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Le Cam maximin tests for symmetry of circular data based on the characteristic function

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Abstract: We consider asymptotic inference for circular data based on the empirical characteristic function. More precisely, we provide tests for reflective symmetry of circular data based on the imaginary part of the empirical characteristic function. We show that the proposed tests enjoy many attractive features. In particular, we obtain that they enjoy the property of being locally and asymptotically maximin in the Le Cam sense under sine-skewed alternatives in the specified mean direction case. To the best of our knowledge, this result provides the first instance of such an optimality property of empirical characteristic functions. For the unspecified mean direction case, we provide corrected versions of the original tests that keep very nice asymptotic power properties. Results are illustrated on a well-known dataset and checked via Monte-Carlo simulations.

Key words and phrases: Directional statistics, characteristic function, reflective symmetry.

1. Introduction

Statistical modeling and corresponding analyses of circular data is a topic that has attracted much attention in the recent years. For instance Jones et al. (2015) introduced copulas for circular distributions while Kato and Jones (2010, 2013, 2015) introduced families of distributions on the circle obtained via different techniques. Density estimation on the circle was considered in García-Portugués *et al.* (2013) while Oliveira *et al.* (2014a, 2014b) provided practical tools to deal with circular data. The popularity of circular statistics finds its roots in various disciplines such as the study of wind directions or animal orientation. Traditional methods to deal with circular data are quite well summarized in the monographs

[1] On sabbatical leave from the University of Athens

Mardia and Jupp (2000) and Jammalamadaka and SenGupta (2001).

Symmetry is definitely one of the most important structural assumptions made on underlying distributions. Circular distributions are not an exception to this rule since (i) most circular distributions are *reflectively symmetric* around a fixed direction and (ii) most inferential procedures actually require the latter symmetry structure in order to be valid or asymptotically valid. Nevertheless, due to their flexibility and their usefulness in practice there has also been a growing interest recently in non-symmetric models such as in Umbach and Jammalamadaka (2009), Kato and Jones (2010), Abe and Pewsey (2011) or Jones and Pewsey (2012). As a direct corollary, testing for symmetry in the circular data context becomes an increasingly important issue. Pewsey (2002, 2004) proposed procedures for testing symmetry around an unspecified and specified mean direction respectively. The proposed tests are based on the second–order trigonometric moments. In the specified mean direction case, the Pewsey (2004) test has been shown to be locally and asymptotically optimal under natural skewed alternatives by Ley and Verdebout (2014) who provided a family of testing procedures.

In general, there exist many different types of multivariate symmetric distributions such as spherically symmetric distributions, elliptically symmetric distributions, etc. A particular symmetry structure often yields to a particular shape of the corresponding characteristic function. For instance, if a random vector $\mathbf{X}$ taking values in $\mathbb{R}^p$ is symmetric around some location parameter $\boldsymbol{\mu}$ (in the sense that $\mathbf{X}-\boldsymbol{\mu} =_d \boldsymbol{\mu}-\mathbf{X}$), the imaginary part of the characteristic function of $\mathbf{X}-\boldsymbol{\mu}$ vanishes. This has been the central idea behind the tests for symmetry proposed in many papers such as in Heathcote et al. (1995), Neuhaus and Zhu (1998), Henze et al. (2003), and Ngatchou–Wandji and Harel (2013). In this paper, we also use the imaginary part of the empirical characteristic function process to provide tests for reflective symmetry of circular data. The resulting procedures enjoy many attractive features. First, they are asymptotically distribution-free which is obviously an important property in the given context. More importantly, we show that, in the specified mean direction case, the procedures based on the empirical characteristic function are locally and asymptotically *maximin* in the Le Cam sense under very general local alternatives. To the best of our

knowledge, this is the first time that such a property holds for procedures based on the empirical characteristic function. We also provide asymptotic procedures in the unspecified location case that are extremely competitive.

The paper is organized as follows. In Section 2, we discuss the properties of the characteristic function of circular random variables. In Section 3, we present our testing procedures in the specified location case and show their optimality properties. In Section 4, we tackle the unspecified location case. Section 5 is devoted to Monte-Carlo simulations, while in Section 6 the procedures are illustrated on a real data set. In Section 7 we conclude the paper. The proofs of the main results of the paper are collected in the Appendix.

2. Properties of the characteristic function

Let θ denote an arbitrary circular random variable with an absolutely continuous circular distribution function $F(t) = \mathbb{P}(\theta \leq t)$. The specificity of such a circular random variable or random angle θ is its periodicity in the sense that letting f stand for the density associated with F,

$$f(t) = f(t + 2k\pi)$$

for any integer k. As on the real line, the distribution of θ is in one–to–one correspondence with the characteristic function (CF) defined as

$$\varphi_\theta(r) := \mathbb{E}\left[e^{ir\theta}\right] = \int_{-\pi}^{\pi} e^{irt} dF(t), r \in \mathbb{R}. \tag{2.1}$$

The CF can also be written in terms of Cartesian coordinates as

$$\varphi_\theta(r) = \mathbb{E}\left[\cos(r\theta)\right] + i\mathbb{E}\left[\sin(r\theta)\right] := \alpha_r + i\beta_r, \tag{2.2}$$

where α_r (resp. β_r) is the real (resp. the imaginary) part of φ_θ. Due to its periodicity, and unlike real–line distributions, the CF of a circular random variable needs to be defined only at *integer values* $r = 0, \pm 1, \pm 2, ...$; see Jammalamadaka and SenGupta (2001).

Now, letting μ stand for the mean direction of θ defined through

$$(\cos(\mu), \sin(\mu))' := \frac{(\mathrm{E}[\cos(\theta)], \mathrm{E}[\sin(\theta)])'}{(\alpha_1^2 + \beta_1^2)^{1/2}},$$

θ is said to be reflectively symmetric around μ if its density f is such that $f(\mu+t) = f(\mu-t)$ for all $t \in [-\pi, \pi)$. Throughout the paper, the class $\mathcal{F}_\mu$ of reflectively symmetric densities around μ will be denoted as

$$\mathcal{F}_\mu \;:=\; \Big\{ f : f(t) > 0 \,\text{a.e.}, f(t+2j\pi) = f(t)\ \forall j \in \mathbb{Z}, f(\mu+t) = f(\mu-t), \\ f \,\text{unimodal at}\, \mu, \int_{-\pi}^{\pi} f(t)dt = 1 \Big\}.$$

Following Jammalamadaka and SenGupta (2001), if θ is reflectively symmetric around μ then the central trigonometric moments

$$\mathbb{E}\left[e^{ir(\theta-\mu)}\right] = \mathbb{E}\left[\cos(r(\theta-\mu))\right] + i\mathbb{E}\left[\sin(r(\theta-\mu))\right] := \bar{\alpha}_r + i\bar{\beta}_r, \tag{2.3}$$

are such that $\bar{\beta}_r = 0$ for all $r \in \mathbb{N}$. This will be the basis of our test statistic for the null hypothesis of symmetry around μ.

3. Optimal tests based on the empirical characteristic function

Let $\theta_1, ..., \theta_n$, be an i.i.d. random sample with mean direction μ. First we consider testing for symmetry around a known center so that we throughout the section assume without loss of generality that $\mu = 0$. The properties of the CF of symmetric circular random variables discussed in the previous section directly lead us to consider the imaginary part of the empirical CF,

$$b_n(r) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sin(r\theta_i) \tag{3.1}$$

for any $r \in \mathbb{N}$. The most simple tests for reflective symmetry are generally based on $b_n(1)$ or $b_n(2)$ such as the tests proposed by Pewsey (2004) and Ley and Verdebout (2014). In particular, Ley and Verdebout (2014) showed that a natural test based on $b_n(r)$ is locally and asymptotically most powerful under *r-skewed alternatives* (see below). In the present paper, we make use of the empirical process $b_n(r)$ to construct new tests for reflective symmetry. Specifically we will use the following result which is a direct consequence of the central limit theorem.

Proposition 1. *Assume that $\theta_1, ..., \theta_n$ is an i.i.d. sequence of circular random variables with density f_0 in $\mathcal{F}_0$. Then we have that the process $\{b_n(r)\}_{r\in\mathbb{N}}$ converges in (finite-dimensional) distribution to a Gaussian process $B(.)$ with mean*

zero and covariance kernel (the expectation is taken under f_0)

$$K(s,t) \quad = \quad \mathbb{E}[\sin(s\theta_1)\sin(t\theta_1)]. \tag{3.2}$$

Proposition 1 above entails that any vector of the form

$$\mathbf{B}^{(n)}_{(k_1,\ldots,k_m)} := (b_n(k_1),\ldots,b_n(k_m))$$

converges weakly to a centered multinormal distribution with a covariance matrix $\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}$ determined by the kernel in (3.2). Letting $\hat{\mathbf{\Sigma}}_{(k_1,\ldots,k_m);f_0}$ stand for a consistent estimator of $\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}$, this suggests to consider test statistics of the form

$$W^{(n)}_{(k_1,\ldots,k_m)} := (\mathbf{B}^{(n)}_{(k_1,\ldots,k_m)})'\hat{\mathbf{\Sigma}}^{-1}_{(k_1,\ldots,k_m);f_0}\mathbf{B}^{(n)}_{(k_1,\ldots,k_m)}. \tag{3.3}$$

Note that a consistent estimator $\hat{\mathbf{\Sigma}}_{(k_1,\ldots,k_m);f_0}$ can simply be obtained by substituting $K(s,t)$ in (3.2) by its empirical counterpart

$$\hat{K}(s,t) := \frac{1}{n}\sum_{i=1}^{n}\sin(s\theta_i)\sin(t\theta_i).$$

Indeed the law of large numbers directly imply that $\hat{\mathbf{\Sigma}}_{(k_1,\ldots,k_m);f_0} - \mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0} = o_{\mathrm{P}}(1)$ as $n \to \infty$ under the null (and therefore also under contiguous alternatives). Test statistics similar to (3.3) have already been suggested by Koutrouvelis (1985) in the case of conventional (non–circular) distributions; see also Csörgő and Heathcote (1987). Also note that if $m = 1$ and $k_1 = r$ say, the test coincides with the test proposed by Ley and Verdebout (2014) which is locally and asymptotically most powerful in the Le Cam sense under r-skewed alternatives. If more that one component is selected in $\mathbf{B}^{(n)}_{(k_1,\ldots,k_m)}$, then the local most powerfulness against a particular alternative as in Abe and Pewsey (2011) and Ley and Verdebout (2014) will be lost but as we explain now, the test will enjoy the property of being locally and asymptotically *maximin* under more general local alternatives.

The alternatives to reflective symmetry considered in Abe and Pewsey (2011) and Ley and Verdebout (2014) are characterized by densities of the form

$$f_\lambda(t) := f_0(t)(1+\lambda\sin(kt)), \quad t \in [-\pi,\pi), k \in \mathbb{N}, \lambda \in [-1,1], \tag{3.4}$$

where f_0 belongs to the class $\mathcal{F}_0$ defined in Section 2. Within this familly of distributions, the null hypothesis of reflective symmetry does coincide with the subset of distributions with $\lambda = 0$.

Now as explained in the previous section we have that $\mathrm{E}[\sin(r\theta_1)] = 0$ for any $r \in \mathbb{N}$ under reflective symmetry. As a consequence, more general alternatives are absolutely continuous distributions with densities of the form

$$f_{\boldsymbol{\lambda}}(t) := f_0(t)(1 + \sum_{i=1}^{m} \lambda_{k_i} \sin(k_i t)) \quad t \in [-\pi, \pi), \tag{3.5}$$

where $(k_1, \ldots, k_m) \in \mathbb{N}^m$ is m-tuple of distinct integers and $\boldsymbol{\lambda} := (\lambda_{k_1}, \ldots, \lambda_{k_m}) \in \mathcal{C}^m := [-\frac{1}{m}, \frac{1}{m}]^m$ is a multivariate skewness parameter. Note that $\boldsymbol{\lambda} \in \mathcal{C}^m$ guarantees that $f_{\boldsymbol{\lambda}}$ in (3.5) is a (positive) density function. In the sequel we write $\mathrm{P}^{(n)}_{\boldsymbol{\lambda};f_0}$ for the joint distribution of a n–tuple $\theta_1, \ldots, \theta_n$, of circular random variables with common density (3.5). Testing for symmetry against such types of alternatives is a more difficult problem since it becomes a multivariate problem as the parameter space attached to the underlying probability space is multidimensional. More precisely, testing for symmetry against those alternatives can be tackled by considering the problem $\mathcal{H}_0 : \boldsymbol{\lambda} = \mathbf{0}$ against $\mathcal{H}_1 : \boldsymbol{\lambda} \neq \mathbf{0}$. We show in the following result that when based on the m–tuple $(k_1, \ldots, k_m)$, $W^{(n)}_{(k_1,\ldots,k_m)}$ is locally and asymptotically *maximin* under local alternatives of the form $\mathrm{P}^{(n)}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}$ for some bounded sequence $\boldsymbol{\ell}^{(n)} = (\ell^{(n)}_{k_1}, \ldots, \ell^{(n)}_{k_m}) \in \mathbb{R}^m$ such that $n^{-1/2}\boldsymbol{\ell}^{(n)} \in \mathcal{C}^m$. A test ϕ^* is called maximin in the class $\mathcal{C}_\alpha$ of level–α tests for $\mathcal{H}_0$ against $\mathcal{H}_1$ if (i) ϕ^* has level α and (ii) the power of ϕ^* is such that

$$\inf_{\mathrm{P}\in\mathcal{H}_1} \mathbb{E}_{\mathrm{P}}[\phi^*] \geq \sup_{\phi\in\mathcal{C}_\alpha} \inf_{\mathrm{P}\in\mathcal{H}_1} \mathbb{E}_{\mathrm{P}}[\phi].$$

We have the following result.

Proposition 2. *Assume that $\theta_1, ..., \theta_n$ is an i.i.d. sequence such that under $\mathrm{P}^{(n)}_{\mathbf{0};f_0}$ with f_0 in $\mathcal{F}_0$, $\boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0}$ has full rank and $\hat{\boldsymbol{\Sigma}}_{(k_1,\ldots,k_m);f_0} - \boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0}$ is $o_{\mathrm{P}}(1)$ as $n \to \infty$. Furthermore let $\boldsymbol{\ell}^{(n)} = (\ell^{(n)}_{k_1}, \ldots, \ell^{(n)}_{k_m})$ be a bounded sequence of $\mathbb{R}^m$ such that (i) $n^{-1/2}\boldsymbol{\ell}^{(n)} \in \mathcal{C}^m$ and (ii) $\boldsymbol{\ell}^{(n)}$ converges to $\boldsymbol{\ell} := \lim_{n\to\infty} \boldsymbol{\ell}^{(n)}$. Then we have that*

(i) $W^{(n)}_{(k_1,\ldots,k_m)}$ is asymptotically chi-square with m degrees of freedom under $\cup_{f_0\in\mathcal{F}}\mathrm{P}_{\mathbf{0};f_0}$;

(ii) $W^{(n)}_{(k_1,\ldots,k_m)}$ is asymptotically chi-square with m degrees of freedom and with non-centrality parameter $\boldsymbol{\ell}'\boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\ell}$ under $\mathrm{P}^{(n)}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}$

(iii) the test $\phi^{(n)}_{\mathrm{MV};(k_1,\ldots,k_m)}$ *that rejects the null hypothesis when* $W^{(n)}_{(k_1,\ldots,k_m)} > \chi^2_{m;1-\alpha}$ *is locally and asymptotically maximin for testing* $\cup_{f_0\in\mathcal{F}}\mathrm{P}_{\mathbf{0};f_0}$ *against*

$$\cup_{f_0\in\mathcal{F}}\mathrm{P}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}.$$

The results obtained in Proposition 2 extend those of Ley and Verdebout (2014) which were obtained under more simple local alternatives of the form (3.4); point (iii) of the Proposition entails that $\phi^{(n)}_{\mathrm{MV}}$ is locally and asymptotically maximin under any reference density $f_0 \in \mathcal{F}$. Since k in (3.4) can not realistically be selected a priori, following Proposition 2 above it is more appropriate to perform the test $\phi^{(n)}_{\mathrm{MV}}$ that rejects the symmetry hypothesis $\mathcal{H}_0$ when $W^{(n)}_{(k_1,\ldots,k_m)} > \chi^2_{m;1-\alpha}$ for some m-tuple $(k_1,\ldots,k_m)$.

A reasonable criterion for selection of the number m and the specific location $(k_1,\ldots,k_m)$ of ECF arguments clearly remains an issue. In fact for the conventional (non–circular) empirical CF the problem dates back to Feigin and Heathcote (1976) and Csörgő and Heathcote (1987), in the case of a single argument ($m = 1$). Feuerverger and McDunnough (1981) provide a fundamental contribution by making a connection of the efficiency of ECF estimation procedures with the efficiency of maximum likelihood, but this refers to estimation rather than testing and moreover assumes a fixed parametric model. On the other hand, and in the context of hypothesis testing, the finite–sample results in Koutrouvelis (1980) and Epps and Singleton (1986) imply that while a larger m may be asymptotically preferable, it is advisable to use a value of $m := m_n$ that depends on the sample size in an increasing fashion. In this connection we note that selecting $m > n$ typically yields an estimator $\hat{\boldsymbol{\Sigma}}_{(k_1,\ldots,k_m);f_0}$ which is singular and it would therefore be impossible to perform the corresponding test in practice, but numerical problems could be encountered even with $m \leq n$, if m and n are of the same order of magnitude.

The only theoretically viable solution to a related problem has been provided by Tenreiro (2009) and corresponds to the optimal choice of the weight parameter a in our test in (4.2), when the weight function $w(r)$ has already been fixed. However, Tenreiro's solution is in the strict parametric context of testing normality, and depends heavily on the direction of departure from the null hypothesis of normality. Such a choice is distant from our context, which

is nonparametric even under the null hypothesis, and too specific for providing guidance in the present situation, but clearly shows the complexity of the problem of choosing m and $(k_1, \ldots, k_m)$. Now in terms of asymptotics, a reasonable approach is to select both m and the m-tuple $(k_1, \ldots, k_m)$ that will maximize the local power under $\mathrm{P}^{(n)}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}$; that is selecting $\tilde{m}$ and the $\tilde{m}$-tuple $(k_1, \ldots, k_{\tilde{m}})$ such that $\boldsymbol{\ell}'\boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\ell}$ is maximal. As mentioned above the problem is very complicated since the local power depends on the perturbation $\boldsymbol{\ell}$ (which is also m-dimensional) and on f_0. One "ad-hoc" possible way to select $(k_1, \ldots, k_{\tilde{m}})$ is to take $(k_1, \ldots, k_{\tilde{m}}) = \mathrm{argmax}_{\cup_{m\in\mathbb{N}}\mathcal{K}_m} \mathrm{tr}(\boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0})/m$ where $\mathcal{K}_m$ is the set of all possible m-tuples of distinct natural numbers. Without providing a solution to this difficult "non-parametric" problem, we just give here some advices. Assume that that the density f_0 is indexed by some positive concentration parameter κ such that when $\kappa = 0$ the distribution is uniform on $\mathcal{S}^1$ and when $\kappa \to \infty$, the distribution tends to a point mass on the location parameter μ. Many well-known densities such as von Mises densities are of this type. For such a $f_0^{(\kappa)}$ say, when $\kappa = 0$, then $\boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0^{(\kappa)}} = 2\mathbf{I}_m$ so that $\beta_{(k_1,\ldots,k_m);f_0^{(\kappa)}} := \mathrm{tr}(\boldsymbol{\Sigma}_{(k_1,\ldots,k_m);f_0^{(\kappa)}})/m$ is constant for any choice of m and $(k_1, \ldots, k_m)$. Therefore, for distributions having a small concentration, $\beta_{(k_1,\ldots,k_m);f_0^{(\kappa)}}$ will not vary much so that one may decide to stick with a small m. Now as κ grows, the variability of the trigonometric moments diminishes especially for small order moments. It is therefore natural to select more higher order moments than lower order moments when the data is much concentrated.

Wrapping up we note that the test $\phi^{(n)}_{\mathrm{MV}}$ still suffers from some drawbacks:

(i) although the new test takes into account the empirical CF process over a fix grid of values $(k_1, \ldots, k_m)$, there still remains the problem of consistency against general alternatives. Such a consistency would be obtained by considering arbitrarily large grids with $m = m_n$ that diverges to ∞. This is a very delicate issue as explained just above;

(ii) the assumption of a known symmetry center is sometimes unrealistic.

In the next section, we consider testing procedures for reflective symmetry that deal with these drawbacks.

4. Tests in the unknown mean direction case

Let $\theta_1, ..., \theta_n$, be independent circular random variable with mean direction μ and consider the centered observations $\vartheta_i = \theta_i - \mu$, $i = 1, \ldots, n$. A well–known nonparametric estimate of the mean direction μ is given by

$$\widehat{\mu} := \arctan\left(\frac{\sum_{i=1}^n \sin\theta_j}{\sum_{i=1}^n \cos\theta_i}\right),$$

see Jammalamadaka and SenGupta (2001). The estimator $\hat{\mu}$ naturally yields to consider estimated versions of the $b_n(r)$'s in (3.1) given by

$$\widehat{b}_n(r) = n^{-1/2}\sum_{i=1}^n \sin(r\widehat{\vartheta}_i),$$

where $\widehat{\vartheta}_i = \theta_i - \widehat{\mu},\ i = 1, ..., n$. In the following result, we study the asymptotic properties of the $\widehat{b}_n(r)$'s under the null hypothesis of symmetry. The proof of the result requires the use of a discretized version of $\hat{\mu}$. More precisely it requires $\hat{\mu}$ to be *locally and asymptotically discrete*: $\hat{\mu}$ only takes a bounded number of distinct values in μ-centered intervals with $O(n^{-1/2})$ radius. It should be noted that such a discretization condition is a purely technical requirement (see for instance Ley *et al.* (2013) and Hallin *et al.* (2013, 2014)), with little practical implications (in fixed-n practice, such discretizations are irrelevant as the discretization radius can be taken arbitrarily large). Therefore, for the sake of simplicity, we tacitly assume in the sequel that $\widehat{\mu}$ is locally and asymptotically discrete.

Proposition 3. *Assume that $\theta_1, ..., \theta_n$, is an i.i.d. sequence of circular random variables with density $f \in \mathcal{F}_\mu$. Then we have that the process $\widehat{b}_n(r)$ converges in (finite-dimensional) distribution to a Gaussian process $\tilde{B}(.)$ with mean zero and covariance kernel (see (2.3) for a definition of the $\bar{\alpha}_r$'s)*

$$\begin{aligned}\tilde{K}(s,t) \quad = \quad & \mathrm{E}[\sin(s\vartheta_1)(\sin(t\vartheta_1)] - \lambda^{-1}s\bar{\alpha}_s\mathrm{E}[\sin(\vartheta_1)(\sin(t\vartheta_1)] \\ & -\lambda^{-1}t\bar{\alpha}_t\mathrm{E}[\sin(\vartheta_1)\sin(s\vartheta_1)] + \lambda^{-2}st\bar{\alpha}_s\bar{\alpha}_t\mathrm{E}[\sin^2(\vartheta_1)],\end{aligned} \tag{4.1}$$

where the expectations are taken under $f \in \mathcal{F}_\mu$.

As in the previous section, it directly follows from Proposition 3 that the vector

$$\widehat{\mathbf{B}}^{(n)}_{(k_1,\ldots,k_m)} := (\widehat{b}_n(k_1), \ldots, \widehat{b}_n(k_m))$$

converges weakly to a centered multinormal distribution with a covariance matrix $\tilde{\mathbf{\Sigma}}_{(k_1,\ldots,k_m)}$ determined by the kernel in (4.1). Note that in (4.1),

$$\tilde{K}(1,1) = (1 - \frac{\bar{\alpha}_1}{\lambda})^2 \mathrm{E}[(\sin^2(\vartheta_1)].$$

Since from standard trigonometry,

$$\begin{aligned} \bar{\alpha}_1 &= \mathbb{E}\left[\cos(\theta_1 - \mu))\right] = \cos(\mu)\mathbb{E}[\cos(\theta_1)] + \sin(\mu)\mathbb{E}[\sin(\theta_1)] \\ &= \lambda\cos^2(\mu) + \lambda\sin^2(\mu) = \lambda, \end{aligned}$$

we readily obtain that $\tilde{K}(1,1) = 0$. This is totally in line with the well–known identity (see e.g. Pewsey 2002)

$$\sum_{i=1}^{n} \sin(\theta_i - \widehat{\mu}) = 0$$

which entails that $\widehat{b}_n(1)$ is equal to zero for any n and therefore has no (asymptotic) variance. As a conclusion, we recommend to choose a m–tuple of indices $(k_1,\ldots,k_m)$ such that $1 \notin (k_1,\ldots,k_m)$. Taking such a m–tuple and letting $\hat{\tilde{\mathbf{\Sigma}}}_{(k_1,\ldots,k_m)}$ stand for a consistent estimator of $\tilde{\mathbf{\Sigma}}_{(k_1,\ldots,k_m)}$ (as in the known μ case, such an estimator can simply be obtained by replacing expectations by empirical means and μ by $\hat{\mu}$ in (4.1)), the test statistic

$$\hat{W}^{(n)}_{(k_1,\ldots,k_m)} := (\hat{\mathbf{B}}^{(n)}_{(k_1,\ldots,k_m)})' \hat{\tilde{\mathbf{\Sigma}}}^{-1}_{(k_1,\ldots,k_m)} \hat{\mathbf{B}}^{(n)}_{(k_1,\ldots,k_m)}$$

is asymptotically chi-square with m degrees of freedom if $1 \notin (k_1,\ldots,k_m)$ and $m-1$ degrees of freedom if $1 \in (k_1,\ldots,k_m)$.

Now, even if the test that rejects the null for large values of $\hat{W}^{(n)}_{(k_1,\ldots,k_m)}$ is asymptotically valid in the μ unspecified case, it still suffers from the drawback that the consistent estimation of $\tilde{\mathbf{\Sigma}}_{(k_1,\ldots,k_m)}$ is an issue when m is large, especially when $m > n$. Furthemore, the asymptotic properties of $\hat{W}^{(n)}_{(k_1,\ldots,k_m)}$ under local alternatives require a careful study of the Fisher information for both the symmetry and location μ parameters which is beyond the scope of the present paper.

Therefore, we rather propose another test that takes into account the full empirical characteristic process $\{\widehat{b}_n(r)\}_{r\in\mathbb{N}}$. The latter rejects the null hypothesis

of reflective symmetry for large values of

$$T_{n,w} = \sum_{r=1}^{\infty} w(r)\widehat{b}_n^2(r), \tag{4.2}$$

where $w(r),\ r \geq 1$, is a sequence of positive weights. It follows easily from the definition of $\widehat{b}_n(r)$ that

$$\widehat{b}_n^2(r) = \frac{1}{2n}\left(\sum_{i,j=1}^{n} \cos(r\widehat{\vartheta}_{i,j,n}^{-}) - \sum_{i,j=1}^{n} \cos(r\widehat{\vartheta}_{i,j,n}^{+})\right)$$

where $\widehat{\vartheta}_{i,j,n}^{\pm} = \widehat{\vartheta}_{i,n} \pm \widehat{\vartheta}_{j,n}$ so that letting $C_w(\vartheta) := \sum_{r=1}^{\infty} w(r)\cos(r\vartheta)$, the test statistic in (4.2) may be rewritten as

$$T_{n,w} = \frac{1}{2n}\sum_{i,j=1}^{n}\left(C_w(\widehat{\vartheta}_{i,j,n}^{-}) - C_w(\widehat{\vartheta}_{i,j,n}^{+})\right).$$

Although the test statistic $T_{n,w}$ is defined through an infinite series, some choices of the weights $w(r)$ make $T_{n,w}$ easy to compute. More precisely, following results in Gradshteyn and Ryzhik (1994), some sequences $w(r)$ which provide closed forms for $T_{n,w}$ are $w(r) = a^r,\ |a| < 1$, which yields

$$C_w(\vartheta) = \frac{1}{2}\left(\frac{1-a^2}{1-2a\cos\vartheta + a^2} - 1\right)$$

and $w(r) = e^{-ar},\ a > 0$, which yields

$$C_w(\vartheta) = \frac{1}{2}\left(\frac{e^a - e^{-a}}{e^a + e^{-a} - 2\cos\vartheta} - 1\right).$$

The following proposition implies the strong (almost sure) consistency of the test statistic $T_{n,w}$ against fixed alternatives.

Proposition 4. *Let $T_{n,w}$ be the the test statistic in (4.2). Then*

$$\frac{T_{n,w}}{n} \longrightarrow \sum_{r=1}^{\infty} w(r)\, b_r^2 := T_w, \text{ a.s. as } n \to \infty. \tag{4.3}$$

Since $T_w = 0$ only under the null hypothesis $\mathcal{H}_0$, (4.3) implies the strong consistency of the test which rejects $\mathcal{H}_0$ for large values of $T_{n,w}$.

The main difficulty with $T_{n,w}$ is that it is not distribution free under the null. In fact it may be argued that $T_{n,w}$ is asymptotically distributed as $\sum_{r=1}^{\infty} w(r)V(r)$, where $V(r)$ is the Gaussian process defined in Proposition 3. Following the ideas of Neuhaus and Zhu (1998), the computation of the critical values of a test based on $T_{n,w}$ can be obtained by permutational arguments. First note that it follows from the proof of Proposition 3 that for all r,

$$\hat{b}_n(r) = n^{-1/2}\sum_{i=1}^{n}\sin(r\vartheta_i) - \frac{r\mathrm{E}[\cos(r\vartheta_1)]}{\sqrt{\mathbb{E}^2[\cos(\theta_1)] + \mathbb{E}^2[\sin(\theta_1)]}} n^{-1/2}\sum_{i=1}^{n}\sin(\vartheta_i) + o_{\mathrm{P}}(1)$$

as $n \to \infty$ under $\mathcal{H}_0$. Letting $\mathbf{e} = (e_1, \ldots, e_n)$ be an i.i.d. sequence of random variables such that $e_i = 1$ with probability .5 and $e_i = -1$ with probability .5, define

$$\tilde{b}_n^{(\mathbf{e})}(r) := n^{-1/2}\sum_{i=1}^{n}\left(\sin(re_i\hat{\vartheta}_i) - \frac{r(\sum_{j=1}^{n}\cos(re_j\hat{\vartheta}_j))}{((\sum_{j=1}^{n}\cos(e_j\hat{\vartheta}_j))^2 + (\sum_{j=1}^{n}\sin(e_j\hat{\vartheta}_j))^2)^{1/2}}\sin(e_i\hat{\vartheta}_i)\right).$$

Using the data $\vartheta_1, \ldots, \vartheta_n$, the critical values of the test based on $T_{n,w}$ at the level α are approximated as follows:

(i) Generate M i.i.d. random sequences of n-dimensional vectors $\mathbf{e}_1, \ldots, \mathbf{e}_M$ distributed as $\mathbf{e}$ above;

(ii) For all $j = 1, \ldots, M$, compute $Q_j := \sum_{r=1}^{m} w(r)(\tilde{b}_n^{(\mathbf{e}_j)}(r))^2$, with m large;

(iii) Select for the critical value the empirical $1-\alpha$ quantile of $Q_1, \ldots, Q_M$.

5. Simulations

In this section, our objective is to compare the properties of the proposed procedures with respect to other well–known tests for the same problem. We performed two sets of simulations: one in the μ–known situation and one in the unknown μ situation.

For the first problem which consists in testing for reflective symmetry around a known center, we generated $N = 2,500$ mutually independent samples of i.i.d. of circular random variables

$$\theta_{\ell;j}^{(\rho)}, \quad \rho = 1, \ldots, 6,\ \ell = 0, \ldots, 3,\ j = 1, \ldots, n = 100,$$

following various skewed distributions with concentration 1 and mean direction 0 as in Abe and Pewsey (2011) and Verdebout and Ley (2014). The $\theta^{(1)}_{\ell;j}$'s are von Mises 1-sine skewed (obtained by taking $k = 1$ in 3.4) with skewness parameter $\lambda = \ell/5$, the $\theta^{(2)}_{\ell;j}$'s are von Mises 2-sine skewed (obtained by taking $k = 2$ in 3.4) with skewness parameter $\lambda = \ell/5$, the $\theta^{(3)}_{\ell;j}$'s are von Mises 4-sine skewed (obtained by taking $k = 4$ in 3.4) with skewness parameter $\lambda = \ell/5$ and the $\theta^{(4)}_{\ell;j}$'s are von Mises 6-sine skewed (obtained by taking $k = 6$ in 3.4) with skewness parameter $\lambda = \ell/5$. The $\theta^{(5)}_{\ell;j}$'s and $\theta^{(6)}_{\ell;j}$ are skewed Möbius distributions as in Kato and Jones (2010) based von Mises with concentration 1 and skewness parameter $\lambda = \ell/5$ respectively with $r = .25$ and $r = .5$ (see Kato and Jones (2010) for details). The value $\ell = 0$ always yields to a reflectively symmetric distribution belonging to the null hypothesis while the values $\ell = 1, 2, 3$ provide distributions that are increasingly skew-symmetric.

The resulting rejection frequencies of the following tests for reflective symmetry all at nominal level 5% are plotted in Figures 1 and 2: the optimal 1–sine skewed test of Ley and Verdebout (2014), the modified runs test of Pewsey (2004), the test $\phi^{(n)}_{\mathrm{MV};(1,2,3)}$ based on $W^{(n)}_{(1,2,3)}$, and the test $\phi^{(n)}_{\mathrm{MV};(1,\ldots,5)}$ based on $W^{(n)}_{(1,\ldots,5)}$. Inspection of Figure 1 and 2 reveals the following features: (i) all the tests reach the correct asymptotic level; (ii) as expected, the optimal 1–sine skewed test of Ley and Verdebout (2014) is optimal under 1-sine skewed alternatives but it is dominated by the tests $\phi^{(n)}_{\mathrm{MV};(1,\ldots,3)}$ and $\phi^{(n)}_{\mathrm{MV};(1,\ldots,5)}$ under all the other alternatives and (iii) $\phi^{(n)}_{\mathrm{MV};(1,\ldots,5)}$ still behaves correctly under 4–sine skewed alternatives, while the same is not true for the test based on $\phi^{(n)}_{\mathrm{MV};(1,\ldots,3)}$. The modified runs test of Pewsey (2004) behaves similarly under any sine-skewed alternative and will clearly be consistent under any alternative. Under skewed Möbius distributions, both the optimal 1–sine skewed test and $\phi^{(n)}_{\mathrm{MV};(1,\ldots,3)}$ behave nicely.

In a second simulation part, we compared the following tests for reflective symmetry around an unspecified symmetry center: the Pewsey (2002) test, our test $\widehat{\phi}^{(n)}_{\mathrm{MV};(2,3)}$ and $\widehat{\phi}^{(n)}_{\mathrm{MV};(2,\ldots,5)}$ based on the asymptotic critical values of $\widehat{W}_{(2,3)}$ and $\widehat{W}_{(2,\ldots,5)}$ respectively and the tests $\phi_{T_{n,1}}$ and $\phi_{T_{n,2}}$ based on $T_{n,w}$ with weights $w_1(r) = .5^r$ and $w_2(r) = e^{-r/2}$ respectively. The critical values of $T_{n,w}$ have been computed using the bootstrap procedure described below Proposition 4 with $M = 10000$ and $m = 40$.

As for the first simulation scheme, we generated $N = 1,500$ mutually independent samples of i.i.d. of circular random variables

$$\theta_{\ell;j}^{(\rho)}, \quad \rho = 1,2,\ \ell = 0,\ldots,3,\ j = 1,\ldots,n,$$

with sine-skewed von Mises distributions with concentration 1 and mean direction 0. The $\theta_{\ell;j}^{(1)}$'s are 2-sine skewed (obtained by taking $k = 2$ in 3.4) with skewness parameter $\lambda = \ell/5$, the $\theta_{\ell;j}^{(2)}$'s are 4-sine skewed (obtained by taking $k = 4$ in 3.4) with skewness parameter $\lambda = \ell/5$. The rejection frequencies of the five tests all at nominal level 5% are plotted in Figure 3 for $n = 100$ and in Figure 4 for $n = 200$. The comparison between the Pewsey (2002) test, and the $\widehat{\phi}_{\mathrm{MV};(2,3)}^{(n)}$, and $\widehat{\phi}_{\mathrm{MV};(2,\ldots,5)}^{(n)}$ tests is very similar as in the μ–specified case. Note however that the tests $\phi_{T_{n,1}}$ and $\phi_{T_{n,2}}$ behave nicely in both cases; the tests clearly seem to detect alternatives of the same magnitude as the other tests (deviations with rate $1/\sqrt{n}$ from the null). All the tests are slightly conservative for $n = 100$, but improve with the higher sample size $n = 200$.

6. Real data illustration

In this section we illustrate the testing procedures of the previous section on a well-known data set from an animal orientation experiment. This data set consists in the directions of 730 red wood ants originally placed in the center of an arena with a black target positioned at an angle of 180° from the zero direction; see Figure 5 for the illustration. The question of interest is whether the directions chosen by the ants are symmetrically distributed around the median direction representing the black target. The data set analyzed in Abe and Pewsey (2011) and Ley and Verdebout (2014) was originally obtained in Jander (1957).

In Ley and Verdebout (2014) it is obtained that the locally and asymptotically most powerfull (LAMP) test against 1-sine-skewed alternatives has a p-value of 0.778, while the LAMP test against 2-sine-skewed alternatives has a p-value of 0.011. We computed our tests based on $W_{(1,2)}^{(n)}$, $W_{(1,2,3)}^{(n)}$ and $W_{(1,\ldots,5)}^{(n)}$ which have p-values 0.012, 0.013 and 0.022 respectively and therefore yield to the rejection of the null hypothesis of reflective symmetry at the nominal level .05.

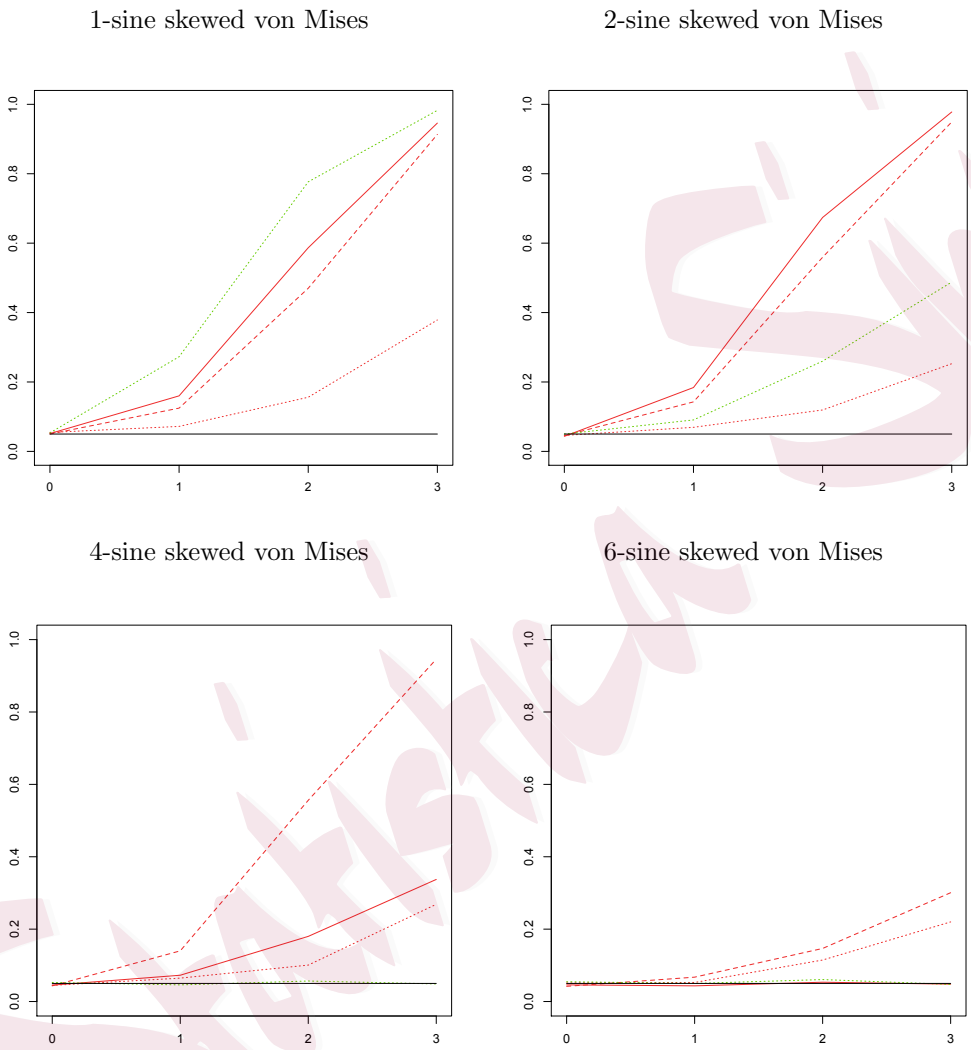

Figure 1: Power curves of (i) the Ley and Verdebout (2014) locally and asymptotically optimal test against 1-skew-FvML alternatives (green curve), (ii) our test $\phi^{(n)}_{\mathrm{MV};(1,2,3)}$ (plain red curve), (iii) our test $\phi^{(n)}_{\mathrm{MV};(1,\ldots,5)}$ (dashed red curve) and (iv) the modified runs test of Pewsey (2004) (dotted red curve). The sample size is $n = 100$.

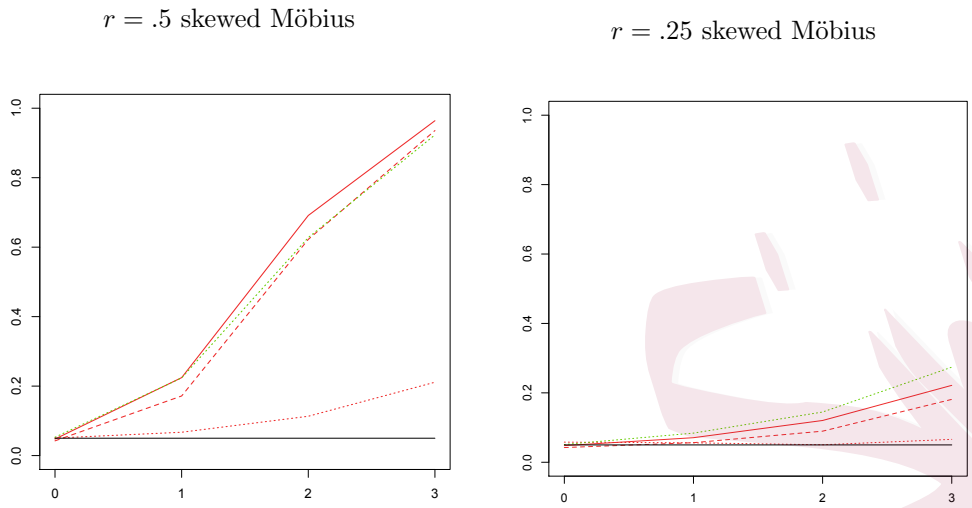

Figure 2: Power curves of (i) the Ley and Verdebout (2014) locally and asymptotically optimal test against 1-skew-FvML alternatives (green curve), (ii) our test with $(k_1, \ldots, k_m) = (1, 2, 3)$ (plain red curve), (iii) our test with $(k_1, \ldots, k_m) = (1, 2, \ldots, 5)$ (dashed red curve) and the modified runs test of Pewsey (2004) (dotted red curve). The sample size is $n = 100$.

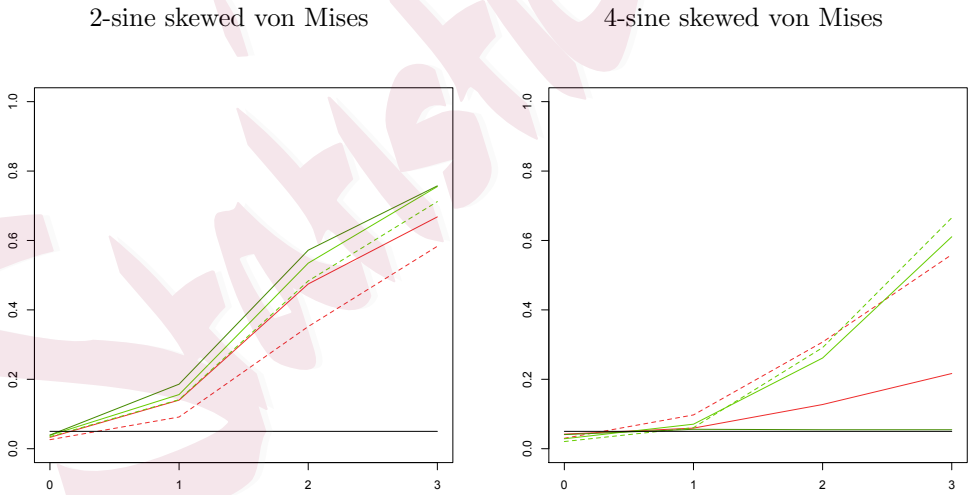

Figure 3: Power curves of the Pewsey (2002) test (dark green plain line), $\hat{\phi}^{(n)}_{\mathrm{MV};(2,3)}$ (red plain line), $\hat{\phi}^{(n)}_{\mathrm{MV};(2,\ldots,5)}$ (red dashed line), $\phi_{T_n,1}$ (light green plain line) and $\phi_{T_n,2}$ (light green pdashed line). The sample size is $n = 100$.

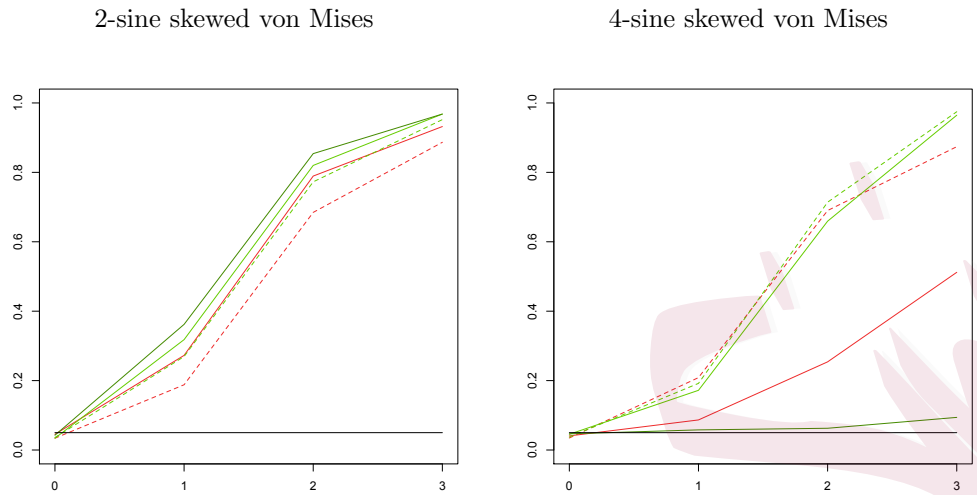

Figure 4: Power curves of the Pewsey (2002) test (dark green plain line), $\hat{\phi}^{(n)}_{\mathrm{MV};(2,3)}$ (red plain line), $\hat{\phi}^{(n)}_{\mathrm{MV};(2,\ldots,5)}$ (red dashed line), $\phi_{T_{n,1}}$ (light green plain line) and $\phi_{T_{n,2}}$ (light green dashed line). The sample size is $n = 200$.

Since symmetry around the black target is the interest, the present problem can be seen as a test for symmetry around a specified direction. We nevertheless performed the tests based on the statistics $\hat{W}^{(n)}_{(1,2,3)}$ and $\hat{W}^{(n)}_{(1,\ldots,5)}$ and the tests $\phi_{T_{n,1}}$ and $\phi_{T_{n,2}}$ based on $T_{n,w}$ with weights $w_1(r) = .5^r$ and $w_2(r) = e^{-r/2}$ respectively, see Section 4 for details. The test based on $\hat{W}^{(n)}_{(1,2,3)}$ has p-value .054, the test based on $\hat{W}^{(n)}_{(1,\ldots,5)}$ has p-value .046; therefore at the nominal level .05, one test rejects the null while the other does not. The two tests $\phi_{T_{n,1}}$ and $\phi_{T_{n,2}}$ also reject the null (the critical values were computed using the bootstrap procedure described below Proposition 4 with $M = 10000$ and $m = 40$.)

7. Conclusion and discussion

We suggest several tests for reflective symmetry based on the empirical characteristic function. In the fixed–location case the new test is asymptotically optimal in the *maxmin* sense against certain alternatives. In the unknown location case we suggested modifications of these optimal procedures as well as a new test which is consistent against each fixed alternative non–symmetric circular distri-

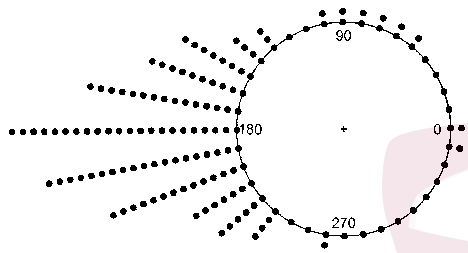

Figure 5: Raw circular plot of the Jander (1957) data set recorded during an orientation experiment with 730 red wood ants. Each dot represents the direction chosen by five ants.

bution. The finite–sample behaviour is investigated via a simulation study, and the suggested tests are shown to perform well in comparison with other powerful symmetry procedures.

Further developments in this line of research are (i) extension of the methods to dimension $p \geq 2$ using the *tangent-normal* decomposition

$$\mathbf{X} = (\mathbf{X}'\boldsymbol{\mu})\boldsymbol{\mu} + (\mathbf{I} - \boldsymbol{\mu}\boldsymbol{\mu}')\mathbf{X},$$

where $(\mathbf{I} - \boldsymbol{\mu}\boldsymbol{\mu}')\mathbf{X}$ has a spherically symmetric around zero distribution under the assumption of rotational symmetry. Then a test of rotational symmetry could be build using a test for spherical symmetry of $(\mathbf{I} - \boldsymbol{\mu}\boldsymbol{\mu}')\mathbf{X}$ along the lines of Henze et al. (2013) and (ii) goodness–of–fit tests for circular distributions based on quadratic forms analogous to those of Section 3 and Section 4; the corresponding test statistics would then involve the real part α_r as well as the imaginary part β_r of the characteristic function.

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Appendix: Proofs

Proof of Proposition 3. First note that

$$\begin{aligned}\hat{b}_n(r) &= n^{-1/2}\sum_{j=1}^{n}(\sin(r\widehat{\vartheta}_j) - \sin(r\vartheta_j) + \sin(r\vartheta_j)) \\ &= W_n(r) + V_n(r),\end{aligned}$$

where

$$V_n(r) := n^{-1/2}\sum_{j=1}^{n}\sin(r\vartheta_j)$$

and

$$W_n(r) := n^{-1/2}\sum_{j=1}^{n}(\sin(r\widehat{\vartheta}_j) - \sin(r\vartheta_j))$$

Since under $\mathcal{H}_0$, $\mathrm{E}[\sin(r\vartheta_j)] = 0$, the central limit theorem directly implies that $V_n(r)$ converges weakly to a Gaussian random variable with mean zero and variance $\mathrm{E}[\sin^2(r\vartheta_j)]$. Now for $W_n(r)$, assume for a moment that $\widehat{\mu}$ is discretized so that $n^{1/2}(\widehat{\mu} - \mu)$ can be replaced by a deterministic sequence in $W_n(r)$, see for instance Kreiss (1987) or Hallin *et al.* (2013, 2014). Then, a Taylor series development directly yields

$$W_n(r) = -r\mathrm{E}[\cos(r\vartheta_j)]n^{1/2}(\widehat{\mu} - \mu) + o_{\mathrm{P}}(1)$$

as $n \to \infty$. To obtain the asymptotic normality of $V_n(r) + W_n(r)$, we need to study the asymptotic joint distribution of $n^{1/2}(\widehat{\mu} - \mu)$ and $V_n(r)$.
Now, applying the delta method, we easily obtain that

$$n^{1/2}(\widehat{\mu} - \mu) = \lambda^{-1}n^{-1/2}\sum_{i=1}^{n}\sin(\vartheta_i) + o_{\mathrm{P}}(1)$$

as $n \to \infty$, where $\lambda = \sqrt{\mathbb{E}^2[\cos(\theta_1)] + \mathbb{E}^2[\sin(\theta_1)]}$. Wrapping up, we obtain that

$$\begin{pmatrix} V_n(r) \\ W_n(r) \end{pmatrix} = \begin{pmatrix} n^{-1/2}\sum_{j=1}^{n}\sin(r(\vartheta_j)) \\ -r\frac{\mathrm{E}[\cos(r(\vartheta_j))]}{\lambda}n^{-1/2}\sum_{j=1}^{n}\sin(\vartheta_j) \end{pmatrix} + o_{\mathrm{P}}(1)$$

as $n \to \infty$ so that the vector $\mathbf{B}_{m,n} := (\hat{b}_n(k_1), \ldots, \hat{b}_n(k_m))'$ for some fixed m is such that

$$\mathbf{B}_{m,n} = \begin{pmatrix} n^{-1/2}\sum_{j=1}^n \sin(k_1\vartheta_j) \\ \vdots \\ n^{-1/2}\sum_{j=1}^n \sin(k_m\vartheta_j) \end{pmatrix} - \lambda^{-1} \begin{pmatrix} k_1\mathbb{E}[\cos(k_1\vartheta_j)] \\ \vdots \\ k_m\ \mathbb{E}[\cos(k_m\vartheta_j)] \end{pmatrix} n^{-1/2}\sum_{i=1}^n \sin(\vartheta_i) + o_{\mathrm{P}}(1).$$

so that $\mathbf{B}_{m,n}$ is asymptotically normal with mean zero and covariance matrix $\mathbf{\Sigma} = (\Sigma_{st})$, where $\Sigma_{st} = K(s,t)$. The result follows. $\square$

Proof of Proposition 2. First note that Point (i) easily follows from Proposition 1. For Points (ii) and (iii), we start the proof by showing that the sequence of models $\{\mathrm{P}^{(n)}_{\boldsymbol{\lambda};f_0}\}$ is locally and asymptotically normal in the vicinity of symmetry. First note that

$$\log \frac{\mathrm{dP}^{(n)}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}}{\mathrm{dP}^{(n)}_{\mathbf{0};f_0}} = \sum_{i=1}^n \log(1 + n^{-1/2}(\boldsymbol{\ell}^{(n)})'\mathbf{S}_i^{(n)}), \tag{1}$$

where $\mathbf{S}_i^{(n)} := (\sin(k_1\theta_i), \ldots, \sin(k_m\theta_i))'$. It easily follows from (1) above, the boundedness of $\mathbf{S}_i^{(n)}$ and from the fact that $\log(1+v) = v - \frac{1}{2}v^2 + o(v^2)$ that

$$\log \frac{\mathrm{dP}^{(n)}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}}{\mathrm{dP}^{(n)}_{\mathbf{0};f_0}} = (\boldsymbol{\ell}^{(n)})'\mathbf{\Delta}^{(n)} - \frac{1}{2}(\boldsymbol{\ell}^{(n)})'\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\ell}^{(n)} + o_{\mathrm{P}}(1) \tag{2}$$

as $n \to \infty$ under $\mathrm{P}_{\mathbf{0};f_0}$, where $\mathbf{\Delta}^{(n)} := n^{-1/2}\sum_{i=1}^n \mathbf{S}_i^{(n)}$ is asymptotically normal with mean zero and covariance $\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}$ still under $\mathrm{P}_{\mathbf{0};f_0}$. It therefore follows from (2) that the sequence of models $\{\mathrm{P}^{(n)}_{\boldsymbol{\lambda};f_0}\}$ is locally and asymptotically normal. Now from local asymptotic normality, a maximin test for $\mathcal{H}_0 : \boldsymbol{\lambda} = \mathbf{0}$ against $\mathcal{H}_1 : \boldsymbol{\lambda} \neq \mathbf{0}$ rejects the null when $(\mathbf{\Delta}^{(n)})'\mathbf{\Sigma}^{-1}_{(k_1,\ldots,k_m);f_0}\mathbf{\Delta}^{(n)}$ exceeds the alpha upper quantile of the chi-square distribution with m degrees of freedom. Since

$$W_n = (\mathbf{\Delta}^{(n)})'\hat{\mathbf{\Sigma}}^{-1}_{(k_1,\ldots,k_m);f_0}\mathbf{\Delta}^{(n)} = (\mathbf{\Delta}^{(n)})'\mathbf{\Sigma}^{-1}_{(k_1,\ldots,k_m);f_0}\mathbf{\Delta}^{(n)} + o_{\mathrm{P}}(1),$$

Point (iii) follows. We now turn to Point (ii). It directly follows from (2) that $((\mathbf{\Delta}^{(n)})', \log \frac{\mathrm{dP}_{n^{-1/2}\boldsymbol{\ell}^{(n)};f_0}}{\mathrm{dP}_{\mathbf{0};f_0}})$ is asymptotically normal with mean $(\mathbf{0}', -\frac{1}{2}\boldsymbol{\ell}'\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\ell})$ and variance

$$\begin{pmatrix} \mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0} & \mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\ell} \\ \boldsymbol{\ell}'\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0} & \boldsymbol{\ell}'\mathbf{\Sigma}_{(k_1,\ldots,k_m);f_0}\boldsymbol{\ell} \end{pmatrix}$$

under $\mathrm{P}_{\mathbf{0};f_0}$. Point (ii) then follows by applying Le Cam's third Lemma. □

Proof of Proposition 4. From the strong law of large numbers we have for $r \geq 1$,

$$n^{-1/2}\widehat{b}_n(r) \longrightarrow b(r), \text{ a.s. as } n \to \infty,$$

and therefore (4.3) follows. Moreover in view of Proposition 1, the almost sure limit T_w in the right–hand side of (4.3) is positive unless $\mathcal{H}_0$ holds true, which in turn implies that

$$T_{n,w} \longrightarrow \infty, \text{ a.s. as } n \to \infty,$$

under any fixed nonsymmetric alternative distribution. □

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