A Skewed Version of the Robbins-Monro-Joseph Procedure
for Binary Response

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Abstract: The Robbins-Monro stochastic approximation procedure has been used for sensitivity testing. Joseph (2004) recognized that it is not well suited for binary data and proposed a modification which gives better performance for p between 0.1 and 0.9. However, for extreme p values, say $p \leq 0.01$ or $p \geq 0.99$, the Joseph version does not perform well. To overcome this difficulty, we propose a modification based on an asymmetric quadratic loss function. The new procedure can speed up convergence by employing different penalties for undershooting and overshooting to reduce the expected loss. Simulation comparisons show the clear advantages of the new procedure for extreme p values.

Key words and phrases: Asymmetric loss function, Bioassay, Extreme quantile, Sensitivity testing, Stochastic approximation.

1. Introduction

Sensitivity testing has been used in bioassays and industries. Their applications include finding the dose level at which 50% of the specimens survive (Wetherill, 1963; Wu, 1985), finding the level of shock necessary to make 99.99% of the explosives to fire (Neyer, 1994), or finding stack force level at which the problem of multi-feeds in paper feeder occurs with 0.1% probability (Joseph and Wu, 2002). The primary interest lies in finding the stimulus level at which response (or success) occurs with probability p. The stimulus level can be the dose of drug or the applied voltage, and the response can be survival or explosion, etc. Let x denote the stimulus level, y the binary response (i.e., success or failure) and $F(x)$ the probability of success at level x. The problem can be formulated as that of finding the pth quantile ζ_p such that $F(\zeta_p) = p$. For this problem, many procedures have been proposed in the literature, e.g., Robbins and Monro (1951), Wu (1985), Neyer (1994), Joseph et al. (2007), Dror and Steinberg (2008) and Wu and Tian (2014). Joseph (2004) proposed a modification in the case of binary data to the Robbins-Monro procedure and showed that it gives

better performance for p between 0.1 and 0.9. However, for extreme quantiles like $p \leq 0.01$ or $p \geq 0.99$, its performance can suddenly deteriorate as demonstrated in the simulation study of Wu and Tian (2014). The main prognosis in Wu and Tian is that the iterative sequence of the Joseph procedure can descend extremely slowly toward a target like $\zeta_{0.99}$ if the starting value of the iteration is guessed far above the target. To mitigate this problem, we propose to use an asymmetric loss function around the target to penalize above target more severely than below target. The minimization of the expected loss will force the iterative sequence to make bigger descending steps.

This paper is organized as follows. In Section 2, the Robbins-Monro procedure and the Joseph modification are reviewed. In Section 3, the proposed procedure is developed. In Section 4, simulation comparisons between the new procedure and the Joseph modification are given. Concluding remarks are given in Section 5.

2. Robbins-Monro procedure and Joseph's modification

Robbins and Monro (1951) proposed the following stochastic approximation procedure:

$$x_{n+1} = x_n - a_n(y_n - p), \tag{2.1}$$

where y_n is the response at the stress level x_n, $\{a_n\}$ is a sequence of positive constants and p is pre-specified by the experimenter. Robbins and Monro (1951) suggested choosing $a_n = c/n$, where c is a constant. Chung (1954), Hodges and Lehmann (1956), and Sacks (1958) recommended the choice $c = \{F'(\zeta_p)\}^{-1}$, where F' denotes the derivative of F. Since $F'(\zeta_p)$ is unknown, an adaptive choice is $c = 1/\hat{\beta}_n$, where $\hat{\beta}_n$ is the regression slope of y_i over x_i. Anbar (1973) and Lai and Robbins (1981) proved that $\hat{\beta}_n$ with proper truncation converges to $F'(\zeta_p)$ and the procedure with the truncated $\hat{\beta}_n$ has the same asymptotic distribution as the procedure with $c = \{F'(\zeta_p)\}^{-1}$. Wetherill (1963), Cochran and Davis (1965) and Young and Easterling (1994) studied the performance of the Robbins-Monro procedure and several modifications through simulations and concluded that the Robbins-Monro procedure and related modifications should be used only for p in the immediate neighborhood of 0.5.

Recognizing that, for binary response y, the Robbins-Monro procedure does not make provision for the binary nature of data, Joseph (2004) proposed the following modification:

$$x_{n+1} = x_n - a_n(y_n - b_n), \tag{2.2}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of constants. Start the iteration at some level x_1

which is believed to be close to $\theta = \zeta_p$ and assume the prior distribution of θ has $E(\Theta) = x_1$ and var$(\Theta) = \tau_1^2$. Joseph proposed to choose a_n and b_n by minimizing $E(z_{n+1}^2)$ subject to the condition $E(z_{n+1}) = 0$, where $z_n = x_n - \Theta$. The solution to this minimization problem is obtained by choosing

$$b_n = \Phi\{\frac{\Phi^{-1}(p)}{(1 + \beta^2\tau_n^2)^{1/2}}\}, \quad a_n = \frac{1}{b_n(1 - b_n)}\frac{\beta\tau_n^2}{(1 + \beta^2\tau_n^2)^{1/2}}\phi\{\frac{\Phi^{-1}(p)}{(1 + \beta^2\tau_n^2)^{1/2}}\},$$

$$\tau_{n+1}^2 = \tau_n^2 - b_n(1 - b_n)a_n^2, \quad \beta = \frac{F'(F^{-1}(p))}{\phi(\Phi^{-1}(p))}\frac{1}{\sigma}, \tag{2.3}$$

where $\Phi(\cdot)$ is the normal distribution function and $\phi(\cdot)$ is the normal density function. For brevity, we shall refer to (2.2)-(2.3) as the Robbins-Monro-Joseph procedure.

A simulation study in Joseph (2004) showed the superior performance of the modification over the original Robbins-Monro procedure. However, these simulations were performed only at different values of p between 0.1 and 0.9. Wu and Tian (2014) performed simulations at $p = 0.9$, $p = 0.99$ and $p = 0.999$ and showed that the performance of the Robbins-Monro-Joseph procedure depends critically on the starting value x_1, especially for $p = 0.99$ and $p = 0.999$. It can perform badly when x_1 is far from the target value ζ_p. A typical example of this poor performance is given in Wu and Tian (2014, p.11). For estimating $\zeta_{0.99}(= 12.3263)$, suppose x_1 is chosen to be 19.3054. In the next 60 iterations the corresponding y_i are all equal to 1 and thus the corresponding x_i sequence makes small decreasing steps. Even the terminating x value $x_{61}(= 17.2733)$ is still much larger than 12.3263. In this situation, the sequence descends very slowly toward ζ_p, which explains why the Robbins-Monro-Joseph procedure has much higher mean square errors than competing procedure in the simulations studied by Wu and Tian (2014). In order to accelerate its convergence toward ζ_p, we need to modify the procedure so that bigger steps can be made and more $y_i = 0$ be observed. One way to achieve this is to use an asymmetric loss function around the target ζ_p to penalize the positive deviation of $x_i - \zeta_p$ more severely than the negative deviation. This will drive the x_i sequence to make bigger descending steps, especially for extreme values of p. The detailed development is given in the next section. Note that asymmetric loss function has been used effectively in deriving an optimization procedure in robust parameter design (Moorhead and Wu, 1998).

3. The new procedure

Consider the procedure in (2.2) with the starting level x_1. Assume the distribution

function $F(x)$ is from a location-scale family with parameters μ and σ, i.e., $F(x) = G((x - \mu)/\sigma)$, where G is a known distribution function. Let $\theta = \zeta_p$ and assume that the prior distribution for θ has $E(\Theta) = \xi$ and $\text{var}(\Theta) = \tau_1^2$. Let $z_n = x_n - \Theta$. Then the procedure (2.2) becomes

$$z_{n+1} = z_n - a_n(y_n - b_n). \tag{3.1}$$

Denote $P(y_n = 1 \mid z_n)$ by $M(z_n)$. Then $M(z_n) = G(z_n/\sigma + G^{-1}(p))$. Consider the asymmetric quadratic loss function

$$L(z) = wz^2, \tag{3.2}$$

with $w = \lambda_1$ for $z \leq 0$ and λ_2 for $z > 0$, where $\lambda_1 > 0$ and $\lambda_2 > 0$. We propose to choose x_1 to minimize $E\{L(z_1)\}$ and a_n and b_n to minimize $E\{L(z_{n+1})\}$. Let $v_n = E(z_n)$ and $\tau_n^2 = \text{var}(z_n)$. From (3.1), we have

$$v_{n+1} = E(z_{n+1}) = v_n - a_n E\{M(z_n)\} + a_n b_n, \tag{3.3}$$

$$\tau_{n+1}^2 = a_n^2[E\{M(z_n)\} - E^2\{M(z_n)\}] - 2a_n[E\{z_n M(z_n)\} - v_n E\{M(z_n)\}] + \tau_n^2. \tag{3.4}$$

Clearly, it is difficult to obtain an exact distribution of z_n or an exact formula for $E\{L(z_n)\}$. We employ an approach similar to those in Joseph (2004, p.463) to approximate the distribution of z_n by $N(v_n, \tau_n^2)$ and $M(z_n)$ by $\Phi\{\Phi^{-1}(p) + \gamma z_n\}$, where

$$\gamma = M'(0)/\phi\{\Phi^{-1}(p)\}. \tag{3.5}$$

The choice of γ in (3.5) is to guarantee that the approximation and M have the same slope at 0. With these approximations, we have

$$E\{L(z_{n+1})\} = \quad (\lambda_1 - \lambda_2)\{(\tau_{n+1}^2 + v_{n+1}^2)\Phi(-\frac{v_{n+1}}{\tau_{n+1}}) - v_{n+1}\tau_{n+1}\phi(-\frac{v_{n+1}}{\tau_{n+1}})\}$$
$$+ \lambda_2(\tau_{n+1}^2 + v_{n+1}^2). \tag{3.6}$$

Choose v_1 to minimize $E\{L(z_1)\}$ for a given τ_1 and let $x_1 = \xi + v_1$. Let $\lambda = \lambda_1/\lambda_2$ and call it the *skewness coefficient*. By minimizing (3.6) with respect to a_n and v_{n+1}, we obtain the optimal a_n and v_{n+1} as the roots of the two equations:

$$\frac{\partial E\{L(z_{n+1})\}}{\partial a_n} = \lambda_2 \tau_{n+1}'[2\lambda\Phi(-\frac{v_{n+1}}{\tau_{n+1}})\tau_{n+1} + 2\tau_{n+1}\{1 - \Phi(-\frac{v_{n+1}}{\tau_{n+1}})\}] = 0, \tag{3.7}$$

$$\frac{\partial E\{L(z_{n+1})\}}{\partial v_{n+1}} = 2\lambda_2(\lambda - 1)\{v_{n+1}\Phi(-\frac{v_{n+1}}{\tau_{n+1}}) - \tau_{n+1}\phi(-\frac{v_{n+1}}{\tau_{n+1}})\} + 2\lambda_2 v_{n+1} = 0, \tag{3.8}$$

where v_{n+1} and τ_{n+1} are defined in (3.3) and (3.4). That the solution to (3.7)-(3.8) is indeed a minimum is proven in Proposition 3.1. It is easy to show from (3.7) that

$$a_n = \frac{E\{z_n M(z_n)\} - v_n E\{M(z_n)\}}{E\{M(z_n)\} - E^2\{M(z_n)\}}. \tag{3.9}$$

Using numerical computation, we can obtain the root v_{n+1} of (3.8). From (3.3), we can obtain the optimal b_n as

$$b_n = E\{M(z_n)\} - (v_n - v_{n+1})/a_n. \tag{3.10}$$

To summarize, the proposed skewed version of the Robbins-Monro-Joseph procedure is defined by equations (3.1), (3.9) and (3.10). At the nth iteration, its step length is the absolute value of $a_n(y_n - b_n)$. Since a_n is given and b_n is close p (as proved in Theorem 3.1), the step length is determined by $|y_n - p|$. For p close to 1, it is much smaller when $y_n = 1$ than when $y_n = 0$. This explains why, in the specific example given after (2.3), all the 60 iterations make small descending steps toward the target when only $y_n = 1$ is observed. By choosing the skewness coefficient λ to be smaller than 1, the skewed procedure forces the iterations to make bigger downward steps than upward steps, which also increases the chance of observing $y_n = 0$. Similarly, for p close to 0, we should choose λ larger than 1 to enforce upward bias. This explains why on the right panel of Fig. 4.1 in Section 4, we choose large values of λ^{-1} while on the left panel we choose large values of λ. A further study will be given in Section 4 to show how this modification can help mitigate the problem encountered in this specific example.

Proposition 3.1. The values of a_n and b_n that minimize $E\{L(z_{n+1})\}$ in (3.6) are unique.

Proof. Since $\lambda > 0$, $\tau_{n+1} > 0$ and $0 < \Phi(-v_{n+1}/\tau_{n+1}) < 1$, the sign of $\partial E\{L(z_{n+1})\}/\partial a_n$ in (3.7) is the same as that of τ'_{n+1}. By taking the derivative of τ^2_{n+1} in (3.4), we have

$$2\tau_{n+1}\tau'_{n+1} = 2a_n[E\{M(z_n)\} - E^2\{M(z_n)\}] - 2[E\{z_n M(z_n)\} - v_n E\{M(z_n)\}]. \tag{3.11}$$

Because $E\{M(Z_n)\} < 1$, the expression on the right of (3.11) is strictly linear in a_n and thus (3.7) has a unique root. Because $\lim\limits_{v_{n+1} \to -\infty} \partial E\{L(z_{n+1})\}/\partial v_{n+1} = -\infty$, $\lim\limits_{v_{n+1} \to \infty} \partial E\{L(z_{n+1})\}/\partial v_{n+1} = \infty$ and $\partial^2 E\{L(z_{n+1})\}/\partial v^2_{n+1} = 2(\lambda - 1)\lambda_2 \Phi(-v_{n+1}/$

$\tau_{n+1}) + 2\lambda_2 \geq 0$, (3.8) has a unique root. A direct computation yields that the determinant of the Hessian matrix is $4[E\{M(z_n)\} - E^2\{M(z_n)\}][(\lambda-1)\lambda_2\Phi(-\upsilon_{n+1}/\tau_{n+1}) + \lambda_2]^2$, which is larger than 0. Therefore, the values a_n and b_n that minimize $E\{L(z_{n+1})\}$ in (3.6) are unique.

Because the $E\{M(z_n)\}$ and $E\{z_nM(z_n)\}$ terms in (3.9) and (3.10) do not have exact formulas, we resort to normal approximation as before (one line above (3.5)) to obtain approximate formulas as follows:

$$
\begin{aligned}
E\{M(z_n)\} &\approx \Phi(\frac{\Phi^{-1}(p) + \gamma\upsilon_n}{\sqrt{1+\gamma^2\tau_n^2}}), \\
E\{z_nM(z_n)\} &\approx \frac{\gamma\tau_n^2}{\sqrt{1+\gamma^2\tau_n^2}}\phi(\frac{\Phi^{-1}(p) + \gamma\upsilon_n}{\sqrt{1+\gamma^2\tau_n^2}}) + \upsilon_n\Phi(\frac{\Phi^{-1}(p) + \gamma\upsilon_n}{\sqrt{1+\gamma^2\tau_n^2}}), \quad (3.12)
\end{aligned}
$$

where γ is given in (3.5). If $\lambda > 1$ (i.e., $\lambda_1 > \lambda_2$), then the penalty for under-shooting is more severe than over-shooting. Therefore, in order to reduce the expected loss $E\{L(z_n)\}$, the corresponding υ_n value should be positive. Similarly, for $\lambda < 1$, $\upsilon_n < 0$ and for $\lambda = 1$, $\upsilon_n = 0$. This intuition is confirmed in the following result.

Proposition 3.2. If $\lambda = 1$, then $\upsilon_n \equiv 0$; if $\lambda > 1$, then $\upsilon_n > 0$; if $\lambda < 1$, then $\upsilon_n < 0$.

Proof. If $\lambda = 1$, the solution of (3.8) is $\upsilon_{n+1} = 0$. If $\lambda > 1$, because $\partial E\{L(z_{n+1})\}/\partial\upsilon_{n+1} < 0$ at $\upsilon_{n+1} = 0$ and because $\partial^2 E\{L(z_{n+1})\}/\partial\upsilon_{n+1}^2 \geq 0$, the root υ_{n+1} of (3.8) is positive. A similar argument can be made for $\lambda < 1$.

The proposed procedure enjoys asymptotic convergence as given in the following result.

Theorem 3.1. For the procedure given in (3.1), (3.9) and (3.10) with the normal approximation (3.12), $z_n \to^p 0$.

Its proof is given in the appendix.

4. Simulation comparisons

We first show how the modified procedure can be used to mitigate the problem encountered in the example in Section 3. This example came from Table 6A in Wu and Tian (2014) with the choice of $\mu_g = 10$ and $\sigma_g = 4$, where μ_g is the same as the mean of the normal distribution and σ_g is the same as γ^{-1} with γ defined in (3.5). Using the same parameters as in Wu and Tian, we generated 1000 simulations for the modified procedure with $\lambda^{-1} = 100$ and 1000. In Table 4.1, the average value and root mean square error (RMSE) of x_{61} and the percentage of $y_i = 0$ among the 1000

simulations are given for each of the three competing versions. Note that the values in the Table 4.1 for $\lambda = 1$ were taken from Table 6A of Wu and Tian. It is seen that as λ^{-1} increases, the RMSE decreases, the final estimate x_{61} gets closer to the target and the percentage of $y_i = 0$ increases. It clearly demonstrates the effectiveness of using the skewness coefficient λ to fine-tune the Robbins-Monro-Joseph procedure.

Next we report a more comprehensive simulation study to compare the performance of the procedure given in (3.1), (3.9), (3.10) with that of the Robbins-Monro-Joseph procedure. Three distributions are considered:

1. Normal distribution: $F(x) = \Phi(x)$,

2. Logistic distribution: $F(x) = (1 + e^{-1.8138x})^{-1}$,

3. Extreme value distribution: $F(x) = exp\{-exp(-x)\}$.

Table 4.1: Average and RMSE of x_{61} for estimating $\zeta_{0.99}(= 12.3263)$, $n = 60$, true model=normal distribution, $\mu_g = 10$, $\sigma_g = 4.0$, and $x_1 = 19.3054$.

λ^{-1}	Average	RMSE	$\%\{y_i = 0\}$
1	17.2733	4.9470	0
100	15.3308	3.0047	0.3%
1000	14.6634	2.3658	37.4%

We choose $\sigma = (1.8138)^{-1}$ for the logistic distribution so that the normal and the logistic distributions have the same standard deviation. This can be important for comparing the performances under the two distributions (Wu and Tian, 2014). As in Joseph (2004), we let $\tau_1 = 1$, $\xi = F^{-1}(p)$ and choose 20 samples to estimate θ and also assume that $M'(0) > 0$ is known. We use the final value of the iterations, x_{21}, as the estimate of θ for both procedures. For the Robbins-Monro-Joseph procedure, the starting level is randomly generated from $N(\xi, \tau_1^2)$ (Joseph, 2004). For the new procedure the starting level is randomly generated from $N(\xi + \upsilon_1, \tau_1^2)$, where υ_1 is chosen by minimizing $E\{L(z_1)\}$. For p between 0.001 and 0.25, we choose $\lambda = 3.5, 10, 50, 100, 1000, 50000$ in the simulation study. And, for p between 0.75 and 0.999, we choose $\lambda^{-1} = 3.5, 10, 50, 100, 1000, 50000$ in the simulation study. For each p, 1000 simulations are performed. The mean square errors (MSEs) of the estimate are calculated. The results for the three distributions are shown in Fig. 4.1, where
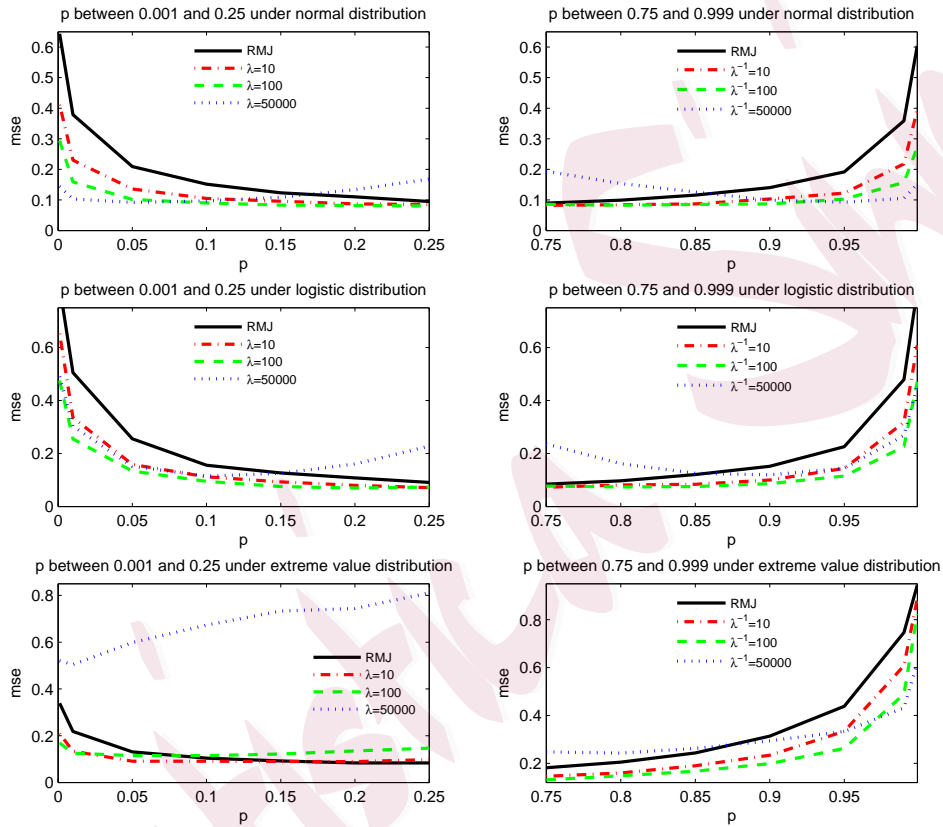
Figure 4.1: Plot of mean square errors of x_{21} for various procedures.

the Robbins-Monro-Joseph procedure is denoted by RMJ and the proposed procedure with $\lambda = 10, 100, 50000$ or $\lambda^{-1} = 10, 100, 50000$ are considered. It can be seen that the new procedure with $\lambda = 10, 100$ or $\lambda^{-1} = 10, 100$ performs better than the Robbins-Monro-Joseph procedure, except for the dashed curve (i.e. $\lambda = 100$) for moderate p values under the extreme value distribution. Result for λ (or λ^{-1}) $= 3.5, 50, 1000$ are omitted in order to keep the clarity of the graphs and these results do not change the overall conclusions. The same simulation study was performed for the skewed logistic distribution $F(x) = (1 + e^{-x})^{-2}$ and cauchy distribution $F(x) = \arctan(x)/\pi + 1/2$. Because the results are similar to those for the logistic distribution, they are omitted here. As λ increases, although the new procedure becomes slightly worse for moderate p values, it still maintains a clear advantage over the Robbins-Monro-Joseph procedure for extreme p values. (These results are given in tables in the supplement.) However when the skewness coefficient λ takes extremely large values like $\lambda = 50000$ in Fig. 4.1, the new procedure can be undesirable, especially under the extreme value distribution.

5. Conclusions and further remarks

In this paper we propose a skewed version of the Robbins-Monro-Joseph procedure for binary data by employing an asymmetric quadratic loss function to allow the procedure to make more downward (or upward) biased moves in its iterations. The simulation study in §4 shows that, for estimating the extreme quantiles, it outperforms the original procedure over a wide range of values for the skewness coefficient λ unless λ or λ^{-1} is exceedingly large. But the study does not tell us exactly what value of λ should be chosen. Because the optimal choice of λ is a complicated function of various factors, it has no explicit form. Thus theory does not help here. Fortunately the simulation study suggests that there is a wide range of λ values that give good performance for the skewed version. To actually find a good λ value will require some tweaking on the part of the experimenter.

A second advantage of the proposed procedure is its flexibility in allocating more runs to above or below target by adjusting the value of λ. This can be more easily seen for p values closer to 0.5. Consider the problem of estimating the effective dose of a compound. It is common to use a sequential design procedure for estimating or approximating the median dose $\zeta_{0.5}$, which is called the ED50 problem in bioassay (Finney, 1978). In this context, the Robbins-Monro-Joseph procedure will generate

roughly equal numbers of $y = 1$ and of $y = 0$. If $y = 1$ means death or damage of an experimental subject, it would be desirable to reduce the number of $y = 1$. This can be easily achieved by using the proposed procedure with an appropriate value of λ larger than 1. If λ is moderately larger than 1, then the loss of estimation efficiency relative to the Robbins-Monro-Joseph procedure will be small. A small simulation study was done to compare the skewed version (with $\lambda = 10$) and the original version (with $\lambda = 1$) in terms of the conflicting goals of estimation efficiency and reducing the number of $y = 1$. In Table 5.1, the results of MSE and $\sharp$ of $\{y_i = 1\}, i = 1, 2, \ldots, 20$ are reported for three distributions (normal, logistic and extreme value). It can be seen that the loss of efficiency by using $\lambda = 10$ ranges from mild (normal) to moderate (logistic and extreme value) while the reduction of $\sharp$ of $\{y_i = 1\}$ is consistently around 30%. Of course a moderately large λ will not have the flexibility in allocating the sample sizes between above target and below the target. The reverse is true if λ is chosen to be much larger than 1. How to make a proper choice between the two objectives will be a subject for further study.

Table 5.1: MSE and average of $\sharp\{y_i = 1\}$ of estimating ED50 with 20 sample for three distributions

	MSE		$\sharp\{y_i = 1\}$	
	$\lambda = 1$	$\lambda = 10$	$\lambda = 1$	$\lambda = 10$
Normal distribution	0.0791	0.0817	10.008	7.093
Logistic distribution	0.0723	0.1149	10.072	7.291
Extreme value distribution	0.1073	0.1424	10.054	7.296

For the sake of clarity, we have focused our discussions on the extreme high quantiles like $p \geq 0.99$. Extreme low quantiles are equally important in practice. For example, measures such as LD_1, which is the dosage required to kill 1% of the test population, are useful for the safety study of compounds. Our discussions and conclusions can be similarly made by changing from p to $1 - p$.

Acknowledgments

Appendix

Joseph (2004) gave the following result about the convergence of the procedure in (3.1).

Lemma A1. If the following conditions hold, then $z_n \to^p 0$.

1. $\sum_{n=1}^{\infty} a_n = \infty$, $\sum_{n=1}^{\infty} a_n^2 < \infty$,

2. $\sum_{n=2}^{\infty} a_n |b_n - p| \sum_{j=1}^{n-1} a_n < \infty$.

Here we show that the sequences $\{a_n\}$ and $\{b_n\}$ given in (3.9) and (3.10) with approximate formula (3.12) satisfy the two conditions in Lemma A1 and thus prove Theorem 3.1.

Proof. Let $\eta_n = \gamma^2 \tau_n^2$. From (3.4), (3.9) and (3.12), we have

$$\eta_{n+1} = \eta_n - \eta_n^2 I\{(\Phi^{-1}(p) - \gamma v_n)/\sqrt{1 + \eta_n}\}/(1 + \eta_n), \tag{F.1}$$

where $I(u) = \phi^2(u)/[\Phi(u)\{1 - \Phi(u)\}]$ is the Fisher information of u in binary data with probability of response equal to $\Phi(u)$. From the proof of Proposition 1 in Joseph (2004), we have $\eta_n \to 0$ or equivalently $\tau_n^2 \to 0$, which together with (3.8) implies $v_n \to 0$. Thus $z_n \to^p 0$ if F is the normal distribution function.

For general F, let $h(\eta_n)$ be the right side of (F.1), $n^* = 1/I(2\Phi^{-1}(p))$ and $\eta^* = 1/I^2(2\Phi^{-1}(p))$. As in the proof of Proposition 2 in Joseph (2004), we can show that there exists an $\tilde{n}$ and some η' and η'' such that $\eta_n \leq \eta'/n$ and $\eta_n \geq \eta''/n$ for all $n \geq \tilde{n}$ and that $a_n \leq \eta'/(\sqrt{2\pi}p(1 - p)n\gamma)$ and $a_n \geq 4\eta''\phi(\Phi^{-1}(p))/\{\gamma(n + 1)\}$ for all $n \geq \tilde{n}$. Thus $\{a_n\}$ satisfies condition (1) of Lemma A1.

Next we verify condition (2). From (3.8) we have $v_{n+1}/\tau_{n+1} = (\lambda - 1)\phi(-v_{n+1}/\tau_{n+1})/\{(\lambda - 1)\Phi(-v_{n+1}/\tau_{n+1}) + 1\}$. Let $g(-v_{n+1}/\tau_{n+1})$ be the right side of the above equation. Because $g(-x) - x = 0$ has a unique root, there exists an M such that $v_{n+1} = M\tau_{n+1}$. From (3.10), we have $a_n(b_n - p) = a_n[\Phi\{(\Phi^{-1}(p) + \gamma v_n)/(1 + \gamma^2\tau_n^2)^{1/2}\} - p] - M(\tau_n - \tau_{n+1})$. Using Taylor series expansion we obtain $\Phi\{(\Phi^{-1}(p) + \gamma v_n)/(1 + \gamma^2\tau_n^2)^{1/2}\} = p + M\gamma\phi\{\Phi^{-1}(p)\}\tau_n + O(\tau_n^2)$. And from (3.4), we have $\tau_n - \tau_{n+1} = a_n^2\Phi\{(\Phi^{-1}(p) + \gamma v_n)/(1 + \eta_n)^{1/2}\}[1 - \Phi\{(\Phi^{-1}(p) + \gamma v_n)/(1 + \eta_n)^{1/2}\}]/(\tau_n + \tau_{n+1})$. Since $\eta_n = \gamma^2\tau_n^2$ was previously shown to be of the order $1/n$, $\tau_n = O(n^{-1/2})$ and also $a_n = O(1/n)$. Thus we obtain $a_n|b_n - p| = O(n^{-3/2})$. This implies that condition (2)

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of Lemma A1 holds. Thus by Lemma A1, $z_n \to^p 0$.

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