{-1}\, \phi_K(t_1)\, dt_1 \\ &\qquad\qquad \times \int \phi_{W0}(t)\, \rho_{j_2}\{\pm(t - t_1/h)\}\, \phi_K(ht - t_1)\, dt \\ &= s_k\, b_1^{-2} \int_{-1}^{1} |\phi_K(t_1)|^2\, dt_1 \cdot \int \phi_{W0}(t)\, dt + o(1)\,, \end{aligned}$$

where the last identity holds if $j_2 = 1$; whereas if $j_1 = 1$ and $j_2 = 2$, $R_6(h) = o(1)$. Now, ϕ_{W0} denotes either $\Re\, \phi_W$ when $k = 1$, or $\Im\, \phi_W$ when $k = 2$, and so, since $\int \phi_W = 2\pi\, f_W(0)$, then $\int \phi_{W0} = 2\pi\, f_W(0)$ when $k = 1$ and equals 0 when $k = 2$. Moreover, $\int |\phi_K|^2 = 2\pi \int K^2$. Therefore, when $w = 0$,

$$R_6(h) = \begin{cases} (2\pi)^2\, s_k^2\, (b_1^2 h)^{-1}\, (\int K^2)\, f_W(0) + o(h^{-1}) & \text{if } j_1 = j_2 = k = 1\,, \\ o(h^{-1}) & \text{otherwise}\,. \end{cases} \tag{F.14}$$

Result (6.11) follows from (F.1), (F.3), (F.5), (F.11), (F.13) and (F.14), which hold in the cases $w \neq 0$ and $w = 0$ respectively.

G Proof of Theorem 2

We treat only the case where $w = 0$. Write $\widehat{F}(t)$ and $F(t)$ for $\widehat{F}_{T\,|\,Q,W,Y}(t\,|\,q, w, y)$ and $F_{T\,|\,Q,W,Y}(t\,|\,q, w, y)$, respectively. It can be proved from Theorem 5 in Appendix D that, if the conditions of Theorem 2 hold, then for each $r \geq 1$,

$$F(t_\alpha) = \alpha = \widehat{F}(\hat{t}_\alpha) = \widehat{F}(t_\alpha) + \sum_{j=1}^{r} \frac{(\hat{t}_\alpha - t_\alpha)^j}{j!}\, \widehat{F}^{(j)}(t_\alpha) + O_p\Big(\big|\hat{t}_\alpha - t_\alpha\big|^{r+1}\Big)\,,$$

where, in the case of part (i) of the theorem, the remainder is of the stated order for each fixed q, w, y and $\alpha \in (0,1)$, and, in the case of part (ii), the remainder is of that order uniformly in q and y in compact sets, and $\alpha \in [\alpha_1, \alpha_2]$. It is straightforward to show that $\widehat{F}(t_\alpha) - F(t_\alpha) = o_p(1)$ and $\widehat{F}'(t_\alpha) - F'(t_\alpha) = o_p(1)$, where, here and immediately below, the remainders are interpreted as in the previous sentence, and therefore it can be proved in succession that $\hat{t}_\alpha - t_\alpha = O_p\{|\widehat{F}(t_\alpha) - F(t_\alpha)|\} = o_p(1)$,

$$\hat{t}_\alpha - t_\alpha = -\{1 + o_p(1)\} \, \frac{\widehat{F}(t_\alpha) - F(t_\alpha)}{F'(t_\alpha)}$$

and

$$\hat{t}_\alpha - t_\alpha = -\frac{\widehat{F}(t_\alpha) - F(t_\alpha)}{F'(t_\alpha)} + \begin{cases} O_p\{(nh)^{-1} + h^{2\ell}\} & \text{for part (i)} \\ O_p\{(n^{1-\eta}h)^{-1} + h^{2\ell}\} & \text{for part (ii),} \end{cases} \tag{G.1}$$

where $\eta > 0$ is arbitrarily small. Parts (i) and (ii) of Theorem 2 follow from (G.1) and parts (i) and (ii), respectively, of Theorem 1.

H Proof of Theorem 3

We treat only the case where $w = 0$. Let F and $\widehat{F}$ be as in the proof of Theorem 2. Note that, as established in Theorem 5, each derivative $\widehat{F}^{(r)}$ converges to the respective $F^{(r)}$ at the same rate, $O_p\{(nh)^{-1/2} + h^\ell\}$ for each q, w and y, or $O_p\{(n^{1-\eta}h)^{-1/2} + h^\ell\}$ uniformly on compacts. Therefore, by Taylor expansion,

$$\alpha = F(t_\alpha) = \widehat{F}(\hat{t}_\alpha) = \widehat{F}(t_\alpha) + \left(\hat{t}_\alpha - t_\alpha\right) \widehat{F}'(t_\alpha) + \tfrac{1}{2}\left(\hat{t}_\alpha - t_\alpha\right)^2 \widehat{F}''(t_\alpha) + \ldots, \tag{H.1}$$

where, here and in (H.2) below, it can be proved from Theorem 2 that the remainder "…" denotes a sum of successive terms of respective sizes $\{(nh)^{-1/2} + h^\ell\}^j$, for $j \geq 3$, and equals $O_p[\{(nh)^{-1/2} + h^\ell\}^{r+1}]$ (or $O_p[\{(n^{1-\eta}h)^{-1/2} + h^\ell\}^{r+1}]$ in a uniform sense) if the last included term is that involving $(\hat{t}_\alpha - t_\alpha)^r$.

In a slight abuse of previous notation, write $\Psi_k(t)$ and $\widehat{\Psi}_k(t)$ for $\Psi_k(t,y,q,w)$ and $\widehat{\Psi}_k(t,y,q,w)$, respectively, and define $\Delta_k = \widehat{\Psi}_k - \Psi_k$. Recall from (2.5) and (2.7) that

$$\begin{aligned}\widehat{F} &= \frac{\Psi_1+\Delta_1}{\Psi_2+\Delta_2} = \Psi_2^{-1}\,(\Psi_1+\Delta_1)\,\big(1-\Psi_2^{-1}\,\Delta_2+\Psi_2^{-2}\,\Delta_2^2-\ldots\big)\\ &= F+\big(\Psi_2^{-1}\,\Delta_1-\Psi_2^{-2}\,\Psi_1\,\Delta_2\big)+\big(\Psi_2^{-3}\,\Psi_1\,\Delta_2^2-\Psi_2^{-2}\,\Delta_1\,\Delta_2\big)+\ldots. \qquad \text{(H.2)}\end{aligned}$$

The advantage of working with this expanded form of $\widehat{F}$ is that it does not involve a random denominator. Write $\widehat{F}_r$ for the version of (H.2) when the expansion on the right-hand side is terminated after terms of size $\{(nh)^{-1/2}+h^\ell\}^r$. For example,

$$\widehat{F}_2 = F+\big(\Psi_2^{-1}\,\Delta_1-\Psi_2^{-2}\,\Psi_1\,\Delta_2\big)+\big(\Psi_2^{-3}\,\Psi_1\,\Delta_2^2-\Psi_2^{-2}\,\Delta_1\,\Delta_2\big)\,. \qquad \text{(H.3)}$$

Since (T,Q,W,Y) is independent of the data $\{(Q_j,W_j,Y_j),\ 1\le j\le n\}$, then, conditionally on Q,W,Y,

$$\begin{aligned}&F_0(\alpha\,|\,q,w,y)\\ &\quad\equiv P\Big(T\le\hat{t}_\alpha\,\Big|\,Q=q,W=w,Y=y\Big)=E\big\{F\big(\hat{t}_\alpha\big)\big\}\\ &\quad= E\Big[\Big\{F(t_\alpha)+\big(\hat{t}_\alpha-t_\alpha\big)\,F'(t_\alpha)+\tfrac{1}{2}\,\big(\hat{t}_\alpha-t_\alpha\big)^2\,F''(t_\alpha)\Big\}\,I(\mathcal{E})\Big]\\ &\quad+O\big\{\delta^3+P\big(\widetilde{\mathcal{E}}\big)\big\}\,, \qquad \text{(H.4)}\end{aligned}$$

where $\mathcal{E}$ represents the event that $|\hat{t}_\alpha-t_\alpha|\le\delta$, $\widetilde{\mathcal{E}}$ denotes the complement of $\mathcal{E}$, and $\delta=\delta(n)$ is a positive sequence decreasing to 0 as $n\to\infty$. Here and below, all expected values are taken conditionally on Q,W,Y. Furthermore, this expansion at (H.4) holds uniformly in t, q and y in any compact subsets of their respective domains, and in $\alpha\in[\alpha_1,\alpha_2]$ for any $0<\alpha_1<\alpha_2<1$.

Recall from (H.1) that $F(t_\alpha)=\widehat{F}(\hat{t}_\alpha)=\alpha$. Using this result, and Taylor-expanding as at (H.1), we deduce that

$$E\Big[\Big\{\big(\hat{t}_\alpha-t_\alpha\big)\,F'(t_\alpha)+\tfrac{1}{2}\,\big(\hat{t}_\alpha-t_\alpha\big)^2\,F''(t_\alpha)\Big\}\,I(\mathcal{E})\Big]$$

$$
\begin{aligned}
&= E\Big[\Big\{\big(\hat{t}_\alpha - t_\alpha\big)\,\widehat{F}_2'(t_\alpha) + \tfrac{1}{2}\big(\hat{t}_\alpha - t_\alpha\big)^2\,\widehat{F}_2''(t_\alpha)\Big\}\,I(\mathcal{E})\Big] \\
&\quad - E\bigg(\Big[\big(\hat{t}_\alpha - t_\alpha\big)\,\big\{\widehat{F}_2'(t_\alpha) - F'(t_\alpha)\big\} + \tfrac{1}{2}\big(\hat{t}_\alpha - t_\alpha\big)^2\,\big\{\widehat{F}_2''(t_\alpha) - F''(t_\alpha)\big\}\Big]\,I(\mathcal{E})\bigg) \\
&= -\alpha - E\Big[\big\{\widehat{F}_2(t_\alpha) - F(t_\alpha)\big\}\Big]\,I(\mathcal{E}) + E\big\{\widehat{F}_2(\hat{t}_\alpha)\,I(\mathcal{E})\big\} + O\big\{\delta^3 + P\big(\widetilde{\mathcal{E}}\big)\big\} \\
&\quad - E\bigg(\Big[\big(\hat{t}_\alpha - t_\alpha\big)\,\big\{\widehat{F}_2'(t_\alpha) - F'(t_\alpha)\big\} + \tfrac{1}{2}\big(\hat{t}_\alpha - t_\alpha\big)^2\,\big\{\widehat{F}_2''(t_\alpha) - F''(t_\alpha)\big\}\Big]\,I(\mathcal{E})\bigg) \\
&= -E\Big[\big\{\widehat{F}_2(t_\alpha) - F(t_\alpha)\big\}\,I(\mathcal{E})\Big] + O\big\{\delta^3 + P\big(\widetilde{\mathcal{E}}\big)\big\} \\
&\quad - E\bigg(\Big[\big(\hat{t}_\alpha - t_\alpha\big)\,\big\{\widehat{F}_2'(t_\alpha) - F'(t_\alpha)\big\} + \tfrac{1}{2}\big(\hat{t}_\alpha - t_\alpha\big)^2\,\big\{\widehat{F}_2''(t_\alpha) - F''(t_\alpha)\big\}\Big]\,I(\mathcal{E})\bigg).
\end{aligned}
$$

Hence, by (H.4),

$$
\begin{aligned}
F_0(\alpha\,|\,q,w,y) =& \alpha - E\Big[\big\{\widehat{F}_2(t_\alpha) - F(t_\alpha)\big\}\,I(\mathcal{E})\Big] - E\bigg(\Big[\big(\hat{t}_\alpha - t_\alpha\big)\,\big\{\widehat{F}_2'(t_\alpha) - F'(t_\alpha)\big\} \\
&+ \tfrac{1}{2}\big(\hat{t}_\alpha - t_\alpha\big)^2\,\big\{\widehat{F}_2''(t_\alpha) - F''(t_\alpha)\big\}\Big]\,I(\mathcal{E})\bigg) + O\big\{\delta^3 + P\big(\widetilde{\mathcal{E}}\big)\big\}\,, \qquad \text{(H.5)}
\end{aligned}
$$

where this identity holds uniformly in q and y in any compact subsets of their respective domains, and in $\alpha \in [\alpha_1, \alpha_2]$ for any $0 < \alpha_1 < \alpha_2 < 1$.

A modification of the Taylor-expansion argument leading to Theorem 2 (see e.g. (G.1)) can be used to show that

$$
\begin{aligned}
&E\Big[\big(\hat{t}_\alpha - t_\alpha\big)\,\big\{\widehat{F}_2'(t_\alpha) - F'(t_\alpha)\big\}\,I(\mathcal{E})\Big] \\
&\quad = -F'(t_\alpha)^{-1}\,E\Big[\big\{\widehat{F}_2(t_\alpha) - F(t_\alpha)\big\}\,\big\{\widehat{F}_2'(t_\alpha) - F'(t_\alpha)\big\}\Big] + O\big\{\delta^3 + P\big(\widetilde{\mathcal{E}}\big)\big\} \\
&\quad = -F'(t_\alpha)^{-1}\,E\Big[\big\{\widehat{F}_1(t_\alpha) - F(t_\alpha)\big\}\,\big\{\widehat{F}_1'(t_\alpha) - F'(t_\alpha)\big\}\Big] + O\big\{\delta^3 + P\big(\widetilde{\mathcal{E}}\big)\big\} + o\big(\delta_1^2\big)\,, \qquad \text{(H.6)}
\end{aligned}
$$

where $\delta_1 = (nh)^{-1/2} + h^\ell$. Similarly but more simply,

$$
E\Big[\big(\hat{t}_\alpha - t_\alpha\big)^2\,\big\{\widehat{F}_2''(t_\alpha) - F''(t_\alpha)\big\}\,I(\mathcal{E})\Big] = O\big\{\delta^3 + P\big(\widetilde{\mathcal{E}}\big)\big\} + o\big(\delta_1^2\big)\,. \qquad \text{(H.7)}
$$

Combining (H.5)–(H.7) we deduce that

$$
F_0(\alpha\,|\,q,w,y) \;=\; \alpha - E\big\{\widehat{F}_2(t_\alpha) - F(t_\alpha)\big\}
$$

$$+F'(t_\alpha)^{-1}\,E\Big[\big\{\widehat{F}_1(t_\alpha)-F(t_\alpha)\big\}\,\big\{\widehat{F}_1'(t_\alpha)-F'(t_\alpha)\big\}\Big]$$
$$+O\big\{\delta^3+P\big(\widetilde{\mathcal{E}}\big)\big\}+o\big(\delta_1^2\big)\,, \tag{H.8}$$

uniformly in the sense described below (H.5).

Define $\widetilde{\Psi}_k(s,y,q,w)=\int\psi_k(s,y,q,w,x)\,\hat{f}_X(x)\,dx$, where ψ_k is as at (2.6), and recall that $\Delta_k=\widehat{\Psi}_k-\Psi_k$, that $\widehat{\Psi}_k(s,y,q,w)=\int\hat{\psi}_k(s,y,q,w,x)\,\hat{f}_X(x)\,dx$, and that $\hat{\psi}_k$ is given by (2.8). It can be proved from these definitions that

$$\widehat{\Psi}_k=\widetilde{\Psi}_k+O_p\big(n^{-1/2}\big)\,,\quad E\big(\widehat{\Psi}_k\big)=E\big(\widetilde{\Psi}_k\big)+O\big(n^{-1}\big)\,, \tag{H.9}$$

$$\begin{aligned}E\big\{\widetilde{\Psi}_k(s,y,q,w)\big\}&=\int\psi_k(s,y,q,w,x)\,E\big\{\hat{f}_X(x)\big\}\,dx\\&=\int\!\!\int\psi_k(s,y,q,w,x+hu)\,K(u)\,f_X(x)\,du\,dx=\int\lambda_k(hu\,|\,s,y,q,w)\,K(u)\,du\\&=\lambda_k(0\,|\,s,y,q,w)+O\big(h^\ell\big)=\Psi_k(s,y,q,w)+O\big(h^\ell\big)\,,\end{aligned} \tag{H.10}$$

in a uniform sense. (Recall that λ_k was defined at (3.3).) For example, in (H.9) uniformity means that $\sup|\widehat{\Psi}(s,y,q,w)-\widetilde{\Psi}(s,y,q,w)|=O_p(n^{-1/2})$ and $\sup|E(\widehat{\Psi}_k)-E(\widetilde{\Psi}_k)|=O(n^{-1})$, where in each case the supremum is taken over s, y and q in any compact subsets of their respective domains. To derive the last identity in (H.10) we used (3.3) and (3.5).

Note that

$$\begin{aligned}&E\Big\{\big|\widehat{\Psi}_k(s,y,q,w)-\widetilde{\Psi}_k(s,y,q,w)\big|^2\Big\}\\&=E\bigg[\int\Big\{\hat{\psi}_k(s,y,q,w,x)-\psi_k(s,y,q,w,x)\Big\}\,\hat{f}_X(x)\,dx\bigg]^2,\\&\le\bigg[\int E\Big\{\hat{\psi}_k(s,y,q,w,x)-\psi_k(s,y,q,w,x)\Big\}^2\,dx\bigg]\int E\big\{\hat{f}_X(x)^2\big\}\,dx\\&=O\bigg[n^a\int E\Big\{\hat{\psi}_k(s,y,q,w,x)-\psi_k(s,y,q,w,x)\Big\}^2\,dx\bigg]\,,\end{aligned} \tag{H.11}$$

uniformly in the sense described in the previous paragraph. To obtain the last identity in (H.11) we used the fact that, by (3.6)(a), $\int E\{\hat{f}_X(x)^2\}\,dx=O(n^a)$ for a constant

$a \geq 0$. Let $D_0, \ldots, D_3$ denote the respective quantities $|\hat{\beta}_0 - \beta_0|$, $|\hat{\beta}_1 - \beta_1|$, $\|\hat{\beta}_2 - \beta_2\|$ and $|\hat{\sigma}_V^2 - \sigma_V^2|$. If

$$\max_{0 \leq j \leq 3} P\big(D_j > n^{-(1-a_1)/2}\big) = O\big(n^{-(1-a_2)}\big)\,, \tag{H.12}$$

where $0 < a_1, a_2 < 1$, then it can be proved by Taylor expansion that

$$\int E\Big\{\hat{\psi}_k(s,y,q,w,x) - \psi_k(s,y,q,w,x)\Big\}^2 dx = O\big(n^{\max(a_1,a_2)-1}\big)\,.$$

Therefore, by (H.11),

$$E\Big\{\big|\widehat{\Psi}_k(s,y,q,w) - \widetilde{\Psi}_k(s,y,q,w)\big|^2\Big\} = O\big(n^{a+\max(a_1,a_2)-1}\big)\,,$$

uniformly in the sense described in the previous paragraph. Hence, provided that

$$n^{a+\max(a_1,a_2)} h = O(1)\,, \tag{H.13}$$

we have:

$$E\Big\{\big|\widehat{\Psi}_k(s,y,q,w) - \widetilde{\Psi}_k(s,y,q,w)\big|^2\Big\} = O\big\{(nh)^{-1}\big\}\,, \tag{H.14}$$

again uniformly. Suppose that, as asserted in (3.6)(b), $n^{a+\varepsilon}\, h = O(1)$ for some $\varepsilon > 0$. By assuming enough finite moments of Q, U, V and X (here we are invoking (3.6)(c)) we can ensure that (H.12) holds for a_1, a_2 in the range $0 < a_1, a_2 \leq \varepsilon$. In this case (H.13), and hence also (H.14), follow from the property $n^{a+\varepsilon}\, h = O(1)$ in (3.6).

Define $\widetilde{\Delta}_k = \widetilde{\Psi}_k - \Psi_k$, and let $\widetilde{\Psi}_k'$ and Ψ_k' be the derivatives of $\widetilde{\Psi}_k$ and Ψ_k with respect to s, so that $\widetilde{\Delta}_k' = \widetilde{\Psi}_k' - \Psi_k'$. Using this notation, and combining (H.3), the second part of (H.9), (H.10) and (H.14), we deduce that

$$\begin{aligned} E\big(\widehat{F}_2\big) - F &= \Psi_2^{-3}\,\Psi_1\,E\big(\widetilde{\Delta}_2^2\big) - \Psi_2^{-2}\,E\big(\widetilde{\Delta}_1\,\widetilde{\Delta}_2\big) + O\big\{(nh)^{-1} + h^\ell\big\} \\ &= O\big\{(nh)^{-1} + h^\ell\big\}\,, \end{aligned} \tag{H.15}$$

$$E\big\{\big(\widehat{F}_1 - F\big)\big(\widehat{F}_1' - F'\big)\big\} = E\Big\{\big(\Psi_2^{-1}\,\widetilde{\Delta}_1 - \Psi_2^{-2}\,\Psi_1\,\widetilde{\Delta}_2\big)\big(\Psi_2^{-1}\,\widetilde{\Delta}_1 - \Psi_2^{-2}\,\Psi_1\,\widetilde{\Delta}_2\big)'\Big\}$$

$$
\begin{aligned}
& + O\big\{(nh)^{-1} + h^\ell\big\} \\
= \; & O\big\{(nh)^{-1} + h^\ell\big\}\,, \qquad \text{(H.16)}
\end{aligned}
$$

where in each case the functions on the left-hand side are evaluated at t_α, and the last identities are derived using standard calculations. Hence, by (H.8), and again in the uniform sense prescribed two paragraphs above,

$$
\begin{aligned}
F_0(\alpha \,|\, q, w, y) - \alpha &= O\Big\{(nh)^{-1} + h^\ell + \delta^3 + P\big(\widetilde{\mathcal{E}}\big)\Big\} + o\big(\delta_1^2\big) \\
&= O\Big\{(nh)^{-1} + h^\ell + \delta^3 + P\big(\widetilde{\mathcal{E}}\big)\Big\}\,. \qquad \text{(H.17)}
\end{aligned}
$$

We know from Theorem 2 that $\hat{t}_\alpha - t_\alpha = O_p\{(nh)^{-1/2} + h^\ell\}$, and so if we define $\delta = \{(nh)^{-1/2} + h^\ell\}\, n^\eta$, where $\eta > 0$ is chosen so small that $\{(nh)^{-1/2}\, n^\eta\}^3 = O\{(nh)^{-1}\}$, then we shall have $\delta^3 = O\{(nh)^{-1/2} + h^\ell\}$. Moreover, Markov's inequality can be used to prove that $P(\widetilde{\mathcal{E}}) = O\{(nh)^{-1/2} + h^\ell\}$. Hence, by (H.17), $F_0(\alpha \,|\, q, w, y) - \alpha = O\{(nh)^{-1/2} + h^\ell\}$, uniformly in q and y in compact subsets of their respective domains. This result is equivalent to (3.8).

Finally we sketch a proof of the variant of Theorem 3 discussed immediately below that theorem. If in Theorem 3 we assume (3.4) then the far right-hand side of (H.10) can be refined to $\Psi_k(s, y, q, w) + c_k\, h^\ell + o(h^\ell)$, where c_k is a constant, and therefore (H.10) becomes

$$
E\big\{\widetilde{\Psi}_k(s, y, q, w)\big\} = \Psi_k(s, y, q, w) + c_k\, h^\ell + o\big(h^\ell\big)\,. \qquad \text{(H.18)}
$$

Furthermore, if in (3.6)(a) we replace $O(h^a)$ by $o(h^a)$, so that $\int E(\hat{f}_X)^2 = o(n^a)$, then (H.13) holds with the right-hand side replaced by $o(1)$, and so (H.14) becomes

$$
E\Big\{\big|\widehat{\Psi}_k(s, y, q, w) - \widetilde{\Psi}_k(s, y, q, w)\big|^2\Big\} = o\big\{(nh)^{-1}\big\}\,. \qquad \text{(H.19)}
$$

Using (H.18) and (H.19) instead of (H.10) and (H.14), respectively, the strings of identities at (H.15) and (H.16) can be refined to

$$
E\big(\widehat{F}_2\big) - F \;=\; \Psi_2^{-3}\, \Psi_1\, E\big(\widetilde{\Delta}_2^2\big) - \Psi_2^{-2}\, E\big(\widetilde{\Delta}_2\, \widetilde{\Delta}_2\big) + o\big\{(nh)^{-1} + h^\ell\big\}
$$

$$
\begin{aligned}
&= d_1\,(nh)^{-1} + d_2\,h^{\ell} + o\big\{(nh)^{-1} + h^{\ell}\big\}\,, && \text{(H.20)}\\
E\big\{\big(\widehat{F}_1 - F\big)\big(\widehat{F}_1' - F'\big)\big\} &= E\Big\{\big(\Psi_2^{-1}\,\widetilde{\Delta}_1 - \Psi_2^{-2}\,\Psi_1\,\widetilde{\Delta}_2\big)\big(\Psi_2^{-1}\,\widetilde{\Delta}_1 - \Psi_2^{-2}\,\Psi_1\,\widetilde{\Delta}_2\big)'\Big\}\\
&\qquad + o\big\{(nh)^{-1} + h^{\ell}\big\}\\
&= d_3\,(nh)^{-1} + o\big\{(nh)^{-1} + h^{\ell}\big\}\,, && \text{(H.21)}
\end{aligned}
$$

where d_1, d_2 and d_3 are constants and, as in (H.15) and (H.16), the functions on the left-hand sides are evaluated at t_α. The remainder $O\{\delta^3 + P(\widetilde{\mathcal{E}})\} + o(\delta_1^2)$ on the right-hand side of (H.8) equals $o\{(nh)^{-1} + h^{\ell}\}$ if δ is chosen appropriately, and so, on substituting (H.20) and (H.21) into (H.8), we obtain:

$$
F_0(\alpha\,|\,q, w, y) = \alpha + \big\{F'(t_\alpha)^{-1}\,d_3 - d_1\big\}\,(nh)^{-1} - d_2\,h^{\ell} + o\big\{(nh)^{-1} + h^{\ell}\big\}\,.
$$

This is the version of (3.8) discussed in the paragraph immediately below Theorem 3.