RESAMPLING METHOD FOR GENERALIZED ONE-PER-STRATUM SAMPLING DESIGNS

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Supplementary Material

The online supplementary material contains a brief description of the construction for $P_{\boldsymbol{S}}$, the Stochastic sampling design of Lahiri and Zhu (2006) (S1), the proofs of Theorem 1 (S3), Theorem 2 (S4) and Theorem 3 (S5), and an additional simulation result for a simple linear regression (S6).

S1 Stochastic sampling design of Lahiri and Zhu (2006)

Let $f(x)$ be a probability density function on the prototype region R_0, and let $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots\}$ be a sequence of independent and identically distributed random vectors with probability density function $f(x)$. Besides, $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots\}$ is independent with the spatial process $\{Z(\boldsymbol{s}) : \boldsymbol{s} \in \mathbb{R}^d\}$. Then, the sampled locations are obtained by

$$\boldsymbol{s}_i = \lambda_n \boldsymbol{X}_i, \quad (i = 1, \ldots, n).$$

S2 Construction of P_S

Denote $(\Omega_n, \mathcal{F}_n, P_n)$ to be the probability space with respect to the proposed one-per-stratum sampling design. Specifically, $\Omega_n = \times_{i=1}^n A_i$ is the product of $A_1, \ldots, A_n$, $\mathcal{F}_n = \times_{i=1}^n \mathcal{G}_i$ is the product σ-algebra of $\mathcal{G}_1, \ldots, \mathcal{G}_n$, P_n is the product probability measure of $P_{n,1}, \ldots, P_{n,n}$, $(A_i, \mathcal{G}_i, P_{n,i})$ is a probability space for the generation of $\boldsymbol{s}_i$, $P_{n,i}$ is the probability with respect to $f_i(\boldsymbol{s})$, and $f_i(\boldsymbol{s})$ is the sampling density function. Then, by the Kolmogorov's consistency theorem, there exists a probability $P_{\boldsymbol{S}}$ on the product space $\Omega_{\boldsymbol{S}} = \times_{n=1}^{\infty} \Omega_n$ equipped with a product σ-algebra, such that $\mathbb{P}_J = P_{\boldsymbol{S}} \circ \xi_J^{-1}$ for all finite positive integer set $J \subset \mathbb{N}_+$, where ξ_J is the canonical projection from $\Omega_{\boldsymbol{S}}$ to the product space $\times_{j\in J}\Omega_j$, $\mathbb{P}_J$ is the product probability measure on the product measurable space $(\times_{j\in J}\Omega_j, \times_{j\in J}\mathcal{F}_j)$, and $\mathbb{N}_+$ is the set of positive integers; see (Klenke, 2014, Section 14.3) for details.

S3 Proof of Theorem 1

Lemma 1. *Suppose that Conditions 2–9 hold. Then, for any $\boldsymbol{a} \in \mathbb{R}^d$ with $\|\boldsymbol{a}\| = 1$,*

$$\sigma_{n,\boldsymbol{a}}^2 \to \boldsymbol{a}^\top H \sigma_\Psi(\mathbf{0}) \boldsymbol{a} + \boldsymbol{a}^\top \left\{ \int Q(\boldsymbol{h}) \sigma_\Phi(\boldsymbol{h}) d\boldsymbol{h} \right\} \boldsymbol{a} \quad a.s.\ (P_{\boldsymbol{S}}),$$

where $\sigma^2_{n,\boldsymbol{a}} = \sum_{i=1}^n \sum_{j=1}^n d_n(\boldsymbol{S}_i) d_n(\boldsymbol{S}_j) \sigma_\Psi(\boldsymbol{S}_i - \boldsymbol{S}_j)$, $d_n(\boldsymbol{s}) = \boldsymbol{a}^\top \Lambda_n^{-1} \boldsymbol{x}(\boldsymbol{s})$, $\boldsymbol{x}(\boldsymbol{s}) = 0$ *if* $\boldsymbol{s} \notin R_n$, *and* $\boldsymbol{S}_i$ *is the random variable associated with its realization* $\boldsymbol{s}_i$.

Proof of Lemma 1. Based on Condition 8, $d \geq 2$ and a similar argument as Lemma 1.3 discussed by Ibragimov (1962), we can show that $\sigma_\Psi(\boldsymbol{h}) = o(\|\boldsymbol{h}\|^{-3/2})$. Thus, we have

$$\int |\sigma_\Psi(\boldsymbol{h})|^{2r+2} \,\mathrm{d}\boldsymbol{h} < \infty, \tag{S3.1}$$

where $r \geq 1$ is a positive integer. Denote $C_\sigma = \int |\sigma_\Psi(\boldsymbol{h})| \mathrm{d}\boldsymbol{h}$, $C_{2\sigma} = \int |\sigma_\Psi(\boldsymbol{h})|^2 \,\mathrm{d}\boldsymbol{h}$, and $C_{4\sigma} = \int |\sigma_\Psi(\boldsymbol{h})|^4 \,\mathrm{d}\boldsymbol{h}$.

For simplicity, denote $h_n(\boldsymbol{x}, \boldsymbol{y}) = d_n(\boldsymbol{x}) d_n(\boldsymbol{y}) \sigma_\Psi(\boldsymbol{x} - \boldsymbol{y})$. Thus, we have $\sigma^2_{n,\boldsymbol{a}} = \sum_{i=1}^n \sum_{j=1}^n h_n(\boldsymbol{S}_i, \boldsymbol{S}_j)$. The expectation of $\sigma^2_{n,\boldsymbol{a}}$ with respect to the one-per-stratum sampling is

$$E_{\boldsymbol{S}}(\sigma^2_{n,\boldsymbol{a}}) = E_{\boldsymbol{S}}\left\{\sum_{i=1}^n h_n(\boldsymbol{S}_i, \boldsymbol{S}_i)\right\} + E_{\boldsymbol{S}}\left\{\sum_{i=1}^n \sum_{j \neq i} h_n(\boldsymbol{S}_i, \boldsymbol{S}_j)\right\}.$$

First, we show

$$E_{\boldsymbol{S}}(\sigma^2_{n,\boldsymbol{a}}) \to \boldsymbol{a}^\top H \sigma_\Psi(\boldsymbol{0}) \boldsymbol{a} + \boldsymbol{a}^\top \left\{\int \sigma_\Psi(\boldsymbol{h}) Q(\boldsymbol{h}) \mathrm{d}\boldsymbol{h}\right\} \boldsymbol{a}. \tag{S3.2}$$

The first part of $E_{\boldsymbol{S}}(\sigma^2_{n,\boldsymbol{a}})$ is

$$E_{\boldsymbol{S}}\left\{\sum_{i=1}^n h_n(\boldsymbol{S}_i, \boldsymbol{S}_i)\right\} = \sigma_\Psi(\boldsymbol{0}) \sum_{i=1}^n \int d_n(\boldsymbol{s})^2 f_i(\boldsymbol{s}) \mathrm{d}\boldsymbol{s} \to \boldsymbol{a}^\top H \boldsymbol{a} \sigma_\Psi(\boldsymbol{0}), \tag{S3.3}$$

where the last convergence holds based on Condition 5. The second part of $E_{\boldsymbol{S}}(\sigma^2_{n,\boldsymbol{a}})$ is

$$
\begin{aligned}
& E_{\boldsymbol{S}}\left\{\sum_{i=1}^{n}\sum_{j\neq i} h_n(\boldsymbol{S}_i,\boldsymbol{S}_j)\right\} \\
= & \sum_{i=1}^{n}\sum_{j\neq i}\int\int d_n(\boldsymbol{x})d_n(\boldsymbol{y})\sigma_\Psi(\boldsymbol{x}-\boldsymbol{y})f_i(\boldsymbol{x})f_j(\boldsymbol{y})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\
= & \sum_{i=1}^{n}\sum_{j\neq i}\int \sigma_\Psi(\boldsymbol{h})\int d_n(\boldsymbol{y}+\boldsymbol{h})d_n(\boldsymbol{y})f_i(\boldsymbol{y}+\boldsymbol{h})f_j(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{h},
\end{aligned}
$$

where the second equality holds by Condition 7, $\sigma_\Psi(\boldsymbol{h}) = o(\|\boldsymbol{h}\|^{-3/2})$ and the Fubini's Theorem (Athreya and Lahiri, 2006, Theorem 5.2.2).

Denote $Q_1(\boldsymbol{h}) = \boldsymbol{a}^\top Q(\boldsymbol{h})\boldsymbol{a}$, and, by Condition 5, we have

$$
\sum_{i=1}^{n}\sum_{j\neq i}\int d_n(\boldsymbol{y}+\boldsymbol{h})d_n(\boldsymbol{y})f_i(\boldsymbol{y}+\boldsymbol{h})f_j(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\to Q_1(\boldsymbol{h}) \quad (n\to\infty). \tag{S3.4}
$$

Next, we show that the left part of (S3.4) is bounded by a constant for $\boldsymbol{h}\in\mathbb{R}^d$. Consider

$$
\begin{aligned}
& \left|\sum_{i=1}^{n}\sum_{j\neq i}\int d_n(\boldsymbol{y}+\boldsymbol{h})d_n(\boldsymbol{y})f_i(\boldsymbol{y}+\boldsymbol{h})f_j(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right| \\
\leq & \sum_{i=1}^{n}\sum_{j\neq i}\int_{A_j}\left|d_n(\boldsymbol{y}+\boldsymbol{h})\mathbb{1}(\boldsymbol{y}+\boldsymbol{h}\in A_i)d_n(\boldsymbol{y})\right| f_i(\boldsymbol{y}+\boldsymbol{h})f_j(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
\leq & \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\left\{\int_{A_j} d_n^2(\boldsymbol{y}+\boldsymbol{h})\mathbb{1}(\boldsymbol{y}+\boldsymbol{h}\in A_i)f_i(\boldsymbol{y}+\boldsymbol{h})^2\mathrm{d}\boldsymbol{y}\right. \\
& \left. +\int_{A_j} d_n^2(\boldsymbol{y})\mathbb{1}(\boldsymbol{y}+\boldsymbol{h}\in A_i)f_j(\boldsymbol{y})^2\mathrm{d}\boldsymbol{y}\right\}, \\
\leq & M_f^2\sum_{i=1}^{n}\int_{A_i} d_n^2(\boldsymbol{y})\mathrm{d}\boldsymbol{y},
\end{aligned}
$$

where the first inequality holds based on Condition 4, and the last inequality is valid based on the following two facts. That is,

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j\neq i}\int_{A_j} d_n^2(\boldsymbol{y}+\boldsymbol{h})\mathbb{1}(\boldsymbol{y}+\boldsymbol{h}\in A_i)f_i(\boldsymbol{y}+\boldsymbol{h})^2\mathrm{d}\boldsymbol{y} \\
\leq\ & M_f\sum_{i=1}^{n}\sum_{j\neq i}\int_{A_j} d_n^2(\boldsymbol{y}+\boldsymbol{h})\mathbb{1}(\boldsymbol{y}+\boldsymbol{h}\in A_i)f_i(\boldsymbol{y}+\boldsymbol{h})\mathrm{d}\boldsymbol{y} \\
=\ & M_f\sum_{i=1}^{n}\int_{\{(R_n\setminus A_i)+\boldsymbol{h}\}\cap A_i} d_n^2(\boldsymbol{y})f_i(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
\leq\ & M_f\sum_{i=1}^{n}\int_{A_i} d_n^2(\boldsymbol{y})f_i(\boldsymbol{y})\mathrm{d}\boldsymbol{y},
\end{aligned} \tag{S3.5}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j\neq i}\int_{A_j} d_n^2(\boldsymbol{y})\mathbb{1}(\boldsymbol{y}+\boldsymbol{h}\in A_i)f_j(\boldsymbol{y})^2\mathrm{d}\boldsymbol{y} \\
\leq\ & M_f\sum_{j=1}^{n}\sum_{i\neq j}\int_{A_j} d_n^2(\boldsymbol{y})\mathbb{1}(\boldsymbol{y}+\boldsymbol{h}\in A_i)f_j(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
=\ & M_f\sum_{j=1}^{n}\int_{\{(R_n\setminus A_j)-\boldsymbol{h}\}\cap A_j} d_n^2(\boldsymbol{y})f_j(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
\leq\ & M_f\sum_{j=1}^{n}\int_{A_j} d_n^2(\boldsymbol{y})f_j(\boldsymbol{y})\mathrm{d}\boldsymbol{y}
\end{aligned} \tag{S3.6}$$

By (S3.3) and (S3.5)–(S3.6), we know that the left part of (S3.4) is bounded by a constant, say C_0, when n is sufficiently large.

Thus, by fact that $|Q_1(\boldsymbol{h})|$ is dominated by a constant and $\int|\sigma_\Phi(\boldsymbol{h})|\mathrm{d}\boldsymbol{h} < \infty$, we have

$$E_{\boldsymbol{S}}\left\{\sum_{i=1}^{n}\sum_{j\neq i}h_n(\boldsymbol{S}_i,\boldsymbol{S}_j)\right\} \to \int \sigma_\Psi(\boldsymbol{h})Q_1(\boldsymbol{h})\mathrm{d}\boldsymbol{h} \tag{S3.7}$$

based on the dominated convergence theorem (Athreya and Lahiri, 2006, Corollary 2.3.13). By (S3.3) and (S3.7), we have shown (S3.2).

Denote $m^2_{0n,\boldsymbol{a}} = \sup\{\left|\boldsymbol{a}^\top \Lambda_n^{-1}\boldsymbol{x}(\boldsymbol{s})\right|^2 : \boldsymbol{s} \in \mathbb{R}^d\}$. By $\|\boldsymbol{a}\| = 1$, Condition 7, and the Hölder's inequality (Athreya and Lahiri, 2006, Theorem 3.1.11), we have

$$m^2_{0n,\boldsymbol{a}} = o(n^{-3/4}). \tag{S3.8}$$

Now, we consider $E_{\boldsymbol{S}}(\sigma^2_{n,\boldsymbol{a}} - E_{\boldsymbol{S}}\sigma^2_{n,\boldsymbol{a}})^4$. Denote

$$\begin{aligned}
D_{1n} &= \sum_{i=1}^{n}[h_n(\boldsymbol{S}_i,\boldsymbol{S}_i) - E_{\boldsymbol{S}}\{h_n(\boldsymbol{S}_i,\boldsymbol{S}_i)\}],\\
D_{2n} &= \sum_{j=1}^{n-1}\sum_{i=j+1}^{n}[h^{(i)}_{1n}(\boldsymbol{S}_j) - E_{\boldsymbol{S}}\{h_n(\boldsymbol{S}_i,\boldsymbol{S}_j)\}],\\
D_{3n} &= \sum_{i=2}^{n} U_i,\\
U_i &= \sum_{j=1}^{i-1}\{h_n(\boldsymbol{S}_i,\boldsymbol{S}_j) - h^{(i)}_{1n}(\boldsymbol{S}_j)\},\\
h^{(i)}_{1n}(\boldsymbol{S}_j) &= E_{\boldsymbol{S}}\{h_n(\boldsymbol{S}_i,\boldsymbol{S}_j)|\boldsymbol{S}_j\}.
\end{aligned}$$

Since $h_n(\boldsymbol{x},\boldsymbol{y}) = h_n(\boldsymbol{y},\boldsymbol{x})$ for $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^d$, we have

$$\sigma^2_{n,\boldsymbol{a}} - E_{\boldsymbol{S}}(\sigma^2_{n,\boldsymbol{a}}) = D_{1n} + 2D_{2n} + 2D_{3n}. \tag{S3.9}$$

Before proceeding, for $r \in \mathbb{N}_+$ and $i = 1, \ldots, n$, consider

$$\begin{aligned} E_{\boldsymbol{S}}\{h_n^{2r}(\boldsymbol{S}_i, \boldsymbol{S}_i)\} &= \int d_n^{4r}(\boldsymbol{s})\sigma_\Psi^{2r}(\mathbf{0})f_i(\boldsymbol{s})\mathrm{d}\boldsymbol{s} \\ &\leq M_f\sigma_\Psi^{2r}(\mathbf{0})m_{0n,\boldsymbol{a}}^{4r}M_A, \qquad \text{(S3.10)} \\ \sum_{i\in J} E_{\boldsymbol{S}}\{h_n^{2r}(\boldsymbol{S}_i, \boldsymbol{S}_j)|\boldsymbol{S}_j\} &= \sum_{i\in J}\int_{A_i} d_n^{2r}(\boldsymbol{s})d_n^{2r}(\boldsymbol{S}_j)\sigma_\Psi^{2r}(\boldsymbol{s}-\boldsymbol{S}_j)f_i(\boldsymbol{s})\mathrm{d}\boldsymbol{s} \\ &\leq M_f m_{0n,\boldsymbol{a}}^{4r}\int |\sigma_\Psi(\boldsymbol{s})|^{2r}\,\mathrm{d}\boldsymbol{s}, \qquad \text{(S3.11)} \end{aligned}$$

where J is a subset of $\{1, \ldots, n\} \setminus \{j\}$ in (S3.11), and recall that $f_i(\boldsymbol{s})$ is zero outside of A_i. For $j = 1, \ldots, n-1$, consider

$$\sum_{i=j+1}^n h_n^{(i)}(\boldsymbol{S}_j) \leq m_{0n,\boldsymbol{a}}^2 M_f \int_{\cup_{i=j+1}^n A_i}|\sigma_\Psi(\boldsymbol{s}-\boldsymbol{S}_j)|\mathrm{d}\boldsymbol{s} \leq m_{0n,\boldsymbol{a}}^2 M_f C_\sigma,$$

$$E_{\boldsymbol{S}}\left(\sum_{i=j+1}^n \left[h_n^{(i)}(\boldsymbol{S}_j) - E_{\boldsymbol{S}}\{h_n^{(i)}(\boldsymbol{S}_j)\}\right]\right)^{2r} \leq E_{\boldsymbol{S}}\left[\sum_{i=j+1}^n \left\{h_n^{(i)}(\boldsymbol{S}_j)\right\}\right]^{2r} \leq C_\sigma^{2r}M_f^{2r}m_{0n,\boldsymbol{a}}^{4r}.$$

For D_{1n}, it is a summation of n independent random variables with mean zero. Thus, we have

$$\begin{aligned} E_{\boldsymbol{S}}(D_{1n}^4) &\leq C_{S1}\left\{\sum_{i=1}^n E_{\boldsymbol{S}}h_n^4(\boldsymbol{S}_i, \boldsymbol{S}_i) + \sum_{i=1}^n\sum_{j\neq i} E_{\boldsymbol{S}}h_n^2(\boldsymbol{S}_i, \boldsymbol{S}_i)E_{\boldsymbol{S}}h_n^2(\boldsymbol{S}_j, \boldsymbol{S}_j)\right\} \\ &\leq C_{S1}\{m_{0n,\boldsymbol{a}}^8 M_f M_A\sigma_\Psi^4(\mathbf{0})n + m_{0n,\boldsymbol{a}}^8\sigma_\Phi^4(\mathbf{0})M_f^2M_A^2n^2\} \\ &\leq C\{M_f, M_A, \sigma_\Psi(\mathbf{0})\}m_{0n,\boldsymbol{a}}^8 n^2, \qquad \text{(S3.12)} \end{aligned}$$

where the second inequality holds by (S3.10), C_{S1} is a constant, and recall that $C\{M_f, M_A, \sigma_\Psi(\mathbf{0})\}$ is a function of M_f, M_A, and $\sigma_\Psi(\mathbf{0})$.

Similarly, we have

$$\begin{aligned} E_{\boldsymbol{S}}(D_{2n}^4) &\leq C\left[\sum_{j=1}^{n-1} E_{\boldsymbol{S}}\left(\sum_{i=j+1}^{n}\left[h_n^{(i)}(\boldsymbol{S}_j) - E_{\boldsymbol{S}}\{h_n^{(i)}(\boldsymbol{S}_j)\}\right]\right)^4 + \right. \\ &\quad \sum_{j=1}^{n-1}\sum_{k\neq j} E_{\boldsymbol{S}}\left(\sum_{i=j+1}^{n}\left[h_n^{(i)}(\boldsymbol{S}_j) - E_{\boldsymbol{S}}\{h_n^{(i)}(\boldsymbol{S}_j)\}\right]\right)^2 \\ &\quad \left. \times E_{\boldsymbol{S}}\left(\sum_{i=k+1}^{n}\left[h_n^{(i)}(\boldsymbol{S}_j) - E_{\boldsymbol{S}}\{h_n^{(i)}(\boldsymbol{S}_j)\}\right]\right)^2\right] \\ &\leq C(C_\sigma^4 M_f^4 m_{0n,\boldsymbol{a}}^8 n + C_\sigma^4 M_f^4 m_{0n,\boldsymbol{a}}^8 n^2) \\ &\leq C(C_\sigma, M_f) m_{0n,\boldsymbol{a}}^8 n^2 \end{aligned} \tag{S3.13}$$

Next, we consider D_{3n}. Note the fact that $E_{\boldsymbol{S}}(U_i|\boldsymbol{S}_1,\ldots,\boldsymbol{S}_{i-1}) = 0$ for $i = 2,\ldots,n$. Thus, $\left\{\sum_{j=2}^{i} U_j, \mathcal{F}_i^{\boldsymbol{S}}\right\}_{i=2}^{n}$ is a martingale, where $\mathcal{F}_i^{\boldsymbol{S}} = \sigma\langle \boldsymbol{S}_1,\ldots,\boldsymbol{S}_i\rangle$.

By Rosenthal's inequality (Hall and Heyde, 1980, Theorem 2.12), we have

$$\begin{aligned} E_{\boldsymbol{S}}(D_{3n}^4) &\leq C\left[E_{\boldsymbol{S}}\left\{\sum_{i=2}^{n} E_{\boldsymbol{S}}(U_i^2|\mathcal{F}_{i-1}^{\boldsymbol{S}})\right\}^2 + \sum_{i=2}^{n} E_{\boldsymbol{S}} U_i^4\right] \\ &\leq C\left(E_{\boldsymbol{S}}\left[(n-1)\sum_{i=2}^{n}\{E_{\boldsymbol{S}}(U_i^2|\mathcal{F}_{i-1}^{\boldsymbol{S}})\}^2\right] + \sum_{i=2}^{n} E_{\boldsymbol{S}} U_i^4\right) \\ &\leq Cn\left\{\sum_{i=2}^{n} E_{\boldsymbol{S}} U_i^4\right\} \\ &\leq C_1 n\left\{\sum_{i=2}^{n} E_{\boldsymbol{S}}\left(E_{\boldsymbol{S}}\left[\{U_i - E_{\boldsymbol{S}}(U_i|\boldsymbol{S}_i)\}^4 |\boldsymbol{S}_i\right] + \{E_{\boldsymbol{S}}(U_i|\boldsymbol{S}_i)\}^4\right)\right\}. \end{aligned} \tag{S3.14}$$

Notice that U_i is a sum of $i-1$ independent random variables given $\boldsymbol{S}_i$, so we have

$$\begin{aligned}
& E_{\boldsymbol{S}}\left[\{U_i - E_{\boldsymbol{S}}(U_i|\boldsymbol{S}_i)\}^4 \,|\boldsymbol{S}_i\right] \\
\leq\; & C\left[\sum_{j=1}^{i-1} E_{\boldsymbol{S}}\{h_n^4(\boldsymbol{S}_i,\boldsymbol{S}_j) \mid \boldsymbol{S}_i\}\right. \\
& \left. + \sum_{j=1}^{i-1}\sum_{k\neq j} E_{\boldsymbol{S}}\{h_n^2(\boldsymbol{S}_i,\boldsymbol{S}_j) \mid \boldsymbol{S}_i\} E_{\boldsymbol{S}}\{h_n^2(\boldsymbol{S}_i,\boldsymbol{S}_k) \mid \boldsymbol{S}_i\}\right] \\
\leq\; & C(M_f m_{0n,\boldsymbol{a}}^8 C_{4\sigma} + M_f^2 m_{0n,\boldsymbol{a}}^8 C_{2\sigma}^2) \\
=\; & C(M_f, C_{2\sigma}, C_{4\sigma}) m_{0n,\boldsymbol{a}}^8
\end{aligned} \tag{S3.15}$$

where the second inequality is based on (S3.11). Besides, we have

$$\begin{aligned}
|E_{\boldsymbol{S}}(U_i|\boldsymbol{S}_i)| &\leq \sum_{j=1}^{i-1} E_{\boldsymbol{S}}\{|h_n(\boldsymbol{S}_i,\boldsymbol{S}_j)| \mid \boldsymbol{S}_i\} + \sum_{j=1}^{i-1} E_{\boldsymbol{S}}\{|h_n(\boldsymbol{S}_i,\boldsymbol{S}_j)|\} \\
&\leq C_\sigma M_f m_{0n,\boldsymbol{a}}^2 + M_A C_\sigma M_f m_{0n,\boldsymbol{a}}^2,
\end{aligned}$$

where the first part in the second inequality can be derived by a similar argument in (S3.11), and the second part is obtained by Condition 2 and integration of (S3.11) over A_i. Therefore,

$$E_{\boldsymbol{S}}\left\{E_{\boldsymbol{S}}(U_i|\boldsymbol{S}_i)\right\}^4 \leq C(C_\sigma, M_f, M_A) m_{0n,\boldsymbol{a}}^8. \tag{S3.16}$$

Thus, by (S3.12)–(S3.16), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} E_{\boldsymbol{S}}\{\sigma_{n,\boldsymbol{a}} - E_{\boldsymbol{S}}(\sigma_{n,\boldsymbol{a}})\}^4 &\leq \sum_{n=1}^{\infty} C(M_f, M_A, \sigma_\Psi(\mathbf{0}), C_\sigma, C_{2\sigma}, C_{4\sigma}) n^2 m_{0n,\boldsymbol{a}}^8 \\
&< \infty,
\end{aligned} \tag{S3.17}$$

where the last equality holds based on (S3.8). Therefore, by the Borel-Cantelli Lemma (Athreya and Lahiri, 2006, Theorem 7.2.2) and Markov's inequality (Athreya and Lahiri, 2006, Proposition 6.2.4), we have proved Lemma 1. □

Theorem 1. *Suppose that Conditions 2–9 hold. For any unit vector* $\boldsymbol{a} \in \mathbb{R}^p$*,*

$$\boldsymbol{a}^\top \Lambda_n^{-1} M_n(\boldsymbol{\beta}_0) \to N(0, \sigma_{\boldsymbol{a}}^2) \tag{S3.18}$$

in distribution almost surely $(P_{\boldsymbol{S}})$*, where*

$$\sigma_{\boldsymbol{a}}^2 = \boldsymbol{a}^\top H \sigma_\Psi(\boldsymbol{0})\boldsymbol{a} + \boldsymbol{a}^\top \left\{ \int \sigma_\Psi(\boldsymbol{h}) Q(\boldsymbol{h}) \mathrm{d}\boldsymbol{h} \right\} \boldsymbol{a}.$$

Proof of Theorem 1. By Lemma 1 and Lemma 1.3 discussed by Ibragimov (1962), we could use a similar blocking argument in Lahiri (2003) to prove this theorem, and we refer readers to Lahiri (2003) for more details. □

Corollary 1. *Suppose the conditions in Theorem 1 hold. Then, we have*

$$\Lambda_n^{-1} M_n(\boldsymbol{\beta}_0) \to N(0, \Sigma_M)$$

in distribution almost surely $(P_{\boldsymbol{S}})$*, where* $\Sigma_M = H\sigma_\Psi(\boldsymbol{0}) + \int \sigma_\Psi(\boldsymbol{h}) Q(\boldsymbol{h}) \mathrm{d}\boldsymbol{h}$.

Lemma 2. *Let* $g : R_n \to \mathbb{R}$ *be a Borel measurable function satisfying* $E[|g\{Z(\boldsymbol{0})\}|] < \infty$ *and* $E[g\{Z(\boldsymbol{0})\}] = 0$ *for* $i = 1, \ldots, n$*. Also, let* $a_{in} =$

$a_{in}(\boldsymbol{S}_i)$, $i = 1, \dots, n$ be $\sigma\langle \boldsymbol{S}_i \rangle$ measurable random variables such that

$$\sum_{i=1}^{n} |a_{in}(\boldsymbol{S}_i)| = O(1), \qquad a.s.\ (P_{\boldsymbol{S}}), \tag{S3.19}$$

and

$$\sum_{i=1}^{n} a_{in}^2(\boldsymbol{S}_i) = o(1), \qquad a.s.\ (P_{\boldsymbol{S}}). \tag{S3.20}$$

Then, $\sum_{i=1}^{n} a_{in}(\boldsymbol{S}_i) g(Z(\boldsymbol{S}_i)) \to 0$ in $P_{\cdot|\boldsymbol{S}}$-probability, a.s. $(P_{\boldsymbol{S}})$.

The proof of Lemma 2 uses the similar steps as discussed by Lahiri (2003), so we omit the details.

Proof of Theorem 1. The proof mainly follows the one in Theorem 3.1 of Lahiri and Mukherjee (2004). We only give the proof for the first part, and the proof for the last two parts is the same.

First, we would like to show that, for any $b \in (0, \infty)$,

$$\sup_{\|\boldsymbol{u}\| \le b} \left\| \Lambda_n^{-1} \{ M_n(\boldsymbol{\beta}_0 + \Lambda_n^{-1} \boldsymbol{u}) - M_n(\boldsymbol{\beta}_0) \} + H E_{\cdot|\boldsymbol{S}} [\Psi' \{ Z(\boldsymbol{0}) \}] \boldsymbol{u} \right\| = o_p(1), \tag{S3.21}$$

and recall that $E_{\cdot|\boldsymbol{S}}(\cdot)$ is the conditional expectation given $\mathcal{S}_n$.

Denote $\boldsymbol{v}_i = \Lambda_n^{-1}\boldsymbol{x}(\boldsymbol{s}_i)$, so we have

$$\begin{aligned}
& \Lambda_n^{-1}\{M_n(\boldsymbol{\beta}_0 + \Lambda_n^{-1}\boldsymbol{u}) - M_n(\boldsymbol{\beta}_0)\} \\
= & \Lambda_n^{-1}\sum_{i=1}^{n}\boldsymbol{x}(\boldsymbol{s})[\Psi\{Z(\boldsymbol{s}_i) - \boldsymbol{v}_i^\top\boldsymbol{u}\} - \Psi\{Z(\boldsymbol{s}_i)\}] \\
= & \Lambda_n^{-1}\sum_{i=1}^{n}\boldsymbol{x}(\boldsymbol{s})\int_{Z(\boldsymbol{s}_i)}^{Z(\boldsymbol{s}_i)-\boldsymbol{v}_i^\top\boldsymbol{u}}\Psi'(t)\mathrm{d}t \\
= & \Lambda_n^{-1}\sum_{i=1}^{n}\boldsymbol{x}(\boldsymbol{s})\int_{0}^{-\boldsymbol{v}_i^\top\boldsymbol{u}}\Psi'\{Z(\boldsymbol{s}_i)+t\}\mathrm{d}t.
\end{aligned}$$

Denote $t_i = \sup\{\left|\boldsymbol{v}_i^\top\boldsymbol{u}\right| : \|\boldsymbol{u}\| \le b\} \le b\,\|\boldsymbol{v}_i\|$, so $t_i = o(1)$ based on Condition 7. By taking conditional expectation, we have

$$\begin{aligned}
& E_{\cdot|\boldsymbol{S}}\sup_{\|\boldsymbol{u}\|\le b}\left\|\Lambda_n^{-1}\{M_n(\boldsymbol{\beta}_0+\Lambda_n^{-1}\boldsymbol{u}) - M_n(\boldsymbol{\beta}_0)\} + \Lambda_n^{-1}\sum_{i=1}^{n}\boldsymbol{x}(\boldsymbol{s}_i)\boldsymbol{v}_i^\top\boldsymbol{u}\Psi'(Z(\boldsymbol{s}_i))\right\| \\
\le & \sum_{i=1}^{n}\|\boldsymbol{v}_i\|\int_0^{\left|\boldsymbol{v}_i^\top\boldsymbol{u}\right|}E_{\cdot|\boldsymbol{S}}\left|\Psi'\{Z(\boldsymbol{s}_i)+t\} - \Psi'\{Z(\boldsymbol{s}_i)\}\right|\mathrm{d}t \\
\le & \sum_{i=1}^{n}\|\boldsymbol{v}_i\|\int_0^{t_i}E_{\cdot|\boldsymbol{S}}\left|\Psi'\{Z(\boldsymbol{s}_i)+t\} - \Psi'\{Z(\boldsymbol{s}_i)\}\right|\mathrm{d}t \\
\le & \frac{b^{1+\gamma}}{(1+\gamma)}\sum_{i=1}^{n}\|\boldsymbol{v}_i\|^{2+\gamma} \\
= & o(1), \qquad \text{(S3.22)}
\end{aligned}$$

where C_γ is a constant, and the third inequality is based on Condition 9, and the last equality is by Condition 7.

Based on Condition 7, we have $\left|\|\boldsymbol{v}_i\|^2 - E_{\boldsymbol{S}}\|\boldsymbol{v}_i\|^2\right| < 2n^{-1/2}$ for $i = 1, \ldots, n$. Therefore, based on Bernstein's inequality (Bennett, 1962), for any $\epsilon > 0$, we have

$$P_{\boldsymbol{S}}\left(\left|\sum_{i=1}^{n}\{\|\boldsymbol{v}_i\|^2 - E_{\boldsymbol{S}}\|\boldsymbol{v}_i\|^2\}\right| > \epsilon\right) \leq \exp\left\{-O(n^{1/2})\right\},$$

where the last inequality is based on Condition 7. Thus, by the Borel-Cantelli Lemma, we have

$$\sum_{i=1}^{n}\|\boldsymbol{v}_i\|^2 - E_{\boldsymbol{S}}\sum_{i=1}^{n}\|\boldsymbol{v}_i\|^2 \to 0 \quad \text{a.s. } (P_{\boldsymbol{S}}), \tag{S3.23}$$

and

$$E_{\boldsymbol{S}}\sum_{i=1}^{n}\|\boldsymbol{v}_i\|^2 = \operatorname{tr}\left(E_{\boldsymbol{S}}\sum_{i=1}^{n}\boldsymbol{v}_i\boldsymbol{v}_i^\top\right) \to \operatorname{tr}(H), \tag{S3.24}$$

where $\operatorname{tr}(A)$ is the trace of a square matrix A, and (S3.24) is based on Condition 5. Thus, we have

$$\sum_{i=1}^{n}\left|\left(\Lambda_n^{-1}\boldsymbol{x}(\boldsymbol{s}_i)\boldsymbol{v}_i^\top\right)_{kl}\right| = O(1) \tag{S3.25}$$

almost surely, where $(A)_{kl}$ is the element in the k-th row and l-th column of a general matrix A.

By noting the fact that $\|\boldsymbol{v}_i\|^4 = o(n^{-1})$ by Condition 7, we have

$$\sum_{i=1}^{n}\left|\left(\Lambda_n^{-1}\boldsymbol{x}(\boldsymbol{s}_i)\boldsymbol{v}_i^\top\right)_{kl}\right|^2 \leq \sum_{i=1}^{n}\|\boldsymbol{v}_i\|^4 = o(1). \tag{S3.26}$$

By (S3.25), (S3.26) and Lemma 2, we have

$$\sup_{\|\boldsymbol{u}\|\leq b}\left\|\Lambda_n^{-1}\sum_{i=1}^{n}\boldsymbol{x}(\boldsymbol{s}_i)\boldsymbol{v}_i^\top\boldsymbol{u}[\Psi'\{Z(\boldsymbol{s}_i)\} - E_{\cdot|\boldsymbol{S}}\Psi'\{Z(\boldsymbol{0})\}]\right\| = o_p(1), \text{ a.s. } (P_{\boldsymbol{S}}). \tag{S3.27}$$

Based on (S3.22) to (S3.24), (S3.27) and the Markov's inequality, we have

$$\sup_{\|\boldsymbol{u}\|\leq b} \left\|\Lambda_n^{-1}\{M_n(\boldsymbol{\beta}_0+\Lambda_n^{-1}\boldsymbol{u})-M_n(\boldsymbol{\beta}_0)\}+HE_{\cdot|\boldsymbol{S}}\Psi'\{Z(\boldsymbol{0})\}\boldsymbol{u}\right\| = o_p(1) \tag{S3.28}$$

a.s. $(P_{\boldsymbol{S}})$ for $b \in (0,\infty)$.

The remaining proof is almost the same with the one shown in Theorem 3.1 of Lahiri (2004). Thus, by Lemma 1, Theorem 1, and

$$\boldsymbol{a}^\top \Lambda_n^{-1} M_n(\boldsymbol{\beta}_0)\boldsymbol{a} \xrightarrow{d} N(0, \boldsymbol{a}^\top \Sigma_M \boldsymbol{a}), \text{ a.s. } (P_{\boldsymbol{S}}), \tag{S3.29}$$

we can get Theorem 1 proved. □

S4 Proof of Theorem 2

of Theorem 2. By Conditions 6–7 and $g(\boldsymbol{s}) = \{\text{vol.}(R_0)\}^{-1}$ for $\boldsymbol{s} \in R_0$, we have, by Lahiri and Zhu (2006),

$$\lambda_n^{d/2}\Lambda_{n,iid}(\hat{\boldsymbol{\beta}}_{n,iid}-\boldsymbol{\beta}) \xrightarrow{d} N(0, \chi_0^{-2}\Sigma_{\boldsymbol{\beta},iid}), \tag{S4.30}$$

where $\hat{\boldsymbol{\beta}}_{n,iid}$ solves (2.2) based on the independent and identically distributed design associated with $g(\boldsymbol{s})$, and

$$\Sigma_{\boldsymbol{\beta},iid} = c^{-1}H_{iid}^{-1}\sigma_\Psi(\boldsymbol{0}) + H_{iid}^{-1}\{\int \sigma_\Psi(\boldsymbol{h})Q_{iid}(\boldsymbol{h})\mathrm{d}\boldsymbol{h}\}H_{iid}^{-1},$$

and recall that $n/\lambda_n^d \to c \in (0,\infty)$.

By $n/\lambda_n^d \to c$, (S4.30) and Slutsky's theorem (Athreya and Lahiri, 2006), we have

$$\sqrt{n}\Lambda_{n,iid}(\hat{\boldsymbol{\beta}}_{n,iid} - \boldsymbol{\beta}) \xrightarrow{d} N(0, c\chi_0^{-2}\Sigma_{\boldsymbol{\beta},iid}) \quad \text{a.s. } (P_{iid}), \tag{S4.31}$$

where P_{iid} is the probability measure for the independent and identically distributed sampling design.

First, we show that the first asymptotic property in Condition 5 holds under the special one-per-stratum sampling design. Consider

$$\begin{aligned} & \sum_{i=1}^{n} \int \boldsymbol{x}(\boldsymbol{s})\boldsymbol{x}(\boldsymbol{s})^\top f_i(\boldsymbol{s})\mathrm{d}\boldsymbol{s} \\ = & \frac{n}{\lambda_n^d \mathrm{vol.}(R_0)} \int_{R_n} \boldsymbol{x}(\boldsymbol{s})\boldsymbol{x}(\boldsymbol{s})^\top \mathrm{d}\boldsymbol{s} \\ = & \frac{n}{\mathrm{vol.}(R_0)} \int_{R_0} \boldsymbol{x}(\lambda_n\boldsymbol{s})\boldsymbol{x}(\lambda_n\boldsymbol{s})^\top \mathrm{d}\boldsymbol{s}. \end{aligned} \tag{S4.32}$$

By (3.3) and (S4.32), we have

$$\Lambda_n^{-1} \left\{ \sum_{i=1}^{n} \int \boldsymbol{x}(\boldsymbol{s})\boldsymbol{x}(\boldsymbol{s})^\top f_i(\boldsymbol{s})\mathrm{d}\boldsymbol{s} \right\} \Lambda_n^{-1} \to H_{iid} \quad \text{as } n \to \infty,$$

where $\Lambda_n = \sqrt{n}\Lambda_{n,iid}$.

Next, for $\boldsymbol{h} \in \mathbb{R}^d$, consider

$$
\begin{aligned}
& \sum_{i=1}^{n} \sum_{j \neq i} \int \boldsymbol{x}(\boldsymbol{y}+\boldsymbol{h}) \boldsymbol{x}(\boldsymbol{y})^{\top} f_i(\boldsymbol{y}+\boldsymbol{h}) f_j(\boldsymbol{y}) \mathrm{d} \boldsymbol{y} \\
= & \left\{\frac{n}{\lambda_n^d \operatorname{vol.}(R_0)}\right\}^2 \sum_{i=1}^{n} \sum_{j \neq i} \int_{A_j \cap (A_i - \boldsymbol{h})} \boldsymbol{x}(\boldsymbol{y}+\boldsymbol{h}) \boldsymbol{x}(\boldsymbol{y})^{\top} \mathrm{d} \boldsymbol{y} \\
= & \left\{\frac{n}{\lambda_n^d \operatorname{vol.}(R_0)}\right\}^2 \sum_{i=1}^{n} \int_{R_n \cap \{A_i^{\mathrm{C}} \cap (A_i - \boldsymbol{h})\}} \boldsymbol{x}(\boldsymbol{y}+\boldsymbol{h}) \boldsymbol{x}(\boldsymbol{y})^{\top} \mathrm{d} \boldsymbol{y} \\
= & \left\{\frac{n}{\lambda_n^d \operatorname{vol.}(R_0)}\right\}^2 \sum_{i=1}^{n} \left(\int_{R_n \cap (A_i - \boldsymbol{h})} - \int_{A_i \cap (A_i - \boldsymbol{h})}\right) \boldsymbol{x}(\boldsymbol{y}+\boldsymbol{h}) \boldsymbol{x}(\boldsymbol{y})^{\top} \mathrm{d} \boldsymbol{y} \\
= & \left\{\frac{n}{\lambda_n^d \operatorname{vol.}(R_0)}\right\}^2 \left\{\int_{R_n} \boldsymbol{x}(\boldsymbol{y}+\boldsymbol{h}) \boldsymbol{x}(\boldsymbol{y})^{\top} \mathrm{d} \boldsymbol{y} - \int_{\cup_{i=1}^{n} \{A_i \cap (A_i - \boldsymbol{h})\}} \boldsymbol{x}(\boldsymbol{y}+\boldsymbol{h}) \boldsymbol{x}(\boldsymbol{y})^{\top} \mathrm{d} \boldsymbol{y}\right\} \\
= & \frac{n^2}{\lambda_n^d} \frac{1}{\{\operatorname{vol.}(R_0)\}^2} \int_{R_0} \boldsymbol{x}(\lambda_n \boldsymbol{y}+\boldsymbol{h}) \boldsymbol{x}(\lambda_n \boldsymbol{y})^{\top} \mathrm{d} \boldsymbol{y} \\
& - \left\{\frac{n}{\lambda_n^d \operatorname{vol.}(R_0)}\right\}^2 \int_{\cup_{i=1}^{n} \{A_i \cap (A_i - \boldsymbol{h})\}} \boldsymbol{x}(\boldsymbol{y}+\boldsymbol{h}) \boldsymbol{x}(\boldsymbol{y})^{\top} \mathrm{d} \boldsymbol{y},
\end{aligned} \tag{S4.33}
$$

where A^{C} is the complement of set A.

By (3.4), we have

$$
\begin{aligned}
& \Lambda_n^{-1} \left\{\sum_{i=1}^{n} \sum_{j \neq i} \int \boldsymbol{x}(\boldsymbol{y}+\boldsymbol{h}) \boldsymbol{x}(\boldsymbol{y})^{\top} f_i(\boldsymbol{y}+\boldsymbol{h}) f_j(\boldsymbol{y}) \mathrm{d} \boldsymbol{y}\right\} \Lambda_n^{-1} \\
= & c \Lambda_{n,iid}^{-1} \left\{\frac{1}{\{\operatorname{vol.}(R_0)\}^2} \int_{R_0} \boldsymbol{x}(\lambda_n \boldsymbol{y}+\boldsymbol{h}) \boldsymbol{x}(\lambda_n \boldsymbol{y})^{\top} \mathrm{d} \boldsymbol{y}\right\} \Lambda_{n,iid}^{-1} \\
& - \Lambda_n^{-1} \left[\left\{\frac{n}{\lambda_n^d \operatorname{vol.}(R_0)}\right\}^2 \int_{\cup_{i=1}^{n} \{A_i \cap (A_i - \boldsymbol{h})\}} \boldsymbol{x}(\boldsymbol{y}+\boldsymbol{h}) \boldsymbol{x}(\boldsymbol{y})^{\top} \mathrm{d} \boldsymbol{y}\right] \Lambda_n^{-1}
\end{aligned} \tag{S4.34}
$$

for $\boldsymbol{h} \in \mathbb{R}^d$.

Based on Theorem 1, we have

$$\sqrt{n}\Lambda_{n,iid}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \xrightarrow{d} N(0, \chi_0^{-2}\Sigma_{\boldsymbol{\beta}}) \quad \text{a.s. } (P_{\boldsymbol{S}}) \tag{S4.35}$$

where $\Sigma_{\boldsymbol{\beta}} = H_{iid}^{-1}\sigma_\Psi(\mathbf{0}) + H_{iid}^{-1}\{\int \sigma_\Psi(\boldsymbol{h})Q(\boldsymbol{h})\mathrm{d}\boldsymbol{h}\}H_{iid}^{-1}$. Thus, by (S4.30), (S4.34) and the fact that the limit of

$$\Lambda_n^{-1}\left[\{n^2\lambda_n^{-2d}\text{vol.}(R_0)^{-2}\}\int_{\cup_{i=1}^n\{A_i\cap(A_i-\boldsymbol{h})\}}\boldsymbol{x}(\boldsymbol{y}+\boldsymbol{h})\boldsymbol{x}(\boldsymbol{y})^\top\mathrm{d}\boldsymbol{y}\right]\Lambda_n^{-1}$$

is positive definitive, we have proved Theorem 2. □

S5 Proof of Theorem 3

Lemma 3. *Suppose that Conditions 1–9 hold. Then,*

$$\left\|\hat{\Sigma}_n - \Sigma_M\right\| \to 0 \quad \text{in } P_{\cdot|\boldsymbol{S}}\text{-probability, a.s. } (P_{\boldsymbol{S}}), \tag{S5.36}$$

where $\hat{\Sigma}_n = \sum_{k\in\mathcal{K}_n} V_*\{\Lambda_n^{-1}S_n^*(\boldsymbol{k},\hat{\boldsymbol{\beta}}_n)\}$, *and recall that* $\Sigma_M = H\sigma_\Psi(\mathbf{0}) + \int \sigma_\Psi(\boldsymbol{h})Q(\boldsymbol{h})\mathrm{d}\boldsymbol{h}$.

Proof of Lemma 3. The argument here is the based on the proof of Lemma 3 of Lahiri and Zhu (2006), and *we only consider the case where* $p = 1$. For higher dimensional space, similar argument can be made.

Denote $\tilde{S}_n(\boldsymbol{l};\boldsymbol{k}) = \sum_{i=1}^n \boldsymbol{x}(\boldsymbol{s}_i)\Psi\{Z(\boldsymbol{s}_i)\}\mathbb{1}(\boldsymbol{s}_i \in B_n(\boldsymbol{l};\boldsymbol{k}))$, where $\boldsymbol{l} \in l_n$ and $\boldsymbol{k} \in \mathcal{K}_n$. Let

$$\tilde{\Sigma}_n = \sum_{\boldsymbol{k}\in\mathcal{K}_n}(|l_n|^{-1}\sum_{\boldsymbol{l}\in l_n}\{\Lambda_n^{-1}\tilde{S}_n(\boldsymbol{l};\boldsymbol{k})\}^2 - [|l_n|^{-1}\sum_{\boldsymbol{l}\in l_n}\{\Lambda_n^{-1}\tilde{S}_n(\boldsymbol{l};\boldsymbol{k})\}]^2).$$

Thus, by Condition 3, Condition 9 and Theorem 1, we have

$$\hat{\Sigma}_n - \tilde{\Sigma}_n \to 0 \quad \text{in } P_{\cdot|\boldsymbol{S}}\text{-probability, a.s. } (P_{\boldsymbol{S}}), \tag{S5.37}$$

and recall that

$$\hat{\Sigma}_n = \sum_{\boldsymbol{k}\in\mathcal{K}_n} (|l_n|^{-1} \sum_{\boldsymbol{l}\in l_n} \{\Lambda_n^{-1}\hat{S}_n(\boldsymbol{l};\boldsymbol{k})\}^2 - [|l_n|^{-1} \sum_{\boldsymbol{l}\in l_n} \{\Lambda_n^{-1}\hat{S}_n(\boldsymbol{l};\boldsymbol{k})\}]^2),$$

where $\hat{S}_n(\boldsymbol{l};\boldsymbol{k}) = \sum_{i=1}^n \boldsymbol{x}(\boldsymbol{s}_i)\Psi\{\hat{Z}(\boldsymbol{s}_i)\}\mathbb{1}\{\boldsymbol{s}_i \in B_n(\boldsymbol{l};\boldsymbol{k})\}$.

By Lemma 2 of Lahiri and Zhu (2006) and Condition 3, we have

$$\begin{aligned} \sum_{\boldsymbol{k}\in\mathcal{K}_n} E_{\cdot|\boldsymbol{S}} \left(|l_n|^{-1} \sum_{\boldsymbol{l}\in l_n} [\{\Lambda_n^{-1}\tilde{S}_n(\boldsymbol{l};\boldsymbol{k})\}^2 - E_{\cdot|\boldsymbol{S}}\{\Lambda_n^{-1}\tilde{S}_n(\boldsymbol{l};\boldsymbol{k})\}^2] \right)^2 &= o(1), \\ \sum_{\boldsymbol{k}\in\mathcal{K}_n} E_{\cdot|\boldsymbol{S}} \left\{ |l_n|^{-1} \sum_{\boldsymbol{l}\in l_n} \Lambda_n^{-1}\tilde{S}_n(\boldsymbol{l};\boldsymbol{k}) \right\}^2 &= o(1). \end{aligned}$$

Thus, it remains to show

$$E_{\boldsymbol{S}} \left[\sum_{\boldsymbol{k}\in\mathcal{K}_n} |l_n|^{-1} \sum_{\boldsymbol{l}\in l_n} E_{\cdot|\boldsymbol{S}} \left\{ \Lambda_n^{-1}\tilde{S}_n(\boldsymbol{l};\boldsymbol{k}) \right\}^2 \right] \to \Sigma_M, \tag{S5.38}$$

$$\sum_{\boldsymbol{k}\in\mathcal{K}_n} |l_n|^{-1} \sum_{\boldsymbol{l}\in l_n} E_{\cdot|\boldsymbol{S}} \left\{ \Lambda_n^{-1}\tilde{S}_n(\boldsymbol{l};\boldsymbol{k}) \right\}^2 \to E_{\boldsymbol{S}} \left[\sum_{\boldsymbol{k}\in\mathcal{K}_n} |l_n|^{-1} \sum_{\boldsymbol{l}\in l_n} E_{\cdot|\boldsymbol{S}} \left\{ \Lambda_n^{-1}\tilde{S}_n(\boldsymbol{l};\boldsymbol{k}) \right\}^2 \right] \tag{S5.39}$$

almost surely as $n \to \infty$. Notice that the proof of (S5.39) is similar with the one in Lemma 1, so we only show (S5.38). Denote

$$\tilde{\Sigma}_{jn} = \sum_{\boldsymbol{k}\in\mathcal{K}_{jn}} |l_n|^{-1} \sum_{\boldsymbol{l}\in l_n} E_{\cdot|\boldsymbol{S}} \left(\Lambda_n^{-1}\tilde{S}_n(\boldsymbol{l};\boldsymbol{k}) \right)^2$$

for $j = 1, 2$. Then,

$$
\begin{aligned}
& E_{\boldsymbol{S}}\left[\sum_{\boldsymbol{k}\in\mathcal{K}_{1n}} |l_n|^{-1} \sum_{\boldsymbol{l}\in l_n} E_{\cdot|\boldsymbol{S}}\left\{\Lambda_n^{-1}\tilde{S}_n(\boldsymbol{l};\boldsymbol{k})\right\}^2\right] \\
= \; & |\mathcal{K}_{1n}|\,|l_n|^{-1} \sum_{\boldsymbol{l}\in l_n}\left[\sum_{i=1}^{n} E_{\boldsymbol{S}}(v_i^2)\sigma_{\Psi}(\boldsymbol{0})\mathbb{1}\{\boldsymbol{s}_i \in B_n(\boldsymbol{l};\boldsymbol{0})\}\right. \\
& \left. + \sum_{i=1}^{n}\sum_{j\neq i} E_{\boldsymbol{S}}(v_i v_j)\sigma_{\Psi}(\boldsymbol{s}_i - \boldsymbol{s}_j)\mathbb{1}\{\boldsymbol{s}_i, \boldsymbol{s}_j \in B_n(\boldsymbol{l};\boldsymbol{0})\}\right] \\
= \; & \Sigma_{11n} + \Sigma_{12n}, \text{ say .}
\end{aligned}
$$

Notice that $|\mathcal{K}_{1n}| = \lambda_n^d b_n^{-d}\text{vol.}(R_0)(1 + o(1))$ and $|l_n| = \lambda_n^d\text{vol.}(R_0)(1 + o(1))$. Denote $R_{2n} = \cup_{\boldsymbol{k}\in(\mathcal{K}_{1n}\cap R_{1n})} R_n(\boldsymbol{k})$, where $R_{1n} = \lambda_n(R_0 \setminus R_0^{b_n\lambda_n^{-1}})$. It can be shown that $\left|\{\boldsymbol{l} \in l_n : \boldsymbol{s} \in \boldsymbol{l} + b_n[0, 1]^d\}\right| = b_n^d\{1 + o(1)\}$ for $\boldsymbol{s} \in R_{2n}$. By Condition 1 and Condition 5, we have

$$
\begin{aligned}
& \Sigma_{11n} \\
= \; & \Lambda_n^{-1}\frac{|\mathcal{K}_{1n}|}{|l_n|}\sigma_{\Psi}(\boldsymbol{0})\left[\sum_{i=1}^{n}\int_{R_{2n}} w_n^2(\boldsymbol{s}) f_i(\boldsymbol{s}) \sum_{\boldsymbol{l}\in l_n}\mathbb{1}\{\boldsymbol{s} \in B_n(\boldsymbol{l};\boldsymbol{0})\}\mathrm{d}\boldsymbol{s}\right]\Lambda_n^{-1}(1 + o(1)) \\
= \; & \sigma_{\Psi}(\boldsymbol{0})H(1 + o(1)). \qquad \text{(S5.40)}
\end{aligned}
$$

Similarly, we have

$$\begin{aligned}
&\Sigma_{12n} \\
=\ & \Lambda_n^{-1}\frac{|\mathcal{K}_{1n}|}{|l_n|}\left[\sum_{i=1}^{n}\sum_{j\neq i}\int\int w_n(\boldsymbol{x})w_n(\boldsymbol{y})f_i(\boldsymbol{x})f_j(\boldsymbol{y})\sigma_\Psi(\boldsymbol{x}-\boldsymbol{y})\right. \\
& \left.\times\sum_{\boldsymbol{l}\in l_n}\mathbb{1}\{\boldsymbol{x},\boldsymbol{y}\in B_n(\boldsymbol{l};\boldsymbol{0})\}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right]\Lambda_n^{-1} \\
=\ & \Lambda_n^{-1}\frac{|\mathcal{K}_{1n}|}{|l_n|}\left[\sum_{i=1}^{n}\sum_{j\neq i}\int_{\|\boldsymbol{h}\|\leq b_n}\sigma_\Psi(\boldsymbol{h})\int_{R_{2n}}w_n(\boldsymbol{y}+\boldsymbol{h})w_n(\boldsymbol{y})f_i(\boldsymbol{y}+\boldsymbol{h})f_j(\boldsymbol{y})\right. \\
& \left.\times\sum_{\boldsymbol{l}\in l_n}\mathbb{1}\{\boldsymbol{y}+\boldsymbol{h},\boldsymbol{y}\in B_n(\boldsymbol{l};\boldsymbol{0})\}\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{h}\right]\Lambda_n^{-1}(1+o(1)) \\
=\ & \int\sigma_\Psi(\boldsymbol{h})Q(\boldsymbol{h})\mathrm{d}\boldsymbol{h}(1+o(1)). \qquad \text{(S5.41)}
\end{aligned}$$

By (S5.40) and (S5.41), we have shown (S5.38), which completes the proof.

□

Proof of Theorem 3. The proof of this theorem extends the one discussed by Lahiri and Zhu (2006) to the proposed sampling design. For convenience, denote $\Phi(\cdot;\Sigma)$ to be the probability measure of $N(\boldsymbol{0},\Sigma)$. Based on Condition 9 and the Taylor's expansion, we have

$$\begin{aligned}
0 &= \sum_{k\in\mathcal{K}_n}\{S_n^*(\boldsymbol{k};\boldsymbol{t})-\hat{c}_n(\boldsymbol{k})\} \\
&= \sum_{k\in\mathcal{K}_n}\{S_n^*(\boldsymbol{k};\hat{\boldsymbol{\beta}}_n)-\hat{c}_n(\boldsymbol{k})\}+\Lambda_n\Gamma_n\lambda_n(\boldsymbol{t}-\hat{\boldsymbol{\beta}}_n)\chi_0+R_n^*(\boldsymbol{t}),
\end{aligned} \qquad \text{(S5.42)}$$

where $\boldsymbol{t}$ is a solution of the first equality, $\Gamma_n = \sum_{i=1}^n \boldsymbol{v}_i \boldsymbol{v}_i^\top$, and $R_n^*(\boldsymbol{t})$ is obtained by subtraction. To be more specific, we have

$$R_n^*(\boldsymbol{t}) = \{R_{1n}^*(\boldsymbol{t}) + R_{2n}^*(\boldsymbol{t}) + R_{3n}^*(\boldsymbol{t})\}\Lambda_n(\boldsymbol{t} - \hat{\boldsymbol{\beta}}_n),$$

where

$$\begin{aligned}
R_{1n}^*(\boldsymbol{t}) &= \sum_{i=1}^n \boldsymbol{x}(\boldsymbol{s}_i)\boldsymbol{x}(\boldsymbol{s}_i)^\top \Lambda_n^{-1} - \sum_{k\in\mathcal{K}_n}\sum_{i=1}^n \boldsymbol{x}(\boldsymbol{s}_i)\boldsymbol{x}(\boldsymbol{s}_i)^\top \Lambda_n^{-1}\mathbb{1}\{\boldsymbol{s}_i \in B_n(I_{\boldsymbol{k}};\boldsymbol{k})\},\\
R_{2n}^*(\boldsymbol{t}) &= \sum_{k\in\mathcal{K}_n}\sum_{i=1}^n \boldsymbol{x}(\boldsymbol{s}_i)\boldsymbol{x}(\boldsymbol{s}_i)^\top \Lambda_n^{-1}\mathbb{1}\{\boldsymbol{s}_i \in B_n(I_{\boldsymbol{k}};\boldsymbol{k})\}\\
&\quad \times \int_0^1 [\Psi'\{\hat{Z}(\boldsymbol{s}_i) - u\boldsymbol{x}(\boldsymbol{s}_i)^\top(\boldsymbol{t} - \hat{\boldsymbol{\beta}}_n)\} - \Psi'\{\hat{Z}(\boldsymbol{s}_i)\}]\mathrm{d}u,\\
R_{3n}^*(\boldsymbol{t}) &= \sum_{k\in\mathcal{K}_n}\sum_{i=1}^n \boldsymbol{x}(\boldsymbol{s}_i)\boldsymbol{x}(\boldsymbol{s}_i)^\top \Lambda_n^{-1}\mathbb{1}\{\boldsymbol{s}_i \in B_n(I_{\boldsymbol{k}};\boldsymbol{k})\}\left[\Psi'\{\hat{Z}(\boldsymbol{s}_i)\} - E\Psi'\{Z(\boldsymbol{0})\}\right]\\
&= \sum_{k\in\mathcal{K}_n}\sum_{i=1}^n \boldsymbol{x}(\boldsymbol{s}_i)\boldsymbol{x}(\boldsymbol{s}_i)^\top \Lambda_n^{-1}\mathbb{1}\{\boldsymbol{s}_i \in B_n(I_{\boldsymbol{k}};\boldsymbol{k})\}\left[\Psi'\{Z(\boldsymbol{s}_i)\} - E\Psi'\{Z(\boldsymbol{0})\}\right]\\
&\quad + o_p(1),
\end{aligned}$$

where the second equality of $R_{3n}^*(\boldsymbol{t})$ holds by Condition 9, (S3.23), (S3.24) and Theorem 1. Besides, based on (S3.23) and (S3.24), we have

$$\Gamma_n = H + o_p(1), \quad \text{a.s. } (P_{\boldsymbol{S}}). \tag{S5.43}$$

By a similar argument in the proof of Theorem 2 (Lahiri and Zhu, 2006) and Lemma 3, we have, for any $\epsilon_0 > 0$,

$$P_{\cdot|\boldsymbol{S}}\left(\sup_{B\in\mathcal{C}}\left|P_*\left[\Lambda_n^{-1}\sum_{k\in\mathcal{K}_n}\{S_n^*(\boldsymbol{k};\hat{\boldsymbol{\beta}}_n) - \hat{c}_n(\boldsymbol{k})\}\in B\right] - \Phi(B;\Sigma_{\boldsymbol{\beta}})\right| > \epsilon_0\right) = o(1), \tag{S5.44}$$

a.s. $(P_{\boldsymbol{S}})$. Now, it remains to prove, for any $\epsilon_n \downarrow 0$,

$$P_{\cdot|\boldsymbol{S}}\left(P_*\left[\left\|\Lambda_n^{-1}\left\{R_{1n}^*(\boldsymbol{t})+R_{2n}^*(\boldsymbol{t})+R_{3n}^*(\boldsymbol{t})\right\}\right\|>\epsilon_n\right]>\epsilon_0\right)=o(1), \quad \text{a.s. } (P_{\boldsymbol{S}}). \tag{S5.45}$$

First, we show

$$E_{\cdot|\boldsymbol{S}}E_*\left\|\Lambda_n^{-1}\left\{R_{1n}^*(\boldsymbol{t})+R_{2n}^*(\boldsymbol{t})+R_{3n}^*(\boldsymbol{t})\right\}\right\|=o(1) \tag{S5.46}$$

with some $\boldsymbol{t}$ such that $\left\|\Lambda_n(\boldsymbol{t}-\hat{\boldsymbol{\beta}}_n)\right\|=O(1)$.

First, consider $\Lambda_n^{-1}R_{1n}^*(\boldsymbol{t})$.

$$\begin{aligned}
& E_*\left[\sum_{\boldsymbol{k}\in\mathcal{K}_{1n}}\sum_{i=1}^n \boldsymbol{v}_i\boldsymbol{v}_i^\top \mathbb{1}(\boldsymbol{s}_i\in R_{2n})\mathbb{1}\{\boldsymbol{s}_i\in B(I_{\boldsymbol{k}};\boldsymbol{k})\}\right]\\
=\;& E_*\left[\sum_{i=1}^n\sum_{\boldsymbol{k}\in\mathcal{K}_{1n}} \boldsymbol{v}_i\boldsymbol{v}_i^\top \mathbb{1}(\boldsymbol{s}_i\in R_{2n})\mathbb{1}\{\boldsymbol{s}_i\in B(I_{\boldsymbol{k}};\boldsymbol{k})\}\right]\\
=\;& \sum_{i=1}^n \boldsymbol{v}_i\boldsymbol{v}_i^\top \mathbb{1}(\boldsymbol{s}_i\in R_{2n})\left|\mathcal{K}_{1n}\right|\frac{b_n^d(1+o(1))}{|l_n|}\\
=\;& \sum_{i=1}^n \boldsymbol{v}_i\boldsymbol{v}_i^\top \mathbb{1}(\boldsymbol{s}_i\in R_{2n})(1+o(1)).
\end{aligned} \tag{S5.47}$$

Besides, by Condition 1 and Condition 7, we have

$$\begin{aligned}
&\sum_{\boldsymbol{k}\in\mathcal{K}_{2n}}\sum_{i=1}^n \|\boldsymbol{v}_i\|\,\mathbb{1}\{\boldsymbol{s}_i\in B(I_{\boldsymbol{k}};\boldsymbol{k})\}\\
&+\sum_{\boldsymbol{k}\in\mathcal{K}_{1n}}\sum_{i=1}^n \|\boldsymbol{v}_i\|\,\mathbb{1}(\boldsymbol{s}_i\notin R_{2n})\mathbb{1}\{\boldsymbol{s}_i\in B(I_{\boldsymbol{k}};\boldsymbol{k})\}=o(1).
\end{aligned} \tag{S5.48}$$

Thus, by (S3.23), (S3.24), (S5.47) and (S5.48), we have

$$E_*\left[\sum_{\boldsymbol{k}\in\mathcal{K}_n}\sum_{i=1}^n \boldsymbol{v}_i\boldsymbol{v}_i^\top \mathbb{1}\{\boldsymbol{s}_i\in B(I_{\boldsymbol{k}};\boldsymbol{k})\}\right]=\Gamma_n+o(1). \tag{S5.49}$$

Denote $\boldsymbol{e}_l$ to be the vector such that all the elements are 0 except that its l-th one is 1, and $l = 1, \ldots, n$. For any $\boldsymbol{e}_l$ and $\boldsymbol{e}_j$,

$$\begin{aligned}
& V_* \left[\boldsymbol{e}_l^\top \sum_{\boldsymbol{k} \in \mathcal{K}_n} \sum_{i=1}^n \boldsymbol{v}_i \boldsymbol{v}_i^\top \mathbb{1}(\boldsymbol{s}_i \in B(I_{\boldsymbol{k}}; \boldsymbol{k})) \boldsymbol{e}_j \right] \\
= \; & |\mathcal{K}_n| \, V_* \left[\boldsymbol{e}_l^\top \sum_{i=1}^n \boldsymbol{v}_i \boldsymbol{v}_i^\top \mathbb{1}(\boldsymbol{s}_i \in B(I_{\boldsymbol{k}}; \boldsymbol{k})) \boldsymbol{e}_j \right] \\
\leq \; & C \, |\mathcal{K}_n| \, E_* \left[\sum_{i=1}^n \|\boldsymbol{v}_i\|^4 \, \mathbb{1}(\boldsymbol{s}_i \in B(I_{\boldsymbol{k}}; \boldsymbol{k})) \right] \\
\leq \; & C \, |\mathcal{K}_n| \sum_{i=1}^n \|\boldsymbol{v}_i\|^4 \, b_n^d / \, |l_n| \\
= \; & o(1),
\end{aligned} \tag{S5.50}$$

where C is a constant, and the last equality holds by Condition 3 and Condition 7.

Thus, by (S5.49) and (S5.50), we have

$$E_* \left\| \Lambda_n^{-1} R_{1n}^*(\boldsymbol{t}) \right\| = o(1). \tag{S5.51}$$

Next, we consider $\Lambda_n^{-1} R_{2n}^*(\boldsymbol{t})$. Since

$$\left\| \Lambda_n^{-1} R_{2n}^*(\boldsymbol{t}) \right\| \leq \sum_{k \in \mathcal{K}_n} \sum_{i=1}^n \|\boldsymbol{v}_i\|^{2+\gamma} \, \mathbb{1}\{\boldsymbol{s}_i \in B_n(I_{\boldsymbol{k}}; \boldsymbol{k})\} \left\| \Lambda_n(\boldsymbol{t} - \hat{\boldsymbol{\beta}}_n) \right\|^\gamma,$$

we have

$$E_* \left\| \Lambda_n^{-1} R_{2n}^*(\boldsymbol{t}) \right\| = o\,(1)\,. \tag{S5.52}$$

where the result holds based on (S3.22), and recall that $\left\| \Lambda_n(\boldsymbol{t} - \hat{\boldsymbol{\beta}}_n) \right\| = O(1)$.

Now, we consider $\Gamma_n^{-1}R_{3n}^*(\boldsymbol{t})$. Denote $W_{jl}(\boldsymbol{s}_i) = \boldsymbol{e}_j^\top \boldsymbol{v}_i \boldsymbol{v}_i^\top \boldsymbol{e}_l [\Psi'\{Z(\boldsymbol{s}_i)\} - \chi_0]$ for $j, l = 1, \ldots, p$.

$$
\begin{aligned}
& E_{\cdot|\boldsymbol{S}}\left(V_*\left[\sum_{\boldsymbol{k}\in\mathcal{K}_n}\sum_{i=1}^n W_{jl}(\boldsymbol{s}_i)\mathbb{1}\{\boldsymbol{s}_i \in B(I_{\boldsymbol{k}};\boldsymbol{k})\}\right]\right) \\
\leq\ & E_{\cdot|\boldsymbol{S}}\left(E_*\left[\sum_{\boldsymbol{k}\in\mathcal{K}_n}\sum_{i=1}^n W_{jl}(\boldsymbol{s}_i)\mathbb{1}\{\boldsymbol{s}_i \in B(I_{\boldsymbol{k}};\boldsymbol{k})\}\right]^2\right) \\
=\ & |l_n|^{-1} E_{\cdot|\boldsymbol{S}}\left(\sum_{\boldsymbol{k}\in\mathcal{K}_n}\sum_{\boldsymbol{x}\in l_n}\left[\sum_{i=1}^n W_{jl}(\boldsymbol{s}_i)\mathbb{1}\{\boldsymbol{s}_i \in B(\boldsymbol{x};\boldsymbol{k})\}\right]^2\right) \\
=\ & o(1),
\end{aligned}
\tag{S5.53}
$$

where the last equality holds based on the result in Lemma 2 of Lahiri and Zhu (2006) by setting $m_n = b_n^d$ based on Condition 3.

Thus, by (S5.51), (S5.52) and (S5.53), we have (S5.46) holds. Therefore, we have

$$
\left\|\Lambda_n^{-1}R_n^*(\boldsymbol{t})\right\| \leq o(1)\left\|\Lambda(\boldsymbol{t}-\hat{\boldsymbol{\beta}}_n)\right\| \tag{S5.54}
$$

for some $\boldsymbol{t}$ such that $\left\|\Lambda(\boldsymbol{t}-\hat{\boldsymbol{\beta}}_n)\right\| = O(1)$.

By Markov's inequality, we can prove (S5.45). Together with (S5.44), Theorem 3 is proved. □

S6 Simulation results by simple linear regression

For comparison, we also consider a naive method using simple linear regression to make inference for the regression parameters. The square root of mean square error and the relative bias for the variance estimator and the coverage rate of the 90% confidence interval, obtained by the Wald method, is summarized in Table 1. When the spatial dependence is weak, reasonable results can be obtained using the simple linear regression since the square root of mean square error and the relative bias for the variance estimator is comparable with those of the resampling method, and the coverage rate is close to 90%. As the spatial dependence becomes stronger, however, the variance is severely underestimated, and the coverage rate is much lower than 90% for both sampling designs.

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Table 1: Summary statistics for the the variance estimator of β_0 and β_1 by the simple linear regression model under the proposed sampling design for different scenarios. "RMES" stands for square root of the mean square error, "RB" for relative bias, "CR" for coverage rate, † for optimal block size, "Uniform" for uniform density function, and "Normal" for bivariate normal density function.

Design	Dependence	Statistics	$n = 400$		$n = 900$	
			β_0	β_1	β_0	β_1
Unifrom	$r = 1$	RMSE	0.53	0.14	0.20	0.04
		RB	-0.18	-0.17	-0.15	-0.13
		CR	0.86	0.86	0.88	0.88
	$r = 3$	RMSE	2.57	0.70	1.25	0.24
		RB	-0.54	-0.53	-0.53	-0.51
		CR	0.72	0.73	0.74	0.74
Normal	$r = 1$	RMSE	0.29	0.07	0.06	0.02
		RB	-0.09	-0.07	-0.01	-0.04
		CR	0.88	0.88	0.90	0.89
	$r = 3$	RMSE	2.49	0.65	1.12	0.24
		RB	-0.52	-0.50	-0.50	-0.51
		CR	0.75	0.75	0.75	0.74