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Lack of Fit Test for Infinite Variation Jumps at High Frequencies

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Abstract: This paper is concerned with testing for infinite variation jumps in addition to a continuous local martingale component driven by Brownian motion using high-frequency data. We developed a lack of fit type test based on the empirical distribution of the "devolatized" increments. Under the null hypothesis that the jump component is of finite variation, the empirical process associated with the "devolatized" increments converges to a Gaussian process in the Skorohod topology. Under the alternative hypothesis that the jumps are of infinite variation, the empirical process explodes instead. Theoretical results and simulation show good performance on the size and power of the test. A real financial data set is analyzed.

Key words and phrases: Itô Semimartingale, Infinite Activity Jumps, Infinite Variation Jumps.

1. Introduction

Itô's semimartingale is of vital importance in stochastic calculus and widely used in empirical studies in finance, environmental science and other

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fields. Mathematically, it is defined on a filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, P)$ and assumes the integral form

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + X_t^d, \tag{1.1}$$

where $\int_0^t b_s ds$ is the drift with b being an optional and càdlàg process, $\int_0^t \sigma_s dW_s$ is a continuous local martingale with σ_s being an adapted process and W being a standard Brownian motion, and X^d is a pure-jump component with the jump activity index (JAI) β defined by

$$\beta = \inf\{r; \sum_{0 \le s \le T} |\Delta_s X|^r < \infty\}, \tag{1.2}$$

where $\Delta_s X = X_s - X_{s-}$ is the jump size of X at time s and T is a fixed time horizon. For the estimation of β, we refer to Aït-Sahalia and Jacod (2009a), Todorov and Tauchen (2010), Todorov and Tauchen (2011) and Jing et al. (2012a). In this paper, we restrict Itô semimartingales to a deterministic JAI. β serves as an indicator of the activity of jumps contained in X^d. The larger the β, the more active the jumps are. In particular, jump processes of finite activity (e.g., compound Poisson process), finite variation, and infinite variation correspond to $\beta = 0$, $0 < \beta < 1$ and $1 < \beta < 2$, respectively.

Model (1.1) contains two kinds of driving forces, the Brownian driving force in the continuous local martingale term and the jump driving force in

X^d including the infinite variation jump driving force ($\beta > 1$) and the finite variation jump driving force ($\beta < 1$). Inference on the types of driving force underlying high-frequency data or time series is of increasing interest in recent years. Barndorff-Nielsen and Shephard (2006), Fan and Wang (2008), and Aït-Sahalia and Jacod (2009b) derived tests for the existence of jumps. Aït-Sahalia and Jacod (2010), Todorov and Tauchen (2014), Jing et al. (2012b), Kong et al. (2015) and Todorov (2015) established tests for the necessity of adding a Brownian force. Aït-Sahalia and Jacod (2011) studied whether the jump component is of finite activity or not when the Brownian force is present. See Jacod and Protter (2012) for recent developments.

In this paper, we are curious on whether it is necessary to add an infinite variation jump term in addition to a continuous local martingale for the purpose of modeling high-frequency data. It is generally accepted that small jumps of infinite variation fluctuate rapidly and play the role of a Brownian motion. But many empirical studies, cf, Aït-Sahalia and Jacod (2009a), show evidence of infinite variation jumps in the presence of the diffusive process. This asks for a statistical method to validate the modeling assumption of no infinite variation jumps given a diffusive term. Statistically, this could be formulated as testing for the following hypotheses on

$\{\inf_{0\leq s\leq T}\sigma_s^2>0\}$,

$$H_0:\beta\leq 1 \quad vs. \quad H_1:\beta>1. \tag{1.3}$$

While much empirical evidence for the presence of infinite variation jumps is based on an estimate of JAI, it is not reliable to propose a test using a point estimator of JAI. This is because the estimation of JAI is challenging in simultaneous presence of a diffusive term, for example, the best convergence rate in Bull (2016) is $n^{\beta/4}$ which is slow for $\beta\leq 1$ and depends on the unknown β. In this paper, we develop a lack of fit type test based on the empirical distribution of the "devolatized" increments. Under H_0, the empirical process associated with the "devolatized" increments converges to a fully specified Gaussian process in any compact subset of R at the rate close to $\sqrt{n}$. Under H_1, it explodes. A comparison of the empirical distribution function with the Gaussian distribution function yields the "Kolmognov-Smirnov type" test statistic, which successfully differentiates the null and alternative hypotheses.

The contribution of this paper is as follows.

- For the first time, we provided a theoretically sound test for the presence of infinite variation jumps in the presence of a diffusive term and a jump component of finite variation. This test has many nice properties. In addition to its good performance on size and power, the test becomes more powerful as JAI increases. This is surprisingly appealing since at the first glance the larger the β the more the jump process behaves like a Brownian motion in path regularities and hence seemingly harder to detect. The reason for the increasing power against β is essentially because of the frequency of small jumps or equivalently the jump intensity around the origin. See the remark below Assumption 1 for more details.

- We established the asymptotic theory of the empirical distribution of the "devolatized" increments of Itô semimartingales with infinitely active or even infinite variation jumps, which is more desirable than those presented in Todorov and Tauchen (2014) allowing only for finitely many jumps. Todorov and Tauchen (2014) lacks the proof of the oscillation property of the empirical process, though the uniform result in (10.41) of that paper turns out to be true. Their equations (10.34) (10.35) (10.36) (10.37) and (10.38) only prove the convergence in moment of small order terms for fixed τ (the argument of the empirical process), which does not imply the uniform convergence in (10.41). Instead, Lemmas 4-5 in the supplement to this paper provide the local uniform convergence of small order terms and the tightness results of the empirical process, which are of their own interests. The proofs are also nontrivial. Another theoretically distinctive feature of our paper is that we estimate the spot volatility based on the realized Laplace transform approach.

The rest of the paper is arranged as follows. In Section 2, we state the assumptions. Main results including the limit theorems of the empirical processes and the size and power performance of the lack of fit test are presented in Section 3. Section 4 is devoted to monte carlo simulations and real data analysis. All the technical proofs are postponed to the supplement.

Throughout the paper, we assume that the available data set is $\{X_{t_i}; 0 \leq i \leq n\}$ which are discretely sampled from X, and are equally spaced in the fixed interval $[0, T]$, i.e., $t_i = i\Delta_n$ with $\Delta_n = T/n$ for $0 \leq i \leq n$. Denote the jth one-step increment by

$$\Delta_j^n X = X_{t_j} - X_{t_{j-1}}, \quad 1 \leq j \leq n.$$

2. Basic Assumptions

First, we make an assumption on the pure-jump process X^d.

Assumption 1. *1. If $\beta > 1$,*

$$X_t^d = \int_0^t \gamma_{s-}^+ dY_s^+ + \int_0^t \gamma_{s-}^- dY_s^- + \int_0^t \int_R \delta(s,z) p(ds,dz),$$

where Y^+ and Y^- are two independent Lévy processes with positive jumps and Lévy triplets $(0, 0, F^{\pm})$, $\gamma^{\pm}$ are two càdlàg adapted pro-

cesses, δ is a càdlàg predictable process, and p is a Poisson random measure on $R_+ \times R$ with intensity $q(dt, dx) = dt \otimes dx$. We assume further that, for some $r < 1$, the Lévy measure satisfies

$$|\overline{F}^{\pm}(x) - \frac{1}{x^{\beta}}| \equiv |F^{\pm}((x,\infty)) - \frac{1}{x^{\beta}}| \leq g(x),\ x \in (0,\infty),$$

with $g(x)$ being a decreasing function s.t. $\int_0^1 x^{r-1} g(x)dx < \infty$ and $\int_1^{\infty} g(x)dx < \infty$.

2. If $\beta \leq 1$, $\gamma^+ = \gamma^- \equiv 0$, i.e.,

$$X_t^d = \int_0^t \int_R \delta(s,z)p(ds,dz).$$

Assumption 1 demonstrates that when $\beta > 1$, X^d has two β-stable-like driving forces. The assumption on the Lévy measure of these two stable-like driving processes is flexible enough. It only requires that the Lévy density is equivalent to that of a stable Lévy process around the origin. The families of tempered stable processes with Lévy density $\frac{e^{-cx}}{x^{1+\beta}} I(x > 0)$ and truncated stable processes with Lévy density $\frac{1}{x^{1+\beta}} I(x \in (0, c])$ are special examples. For the latter two examples, the jumps have moments of any polynomial order. The coefficient processes $\gamma_t^{\pm}$ can be either positive or negative, therefore, X^d allows for asymmetric jumps in its trajectory, which captures the empirical feature (downward jumps occur more frequently than upward jumps) in financial markets due to risk aversion. When the jump activity index is smaller than 1, X^d is completely nonparametric.

We also need the following assumptions on the structure of σ and $\gamma^{\pm}$.

Assumption 2. *1. σ_t is a Itô semimartingale of the form*

$$\sigma_t = \sigma_0 + \int_0^t b_s^{\sigma} ds + \int_0^t H_s^{\sigma} dW_s + \int_0^t H_s'^{\sigma} dW_s' + \int_0^t \int_R \delta^{\sigma}(s,x)\tilde{p}(ds,dx),$$

where all the integrands are optional processes satisfying the integrable condition of Itô's sense, and $\tilde{p}$ is another Poisson random measure independent of W with intensity $\tilde{q}(dt,dx) = dt \otimes dx$. W and W' are two independent Brownian motions that are further independent of (p, Y^+, Y^-).

2. $\gamma_t^{\pm}$ are Itô semimartingales of the form

$$\begin{aligned}\gamma_t^{\pm} &= \gamma_0^{\pm} + \int_0^t b_s^{\gamma^{\pm}} ds + \int_0^t H_s^{\gamma^{\pm}} dW_s + \int_0^t H_s'^{\gamma^{\pm}} dW_s' \\ &\quad + \int_0^t \int_R \delta^{\gamma^{\pm}}(s,x)\bar{p}(ds,dx),\end{aligned}$$

where all the integrands are optional processes satisfying the integrable condition of Itô's sense, and $\bar{p}$ is another Poisson random measure independent of W with intensity $\bar{q}(dt,dx) = dt \otimes dx$.

Assumption 2 is a standard assumption in the literature which allows for the "leverage" effect due to the common driving force W in X, σ, and

$\gamma^{\pm}$. $\tilde{p}$ and $\bar{p}$ are two new Poisson random measures that may not necessarily equal to or be independent of each other, or equal to or be independent of p.

Assumption 3. *We have a sequence τ_n of stopping times increasing to infinity, a sequence a'_n of numbers, and a nonnegative Lebesgue-integrable function J on R, such that the processes b, H^{σ}, $H^{\gamma^{\pm}}$, $\gamma^{\pm}$ are càdlàg adapted, the coefficients δ, δ^{σ} and $\delta^{\gamma^{\pm}}$ are predictable, the processes b^{σ}, H'^{σ}, $b^{\gamma^{\pm}}$ and $H'^{\gamma^{\pm}}$ are progressively measurable, and for some constant $r < 1$,*

$$t < \tau_n \Rightarrow |\delta(t,z)|^r \wedge 1 \leq a'_n J(z), \ \ |\delta^{\sigma}(t,z)| \wedge 1 \leq a'_n J(z),$$

$$|\delta^{\gamma^{\pm}}(t,z)| \wedge 1 \leq a'_n J(z);$$

$$t < \tau_n, V = b, b^{\sigma}, H^{\sigma}, H'^{\sigma}, b^{\gamma^{\pm}}, H^{\gamma^{\pm}}, H'^{\gamma^{\pm}}, |\sigma|, |\sigma|^{-1} \Rightarrow |V_t| \leq a'_n,$$

$$V = b, H^{\sigma}, H'^{\sigma}, \delta^{\sigma}, H^{\gamma^{\pm}}, H'^{\gamma^{\pm}}, \delta^{\gamma^{\pm}}$$

$$\Rightarrow |E(V_{(t+s)\wedge\tau_n} - V_{t\wedge\tau_n}|\mathcal{F}_t)| + E(|V_{(t+s)\wedge\tau_n} - V_{t\wedge\tau_n}|^2|\mathcal{F}_t) \leq a'_n s.$$

If $\beta = 1$, we further assume $E[(\delta((t+s)\wedge\tau_n, x) - \delta(t\wedge\tau_n, x))^2|\mathcal{F}_t] \leq a'_n s^{1+\epsilon}$ uniformly for $x \in R$ and any $\epsilon > 0$.

Assumption 3 is a rather general assumption which is satisfied by the multifactor stochastic volatility model that is widely used in financial econometrics, e.g., the popular affine jump diffusion model in Duffie et al. (2000). This assumption also requires that the jumps in σ and $\gamma^{\pm}$ are of finite variation, which is technically required in proving Lemma 1 and (S3.25) in the supplement.

3. Methodology and Main Results

3.1. Motivational Example

As mentioned in the introduction, our test statistic is based on the empirical process of the "devolatized" increments. As a motivation for constructing the empirical process, consider a simple example of Itô process, $X_t = \sigma_t^0 W_t + \gamma_{\alpha,t} Y_{\alpha,t} + \gamma_{\beta,t} Y_{\beta,t}$ where σ_t^0, $\gamma_{\alpha,t}$ and $\gamma_{\beta,t}$ are three deterministic smooth functions, and $Y_{\alpha,t}$ and $Y_{\beta,t}$ are, respectively, symmetric α and β stable Lévy processes with $\alpha < 1$ and $\beta > 1$ representing jump processes of finite variation and infinite variation, respectively. By smoothness and self-similarity, we have

$$\frac{\Delta_j^n X}{\sqrt{\Delta_n}} \approx \sigma_{j\Delta_n}^0 \mathcal{N}_j(0,1) + \gamma_{\alpha,j\Delta_n} \Delta_n^{\frac{1}{\alpha}-\frac{1}{2}} S_j^{\alpha} + \gamma_{\beta,j\Delta_n} \Delta_n^{\frac{1}{\beta}-\frac{1}{2}} S_j^{\beta}, \tag{3.1}$$

where $\mathcal{N}_j(0,1)$'s, S_j^{α}'s and S_j^{β}'s form sequences of i.i.d. standard normal, symmetric α and β stable variables, respectively. Seen from (3.1), we have the following observations on the function $P(\frac{\Delta_j^n X}{\sqrt{\Delta_n}} > x)$ defined on the real line.

- The function $P(\frac{\Delta_j^n X}{\sqrt{\Delta_n}} > x)$ is not time invariant due to time varying of σ_t^0. So, it is beneficial to standardize the increments by estimated spot volatilities.

- For fixed x, the function $P(\frac{\Delta_j^n X}{|\sigma^0_{j\Delta_n}|\sqrt{\Delta_n}} > x) \sim 1 - \Phi(x)$ where $\Phi(x)$ is the c.d.f. of standard normal random variable and hence is mainly contributed by the Brownian motion. The α stable Lévy motion is completely dominated by the other two terms and it reduces to 0 even faster than $\sqrt{\Delta_n}$. Although the β stable Lévy motion is negligible, its rate converging to 0 is lower than $\sqrt{\Delta_n}$ resulting in a non-negligible bias ('signal' under H_1) that can not be balanced by the asymptotic variance (typically of order close to $\sqrt{\Delta_n}$). So under the null hypothesis, the devolatized increments look like that of a standard Brownian motion in distribution. While under H_1, the unbalanced bias term due to $\gamma_{\beta,j\Delta_n}\Delta_n^{\frac{1}{\beta}-\frac{1}{2}} S_j^{\beta}$ stands out resulting in good power which increases against β.

- The upmost tail for $x \sim c\Delta_n^{-\varpi}$ for some constants $c > 0$ and $0 < \varpi < 1/2$ is dominated by β (α under H_0) stable Lévy motion if it is present, since $P(\Delta_j^n Y_\beta/\sqrt{\Delta_n} > x) \sim c' x^{-\beta}\Delta_n^{1-\beta/2}$ ($c' x^{-\alpha}\Delta_n^{1-\alpha/2}$ under H_0) for some constant $c' > 0$ while $P(\Delta_j^n W/\sqrt{\Delta_n} > x) \sim \frac{1}{x}\phi(x)$ where $\phi(x)$ is the p.d.f. of the standard normal distribution. So it is better not to consider too large values of x, otherwise the size may not be safeguarded.

Certainly, the model under the Assumptions in last section is much more general than the toy one in the motivation. In the sequel, we will extend the above findings to the more general settings.

3.2. Method of Devolatizing the Increments

To estimate the spot volatility, we use the local method and thus split the interval into non-overlapping shrinking blocks with each block length equal to $2v_n$ consisting of $2k_n$ intervals of length Δ_n, where k_n is some integer depending on n. Aggregation of local estimates is widely used in other contexts, cf, Mykland and Zhang (2009) Mykland et al. (2012), and Todorov and Tauchen (2012). Back to the toy example, the characteristic function of the **symmetrized** increment is

$$\begin{aligned} & E\exp\left(\sqrt{-1}u\frac{X_{t+2\Delta_n}-X_{t+\Delta_n}-(X_{t+\Delta_n}-X_t)}{\sqrt{\Delta_n}}\right) \\ \approx\ & Ecos\left(u\frac{X_{t+2\Delta_n}-X_{t+\Delta_n}-(X_{t+\Delta_n}-X_t)}{\sqrt{\Delta_n}}\right) \\ \approx\ & \exp\left(-u^2\sigma_t^{02}-2|u\gamma_{\alpha,t}|^{\alpha}\Delta_n^{1-\alpha/2}-2|u\gamma_{\beta,t}|^{\beta}\Delta_n^{1-\beta/2}\right). \end{aligned}$$

This motivates us to compute the following sample analogue in the jth shrinking block,

$$\begin{aligned} L_j(u) &= \frac{1}{k_n}\sum_{l=1}^{k_n} cos\left(u\frac{\Delta^n_{2jk_n+2l}X-\Delta^n_{2jk_n+2l-1}X}{\sqrt{\Delta_n}}\right), \\ c_j(u) &= -\frac{1}{u^2}\log\left(L_j(u)\vee\frac{c}{\sqrt{k_n}}\right),\quad 0\le j\le[n/(2k_n)]-1, \end{aligned}$$

where the lower threshold $\frac{c}{\sqrt{k_n}}$ is to assure that the logarithmic function makes sense. And our local estimate of $\sigma_j^2 \equiv \sigma^2_{2jk_n\Delta_n}$ is

$$\hat{\sigma}_j^2(u) = c_j(u) - \frac{1}{u^2 k_n}(sinh(u^2 c_j(u)))^2,$$

where the subtracted term is used to correct the bias due to the jumps. This local estimate was used in Jacod and Todorov (2014) to get an efficient estimator of the integrated volatility, and Kong *et al.* (2015) to test for the presence of a Brownian force. There are some other methods to estimate the spot volatility, cf, Todorov and Tauchen (2014), Jacod and Rosenbaum (2013), Fan and Wang (2007), Li and Xiu (2016), and Li et al. (2017). For long-memory volatility models that are driven by fractional Brownian motion, we refer to Comte and Renault (1996) and Comte and Renault (1998). The major advantage of this Laplace-transform-based local estimator is that it can easily separate the effect of the Brownian force and the stable-like driving force.

For properly chosen m_n and u_n, the empirical distribution function of the devolatized increments is defined as,

$$\hat{F}_n(u_n, \tau) = \frac{1}{[n/(2k_n)]m_n} \sum_{j=1}^{[n/(2k_n)]} \sum_{i=2jk_n+1}^{2jk_n+m_n} I\left(\frac{\Delta_i^n X}{\sqrt{\hat{\sigma}^2_{j-1}(u_n)\Delta_n}} \leq \tau \right), \quad (3.2)$$

for $\tau \in R$.

3.3. Empirical Processes and Their Limiting Properties

To define the empirical process, we need some more notations. Let

$\tilde{\Phi}^n_{j,i}(\tau)$ be the c.d.f. of

$$\frac{\Delta^n_{2jk_n+i}W}{\sqrt{\Delta_n}} + \frac{\gamma^+_{t_{2jv_n+(i-1)\Delta_n}}\Delta^n_{2jk_n+i}Y^+ + \gamma^-_{t_{2jv_n+(i-1)\Delta_n}}\Delta^n_{2jk_n+i}Y^-}{|\sigma_{2(j-1)k_n\Delta_n}|\sqrt{\Delta_n}},$$

conditional on $\mathcal{F}_{2jv_n+(i-1)\Delta_n}$, and

$$\overline{\Phi}_n(\tau) = \frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}\tilde{\Phi}^n_{j,i}(\tau).$$

When the JAI of X^d is no larger than 1, $\bar{\Phi}_n(\tau)$ reduces to $\Phi(\tau)$.

Now we define the empirical process as

$$\hat{Y}_n(\tau) = \sqrt{[n/(2k_n)]m_n}\left(\hat{F}_n(u_n,\tau) - \Phi(\tau)\right).$$

Before stating the main theorems, we introduce some notations. Let

$$U_t(u) = \exp\left(-u^2\sigma_t^2 - 2\Delta_n^{1-\beta/2}u^\beta a_t\right), \text{where } a_t = \chi(\beta)(|\gamma_t^+|^\beta + |\gamma_t^-|^\beta)$$

with $\chi(\beta) = \int_0^\infty y^{-\beta}sin(y)dy$, and $\xi_j(u) = L_j(u)/U_{2jk_n\Delta_n}(u) - 1$. Define

$$\hat{Z}^n_1(\tau) = \frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]-1}\sum_{i=1}^{m_n}\left(I(\frac{\Delta^n_{2jk_n+i}W}{\sqrt{\Delta_n}} \le \tau) - \Phi(\tau)\right),$$

and

$$\hat{Z}^n_2(\tau) = \frac{1}{[n/(2k_n)]}\sum_{j=1}^{[n/(2k_n)]-1}\left(\frac{1}{2}\tau\Phi'(\tau)\frac{\xi_{j-1}(u_n)}{u_n^2\sigma^2_{2k_n(j-1)\Delta_n}}\right).$$

We first state a functional central limit theorem of Donsker's type on the empirical process when $\beta \le 1$.

Theorem 1. *Suppose k_n, $u_n \downarrow 0$, m_n, and Δ_n satisfy* $k_n\Delta_n^{1/2} \to 0$,

$$k_n\Delta_n^{1/2-\epsilon} \to \infty,\ \sup_n \frac{k_n\Delta_n^{1/2}}{u_n^4} < \infty,\ \frac{k_n}{m_n(\log n)^4} \to \infty,\ \frac{k_n\Delta_n^{\epsilon}}{m_n} \to 0, \tag{3.3}$$

for any $\epsilon > 0$, and $\beta \le 1$ ($\gamma^+ = \gamma^- \equiv 0$). Under Assumptions 1-3, we have that $\hat{Y}_n(\tau)$ and

$$\sqrt{[n/(2k_n)]m_n}\left(\hat{Z}_1^n(\tau) + \hat{Z}_2^n(\tau) - \frac{1}{4Tk_n}\tau^2(\Phi''(\tau) - \Phi'(\tau))\right) \tag{3.4}$$

converge weakly to the same Gaussian process in Skorohod's topology for all $\tau \in \mathcal{A}_c$ where $\mathcal{A}_c$ is any compact subset of R, and

$$(\sqrt{[n/(2k_n)]m_n}\hat{Z}_1^n(\tau), \sqrt{[n/(2k_n)]2k_n}\hat{Z}_2^n(\tau)) \Rightarrow (Z_1(\tau), Z_2(\tau)), \tag{3.5}$$

functionally in $\tau \in \mathcal{A}_c$ in the sense of the Skorohod topology, where $Z_1(\tau)$ and $Z_2(\tau)$ are two independent centered Gaussian processes with covariance functions

$$Cov(Z_1(\tau_1), Z_1(\tau_2)) = \Phi(\tau_1 \wedge \tau_2) - \Phi(\tau_1)\Phi(\tau_2),\ \tau_1, \tau_2 \in R, \tag{3.6}$$

$$Cov(Z_2(\tau_1), Z_2(\tau_2)) = \frac{\tau_1\Phi'(\tau_1)\tau_2\Phi'(\tau_2)}{T}. \tag{3.7}$$

When $\beta > 1$, however, we failed in obtaining the functional central limit theorem. Nevertheless we have the following pointwise central limit theorem for $\hat{Y}_n(\tau)$. This is already enough in the context of detecting infinite variation jumps.

Theorem 2. *Suppose that $\beta > 1$ and all other conditions in Theorem 1 hold, then*

$$\begin{aligned}\hat{Y}_n(\tau) &= \sqrt{[n/(2k_n)]m_n}\left(\hat{Z}_1^n(\tau)+\hat{Z}_2^n(\tau)-\frac{1}{4Tk_n}\tau^2(\Phi''(\tau)-\Phi'(\tau))\right)\\ &\quad +\sqrt{[n/(2k_n)]m_n}\left(\overline{\Phi}_n(\tau)-\Phi(\tau)\right)\\ &\quad +\frac{m_n^{1/2}\tau\Phi'(\tau)u_n^{\beta-2}\Delta_n^{1-\beta/2}}{[n/(2k_n)]^{1/2}}\sum_{j=1}^{[n/(2k_n)]}\frac{a_{j\Delta_n}}{\sigma^2_{(j-1)\Delta_n}}+o_p(1), \qquad (3.8)\end{aligned}$$

pointwise in $\tau \in R$.

Remark 1. In Theorem 2, $\overline{\Phi}_n(\tau)$ depends on the mixed distribution of the Brownian force and the β-stable-like Lévy process. In the limiting sense, under the conditions in Theorem 2, we have (proven in the supplement),

$$\begin{aligned}&\overline{\Phi}_n(\tau)\\ =&\ \Phi(\tau)-\frac{\Phi'(\tau)\Delta_n^{\frac{1}{\beta}-\frac{1}{2}}}{[\frac{n}{2k_n}]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}\frac{\gamma^-_{j,i-1}EY_1^-+\gamma^+_{j,i-1}EY_1^+}{|\sigma_{2(j-1)k_n\Delta_n}|}+O_p(\Delta_n^{1-\frac{\beta}{2}-\epsilon})\\ =&\ \Phi(\tau)-\frac{\Phi'(\tau)\Delta_n^{\frac{1}{\beta}-\frac{1}{2}}}{T}\int_0^T\frac{\gamma_s^+EY_1^++\gamma_s^-EY_1^-}{|\sigma_s|}ds+O_p(\Delta_n^{(1-\frac{\beta}{2}-\epsilon)\wedge(\frac{1}{\beta}-\frac{1}{4}+\epsilon)}).\end{aligned}$$

(3.9)

Theorem 2 demonstrates that there are two major sources of the bias for the empirical process. One is the increment of the driving jump process, $\Delta^n_{2jk_n+i}Y^{\pm}$, which distorts the empirical distribution function away from $\Phi(\tau)$, see (3.9); A second one is the estimation error of the instantaneous volatility using the Laplace-transform-based procedure, see (3.8). If the jump component is of finite variation, both biases vanish asymptotically, but they explode otherwise. If $\beta > 1$, the first bias dominates the second in the explosive rate.

Remark 2. (3.3) provides guidelines for choosing k_n, m_n and u_n. If one sets $u_n = c_1/\log n$, $k_n = [c_2\sqrt{n}/(\log n)^4]$ and $m_n = [c_1 k_n/(\log n)^{4+\epsilon}]$ for some $\epsilon > 0$, (3.3) is satisfied. This implies that the convergence rate of the empirical distribution function of the "devolatized" increments is almost $\sqrt{n}$.

For completeness of the theory on the empirical distribution of the "devolatized" increments, though irrelevant to the testing background here, we also present a result for $\hat{F}_n(u_n, \tau)$, when the Brownian force does not exist but the β-stable-like driving process is present.

Theorem 3. *Assume Assumptions 1-3 except that $|\sigma| \equiv 0$, and $|\gamma^+|^{-1}$ and $|\gamma^-|^{-1}$ are strictly positive. Suppose that (3.3) holds. Then we have*

$$\hat{F}_n(u_n, \tau) \to^P 1, \tag{3.10}$$

pointwise in $\tau \in R$.

3.4. Test Statistics

Tests for infinite variation jumps via estimating the jump activity index directly with simultaneous presence of diffusive process suffer from the difficulty of separating the continuous term and the jump term, and typically some semi-parametric assumption for the jumps under H_0 would be needed, see for example, Aït-Sahalia and Jacod (2009a) and Jing et al. (2012a). Theorems 1-2, as well as their remarks motivate us to propose a test statistic of the "Kolmogrov-Smirnov" type as

$$\mathcal{T}^n_{\mathcal{A}_c} \equiv \sqrt{[n/(2k_n)]m_n} \sup_{\tau\in\mathcal{A}_c} \left|\hat{F}(u_n,\tau) - \Phi(\tau)\right| \tag{3.11}$$

where $\mathcal{A}_c$ is a compact subset in R. Then under H_0, by Theorem 1,

$$\mathcal{T}^n_{\mathcal{A}_c} =^{\mathcal{L}} \sqrt{[n/(2k_n)]m_n} \sup_{\tau\in\mathcal{A}_c} \left|\hat{Z}^n_1(\tau) + \hat{Z}^n_2(\tau) - \frac{\tau^2(\Phi''(\tau) - \Phi'(\tau))}{4k_nT}\right| + o_p(1). \tag{3.12}$$

Then by (3.12) and Theorem 1, we can approximate the $(1-\alpha)$ quantile of the null distribution by that of the distribution of

$$\sup_{\tau\in\mathcal{A}_c} \left|Z_1(\tau) + \sqrt{\frac{m_n}{2k_n}} Z_2(\tau) - \frac{\sqrt{[n/2k_n]m_n}}{4k_nT}\tau^2(\Phi''(\tau) - \Phi'(\tau))\right|, \tag{3.13}$$

which is denoted by $Q_n(\alpha,\mathcal{A}_c)$ and can be estimated via simulation. The test is thus

$$\mathcal{T}^n_{\mathcal{A}_c} > Q_n(\alpha,\mathcal{A}_c) \Leftrightarrow \text{Rejecting } H_0. \tag{3.14}$$

By Theorems 1-2, and their remarks, we soon have the following conclusion.

Theorem 4. *Under the conditions in Theorem 2, we have*

$$P(\mathcal{T}^n_{\mathcal{A}_c} > Q_n(\alpha,\mathcal{A}_c) \mid \beta \le 1) \to \alpha, \tag{3.15}$$

and on $\{\int_0^T \frac{\gamma_s^+ EY_1^+ + \gamma_s^- EY_1^-}{|\sigma_s|} ds \neq 0\}$,

$$P(\mathcal{T}^n_{\mathcal{A}_c} > Q_n(\alpha, \mathcal{A}_c) \mid \beta > 1) \to 1. \tag{3.16}$$

Theorem 4 demonstrates that the test (3.14) has the asymptotic size close to the nominal level and the asymptotic power close to 1. As seen in (3.9), the larger the β, the higher the detection power.

4. Numerical Studies

4.1. Monte Carlo Experiments

In this section, we conduct simulation studies to check the finite sample performance of the test. We generate data from the following stochastic volatility model for 5,000 times,

$$X_t = X_0 + \int_0^t \sqrt{c_s} dW_s + 0.5 Y_t, \quad 0 \le t \le T, \tag{4.1}$$

$$c_t = c_0 + \int_0^t 0.03(1.0 - c_s) ds + 0.15 \int_0^t \sqrt{c_s} dW'_s, \tag{4.2}$$

where Y_t is a skewed β stable Lévy process with the negative jumps appearing twice as much as the positive jumps to capture the market stylized feature caused by risk aversion. The volatility c_t is a square root diffusion process which is widely used in financial applications. The parameters in c_t are specified as in Jacod and Todorov (2014). In order to incorporate the leverage effect, we set $corr(dW, dW') = -0.5$. We also set $u_n = 0.5$ in estimating c_t. To illustrate the effect of the microstructure noise and find a good sampling frequency, we add a noise term onto X at the observation times. That is, what we observe is

$$\tilde{X}_{t_i} = X_{t_i} + \epsilon_{t_i}, \text{ where } \epsilon_{t_i} \sim \mathcal{N}(0, 0.035).$$

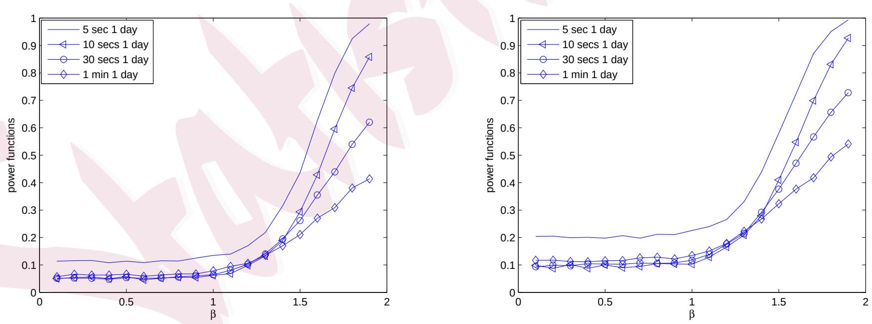
In the simulation studies, our tests are carried on the data set $\{\tilde{X}_{t_i}; i = 1, ..., n\}$. We take $\mathcal{A}_c$ as a set of grid points in $[-1, 1]$ with step length equal to 0.2. On the same $\mathcal{A}_c$, we use monte carlo method to find $Q_n(\alpha, \mathcal{A}_c)$ for $\alpha = 0.05$ *and* 0.10.

We first sample the data every 5, 10 or 30 seconds, or one minute in a single day ($T = 1$). Correspondingly the sample sizes are 4680, 2340, 780 and 390, respectively. The conditions in Theorem 1 imply that for finite sample k_n should be smaller than $\sqrt{n}$ and m_n should be smaller than k_n. Hence for the sampling frequencies mentioned above we set the pairs of (k_n, m_n) to be $(60, 28)$, $(40, 20)$, $(26, 18)$ and $(18, 12)$ with k_n/m_n ranging from 1.5 to 2.15 and having an increasing trend. Figure 1 displays the power functions of the test against β, from which we observe the following.

- Due to the bias caused by the microstructure noise, our test can not control type I error when the sampling frequency is as high as every 5 seconds;

- When the sampling frequencies are equal to or below every 10 seconds, our test is quite robust to the microstructure noise, and we observe 1) for $\beta < 1$, the three curves in both panels are close to the nominal level which is consistent to Theorem 4; 2) for $\beta > 1$, the four curves in both panels increase, demonstrating that the larger the β, the more powerful our test is, which is consistent to Remark 1; and 3) as the sample sizes increase, the performance of our test improves which is consistent to Theorem 4.

Figure 1: The power functions of the test for infinite variation jumps for five-second (solid), ten-second (left triangle), thirty-second (circle) and one-minute (square) data in a single day ($T = 1$); Left panel: $\alpha = 0.05$; Right panel: $\alpha = 0.10$.



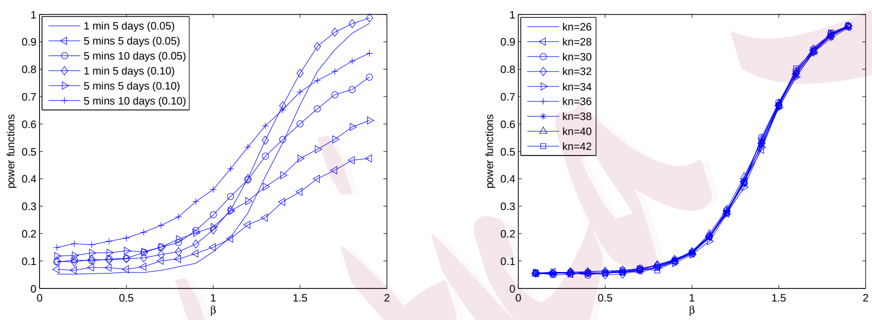
To check the performance of the test for sparser sampling schemes, we generate five days ($T = 5$) of one-minute and five-minute data and ten days ($T = 10$) of five-minute data from the noise contaminated model. We set (k_n, m_n) as $(39, 20)$, $(12, 8)$ and $(25, 15)$ by the same principle as stated above. The power functions are plotted in the left panel of Figure 2. Except for five days of one-minute data, the power functions indicate over rejection of our test due to the increase of discretization error as Δ_n increases. This shows why the power function for five days of one-minute data performs better than that for five days of five-minute data. The power function for five days of five-minute data is below that for ten days of five-minute data possibly due to the adverse effect of the microstructure noise which needs deeper investigation. This and the findings in Figure 1 show that sampling up to half a minute in a single day or every one minute in five days is safe for killing the noise in testing for infinite variation jumps.

Next, we do a sensitivity study on the tuning parameter k_n and m_n. We let $k_n = 26, 28, 30, 32, 34, 36, 38, 40, 42$ (m_n changes correspondingly as 13, 15, 17, 19, 21, 23, 25, 23, 22). For each k_n, we plot a power function in the right panel of Figure 2. From the figure, it is not easy to tell the differences among all power functions. This demonstrates that our test is not sensitive to the choice of k_n and m_n.

4.2. Real Data Analysis

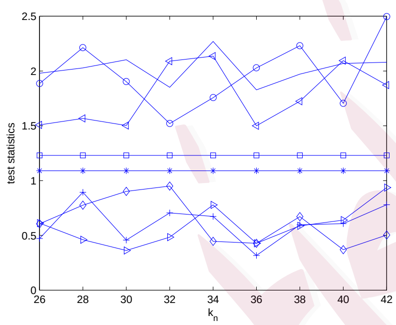
In this section, we implement our test to two constituent stocks of the S&P 500 index. The test is first carried on the every-one-minute

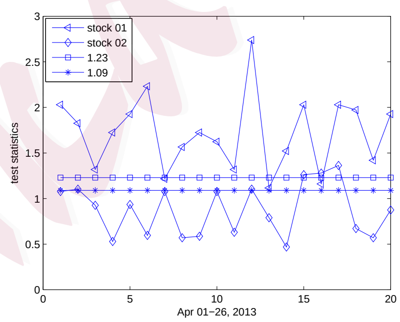
Figure 2: Left panel: the power functions of the test for infinite variation jumps for one-minute data in five days (solid for $\alpha = 0.05$ and diamond for $\alpha = 0.10$, $T = 5$), five-minute data in five days (left triangle for $\alpha = 0.05$ and right triangle for $\alpha = 0.10$, $T = 5$) and ten days (circle for $\alpha = 0.05$ and plus for $\alpha = 0.10$, $T = 10$); Right panel: stability analysis of the power function against k_n for five days of one-minute data; $\alpha = 0.05$.



data sets in Apr 8th, 11th and 24th in 2013 for stock 01 and stock 02 ($T = 1$ and $n = 390$). We aim to plot the observed test statistics against different values of $k_n = 42, 40, 38, 36, 34, 32, 30, 28, 26$ (correspondingly $m_n = 15, 13, 13, 12, 11, 10, 10, 9, 9$). We set other parameters as in the simulation studies. The results are illustrated in the left panel of Figure 3. The finding is that for all three days and any value of k_n, there are strong evidences for infinite variation jumps in the continuous-time price dynamics of stock 01. But the evidences are not significant for stock 02.

Figure 3: Left panel: the observed test statistics against $k_n = 26, 28, 30, 32, 34, 36, 38, 40, 42$ for stock 01 in Apr 8th (circle), Apr 11th (left triangle), and Apr 24 (solid) in 2013, and for stock 02 in Apr 8th (plus), Apr 11th (right triangle), and Apr 24 (diamond) in 2013; Right panel: test statistics in 20 consecutive trading days from Apr 1st to 26th in 2013 for stock 01 (left triangle) and stock 02 (diamond); The upper 0.05, *and* 0.10 quantiles are 1.23 (square) and 1.09 (star; two digital decimals are left for all k_n), respectively.





To study the existence of infinite variation jumps across different days, we do the test in 20 consecutive trading days from Apr 1st to 26th in 2013 for both stocks. The daily test statistics are plotted in the right panel of Figure 3. We find that in all days existence of infinite variation jumps is significant (above upper 0.10 quantile) for stock 01, while for stock 02 only 20% of the days have strong evidence for infinite variation jumps.

Supplementary Materials

The supplement contains the technical proofs of the main results and some auxiliary lemmas which are of their own interests.

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