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Complete List of Authors	Rakhi Singh Ashish Das and Feng-Shun Chai
Corresponding Author	Ashish Das
E-mail	ashishdas.das@gmail.com
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Optimal Paired Choice Block Designs

Rakhi Singh[1], Ashish Das[2] and Feng-Shun Chai[3]

[1]*IITB-Monash Research Academy,* [2]*Indian Institute of Technology Bombay and* [3]*Academia Sinica*

Abstract: Choice experiments help manufacturers, service-providers, policy-makers and other researchers in taking business decisions. Traditionally, while using designs for discrete choice experiments, every respondent is shown the same collection of choice pairs (that is, the choice design). Also, as the attributes and/or the number of levels under each attribute increases, the number of choice pairs in an optimal paired choice design increases rapidly. Moreover, in the literature under the utility-neutral setup, random subsets of the theoretically obtained optimal designs are often allocated to respondents. The question therefore is whether one can do better than a random allocation of subsets. To address these concerns, in the linear paired comparison model (or, equivalently the multinomial logit model), we first incorporate the fixed respondent effects (also referred to as the block effects) and then obtain optimal designs for the parameters of interest. Our approach is simple and theoretically tractable, unlike other approaches which are algorithmic in nature. We present several constructions of optimal block designs for estimating main effects or main plus two-factor interaction effects. Our results show when and how an optimal design for the model without blocks can be split into blocks so as to retain the optimality properties under the block model.

Key words and phrases: Choice experiment, Hadamard matrix, Linear paired comparison model, Multinomial logit model, Orthogonal array, Utility-neutral setup.

1. Introduction

Choice experiments mimic situations where individuals have to choose between a number of competing options. The goal is to quantify the influence of the attributes which characterize the choice options through a choice experiment. In a choice experiment, respondents are shown multiple choice sets of options and from each set they choose the preferred option. Considering choice sets of size two and r given respondents, a paired choice experiment is usually perceived as showing the same set of N choice pairs to each of the r respondents. The respondents are asked to give their preference among the two options for each of the N choice pairs shown to them. Each option in a choice pair is described by a set of k attributes, where for $i = 1, \ldots, k$, the ith attribute has v_i levels, $v_i \geq 2$. We represent the v_i levels by $0, \ldots, v_i - 1$. In a choice experiment, a paired choice design d is an allocation of choice pairs among r respondents such that each respondent observes N choice pairs. Such paired choice designs are often analyzed under the multinomial logit model.

One objective of a choice experiment is to optimally or efficiently estimate the parameters of interest which essentially consists of either only the main effects or the main plus two-factor interaction effects of the k attributes. D-optimal designs have been obtained in the literature either under the utility-neutral setup or using the locally D-optimal/Bayesian approach. D-optimal designs have been obtained theoretically under the utility-neutral setup, for example, see Graßhoff et al. (2003), Graßhoff et al. (2004), Street and Burgess (2007), Street and Burgess (2012), Demirkale, Donovan, and Street (2013), Bush (2014), Großmann and Schwabe (2015) and Singh, Chai, and Das (2015). In contrast, in the locally-optimal and the Bayesian approach, D-optimal designs have been obtained using computer algorithms (see, Huber and Zwerina (1996), Sándor and Wedel (2001), Sándor and Wedel (2002), Sándor and Wedel (2005), Kessels, Goos, and Vandebroek (2006), Kessels, Goos, and Vandebroek (2008), Kessels et al. (2008), Kessels et al. (2009), Yu, Goos, and Vandebroek (2009)). In this paper, we follow the utility-neutral approach.

Traditionally, in a choice experiment, respondents are shown the same collection of N choice pairs under the assumption that the respondents are alike. A choice experiment with the inherent premise that the respondents are alike is not quite practical since respondents, being a random sample from a population, are more likely to be heterogeneous. Kessels, Goos, and Vandebroek (2008) also noted that heterogeneity leads to responses from different respondents being different.

In a paired choice experiment, there is always a constraint on the maximum number of choice pairs that can be shown to each respondent so as to maintain overall response quality. A major concern with the traditional optimal paired choice designs is that the number of choice pairs in the design increases rapidly as k and/or v_i's are moderately increased.

Attempts have been made to address the issue of heterogeneity through different models and approaches. Sándor and Wedel (2002) have addressed the heterogeneity in respondents by constructing designs through a computer-intensive algorithmic approach under the so called mixed logit model. In their approach, same set of N choice pairs are shown to every respondent. Subsequently, Sándor and Wedel (2005) demonstrated that the use of different choice designs for different respondents and the random allocation of respondents to these designs yields substantially higher efficiency than the designs obtained in Sándor and Wedel (2002). Later Kessels, Goos, and Vandebroek (2008), for catering to heterogeneity in conjoint experiments, introduced a random respondent effects model for estimating the main effects and used algorithmic methods for constructing D-optimal designs. The conjoint designs under their setup consists of identifying as many sets of options as there are respondents. Therefore, the approach, though similar, is not applicable to our setup.

Often in practice, there is a pool of choice sets and respondents are allocated a random subset of choice sets (Street and Burgess, 2007). This process is continued until all choice sets are used once. Thereafter the process is started again. To address the *ad hoc* approach in the random allocation of choice sets, we use an additional fixed-effect term in the model to systematically split the pool of choice sets. In experimental design theory, the concept of blocking, as a tool to eliminate systematic heterogeneity in the experimental material, has been used extensively. Following the same approach, we consider the respondents as blocks. Thus, in contrast to the computer-intensive algorithmic approaches of Sándor and Wedel (2005) and Kessels, Goos, and Vandebroek (2008), we treat the respondent heterogeneity as a nuisance factor by including respondent-level block effect terms in the model and then design experiments to optimally estimate the parameters of interest after eliminating the respondent (block) effects. Adopting such an approach also enables the experimenter to get optimal designs with reasonable number of choice pairs $s(< N)$ shown to each of the r respondents. Later in Section 2, we discuss the kind of heterogeneity that is being taken care of in our approach and the seemingly similar approaches.

In what follows, a design with b blocks each of size s is generated and that each block is associated to a respondent. Usually t copies of a proposed design is used for larger numbers of respondents $r = tb$, since replicating the design does not affect its optimality. We therefore, restrict ourselves to optimal paired choice block designs with b blocks each of size s with $N = bs$.

In this context, the traditional paired choice designs reduce to $b = 1, s = N$ and $r = t$ where s is necessarily atleast the number of model parameters. However, for $b > 1$, the block size s can be smaller than the number of model parameters, but the paired choice design with b blocks can still estimate all model parameters. In order to estimate the model parameters, we provide optimal designs with block sizes that are flexible and practical under our setup.

In Section 2, treating respondent heterogeneity as a nuisance factor and incorporating the fixed respondent (block) effects in the model, we obtain the information matrix for estimating the parameters of interest after eliminating the respondent (block) effects. In Section 3, under the main effects block model, we provide optimal paired choice block designs for estimating the main effects for symmetric and asymmetric attributes.

We also give a simple solution to the problem of identifying generators in the constructions of optimal paired choice designs. In Section 4, under a broader main effects block model, we provide optimal paired choice block designs for symmetric and asymmetric attributes. The broader main effects model constitutes the main effects and the two-factor interaction effects with interest lying only in the estimation of the main effects. Finally, in Section 5, we provide optimal paired choice block designs for estimating the main plus two-factor interaction effects. Finally, we provide a Discussion in Section 6.

2. Preliminaries and the model incorporating respondent effects

Most of the work on optimal choice designs is based on the multinomial logit model approach of either Huber and Zwerina (1996) or that followed in Street and Burgess (2007). Großmann and Schwabe (2015) observed that the two approaches are equivalent for the purpose of finding optimal designs. We work with the multinomial logit model approach of Huber and Zwerina (1996). The multinomial logit model supposes that the probability of preferring option 1 over option 2 in the ith choice pair can be expressed as $\pi_{12i} = e^{u_{1i}}/(e^{u_{1i}} + e^{u_{2i}})$, where u_{1i} and u_{2i} represent the systematic part of the utilities attached to the two options in choice pair i. Similarly $\pi_{21i} =$

$1-\pi_{12i}$ is the probability that option 2 is preferred over option 1. It follows that for the ith choice pair, the choice probabilities depend only on the utility difference $u_{1i}-u_{2i}$. For a design d with N choice pairs, since options are described by k attributes, the utilities are modeled using the linear predictor $u_j = P_{pj}\theta$, where θ is a $p \times 1$ vector representing the parameters of interest, P_{pj} is an $N \times p$ effects-coded matrix for the jth option, and $u_j = (u_{ji})$ is an $N \times 1$ utility vector for the jth option, $j = 1, 2$. The utility difference $u_1 - u_2 = (P_{p1} - P_{p2})\theta = P_p\theta$ is then a linear function of the parameter vector θ. For the purpose of deriving optimal designs, it is often assumed that $\theta = 0$. This indifference or the utility-neutral assumption means that the two options in a choice set are equally attractive and leads to a considerable simplification of the information matrix and the design problem. Under the utility-neutral multinomial logit model, the Fisher information matrix is $(1/4)P_p'P_p$ (see, Großmann and Schwabe (2015)).

Simultaneously, Graßhoff et al. (2003) and Graßhoff et al. (2004) studied linear paired comparison designs which are analyzed under the linear paired comparison model. The observed utility difference Z between the two options again depends on the difference matrix $P_p = P_{p1}-P_{p2}$. More precisely, the response is described by the model, $Z = u_1-u_2+\epsilon = (P_{p1}-P_{p2})\theta+\epsilon = P_p\theta + \epsilon$, where ϵ is the random error vector. The matrix $C = P_p'P_p$ is the information matrix under the linear paired comparison model. Since C is proportional to the information matrix under the utility neutral multinomial logit model, it follows that the designs optimal under the linear paired comparison model are also optimal under the multinomial logit model and vice versa.

We discuss only D-optimality since, as noted in Großmann and Schwabe (2015), most of the optimality results for choice designs and linear paired comparison designs are available for the D-criterion. A D-optimal design has the maximum determinant of the information matrix among all competing designs.

For paired choice experiments, the multinomial logit model as well as the linear paired comparison model are based on the utility difference $u_1 - u_2$. By incorporating respondent effects, the relevant utility differences under the block model, with blocks being the respondents, becomes

$$u_1 - u_2 = (P_{p1} - P_{p2})\theta + W\beta = P_p\theta + W\beta, \tag{2.1}$$

where $\beta = (\beta_1, \ldots, \beta_b)'$ represents the $b \times 1$ vector of block effects, and $W = (w_{ij})$ is an $N \times b$ incidence matrix with $w_{ij} = 1$ if the ith choice pair belongs to the jth block and 0 otherwise. Without loss of generality, we take $W = I_b \otimes 1_s$, where I_a and 1_a denotes the identity matrix of order a and the $a \times 1$ vector of all ones, respectively. Here, $\otimes$ denotes the Kronecker product.

Note that (2.1) corresponds to a paired choice block design with b blocks each of size s and that such b blocks are repeated t times to accommodate for $r = tb$ respondents. Each of the r respondents is associated to a single block of the design.

Unlike Sándor and Wedel (2005) and Kessels, Goos, and Vandebroek (2008), where an assumed distribution on the model parameters takes care of the respondent effects, our approach, following the standard block design theory, has been to consider β_j as a fixed-effects term. While the vast literature on theoretically obtained D-optimal designs for choice experiments rests on a multinomial logit model without any respondent effects, our fixed-effects block model attempts to obtain the optimal block designs theoretically under the utility-neutral setup.

In either the multinomial logit model or the linear paired comparison model, including respondent effects β can be regarded as adding b two-level attributes to the set of p predictor variables. Then, the corresponding difference matrix for the pairs, in b blocks, has an additional component and can be written as (P_p, W). Thus, under the utility-neutral multinomial logit block model, it follows that the information matrix for estimating θ

and β is

$$M = \frac{1}{4}\begin{bmatrix} C & P_p'W \\ W'P_p & W'W \end{bmatrix} \tag{2.2}$$

where $C = P_p'P_p$, as defined earlier. Moreover, upto a constant factor of $1/4$, M coincides with the information matrix in the linear paired comparison block model. Thus, optimal designs under the linear paired comparison block model are also optimal under the utlity neutral multinomial logit block model. The information matrix for estimating θ under the linear paired comparison block model after eliminating the block effects is

$$\tilde{C} = C - P_p'W(W'W)^{-1}W'P_p = C - (1/s)P_p'WW'P_p. \tag{2.3}$$

This follows from the standard linear model theory where a parameter vector is partitioned into a parameter vector of interest and the nuisance parameters (see, for example, Page 68 of Haines (2015)).

A paired choice block design is connected if all the parameters of interest are estimable, and this happens if and only if $\tilde{C}$ has rank p. In what follows, the class of all connected paired choice block designs with k attributes in b blocks each of size s is denoted by $\mathcal{D}_{k,b,s}$. From (2.3), since $C - \tilde{C}$ is a non-negative definite matrix, if in the class of unblocked designs with $N = bs$, a paired choice design d is D-optimal, then d, considered as a design in $\mathcal{D}_{k,b,s}$, is also D-optimal, provided $\tilde{C} = C$.

It is observed that eliminating respondent effects simultaneously controls the within-pair order effects (see, Goos and Großmann (2011) and Bush, Street, and Burgess (2012)).

3. Optimal block designs under the main effects model

Under the main effects block model, from (2.1) it follows that $u_1 - u_2 = (P_{M1} - P_{M2})\tau + W\beta = P_M\tau + W\beta$, where τ is a $\sum_{i=1}^{k}(v_i - 1) \times 1$ parameter vector for main effects, P_{Mj} is an $N \times \sum_{i=1}^{k}(v_i - 1)$ effects-coded matrix of the main effects for the jth option, $j = 1, 2$, and $P_M = P_{M1} - P_{M2}$. In a row of P_{Mj} is embedded the effects-coded row vector of length $v_i - 1$ for the ith attribute. The effects coding for level l is represented by a unit vector with 1 in the $(l + 1)$th position for $l = 0, \ldots, v_i - 2$, and for level $v_i - 1$ is represented by -1 in each of the $v_i - 1$ positions, $i = 1, \ldots, k$. For example, for $v = 3$, effects-coded vectors for $l = 0, 1, 2$ are $(1\ 0), (0\ 1)$ and $(-1\ \ -1)$, respectively.

From (2.3), the information matrix for estimating the main effects after eliminating the block effects is

$$\tilde{C}_M = C_M - (1/s)P_M' W W' P_M, \tag{3.1}$$

where $C_M = P_M' P_M$ is the information matrix for estimating the main effects under the unblocked model. From (3.1), it follows that a necessary and sufficient condition for $\tilde{C}_M = C_M$ to hold is $W'P_M = 0$. Therefore, by suitably blocking the choice pairs of an optimal paired choice design into b blocks such that $W'P_M = 0$, one can obtain an optimal paired choice block design. We provide a simple condition to achieve the same, proof of which is provided in the Supplementary Material.

Theorem 1. $\tilde{C}_M = C_M$ *if for each block, the levels of every attribute appear equally often in the first option as well as in the second option.*

This property of every level of an attribute appearing the same number of times in the first and second option of pairs is also known as *position-balance* (see, Großmann and Schwabe (2015)).

An orthogonal array $OA(n, k, v_1 \times \cdots \times v_k, t)$, of strength t, is an $n \times k$ array with elements in the ith column from a set of v_i distinct symbols $\{0, 1, \ldots, v_i - 1\}$ $(i = 1, \ldots, k)$, such that all possible combinations of symbols appear equally often as rows in every $n \times t$ subarray. An orthogonal array is symmetric if $v_i = v$ for all i and the corresponding OA is denoted by $OA(n, k, v^k, t)$, else it is an asymmetric orthogonal array.

Street and Burgess (2007), Demirkale, Donovan, and Street (2013) and Bush (2014) provide the $OA + G$ method for constructing optimal paired choice designs using orthogonal arrays and generators G. Let G be a collection of h generators $G_1, \ldots, G_h$ where $G_j = (g_{j_1}, g_{j_2}, \ldots, g_{j_k})$. The $OA + G$

method gives a paired choice design $(A, B_j), j = 1, \ldots, h$ where $A = (A_{li})$ is an $OA(n_1, k, v_1 \times \cdots \times v_k, t)$ and $B_j = (B^j_{li})$ with $B^j_{li} = A_{li} + g_{ji}$ reduced mod v_i, $l = 1, \ldots, n_1$, $i = 1 \ldots, k$, $j = 1, \ldots, h$. This method depends on the availability of the required orthogonal array, which may not always exist. The SAS link `http://support.sas.com/techsup/technote/ts723.html`, the Sloane link `http://neilsloane.com/oadir/` and Hedayat, Sloane, and Stufken (1999) provide a comprehensive summary of orthogonal arrays and their constructions.

In the literature, arriving at the generators G has been usually through a trial-and-error approach, and no general results on the structure of such generators appear to exist. In fact, Bush (2014) highlights the complexities involved in choosing the sets of generators. We present a simple result that systematically provides the h generators, proof of which is provided in the Supplementary Material. Let $lcm(a_1, \ldots, a_k)$ denotes the least common multiple of $a_1, \ldots, a_k$.

Theorem 2. *Number of generators for the optimal paired choice design with k attributes is $h = lcm(h_1, \ldots, h_k)$ where $h_i = v_i - 1$ for v_i even and $h_i = (v_i - 1)/2$ for v_i odd, $i = 1, \ldots, k$. The generators are then given by $G_j = (g_{j_1}, g_{j_2}, \ldots, g_{j_k})$, where g_{j_i} takes each of the values from the set $\{1, \ldots, h_i\}$ with frequency h/h_i, $j = 1, \ldots, h$, $i = 1, \ldots, k$.*

Note that Theorem 2 provides generators for the unblocked paired choice designs. As in Street and Burgess (2007), we use several sets of generators to create the final design and that the number of generators given in Theorem 2 may not be the smallest possible.

Example 1. Suppose there are three attributes with $v_1 = 2, v_2 = 3$ and $v_3 = 4$. Then we have $h_1 = 1$, $g_{j_1} = 1$; $h_2 = 1$, $g_{j_2} = 1$; and $h_3 = 3$, $g_{j_3} = 1, 2, 3$. Thus, $h = lcm(1, 1, 3) = 3$. This leads to the generators $G_1 = (111), G_2 = (112)$ and $G_3 = (113)$. Thus, for a given $OA(24, 3, 2 \times 3 \times 4, 2)$, the corresponding optimal paired choice design with parameters k, $v_1 = 2, v_2 = 3, v_3 = 4$, $b = 1$, $N = s = hn_1 = 3 \times 24 = 72$, is obtained using the $OA + G$ method of construction with three generators. The corresponding design is given in the Supplementary Material.

Example 2. As another example, suppose there are two attributes with $v_1 = 4$ and $v_2 = 5$. Then we have $h_1 = 3$, $g_{j_1} = 1, 2, 3$ and $h_2 = 2$, $g_{j_2} = 1, 2$. Thus, $h = lcm(3, 2) = 6$. This leads to the six generators $G_1 = (11), G_2 = (12)$, $G_3 = (21)$, $G_4 = (22)$, $G_5 = (31)$ and $G_6 = (32)$ which will give an optimal paired choice design when used in conjunction with $OA(20, 2, 4 \times 5, 2)$.

In general, for a given $OA(n_1, k, v_1 \times \cdots \times v_k, 2)$, the corresponding optimal paired choice design d_1 with parameters k, $v_1, \ldots, v_k$, $b = 1$, $N = s =$

hn_1, is obtained using the $OA+G$ method of construction with generators $G_j, j = 1, \ldots, h$. When $N = s$ is large, we find that practitioners advocate allocation of the choice pairs into more than one blocks either randomly or using a spare attribute (see, Street and Burgess (2007), Bliemer and Rose (2011)). Based on Theorem 1, it follows that under our block model, we can retain optimality of the design obtained through the $OA+G$ method if blocking is done using a column corresponding to an attribute. Any other blocking approach may jeopardize the characteristics of the design. We now provide four theorems and their constructions, detailed proofs of which are provided in the Supplementary Material.

Theorem 3. *For $\delta \geq 1$ and an $OA(n_1, k+1, v_1 \times \cdots \times v_k \times \delta, 2)$, there exists an optimal paired choice block design $d_2 \in \mathcal{D}_{k,b,s}$ with parameters k, $v_1, \ldots, v_k$, $b = h\delta$, $s = n_1/\delta$, where $h = lcm(h_1, \ldots, h_k)$.*

Construction. *For a given $OA(n_1, k+1, v_1 \times \cdots \times v_k \times \delta, 2)$, corresponding to the k attributes at levels $v_i, i = 1, \ldots, k$, let d_1 be the design constructed through $OA+G$ method using $h = lcm(h_1, \ldots, h_k)$ generators from Theorem 2. Then d_1 with parameters k, $v_i, i = 1, \ldots, k$, $b = 1$, $s = hn_1$ is an optimal paired choice design. From d_1, the choice pairs obtained through each of the h generators constitute a block of size n_1. Finally, we use the δ symbols of the $(k+1)$th column of the orthogonal array for further blocking. This gives* *us a paired choice block design d_2 with parameters k, $v_1, \ldots, v_k$, $b = h\delta$, $s = n_1/\delta$.*

Example 3. From an $OA(24, 15, 2^{13} \times 3 \times 4, 2)$, for estimating the main effects of $k = 14$ attributes of which 13 attributes are at 2 levels and 1 attribute is at 3 levels, an optimal paired choice block design can be constructed for $\delta = 4, h = 1, k = 14, b = 4, s = 6$ are optimal. As an illustration, we give a $2^4 \times 3$ paired choice block design d_2 with parameters $k = 5, b = 4, s = 6$.

$d_2 =$

B_1	B_2	B_3	B_4
(00000, 11111)	(01102,10010)	(10112,01000)	(10001,01112)
(11010,00101)	(11110,00001)	(00111,11002)	(00012,11100)
(01101,10012)	(11011,00102)	(01002,10110)	(10100,01011)
(11002,00110)	(00100,11011)	(11101,00012)	(01011,10102)
(10111,01002)	(10012,01100)	(01010,10101)	(01110,10001)
(00112,11000)	(00001,11112)	(10000,01111)	(11102,00010)

It is noted that when the attributes have mixed levels greater than 3, the $OA + G$ method leads to choice designs with a large number of choice pairs. However, blocking still helps in reducing the number of choice pairs shown to respondent from $N = s = 96$ to $s = 24$. For example, an $OA(32, 11, 2^3 \times 4^7 \times 8, 2)$ can be used to construct a paired choice block design having three 2-level attributes and seven 4-level attributes in $N = 96$ choice pairs with $b = 24$ and $s = 4$.

For many parameter sets corresponding to k attributes each at v levels, Graßhoff et al. (2004) and Demirkale, Donovan, and Street (2013) have provided constructions of optimal paired choice designs with a reduced number of choice pairs in comparison to the $OA + G$ method of construction. We now show how an optimal paired choice block design can be constructed starting from their designs.

Theorem 4. *For a Hadamard matrix H_m, an optimal paired choice design d_3 with parameters $k, v, b = 1, s = mv(v-1)/2$, $k \leq m$ exists. Furthermore for v odd, a paired choice block design d_4 with parameters $k, v, b = m(v-1)/2, s = v$ exists, which is optimal in $\mathcal{D}_{k,b,s}$.*

Construction. *For a given H_m, an optimal paired choice design d_3 is obtained through Theorem 3 of Graßhoff et al. (2004) with parameters $k, v, b = 1, s = mv(v-1)/2$. Moreover, for v odd, the choice pairs corresponding to each of the rows of $\{H_m, -H_m\}$ forms a block and the design so obtained is an optimal paired choice block design. Now, using a result from Dey (2009), $v(v-1)/2$ combinations involving v levels taken two at a time can be grouped into $(v-1)/2$ replicate each comprising v elements. Therefore, the blocks generated by each row of H_m can be further broken into $(v-1)/2$ blocks each of size v, which gives the optimal paired choice block design d_4.*

Example 4. Consider $v = 3$ with combinations $(0, 1), (1, 2), (2, 0)$ and the

Hadamard matrix H_4. An optimal paired choice design d_3 with parameters $k = 4, v = 3, b = 1, s = 12$ exists. Furthermore, since v is odd, an optimal paired choice block design d_4 is constructed with parameters $k = 4, v = 3, b = 4, s = 3$ by considering choice pairs generated by each row of $\{H_4, -H_4\}$ as a block.

$d_4 =$

B_1	B_2	B_3	B_4
(0000,1111)	(0101,1010)	(0011,1100)	(0110,1001)
(1111,2222)	(1212,2121)	(1122,2211)	(1221,2112)
(2222,0000)	(2020,0202)	(2200,0022)	(2002,0220)

Theorem 5. *For an $OA(n_2, k+1, v^k \times v_{k+1}, 2)$ with $v_{k+1} = n_2/v$, an optimal paired choice design d_5 with parameters $k, v, b = 1, s = n_2(v-1)/2$ exists. Furthermore for v odd, a paired choice block design d_6 with parameters $k, v, b = n_2(v-1)/2v, s = v$ exists, which is optimal in $\mathcal{D}_{k,b,s}$.*

Construction. *For a given $OA(n_2, k+1, v^k \times v_{k+1}, 2)$ with $v_{k+1} = n_2/v$, an optimal paired choice design d_5 is obtained through Construction 3.2 of Demirkale, Donovan, and Street (2013) with parameters $k, v, b = 1, s = v_{k+1}\binom{v}{2}$. Moreover, for v odd, the choice pairs corresponding to each of the parallel sets of the orthogonal array forms a block and the design so obtained is an optimal paired choice block design. Now, following Dey (2009), the blocks generated by each parallel set can be further broken into $(v-1)/2$ blocks each of size v, which gives the optimal paired choice block design d_6.*

Theorem 6. *For $\delta \geq 1$ and an $OA(n_3, k+1, m_1 \times \cdots \times m_k \times \delta, 2)$ with $m_i = v_i(v_i - 1)/2$ for some odd v_i, an optimal paired choice block design d_8 with parameters $k, v_i, \ldots, v_k, b = \delta, s = n_3/\delta$ exists.*

Construction. *For a given $OA(n_3, k+1, m_1 \times \cdots \times m_k \times \delta, 2)$ with $m_i = v_i(v_i - 1)/2$ for some odd v_i, an optimal paired choice design d_7 is obtained through Theorem 4 of Graßhoff et al. (2004) with parameters $k, v_i, \ldots, v_k, b = 1, s = n_3$. Then, similar to construction of Theorem 3, we use the δ (≥ 1) symbols of the $(k+1)$th column of the orthogonal array for blocking. This gives us an optimal paired choice block design d_8 with parameters $k, v_i, \ldots, v_k, b = \delta, s = n_3/\delta$. Note that this method of blocking is applicable only for odd v_i.*

Table 1 highlights the flexibility in the number of blocks while blocking the traditional optimal symmetric paired choice designs as listed in Table 2 of Demirkale, Donovan, and Street (2013). We list the values of s and b corresponding to the optimal designs obtained through Theorem 3 and Theorem 4. It is observed that in the parameter range of Table 1, Theorems 5 and 6 do not provide any additional designs that are not obtainable from Theorem 3 and Theorem 4. Some of the traditional optimal paired choice designs, marked $*$, are not optimal under the block setup for blocks of size

$s = N$ and $b = 1$ since the design matrices are not orthogonal to the vector of all ones. However, by having $b > 1$, optimal designs having blocks of size $s = N/b$ are feasible using Theorem 3.

Note that, from a given optimal paired choice design in $\mathcal{D}_{k,b,s}$, we can randomly group the b blocks into b/x blocks each of size xs to obtain optimal paired choice designs in $\mathcal{D}_{k,b/x,sx}$. In Table 1, the designs with $x = 1$ are first obtained using the Theorems as mentioned in the corresponding column headers whereas the designs with $x > 1$ are obtained thereafter through random grouping. One could obtain a table similar to Table 1, for optimal asymmetric paired choice designs based on a list of more than 600 orthogonal arrays with $n \leq 100$.

Table 1: Optimal designs in $\mathcal{D}_{k,b,s}$

v	k	Traditional (s,1)	Theorem 3 (s, b)	Theorem 4 (s, b)
2	3	4	(4,1)	
2	4	4*	(4x,2/x), x=1,2	
2	5-6	8	(4x,2/x), x=1,2 (6x,2/x), x=1,2	
2	7	8	(8,1) (6x,2/x), x=1,2 (4x,4/x), x=1,2,4	
2	8	8*	(6x,2/x), x=1,2 (4x,4/x), x=1,2,4	

2	9-10	12	(6x,2/x), x=1,2 (4x,4/x), x=1,2,4 (10x,2/x), x=1,2	
2	11	12	(12,1) (4x,4/x), x=1,2,4 (10x,2/x), x=1,2 (6x,4/x), x=1,2,4	
2	12	12*	(4x,4/x), x=1,2,4 (10x,2/x), x=1,2 (6x,4/x), x=1,2,4	
3	3	9,12	(3x,3/x), x=1,3	(3x,4/x), x=1,2,4
3	4	9,12,18	(9,1) (3x,6/x), x=1,2,3,6	(3x,4/x), x=1,2,4
3	5,6	18,24	(3x,6/x), x=1,2,3,6	(3x,8/x), x=1,2,4,8
3	7	18,24,27	(9x,2/x), x=1,2 (3x,9/x), x=1,3,9	(3x,8/x), x=1,2,4,8
3	8	24,27	(3x,9/x), x=1,3,9	(3x,8/x), x=1,2,4,8
3	9	27,36	(3x,9/x), x=1,3,9	(3x,12/x), x=1,2,3,4,6,12
3	10-12	27,36	(9x,3/x), x=1,3 (3x,12/x), x=1,2,3,4,6,12	(3x,12/x), x=1,2,3,4,6,12
4	3-4	24*,28	(4x,12/x), x=1,2,3,4,6,12	
4	5	48	(16x,3/x), x=1,3 (4x,24/x), x=1,2,3,4,6,8,12,24	
4	6-8	48*,96	(4x,24/x), x=1,2,3,4,6,8,12,24	
4	9	72*,96	(16x,6/x), x=1,2,3,6 (4x,36/x), x=1,2,3,4,6,9,12,18,36	
4	10-12	72*,144	(4x,36/x), x=1,2,3,4,6,9,12,18,36	
5	3-4	40,50	(5x,10/x), x=1,2,5,10	(5x,8/x), x=1,2,4,8

5	5	50,80	(5x,10/x), x=1,2,5,10	(5x,16/x), x=1,2,4,8,16
5	6	50,80,100	(25x,2/x), x=1,2 (5x,20/x), x=1,2,4,5,10,20	(5x,16/x), x=1,2,4,8,16
5	7-8	80,100	(5x,20/x), x=1,2,4,5,10,20	(5x,16/x), x=1,2,4,8,16
5	9-10	100,120	(5x,20/x), x=1,2,4,5,10,20	(5x,24/x), x=1,2,3,4,6,12,24
6	3	60*,180	(12x,15/x), x=1,3,5,15 (18x,10/x), x=1,2,5,10	
6	4	60*,180*,360	(6x,60/x), x=1-6,10,12,15,20,30,60	
6	5-6	120*,180*,360	(6x,60/x), x=1-6,10,12,15,20,30,60	
7	3-4	84,147	(7x,21/x), x=1,3,7,21	(7x,12/x), x=1,2,3,4,6,12
7	5-7	147,168	(7x,21/x), x=1,3,7,21	(21x,8/x), x=1,2,4,8
7	8	147,168,294	(49x,3/x), x=1,3 (7x,42/x), x=1,2,3,6,7,14,21,42	(21x,8/x), x=1,2,4,8

4. Optimal block designs under the broader main effects model

In this section, we consider estimation of the main effects under the broader main effects model for an asymmetric paired choice design where the ith attribute is at v_i levels, $i = 1, \ldots, k$. The broader main effects model constitutes the main effects and the two-factor interaction effects with interest lying only in the estimation of the main effects. For the symmetric paired choice designs, Graßhoff et al. (2003) characterized optimal paired choice designs under the broader main effects model. More recently, for $v_i = 2$, Singh, Chai, and Das (2015) obtained optimal designs under such a model.

With the introduction of the respondent effects, from (2.1), the relevant utility differences become

$$u_1 - u_2 = (P_{M1} - P_{M2})\tau + (P_{I1} - P_{I2})\gamma + W'\beta = P_M\tau + P_I\gamma + W'\beta, \quad (4.1)$$

where γ is a $\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(v_i-1)(v_j-1)\times 1$ parameter vector for the two-factor interaction effects, P_{Ij} is an $N\times\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(v_i-1)(v_j-1)$ effects-coded matrix of the two-factor interaction effects for the jth option, $j=1,2$, and $P_I = P_{I1} - P_{I2}$. Let $P_{Ij} = (P_{Ij}^{1'}, \ldots, P_{Ij}^{n'})'$ where P_{Ij}^l corresponds to the lth choice pair in P_{Ij}. Also, let $P_{Mj(i)}^l$ represent the columns of P_{Mj} corresponding to the lth choice pair and ith attribute. Then, $P_{Ij}^l = (P_{Mj(1)}^l \otimes P_{Mj(2)}^l, P_{Mj(1)}^l \otimes P_{Mj(3)}^l, \ldots, P_{Mj(k-1)}^l \otimes P_{Mj(k)}^l)$.

The information matrix for estimating the main effects after eliminating the two-factor interaction effects and the block effects is

$$\tilde{C}_B = C_M - [P_M'P_I \; P_M'W]\begin{bmatrix} P_I'P_I & P_I'W \\ P_I'W & W'W \end{bmatrix}^{-}\begin{bmatrix} P_I'P_M \\ W'P_M \end{bmatrix}. \quad (4.2)$$

Therefore, a paired choice design which is optimal under the main effects model is also optimal under the broader main effects block model if $\tilde{C}_B = C_M$, that is, if $P_I'P_M = 0$ and $W'P_M = 0$. The designs in Theorem 3 satisfy $W'P_M = 0$ and for symmetric designs with $v = 2$, it follows from

Singh, Chai, and Das (2015) that the designs additionally satisfy $P_I'P_M = 0$. Therefore, in particular, for symmetric designs with $v = 2$, the paired choice block designs of Theorem 3 are also optimal under the broader main effects block model.

We now give the following construction for optimal paired choice block designs under the broader main effects model.

Theorem 7. *Under the broader main effects model, for an $OA(n_1, k, v_1 \times \cdots \times v_k, 3)$ and $h = lcm(v_1, \ldots, v_k)$, there exists a paired choice block design d_1^B with parameters $k, v_1, \ldots, v_k, b = 1, s = hn_1$, which is optimal in $\mathcal{D}_{k,b,s}$.*

Construction. *We obtain d_1^B through the $OA + G$ method of construction using h generators as in Theorem 2. Detailed proof is provided in the Supplementary Material.*

Theorem 8. *Under the broader main effects model, for $\delta \geq 1$ and an $OA(n_1, k+1, v_1 \times \cdots \times v_k \times \delta, 3)$, there exists a paired choice block design d_2^B with parameters $k, v_1, \ldots, v_k, b = h\delta, s = n_1/\delta$, which is optimal in $\mathcal{D}_{k,b,s}$.*

Construction. *On lines similar to Theorem 3, the construction here is based on using sets of generators, from Theorem 2, on an orthogonal array of strength 3.*

We now provide another method to obtain symmetric optimal paired choice block designs with $s = v; v \geq 3$.

Theorem 9. *For an $OA(n_1, k-1, v^{k-1}, 3)$, there exists a paired choice block design d_3^B with parameters $k, v \geq 3, s = v, b = hn_1$, which is optimal in $\mathcal{D}_{k,b,s}$.*

Construction. *We adopt the following method of construction.*

(i) Following Theorem 8, construct d_2^B from an $OA(n_1, k, v^{k-1} \times 1, 3)$ for $k-1$ attributes each at v levels. While constructing d_2^B, the h generators, as in Theorem 2, are $(k-1)$-tuples of the form $(1\dots1), \cdots, (v-1\dots v-1)$ for v even ($h = (v-1)$), and of the form $(1\dots1), \cdots, ((v-1)/2 \dots (v-1)/2)$ for v odd ($h = (v-1)/2$). Then, for each choice pair, add the kth attribute at level 0 in the option 1 and similarly, the kth attribute in the second option is generated using the same generator as that used for the other $k-1$ attributes.

(ii) For each of the h generators, generate $v-1$ additional copies of the design obtained in (i) by adding 1 (mod v),$\dots,(v-1)$ (mod v) in every attribute under both the options. Note that every copy in (ii) is just the recoding of the design obtained in (i), and hence the resultant design with parameters $k, v, s = hn_1v, b = 1$ is also optimal.

(iii) Finally for each of the h generators, the ith block of size v comprises of the ith row from each of the v copies created in (ii), $i = 1, \ldots, n_1$.

The hn_1 blocks so obtained with $s = v$ forms the required optimal design d_3^B. The design so obtained has distinct choice pairs in every block.

5. Optimal block designs for estimating the main plus two-factor interaction effects

The literature on optimal paired choice designs for estimating the main plus two-factor interaction effects is very limited since such designs require a large number of choice pairs to be shown to every respondent. Graßhoff et al. (2003), Street and Burgess (2004) and Großmann, Schwabe, and Gilmour (2012) have provided optimal and/or efficient paired choice designs under this setup for k attributes each at two levels. In this Section, we consider each of the k attributes to be at two levels. Let $q = \lceil k/2 \rceil$, where $\lceil z \rceil$ represents the smallest integer greater than or equal to z. The construction method of Street and Burgess (2007) entails starting with an orthogonal array $OA(n_1, k, 2^k, 4)$ as a set of n_1 first options, and then taking the foldover of α attributes in the second option, keeping the rest of the $k - \alpha$ attributes same for each of the n_1 choice pairs. Here $\alpha = q$ for

k odd and $\alpha = q$ and $q+1$ for k even. This process is repeated for $\binom{k}{\alpha}$ possible combinations of the attributes. Here, the foldover of an attribute in the second option of a choice pair means that the attribute level in the second option is different from that in the first. Such a paired choice design d_1^I with parameters $k, v, s, b = 1$ is optimal where $s = n_1\binom{k}{q}$ for k odd and $s = n_1\binom{k+1}{q+1}$ for k even.

Incorporating respondent effects, the model is as given in (4.1). However, in contrast to Section 4, interest here lies in the estimation of both the main-effects and the two-factor interaction effects. The information matrix for estimating the main plus two-factor interaction effects under the multinomial logit model incorporating respondent effects is

$$\tilde{C}_I = \begin{bmatrix} C_M & P_M' P_I \\ P_I' P_M & P_I' P_I \end{bmatrix} - (1/s)[P_M' W \;\; P_I' W] \begin{bmatrix} W' P_M \\ W' P_I \end{bmatrix}. \tag{5.1}$$

As earlier, in order to achieve optimal paired choice block designs, we start with an optimal paired choice design d_1^I and enforce blocking such that $W'P_M = 0$ and $W'P_I = 0$. We provide a simple condition to achieve the same, proof of which is provided in the Supplementary Material.

Let pair (a_1, b_1) means that a_1 and b_1 are the levels corresponding to an attribute for the first and second options, respectively. Similarly, let pair (a_1a_2, b_1b_2) means that a_1a_2 and b_1b_2 are the levels corresponding to the two attributes for the first and second options, respectively.

Theorem 10. *$W'P_M = 0$ and $W'P_I = 0$ if and only if for every block,*

(i) the frequency of the pair $(1, 0)$ is same as the frequency of the pair $(0, 1)$ for every attribute;

(ii) the frequency of the pairs from the set $\{(01, 00)$, $(01, 11)$, $(10, 00)$, $(10, 11)\}$ is same as the frequency of the pairs from the set $\{(00, 01)$, $(00, 10)$, $(11, 01)$, $(11, 10)\}$ for every two attributes.

We now provide a method of construction for optimal paired choice block designs with $s = 4$.

Theorem 11. *For $k > 4$, there exists a paired choice block design d_2^I with parameters $k, v = 2, s = 4, b$, which is optimal in $\mathcal{D}_{k,b,s}$. Here $b = 2^{k-3}\binom{k}{q}$ for k odd and $b = 2^{k-3}\binom{k+1}{q+1}$ for k even.*

Construction. *Let F be a set of $\binom{k}{\alpha}$ attribute indices of size $\alpha = q$ obtained from the attribute labels $1, \cdots, k$ taking α labels at a time such that $2 \leq \alpha \leq k - 2$. For an element $f = (f_1, \ldots, f_i, \ldots, f_\alpha)$ of F, let $f' = \{1, \ldots, k\} - f = (f'_1, \ldots, f'_j, \ldots, f'_{(k-\alpha)})$ be the complement of f. Keeping in view the construction of the design d_1^I, we adopt the steps (i)-(v) to construct an optimal paired choice block design d_2^I for k attributes.*

(i) Write the complete factorial involving 2^{α} *combinations. Divide this set into two-halves such that the second half is a foldover of the first half.*

(ii) Write the complete factorial involving $2^{k-\alpha}$ *combinations. Divide this set into two-halves such that the second half is a foldover of the first half.*

(iii) Take one combination from the first half of (i), say a*, and two combinations from the first half of (ii), say* b *and* c*. Let* a'*,* b' *and* c' *be the foldovers of* a*,* b *and* c*, respectively. Corresponding to the element* f *of* F*, make a block having choice pairs* $(ab, a'b)$*,* $(ab', a'b')$*,* $(a'c, ac)$*,* $(a'c', ac')$*. Here, in a choice pair, the option* ab *implies that if* $a = a_1 \cdots a_i \cdots a_{\alpha}$ *and* $b = b_1 \cdots b_j \cdots b_{k-\alpha}$*, then* a_i *corresponds to the attribute index* f_i *and* b_j *corresponds to the attribute index* f'_j*.*

(iv) Repeat (iii) for each of the $2^{\alpha-1}$ *combinations in the first half of (i) using the same* b *and* c *as in (iii). Then, repeat the entire process for two different combinations from the first half of (ii).*

(v) Repeating (i)-(iv) for every element f *of* F *corresponding to* $\alpha = q$ *for* k *odd and* $\alpha = q$ *and* $q+1$ *for* k *even, an optimal paired choice block design* d_2^I *is obtained with parameters* $k, v = 2, s = 4, b$ *where* $b = 2^{k-3}\binom{k}{q}$ *for* k *odd and* $b = 2^{k-3}\binom{k+1}{q+1}$ *for* k *even.*

Example 5. Let $k = 4, v = 2, b = 10, s = 8$. For $k = 4$, α takes the values 2 and 3. Since $\alpha = 3 > 2 = k - 2$, Theorem 11 does not allow to achieve d_2^I from d_1^I. However, for $\alpha = 2$, the proposed construction method still holds, for which we get 12 blocks each of size 4, as below.

B_1	B_2	B_3	B_4	B_5	B_6
(0000,1100)	(0100,1000)	(0000,1010)	(0010,1000)	(0000,1001)	(0001,1000)
(0011,1111)	(0111,1011)	(0101,1111)	(0111,1101)	(0110,1111)	(0111,1110)
(1101,0001)	(1001,0101)	(1011,0001)	(1001,0011)	(1011,0010)	(1010,0011)
(1110,0010)	(1010,0110)	(1110,0100)	(1100,0110)	(1101,0100)	(1100,0101)
B_7	B_8	B_9	B_{10}	B_{11}	B_{12}
(0000,0110)	(0010,0100)	(0000,0101)	(0001,0100)	(0000,0011)	(0001,0010)
(1001,1111)	(1011,1101)	(0110,0011)	(0111,0010)	(1100,1111)	(1101,0010)
(0111,0001)	(0101,0011)	(1011,1110)	(1010,1111)	(0111,0100)	(0110,0101)
(1110,1000)	(1100,1010)	(1101,1000)	(1100,1001)	(1011,1000)	(1010,1001)

For $\alpha = 3$, we provide a design in 4 blocks each of size 8, as below.

B_{13}	B_{14}	B_{15}	B_{16}
(0000,1110)	(0000,1011)	(0000,1101)	(0000,0111)
(0110,1000)	(0011,1000)	(0101,1000)	(0011,0100)
(1010,0100)	(1010,0001)	(1100,0001)	(0101,0010)
(1100,0010)	(1001,0010)	(1001,0100)	(0110,0001)
(0001,1111)	(0100,1111)	(0010,1111)	(1000,1111)
(0111,1001)	(0111,1100)	(0111,1010)	(1011,1100)
(1011,0101)	(1110,0101)	(1110,0011)	(1101,1010)
(1101,0011)	(1101,0110)	(1011,0110)	(1110,1001)

We form 6 blocks each of size 8 by combining blocks B_i and B_{i+6},

$i = 1, \ldots, 6$, which in combination with the 4 blocks B_i, $i = 13, \ldots, 16$ gives the optimal design with parameters $k = 4, b = 10, s = 8$.

6. Discussion

In situations where an optimal design has more choice pairs than a respondent can complete, the N choice pairs can be split among the respondents (blocks) either randomly or using a spare attribute, if there is one available (see, Street and Burgess (2007)). To this effect, we have instances of respondents being considered as blocks in various choice experiments, although without much theoretical rigor. Bliemer and Rose (2011) reported that 64% of studies used a blocking column to allocate choice sets to respondents, 13% assigned choice sets randomly to respondents, 5% studies provided the full factorial to each respondent and for the remaining 18% of the studies, it could not be determined how choice sets were assigned to respondents.

With an objective to assess the main or interaction effects, wherever practical, the same set of N optimal choice pairs are shown to every respondent. As such there are no theoretical results on optimal designs, under the utility-neutral setup, where different respondent sees smaller and different designs. In contrast, the approach that is adopted here allows the construction of optimal designs with smaller and flexible number of choice pairs, to be shown to every respondent. Even in situations where simple techniques like blocking using a spare attribute can not be used, we provide optimal paired choice block designs.

In contrast to the approaches of Sándor and Wedel (2005) and Kessels, Goos, and Vandebroek (2008), following the block design theory, we adopt the fixed-effects block model for obtaining optimal designs. The approach adopted here treats respondent heterogeneity as a nuisance factor by including respondent-level fixed-effect terms in the model and enables the derivation of analytical results. Though there is no guarantee that the optimal block designs obtained under this setup and the heterogeneous designs obtained by Sándor and Wedel (2005) would be same, it would require a separate study to compare optimal designs obtained under the two approaches.

Furthermore, unlike their designs, which are available only for situations when estimation of the main effects is of interest, we have provided optimal paired choice block designs not only under the main effects model but also under the broader main effects model and under the main plus two-factor interaction effects model.

Supplementary Materials

Supplementary Material available online includes proofs.

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IITB-Monash Research Academy, Mumbai, India

E-mail: agrakhi@gmail.com

Indian Institute of Technology Bombay, Mumbai, India

E-mail: ashish@math.iitb.ac.in

Academia Sinica, Taipei, Taiwan

E-mail: fschai@stat.sinica.edu.tw