Statistica Sinica

SUPPLEMENTARY MATERIAL

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The supplementary material contains the proofs of the theoretical results presented in Section 2 - Sections 4, as well as some additional simulation results.

S1 Appendix A: Technical Details

Proof of Proposition 1

Obviously, $CD(X,Y)$ is nonnegative and equal to zero if X and Y are draw from the single distribution. Next, we will verify that if $CD(X,Y)=0$, then X and Y are identically distributed. Recall

$$
\begin{aligned}
& CD(X,Y) \\
&= E\left[\left\|E\left(e^{i\frac{\langle X'', X-X'\rangle}{\sqrt{Var\langle X'', X-X'\rangle}}}\middle| X-X'\right)-E\left(e^{i\frac{\langle Y, X-X'\rangle}{\sqrt{Var\langle X'', X-X'\rangle}}}\middle| X-X'\right)\right\|^2\right] \\
&+ E\left[\left\|E\left(e^{i\frac{\langle X, Y-Y'\rangle}{\sqrt{Var\langle Y'', Y-Y'\rangle}}}\middle| Y-Y'\right)-E\left(e^{i\frac{\langle Y'', Y-Y'\rangle}{\sqrt{Var\langle Y'', Y-Y'\rangle}}}\middle| Y-Y'\right)\right\|^2\right] \\
&:= A+C,
\end{aligned}
$$

thus $CD(X,Y)=0$ implies $A=0$ and $C=0$.

Let $F(\mathbf{s})$ be the distribution function of $X-X'$, and S_1 is the support of $X-X'$ consisting of the points such that $F(\mathbf{s})>0$, then

$$E\left[\left\|E\left(e^{i\frac{\langle X'',X-X'\rangle}{\sqrt{Var\langle X'',X-X'\rangle}}}\middle| X-X'\right)-E\left(e^{i\frac{\langle Y,X-X'\rangle}{\sqrt{Var\langle X'',X-X'\rangle}}}\middle| X-X'\right)\right\|^2\right]$$
$$=\int_{S_1}\left\|E\left(e^{i\frac{\langle X'',X-X'\rangle}{\sqrt{Var\langle X'',X-X'\rangle}}}\middle| X-X'=\mathbf{s}\right)-E\left(e^{i\frac{\langle Y,X-X'\rangle}{\sqrt{Var\langle X'',X-X'\rangle}}}\middle| X-X'=\mathbf{s}\right)\right\|^2 dF(\mathbf{s}).$$

Next, set $\mathbf{t}=\mathbf{s}/\sqrt{Var\langle X'',X-X'\rangle}$, and let S be the support consisting of the points such that $F(\mathbf{t})>0$, X,X',X'' and Y are mutually independent indicating that

$$A=\int_S\left\|e^{i\langle X,\mathbf{t}\rangle}-e^{i\langle Y,\mathbf{t}\rangle}\right\|^2 dF(\mathbf{t})=0.$$

Since $\left\|e^{i\langle X,\mathbf{t}\rangle}-e^{i\langle Y,\mathbf{t}\rangle}\right\|^2$ is nonnegative, we have

$$\left\|e^{i\langle X,\mathbf{t}\rangle}-e^{i\langle Y,\mathbf{t}\rangle}\right\|^2=0,\quad a.s.$$

Therefore

$$e^{i\langle X,\mathbf{t}\rangle}=e^{i\langle Y,\mathbf{t}\rangle},\quad a.s.$$

That is, X and Y are identically distributed. Similarly, we can verify that X and Y are draw from the single distribution if $C=0$.

This concludes the proof.

Proof of Proposition 2

By the definition of $U_{n,m}$ and $T_{n,m}$, it follows that proving $U_{n,m}=T_{n,m}$ is equivalent to proving $A_{n,m}=\widetilde{A}_{n,m}$, and $C_{n,m}=\widetilde{C}_{n,m}$. We only prove the former, similar arguments hold for $C_{n,m}=\widetilde{C}_{n,m}$.

Notice that

$$
\begin{aligned}
&A_{n,m}\\
&= \frac{1}{\binom{n}{4}\binom{m}{2}} \sum_{j<q<k<k'}^{n} \sum_{l<l'}^{m} \psi_A^s(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'})\\
&= \frac{4! \cdot 2!}{n(n-1)(n-2)(n-3)} \frac{1}{m(m-1)} \sum_{j<q<k<k'}^{n} \sum_{l<l'}^{m} \psi_A^s(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}),
\end{aligned}
$$

and

$$
\psi_A^s(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}) = \frac{1}{4!2!} \sum_{\tau \in \pi(j,q,k,k')} \sum_{\gamma \in \pi(l,l')} \psi_A(X_{\tau(1)}, X_{\tau(2)}, X_{\tau(3)}, X_{\tau(4)}; Y_{\gamma(1)}, Y_{\gamma(2)}),
$$

thus

$$
A_{n,m} = \frac{1}{n(n-1)(n-2)(n-3)} \frac{1}{m(m-1)} \sum_{j,q,k,k'}^{*} \sum_{l,l'}^{*} \psi_A(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}).
$$

Further since

$$
\begin{aligned}
&\psi_A(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'})\\
&= cos\frac{\langle X_k - X_{k'}, X_j - X_q\rangle}{\sqrt{U_n}} + cos\frac{\langle Y_l - Y_{l'}, X_j - X_q\rangle}{\sqrt{U_n}}\\
&- cos\frac{\langle X_k - Y_{l'}, X_j - X_q\rangle}{\sqrt{U_n}} - cos\frac{\langle X_{k'} - Y_l, X_j - X_q\rangle}{\sqrt{U_n}},
\end{aligned}
$$

direct calculation shows that $A_{n,m} = \widetilde{A}_{n,m}$.

This concludes the proof.

Proof of Theorem 1

Recall that

$$
U_n = \frac{1}{\binom{n}{3}} \sum_{u<v<s}^{n} \frac{1}{3}(\langle X_u, X_v - X_s\rangle^2 + \langle X_v, X_u - X_s\rangle^2 + \langle X_s, X_v - X_u\rangle^2),
$$

hence

$$
EU_n = Var\langle X_1, X_2 - X_3\rangle^2.
$$

Using Lemma A in Serfling (1980)(Section 5.2), we have

$$U_n = Var\langle X'', X - X'\rangle + O_p(1/\sqrt{n}). \tag{01}$$

Next, Invoking (01) and condition $\sup_{1\leqslant i\leqslant p} EX_i^2 < \infty$, one can obtain

$$U_n = O_P(1). \tag{02}$$

Further since

$$\frac{1}{\sqrt{U_n}} - \frac{1}{\sqrt{Var\langle X'', X - X'\rangle}} = -\frac{1}{2\sqrt{\xi_n^3}}(U_n - Var\left(\langle X_1 - X_2, X_3\rangle\right)),$$

where ξ_n is between U_n and $Var\langle X'', X - X'\rangle$, thus by (01) and (02), we can deduce that

$$\frac{1}{\sqrt{U_n}} - \frac{1}{\sqrt{Var\langle X'', X - X'\rangle}} = O_P(1/\sqrt{n}). \tag{03}$$

(03) together with $\langle X_1 - X_2, Z_1^* - Z_2^*\rangle = O_P(1)$ implies that

$$\frac{\langle X_1 - X_2, Z_1^* - Z_2^*\rangle}{\sqrt{U_n}} = \frac{\langle X_1 - X_2, Z_1^* - Z_2^*\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} + O_p\left(\frac{1}{\sqrt{n}}\right),$$

where $\langle X_1 - X_2, Z_1^* - Z_2^*\rangle = O_P(1)$ follows from $X = EX + O_p(\sqrt{VarX})$, $E(< X_1 - X_2, Z_1^* - Z_2^* >) = 0$ and $Var(< X_1 - X_2, Z_1^* - Z_2^* >) < \infty$.

Similar to the proof of the above results, we can obtain the other conclusions in Theorem 1 are indeed true.

This concludes the proof. □

Proof of Theorem 2

Recall that

$$\begin{aligned}
&\psi_A(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}) \\
&= cos\frac{\langle X_k - X_{k'}, X_j - X_q\rangle}{\sqrt{U_n}} - cos\frac{\langle X_k - Y_{l'}, X_j - X_q\rangle}{\sqrt{U_n}} \\
&- cos\frac{\langle X_{k'} - Y_l, X_j - X_q\rangle}{\sqrt{U_n}} + cos\frac{\langle Y_l - Y_{l'}, X_j - X_q\rangle}{\sqrt{U_n}},
\end{aligned} \tag{04}$$

let $f(x) = cosx$, and represent

$$f\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{U_n}}\right) = cos\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{U_n}}\right).$$

Based on Theorem 1, and applying the first order Taylor expansion of $f(x)$ around

$$f\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right),$$

it can be seen that

$$cos\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{U_n}}\right) = cos\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right) + O_P\left(\frac{1}{\sqrt{n}}\right). \tag{05}$$

Similarly

$$cos\left(\frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{U_n}}\right) = cos\left(\frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right) + O_P\left(\frac{1}{\sqrt{n}}\right), \tag{06}$$

and

$$cos\left(\frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{U_n}}\right) = cos\left(\frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right) + O_P\left(\frac{1}{\sqrt{n}}\right), \tag{07}$$

$$cos\left(\frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{U_n}}\right) = cos\left(\frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right) + O_P\left(\frac{1}{\sqrt{n}}\right). \tag{08}$$

Now combining (04), (05) (06), (07) and (08), it can be concluded that

$$A_{n,m} = A_{n,m,1} + O_P\left(\frac{1}{\sqrt{n}}\right), \tag{09}$$

where

$$A_{n,m,1} = \frac{1}{\binom{n}{4}\binom{m}{2}} \sum_{j<q<k<k'}^{n} \sum_{l<l'}^{m} \phi_A^s(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}),$$

and

$$\phi_A^s(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}) = \frac{1}{4!2!} \sum_{\tau \in \pi(j,q,k,k')} \sum_{\gamma \in \pi(l,l')} \phi_A(X_{\tau(1)}, X_{\tau(2)}, X_{\tau(3)}, X_{\tau(4)}; Y_{\gamma(1)}, Y_{\gamma(2)}),$$

$$\begin{aligned} &\phi_A(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}) \\ &= cos\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right) - cos\left(\frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right) \\ &- cos\left(\frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right) + cos\left(\frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right). \end{aligned}$$

Note $A_{n,m,1}$ is a general U-statistic (see Randles (1982)), and $E \mid \phi_A(X_1, X_2, X_3, X_4; Y_3, Y_4) \mid \leq 4$, these facts along with Theorem 3.2.1 in Koroljuk (1994), yield

$$A_{n,m,1} \xrightarrow{a.s.} E\phi_A^s(X_1, X_2, X_3, X_4; Y_3, Y_4). \tag{010}$$

In addition, direct calculation shows that

$$\begin{aligned} &E\phi_A(X_1, X_2, X_3, X_4; Y_3, Y_4) \\ &= E\{E[\phi_A(X_1, X_2, X_3, X_4; Y_3, Y_4)|X_1 - X_2]\} \\ &= E\Bigg\{\left[E\left(cos\left(\frac{\langle X_1 - X_2, X_3\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right)\middle|X_1 - X_2\right) - E\left(cos\left(\frac{\langle X_1 - X_2, Y_3\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right)\middle|X_1 - X_2\right)\right]^2 \\ &+ \left[E\left(sin\left(\frac{\langle X_1 - X_2, X_3\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right)\middle|X_1 - X_2\right) - E\left(sin\left(\frac{\langle X_1 - X_2, Y_3\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right)\middle|X_1 - X_2\right)\right]^2\Bigg\} \\ &= A, \end{aligned}$$

this fact together with (09) and (010) implies that

$$A_{n,m} \xrightarrow{a.s.} A.$$

In a similar way, we can obtain

$$C_{n,m} \xrightarrow{a.s.} C,$$

hence

$$U_{n,m} \xrightarrow{a.s.} A + C.$$

This concludes the proof. □

Proof of Theorem 3

Consider $f(x) = cosx$, and represent

$$f\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{U_n}}\right) = cos\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{U_n}}\right),$$

by using the second order Taylor expansion of $cos\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{U_n}}\right)$ with Theorem 1, one has

$$\begin{aligned}
&cos\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{U_n}}\right) - cos\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right) \qquad (011)\\
&= -sin\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{U_n}} - \frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)\\
&- \frac{1}{2}cos\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{U_n}} - \frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)^2\\
&+ O_P\left(\frac{1}{n\sqrt{n}}\right).
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}
&cos\left(\frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{U_n}}\right) - cos\left(\frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right) \qquad (012)\\
&= -sin\frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{U_n}} - \frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)\\
&- \frac{1}{2}cos\frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{U_n}} - \frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)^2\\
&+ O_P\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned}$$

and

$$\begin{aligned}
&cos\left(\frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{U_n}}\right) - cos\left(\frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right) \qquad (013)\\
&= -sin\frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{U_n}} - \frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)\\
&- \frac{1}{2}cos\frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{U_n}} - \frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)^2\\
&+ O_P\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned}$$

$$\begin{aligned}
&cos\left(\frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{U_n}}\right) - cos\left(\frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{Var\left(\langle X - X', X''\rangle\right)}}\right) \qquad (014)\\
&= -sin\frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{U_n}} - \frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)\\
&- \frac{1}{2}cos\frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{U_n}} - \frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)^2\\
&+ O_P\left(\frac{1}{n\sqrt{n}}\right).
\end{aligned}$$

Based on (04), (011), (012), (013) and (014), it can be further seen that

$$\begin{aligned}
A_{n,m} = A_{n,m,1} - A_{n,m,2}\cdot\left(\frac{1}{\sqrt{U_n}} - \frac{1}{\sqrt{Var\langle X'', X - X'\rangle}}\right) \qquad (015)\\
- \frac{A_{n,m,3}}{2}\cdot\left(\frac{1}{\sqrt{U_n}} - \frac{1}{\sqrt{Var\langle X'', X - X'\rangle}}\right)^2 + O_P\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned}$$

where

$$A_{n,m,2} = \frac{1}{\binom{n}{4}\binom{m}{2}}\sum_{1\le j<q<k<k'\le n}\sum_{1\le l<l'\le m} H_A^s(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}),$$

and

$$A_{n,m,3} = \frac{1}{\binom{n}{4}\binom{m}{2}}\sum_{1\le j<q<k<k'\le n}\sum_{1\le l<l'\le m} h_A^s(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}).$$

For $H_A^s(W_j, W_q, W_k, W_{k'}, W_l, W_{l'})$ and $h_A^s(W_j, W_q, W_k, W_{k'}, W_l, W_{l'})$, we define them as

$$H_A^s(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}) = \frac{1}{4!2!} \sum_{\tau \in \pi(j,q,k,k')} \sum_{\gamma \in \pi(l,l')} H_A(X_{\tau(1)}, X_{\tau(2)}, X_{\tau(3)}, X_{\tau(4)}; Y_{\gamma(1)}, Y_{\gamma(2)}),$$

and

$$h_A^s(W_j, W_q, W_k, W_{k'}, W_l, W_{l'}) = \frac{1}{4!2!} \sum_{\tau \in \pi(j,q,k,k')} \sum_{\gamma \in \pi(l,l')} h_A(X_{\tau(1)}, X_{\tau(2)}, X_{\tau(3)}, X_{\tau(4)}; Y_{\gamma(1)}, Y_{\gamma(2)}),$$

where

$$\begin{aligned} & H_A(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}) \\ = & \, sin \frac{\langle X_j - X_q, X_k - X_{k'} \rangle}{\sqrt{Var\langle X'', X - X' \rangle}} \langle X_j - X_q, X_k - X_{k'} \rangle - sin \frac{\langle X_j - X_q, X_k - Y_{l'} \rangle}{\sqrt{Var\langle X'', X - X' \rangle}} \langle X_j - X_q, X_k - Y_{l'} \rangle \\ & - sin \frac{\langle X_j - X_q, X_{k'} - Y_l \rangle}{\sqrt{Var\langle X'', X - X' \rangle}} \langle X_j - X_q, X_{k'} - Y_l \rangle + sin \frac{\langle X_j - X_q, Y_l - Y_{l'} \rangle}{\sqrt{Var\langle X'', X - X' \rangle}} \langle X_j - X_q, Y_l - Y_{l'} \rangle , \end{aligned}$$

and

$$\begin{aligned} & h_A(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}) \\ = & \, cos \frac{\langle X_j - X_q, X_k - X_{k'} \rangle}{\sqrt{Var\langle X'', X - X' \rangle}} \langle X_j - X_q, X_k - X_{k'} \rangle^2 - cos \frac{\langle X_j - X_q, X_k - Y_{l'} \rangle}{\sqrt{Var\langle X'', X - X' \rangle}} \langle X_j - X_q, X_k - Y_{l'} \rangle^2 \\ & - cos \frac{\langle X_j - X_q, X_{k'} - Y_l \rangle}{\sqrt{Var\langle X'', X - X' \rangle}} \langle X_j - X_q, X_{k'} - Y_l \rangle^2 + cos \frac{\langle X_j - X_q, Y_l - Y_{l'} \rangle}{\sqrt{Var\langle X'', X - X' \rangle}} \langle X_j - X_q, Y_l - Y_{l'} \rangle^2 . \end{aligned}$$

Following the calculations in $A_{n,m}$, we can easily show that

$$\begin{aligned} C_{n,m} = & \, C_{n,m,1} - C_{n,m,2} \cdot \left(\frac{1}{\sqrt{U_m}} - \frac{1}{\sqrt{Var\langle Y'', Y - Y' \rangle}} \right) \\ & - \frac{C_{n,m,3}}{2} \cdot \left(\frac{1}{\sqrt{U_m}} - \frac{1}{\sqrt{Var\langle Y'', Y - Y' \rangle}} \right)^2 + O_P \left(\frac{1}{m\sqrt{m}} \right), \end{aligned} \tag{016}$$

where

$$C_{n,m,1} = \frac{1}{n^2 m^4} \sum_{k,k'=1}^{n} \sum_{j,q,l,l'=1}^{m} \phi_C^s(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}),$$

$$C_{n,m,2} = \frac{1}{n^2m^4} \sum_{k,k'=1}^{n} \sum_{j,q,l,l'=1}^{m} H_C^s(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}),$$

$$C_{n,m,3} = \frac{1}{n^2m^4} \sum_{k,k'=1}^{n} \sum_{j,q,l,l'=1}^{m} h_C^s(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}).$$

For $\phi_C^s(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'})$, $H_C^s(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'})$ and $h_C^s(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'})$, defined as follows

$$\phi_C^s(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}) = \frac{1}{2!4!} \sum_{\tau \in \pi(k,k')} \sum_{\gamma \in \pi(j,q,l,l')} \phi_C(X_{\tau(1)}, X_{\tau(2)}; Y_{\gamma(1)}, Y_{\gamma(2)}, Y_{\gamma(3)}, Y_{\gamma(4)}),$$

$$H_C^s(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}) = \frac{1}{2!4!} \sum_{\tau \in \pi(k,k')} \sum_{\gamma \in \pi(j,q,l,l')} H_C(X_{\tau(1)}, X_{\tau(2)}; Y_{\gamma(1)}, Y_{\gamma(2)}, Y_{\gamma(3)}, Y_{\gamma(4)}),$$

$$h_C^s(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}) = \frac{1}{2!4!} \sum_{\tau \in \pi(k,k')} \sum_{\gamma \in \pi(j,q,l,l')} h_C(X_{\tau(1)}, X_{\tau(2)}; Y_{\gamma(1)}, Y_{\gamma(2)}, Y_{\gamma(3)}, Y_{\gamma(4)}),$$

and

$$\begin{aligned}
&\phi_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}) \\
&= cos\frac{\langle X_k - X_{k'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} + cos\frac{\langle Y_l - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} \\
&- cos\frac{\langle X_k - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} - cos\frac{\langle X_{k'} - Y_l, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}},
\end{aligned}$$

$$\begin{aligned}
&H_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}) \\
&= sin\frac{\langle X_k - X_{k'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} \langle X_k - X_{k'}, Y_j - Y_q\rangle + sin\frac{\langle Y_l - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} \langle Y_l - Y_{l'}, Y_j - Y_q\rangle \\
&- sin\frac{\langle X_k - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} \langle X_k - Y_{l'}, Y_j - Y_q\rangle - sin\frac{\langle X_{k'} - Y_l, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} \langle X_{k'} - Y_l, Y_j - Y_q\rangle,
\end{aligned}$$

$$\begin{aligned}
&h_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}) \\
&= cos\frac{\langle X_k - X_{k'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} \langle X_k - X_{k'}, Y_j - Y_q\rangle^2 + cos\frac{\langle Y_l - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} \langle Y_l - Y_{l'}, Y_j - Y_q\rangle^2 \\
&- cos\frac{\langle X_k - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} \langle X_k - Y_{l'}, Y_j - Y_q\rangle^2 - cos\frac{\langle X_{k'} - Y_l, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} \langle X_{k'} - Y_l, Y_j - Y_q\rangle^2.
\end{aligned}$$

Next, we will study the asymptotic distribution of $A_{n,m,1} + C_{n,m,1}$. Since $A_{n,m,1}$ is a generalized two sample U-statistic, using H-decomposition (Koroljuk (1994), Section 3.2) to $A_{n,m,1}$, one can obtain

$$\begin{aligned} A_{n,m,1} &= \frac{1}{\binom{n}{4}\binom{m}{2}} \sum_{j<q<k<k'}^{n} \sum_{l<l'}^{m} \phi_A^s(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}) \\ &= \sum_{d_1=0}^{4} \sum_{d_2=0}^{2} \binom{4}{d_1}\binom{2}{d_2} H_A^{(d_1,d_2)}, \end{aligned} \tag{017}$$

where $H_A^{(d_1,d_2)}$ is a two sample U-statistic with the kernel $g^{(d_1,d_2)}$ from (018),

$$\begin{aligned} &g^{(d_1,d_2)}(x_{i_1}, \ldots, x_{i_{d_1}}; y_{j_1}, \ldots, y_{j_{d_2}}) \\ &= \sum_{r_1=0}^{d_1} \sum_{r_2=0}^{d_2} (-1)^{(d_1+d_2)-(r_1+r_2)} \sum_{1 \le i_1 < \cdots < i_{r_1} \le d_1} \sum_{1 \le j_1 < \cdots < j_{r_2} \le d_2} \phi_A^s(x_{i_1}, \cdots, x_{i_{r_1}}; y_{j_1}, \cdots, y_{j_{r_2}}). \end{aligned} \tag{018}$$

Due to the fact that under the null hypothesis, $A = 0$,

and

$$\begin{aligned} &E[\phi_A(\mathbf{x}, X_1, X_2, X_3; Y_2, Y_3)] = E[\phi_A(X_1, \mathbf{x}, X_2, X_3; Y_2, Y_3)] \\ &= E[\phi_A(X_1, X_2, \mathbf{x}, X_3; Y_2, Y_3)] = E[\phi_A(X_1, X_2, X_3, \mathbf{x}; Y_2, Y_3)] = 0, \end{aligned}$$

$$E[\phi_A(X_1, X_2, X_3, X_4; \mathbf{y}, Y)] = [\phi_A(X_1, X_2, X_3, X_4; Y, \mathbf{y})] = 0.$$

$$\begin{aligned} &E[\phi_A(\mathbf{x}, \mathbf{x}', X_1, X_2; Y_1, Y_2)] = E[\phi_A(\mathbf{x}, X_1, \mathbf{x}', X_2; Y_1, Y_2)] = E[\phi_A(\mathbf{x}, X_1, X_2, \mathbf{x}'; Y_1, Y_2)] \\ &= E[\phi_A(X_1, \mathbf{x}, \mathbf{x}', X_2; Y_1, Y_2)] = E[\phi_A(X_1, \mathbf{x}, X_2, \mathbf{x}'; Y_1, Y_2)] = 0, \end{aligned}$$

$$\begin{aligned} &E[\phi_A(\mathbf{x}, X_1, X_2, X_3; \mathbf{y}, Y)] = E[\phi_A(\mathbf{x}, X_1, X_2, X_3; Y, \mathbf{y})] = E[\phi_A(X_1, \mathbf{x}, X_2, X_3; \mathbf{y}, Y)] \\ &= E[\phi_A(X_1, \mathbf{x}, X_2, X_3; Y, \mathbf{y})] = E[\phi_A(X_1, X_2, \mathbf{x}, X_3; \mathbf{y}, Y)] = E[\phi_A(X_1, X_2, X_3, \mathbf{x}; Y, \mathbf{y})] = 0. \end{aligned}$$

Therefore by the above results and (017), it can be shown that

$$
\begin{aligned}
&A_{n,m,1} \\
&= \frac{1}{\binom{n}{2}\binom{m}{2}} \times \sum_{k<k'}^{n}\sum_{l<l'}^{m}(\phi_A^{(2,0)}(X_k, X_{k'}) + \phi_A^{(1,1)}(X_k, Y_l) + \phi_A^{(1,1)}(X_{k'}, Y_{l'}) + \phi_A^{(0,2)}(Y_l, Y_{l'})) \\
&+ O_p\left(\frac{1}{n\sqrt{m}}\right) + O_p\left(\frac{1}{m\sqrt{n}}\right) + O_p\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&\phi_A^{(2,0)}(\mathbf{x}, \mathbf{x}') \\
&= E\left(cos\frac{\langle X_1 - X_2, \mathbf{x} - \mathbf{x}'\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) + E\left(cos\frac{\langle X_1 - X_2, Y_1 - Y_2\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) \\
&- E\left(cos\frac{\langle X_1 - X_2, \mathbf{x} - Y_2\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) - E\left(cos\frac{\langle X_1 - X_2, \mathbf{x}' - Y_1\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\phi_A^{(1,1)}(\mathbf{x}, \mathbf{y}) \\
&= E\left(cos\frac{\langle X_1 - X_2, \mathbf{x} - X\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) + E\left(cos\frac{\langle X_1 - X_2, Y - \mathbf{y}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) \\
&- E\left(cos\frac{\langle X_1 - X_2, \mathbf{x} - \mathbf{y}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) - E\left(cos\frac{\langle X_1 - X_2, X - Y\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\phi_A^{(0,2)}(\mathbf{y}, \mathbf{y}') \\
&= E\left(\frac{cos\langle X_1 - X_2, X_3 - X_4\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) + E\left(\frac{cos\langle X_1 - X_2, \mathbf{y} - \mathbf{y}'\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) \\
&- E\left(\frac{cos\langle X_1 - X_2, X_3 - \mathbf{y}'\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) - E\left(\frac{cos\langle X_1 - X_2, X_4 - \mathbf{y}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right).
\end{aligned}
$$

Let

$$
Q_A(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') = \phi_A^{(2,0)}(\mathbf{x}, \mathbf{x}') + \phi_A^{(1,1)}(\mathbf{x}, \mathbf{y}) + \phi_A^{(1,1)}(\mathbf{x}', \mathbf{y}') + \phi_A^{(0,2)}(\mathbf{y}, \mathbf{y}'),
$$

then

$$A_{n,m,1} = \frac{1}{\binom{n}{2}\binom{m}{2}} \sum_{k<k'}^{n} \sum_{l<l'}^{m} Q_A(X_k, X_{k'}; Y_l, Y_{l'}) + O_p\left(\frac{1}{n\sqrt{m}} + \frac{1}{m\sqrt{n}} + \frac{1}{n\sqrt{n}}\right). \qquad (019)$$

Similar to the proof of $A_{n,m,1}$, we have

$$C_{n,m,1} = \frac{1}{\binom{n}{2}\binom{m}{2}} \sum_{k<k'}^{n} \sum_{l<l'}^{m} Q_C(X_k, X_{k'}; Y_l, Y_{l'}) + O_p\left(\frac{1}{n\sqrt{m}} + \frac{1}{m\sqrt{n}} + \frac{1}{m\sqrt{m}}\right), \qquad (020)$$

where

$$Q_C(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') = \phi_C^{(2,0)}(\mathbf{x}, \mathbf{x}') + \phi_C^{(1,1)}(\mathbf{x}, \mathbf{y}) + \phi_C^{(1,1)}(\mathbf{x}', \mathbf{y}') + \phi_C^{(0,2)}(\mathbf{y}, \mathbf{y}'),$$

and

$$\begin{aligned}
&\phi_C^{(2,0)}(\mathbf{x}, \mathbf{x}') \\
&= E\left(cos\frac{\langle Y_1 - Y_2, \mathbf{x} - \mathbf{x}'\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) + E\left(cos\frac{\langle Y_1 - Y_2, Y_3 - Y_4\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) \\
&- E\left(cos\frac{\langle Y_1 - Y_2, \mathbf{x} - Y_4\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) - E\left(cos\frac{\langle Y_1 - Y_2, \mathbf{x}' - Y_3\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right),
\end{aligned}$$

$$\begin{aligned}
&\phi_C^{(1,1)}(\mathbf{x}, \mathbf{y}) \\
&= E\left(cos\frac{\langle Y_1 - Y_2, \mathbf{x} - X\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) + E\left(cos\frac{\langle Y_1 - Y_2, Y - \mathbf{y}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) \\
&- E\left(cos\frac{\langle Y_1 - Y_2, \mathbf{x} - \mathbf{y}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) - E\left(cos\frac{\langle Y_1 - Y_2, X - Y\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right),
\end{aligned}$$

$$\begin{aligned}
&\phi_C^{(0,2)}(\mathbf{y}, \mathbf{y}') \\
&= E\left(cos\frac{\langle Y_1 - Y_2, X_1 - X_2\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) + E\left(cos\frac{\langle Y_1 - Y_2, \mathbf{y} - \mathbf{y}'\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) \\
&- E\left(cos\frac{\langle Y_1 - Y_2, X_1 - \mathbf{y}'\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right) - E\left(cos\frac{\langle Y_1 - Y_2, X_2 - \mathbf{y}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right).
\end{aligned}$$

On the other hand, under the null hypothesis it can be verified that

$$Q_A(\mathbf{x},\mathbf{y};\mathbf{x}',\mathbf{y}') = Q_C(\mathbf{x},\mathbf{y};\mathbf{x}',\mathbf{y}'),$$

set

$$Q(x,\mathbf{y};\mathbf{x}',\mathbf{y}') = Q_A(\mathbf{x},\mathbf{y};\mathbf{x}',\mathbf{y}') = Q_C(\mathbf{x},\mathbf{y};\mathbf{x}',\mathbf{y}'),$$

(019) and (020) imply that

$$\begin{aligned} & A_{n,m,1} + C_{n,m,1} \qquad (021) \\ = & \frac{2}{\binom{n}{2}\binom{m}{2}} \sum_{k<k'}^{n} \sum_{l<l'}^{m} Q(X_k, X_{k'}; Y_l, Y_{l'}) + O_p\left(\frac{1}{n\sqrt{m}} + \frac{1}{m\sqrt{n}} + \frac{1}{m\sqrt{m}} + \frac{1}{n\sqrt{n}}\right). \end{aligned}$$

Now by Theorem 1.1 in Neuhaus (1977), one has

$$Q(\mathbf{x},\mathbf{y};\mathbf{x}',\mathbf{y}') = \sum_{k=1}^{\infty} \lambda_k f_k(\mathbf{x},\mathbf{y}) f_k(\mathbf{x}',\mathbf{y}'),$$

and

$$\begin{aligned} & \frac{mn}{m+n} \times \left(\frac{2}{\binom{n}{2}\binom{m}{2}} \sum_{k<k'}^{n} \sum_{l<l'}^{m} Q(X_k, X_{k'}; Y_l, Y_{l'}) \right) \qquad (022) \\ \xrightarrow{D} & \sum_{k=1}^{\infty} 2\lambda_k [(a_k(\theta) Z_{1k} + b_k(\theta) Z_{2k})^2 - (a_k^2(\theta) + b_k^2(\theta))], \end{aligned}$$

where

$$a_k^2(\theta) = (1-\theta) E_X [E_Y f_k(X,Y)]^2, \quad b_k^2(\theta) = \theta E_Y [E_X f_k(X,Y)]^2,$$

$$Z_{1k}, Z_{2k} \overset{i.i.d}{\sim} N(0,1), k = 1, 2, \cdots.$$

Combining (021) and (022), we have

$$\frac{mn}{m+n} \times (A_{n,m,1} + C_{n,m,1}) \xrightarrow{D} \sum_{k=1}^{\infty} 2\lambda_k [(a_k(\theta) Z_{1k} + b_k(\theta) Z_{2k})^2 - (a_k^2(\theta) + b_k^2(\theta))]. \qquad (023)$$

In a similar way, we have $\frac{nm}{n+m} A_{n,m,2}$, $\frac{nm}{n+m} C_{n,m,2}$, $\frac{nm}{n+m} A_{n,m,3}$ and $\frac{nm}{n+m} C_{n,m,3}$ are all asymptotically chi-squared. These results together with (015), (016), (023) and Theorem 1, as well as Slutsky's theorem, imply that

$$\frac{nm}{n+m} U_{n,m} \xrightarrow{D} \sum_{k=1}^{\infty} 2\lambda_k [(a_k(\theta) Z_{1k} + b_k(\theta) Z_{2k})^2 - (a_k^2(\theta) + b_k^2(\theta))].$$

This concludes the proof. □

Proof of Theorem 4

Similar to the proof of (015) and (016), one can verify

$$\begin{aligned} & A_{n,m} + C_{n,m} \qquad (024) \\ = & (A_{n,m,1} + C_{n,m,1}) - A_{n,m,2} \left(\frac{1}{\sqrt{U_n}} - \frac{1}{\sqrt{Var\langle X'', X - X' \rangle}} \right) \\ & - C_{n,m,2} \left(\frac{1}{\sqrt{U_m}} - \frac{1}{\sqrt{Var\langle Y'', Y - Y' \rangle}} \right) + O_P\left(\frac{1}{n}\right) + O_P\left(\frac{1}{m}\right). \end{aligned}$$

Next, for the convenience of proving, we first establish the limit behavior of $\sqrt{\frac{nm}{n+m}}(A_{n,m,1} + C_{n,m,1})$. By (017), we can get

$$\begin{aligned} & A_{n,m,1} + C_{n,m,1} \\ = & \sum_{d_1=0}^{4} \sum_{d_2=0}^{2} \binom{4}{d_1} \binom{2}{d_2} H_A^{(d_1,d_2)} + \sum_{d_1=0}^{2} \sum_{d_2=0}^{4} \binom{2}{d_1} \binom{4}{d_2} H_C^{(d_1,d_2)} \\ = & A + \frac{1}{n} \sum_{k=1}^{n} (\phi_{A,1}^{(1,0)}(X_k) + \phi_{A,2}^{(1,0)}(X_k) + \phi_{A,3}^{(1,0)}(X_k) + \phi_{A,4}^{(1,0)}(X_k)) \\ & + \frac{1}{m} \sum_{l=1}^{m} (\phi_{A,1}^{(0,1)}(Y_l) + \phi_{A,2}^{(0,1)}(Y_l)) + C + \frac{1}{n} \sum_{k=1}^{n} (\phi_{C,1}^{(1,0)}(X_j) + \phi_{C,2}^{(1,0)}(X_j)) \\ & + \frac{1}{m} \sum_{l=1}^{m} (\phi_{C,1}^{(0,1)}(Y_l) + \phi_{C,2}^{(0,1)}(Y_l) + \phi_{C,3}^{(0,1)}(Y_l) + \phi_{C,4}^{(0,1)}(Y_l)) + O_p\left(\frac{1}{n} + \frac{1}{m} + \frac{1}{\sqrt{nm}}\right), \end{aligned}$$

where

$$\phi_{A,i}^{(1,0)}(x) = E(\phi_A(X_1, X_2, X_3, X_4; Y_1, Y_2) \mid X_i = \mathbf{x}) - A, \qquad i = 1,2,3,4,$$

$$\phi_{A,i}^{(0,1)}(y) = E(\phi_A(X_1, X_2, X_3, X_4; Y_1, Y_2) \mid Y_i = \mathbf{y}) - A, \qquad i = 1,2,$$

$$\phi_{C,i}^{(1,0)}(x) = E(\phi_C(X_1, X_2; Y_1, Y_2, Y_3, Y_4) \mid X_i = \mathbf{x}) - C, \qquad i = 1,2,$$

$$\phi_{C,i}^{(0,1)}(y) = E(\phi_C(X_1, X_2; Y_1, Y_2, Y_3, Y_4) \mid Y_i = \mathbf{y}) - C, \qquad i = 1,2,3,4.$$

Let

$$g^{(1,0)}(X_k) = \phi_{A,1}^{(1,0)}(X_k) + \phi_{A,2}^{(1,0)}(X_k) + \phi_{A,3}^{(1,0)}(X_k) + \phi_{A,4}^{(1,0)}(X_k) + \phi_{C,1}^{(1,0)}(X_k) + \phi_{C,2}^{(1,0)}(X_k),$$

$$g^{(0,1)}(Y_l) = \phi_{A,1}^{(0,1)}(Y_l) + \phi_{A,2}^{(0,1)}(Y_l) + \phi_{C,1}^{(0,1)}(Y_l) + \phi_{C,2}^{(0,1)}(Y_l) + \phi_{C,3}^{(0,1)}(Y_l) + \phi_{C,4}^{(0,1)}(Y_l),$$

then

$$A_{n,m,1} + C_{n,m,1} = A + C + \frac{1}{n}\sum_{k=1}^{n} g^{(1,0)}(X_k) + \frac{1}{m}\sum_{l=1}^{m} g^{(0,1)}(Y_l) + O_p\left(\frac{1}{n} + \frac{1}{m} + \frac{1}{\sqrt{nm}}\right),$$

hence

$$\begin{aligned}
&\sqrt{\frac{nm}{n+m}}(A_{n,m,1} + C_{n,m,1} - CD(X,Y)) \\
&= \sqrt{\frac{nm}{n+m}}\frac{1}{n}\sum_{k=1}^{n} g^{(1,0)}(X_k) + \sqrt{\frac{nm}{n+m}}\frac{1}{m}\sum_{l=1}^{m} g^{(0,1)}(Y_l) + o_p(1).
\end{aligned}$$

Notice that

$$Eg^{(1,0)}(X_k) = Eg^{(0,1)}(Y_l) = 0,$$

denote

$$Var(g^{(1,0)}(X_k)) = \delta_{1,0}^2, \qquad Var(g^{(0,1)}(Y_l)) = \delta_{0,1}^2,$$

combining the above results, together with the central limit theorem and Slutsky's theorem, we can obtain

$$\sqrt{\frac{nm}{n+m}}(A_{n,m,1} + C_{n,m,1} - CD(X,Y)) \longrightarrow^d N(0, (1-\theta)\delta_{1,0}^2 + \theta\delta_{0,1}^2). \tag{025}$$

On the other hand, similar to the proof of (010), one has

$$A_{n,m,2} \xrightarrow{a.s.} EH_A(X_1, X_2, X_3, X_4; Y_3, Y_4),$$

and

$$C_{n,m,2} \xrightarrow{a.s.} EH_C(X_3, X_4; Y_1, Y_2, Y_3, Y_4).$$

So by Theorem 1, it can be obtained that

$$\sqrt{\frac{nm}{n+m}} A_{n,m,2} \cdot \left(\frac{1}{\sqrt{U_n}} - \frac{1}{\sqrt{Var\langle X'', X - X' \rangle}} \right) \xrightarrow{a.s.} \sqrt{1-\theta} EH_A(X_1, X_2, X_3, X_4; Y_3, Y_4), \tag{026}$$

and

$$\sqrt{\frac{nm}{n+m}} C_{n,m,2} \cdot \left(\frac{1}{\sqrt{U_m}} - \frac{1}{\sqrt{Var\langle Y'', Y - Y' \rangle}} \right) \xrightarrow{a.s.} \sqrt{\theta} EH_C(X_3, X_4; Y_1, Y_2, Y_3, Y_4). \tag{027}$$

Denote

$$\tilde{\zeta} = -\sqrt{1-\theta} EH_A(X_1, X_2, X_3, X_4; Y_3, Y_4) - \sqrt{\theta} EH_C(X_3, X_4; Y_1, Y_2, Y_3, Y_4),$$

based on (024), (025), (026) and (027), as well as Slutsky's theorem, one can show that

$$\sqrt{\frac{nm}{n+m}}(U_{n,m} - CD(X,Y)) \xrightarrow{D} N(0, (1-\theta)\delta_{1,0}^2 + \theta\delta_{0,1}^2) + \tilde{\zeta}.$$

This concludes the proof. □

Proof of Theorem 5

To prove Theorem 5, we proceed with our proof in the following steps.

(i) Using the Chebyshev's inequality to prove

$$\frac{U_n}{Var(\langle X - X', X'' \rangle)} - 1 = O_P\left(\frac{1}{\sqrt{n}}\right). \tag{028}$$

(ii) Applying the Lagrange mean value theorem, further by (028), to show that

$$\sqrt{\frac{Var(\langle X - X', X'' \rangle)}{U_n}} - 1 = O_P\left(\frac{1}{\sqrt{n}}\right). \tag{029}$$

(iii) Invoking assumption **(A2)** and (029) to verify

$$\frac{\langle X_1 - X_2, Z_1^* - Z_2^* \rangle}{\sqrt{U_n}} = \frac{\langle X_1 - X_2, Z_1^* - Z_2^* \rangle}{\sqrt{Var\langle X'', X - X' \rangle}} + O_P\left(\frac{1}{\sqrt{n}}\right).$$

Following the same steps as the above proof, we begin with (i), and then (ii) and (iii).

To prove (028), by applying Cauchy-Schwarz inequality, it is sufficient to show that

$$E\left(\frac{U_n}{Var\langle X'', X - X' \rangle}\right) = 1, \tag{030}$$

and

$$Var\left(\frac{U_n}{Var\langle X'', X - X' \rangle}\right) = O\left(\frac{1}{n}\right). \tag{031}$$

(030) is easy to verify, we only prove (031).

Note

$$Var\left(\frac{U_n}{Var\langle X'', X - X' \rangle}\right) = \frac{E(U_n^2)}{(Var\langle X'', X - X' \rangle)^2} - 1,$$

to proceed, we will first calculate $E(U_n^2)$.

Based on the definiton of U_n, we can deduce that

$$
\begin{aligned}
E(U_n)^2 &= \frac{1}{n^2(n-1)^2(n-2)^2} \sum_{i_1,j_1,k_1}^{*} \sum_{i_2,j_2,k_2}^{*} E[\varphi(X_{i_1}, X_{j_1}, X_{k_1})\varphi(X_{i_2}, X_{j_2}, X_{k_2})] \qquad (032) \\
&= \frac{(n-3)(n-4)(n-5)}{n(n-1)(n-2)} E[\varphi(X_1, X_2, X_3)\varphi(X_4, X_5, X_6)] \\
&+ \frac{9(n-3)(n-4)}{n(n-1)(n-2)} E[\varphi(X_1, X_2, X_3)\varphi(X_1, X_4, X_5)] \\
&+ \frac{18(n-3)}{n(n-1)(n-2)} E[\varphi(X_1, X_2, X_3)\varphi(X_1, X_2, X_4)] \\
&+ \frac{6}{n(n-1)(n-2)} E[\varphi(X_1, X_2, X_3)]^2.
\end{aligned}
$$

Next, for the convenience of calculations, write

$$
\mathcal{I}_3 = \{i_1, j_1, k_1\} \cap \{i_2, j_2, k_2\},
$$

and consider the cases as below,

- $\mathcal{C}_a = \{i_1, j_1, k_1, i_2, j_2, k_2 : \#|\mathcal{I}_3| = 0\}$,
- $\mathcal{C}_b = \{i_1, j_1, k_1, i_2, j_2, k_2 : \#|\mathcal{I}_3| = 1\}$,
- $\mathcal{C}_c = \{i_1, j_1, k_1, i_2, j_2, k_2 : \#|\mathcal{I}_3| = 2\}$,
- $\mathcal{C}_d = \{i_1, j_1, k_1, i_2, j_2, k_2 : \#|\mathcal{I}_3| = 3\}$.

For $\mathcal{C}_a$, it is not difficult to see that

$$
E[\varphi(X_1, X_2, X_3)\varphi(X_4, X_5, X_6)] = [Var\langle X''.X - X'\rangle]^2. \qquad (033)
$$

Similarly, for $\mathcal{C}_b$ direct calculation shows

$$
\begin{aligned}
&E[\varphi(X_1, X_2, X_3)\varphi(X_1, X_4, X_5)] \\
&= \frac{4}{9} E[\langle X_1, X_2 - X_3\rangle^2 \langle X_4, X_5 - X_1\rangle^2] + \frac{4}{9} E[\langle X_3, X_2 - X_1\rangle^2 \langle X_4, X_5 - X_1\rangle^2] \\
&+ \frac{1}{9} E[\langle X_1, X_2 - X_3\rangle^2 \langle X_1, X_4 - X_5\rangle^2].
\end{aligned}
$$

For $\mathcal{C}_c$ and $\mathcal{C}_d$, similar arguments can also be used to show

$$\begin{aligned}
&E[\varphi(X_1, X_2, X_3)\varphi(X_1, X_2, X_4)] \\
&= \frac{2}{9}E[\langle X_1, X_2 - X_3\rangle^2\langle X_1, X_2 - X_4\rangle^2] + \frac{2}{9}E[\langle X_1, X_2 - X_3\rangle^2\langle X_2, X_1 - X_4\rangle^2] \\
&+ \frac{2}{9}E[\langle X_1, X_2 - X_3\rangle^2\langle X_4, X_2 - X_1\rangle^2] + \frac{2}{9}E[\langle X_2, X_1 - X_3\rangle^2\langle X_4, X_2 - X_1\rangle^2] \\
&+ \frac{1}{9}E[\langle X_3, X_2 - X_1\rangle^2\langle X_4, X_2 - X_1\rangle^2],
\end{aligned}$$

and

$$E[\varphi(X_1, X_2, X_3)]^2 = \frac{1}{3}E[\langle X_1, X_2 - X_3\rangle^4] + \frac{2}{3}E[\langle X_1, X_2 - X_3\rangle^2\langle X_2, X_1 - X_3\rangle^2].$$

Now by using Cauchy-Schwarz inequality and the assumption **(A2)**, it can be seen that

$$E[\langle X_1, X_2 - X_3\rangle^2\langle X_4, X_5 - X_1\rangle^2] \leq \sqrt{E\langle X_1, X_2 - X_3\rangle^4 E\langle X_4, X_5 - X_1\rangle^4} = O(p^2),$$

also note

$$E[\langle X_3, X_2 - X_1\rangle^2\langle X_4, X_5 - X_1\rangle^2] = O(p^2),$$

and

$$E[\langle X_1, X_2 - X_3\rangle^2\langle X_1, X_4 - X_5\rangle^2] = O(p^2),$$

based on the previous observation, we can conclude

$$E[\varphi(X_1, X_2, X_3)\varphi(X_1, X_4, X_5)] = O(p^2). \tag{034}$$

Similarly

$$E[\varphi(X_1, X_2, X_3)\varphi(X_1, X_2, X_4)] = O(p^2), \tag{035}$$

and

$$E[\varphi(X_1, X_2, X_3)]^2 = O(p^2). \tag{036}$$

In addition,

$$Var\langle X'', X - X'\rangle = 2\mu_X^\tau \Sigma_X \mu_X + 2tr\Sigma_X^2,$$

thus by assumption **(A1)**, we have

$$Var\langle X'', X - X'\rangle = O(p). \tag{037}$$

Now plugging (033), (034), (035), (036) and (037) into (032), we have

$$\frac{E(U_n)^2}{Var\langle X'', X - X'\rangle^2} = 1 + O\left(\frac{1}{n}\right).$$

Hence

$$Var\left(\frac{U_n}{Var\langle X'', X - X'\rangle}\right) = O\left(\frac{1}{n}\right).$$

(ii) Consider $f(t) = \frac{1}{\sqrt{t}}$, by using the Langrange mean value theorem, it is easy to be seen

$$\sqrt{\frac{Var\langle X'', X - X'\rangle}{U_n}} - 1 = -\frac{1}{2\sqrt{\eta_n^3}}\left(\frac{U_n}{Var\langle X'', X - X'\rangle} - 1\right),$$

where η_n is between 1 and $U_n/Var(\langle X'', X - X'\rangle)$. So, by this result and (028), we can deduce that (029) is indeed true.

(iii) Notice that

$$\begin{aligned} &\frac{\langle X_1 - X_2, Z_1^* - Z_2^*\rangle}{\sqrt{U_n}} - \frac{\langle X_1 - X_2, Z_1^* - Z_2^*\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} \\ &= \frac{\langle X_1 - X_2, Z_1^* - Z_2^*\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\sqrt{\frac{Var\langle X'', X - X'\rangle}{U_n}} - 1\right), \end{aligned} \tag{038}$$

to proceed, let us now turn to study the asymptotic behavior of $\frac{\langle X_1 - X_2, Z_1^* - Z_2^*\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}$. First based on the assumption **(A1)**, we can conclude that

$$Var(\langle X_1 - X_2, Z_1^* - Z_2^*\rangle) = O(p),$$

then by the formula

$$X = EX + O_P(Var(X)),$$

it follows that

$$\langle X_1 - X_2, Z_1^* - Z_2^* \rangle = O_P(\sqrt{p}).$$

Combining the result with (037) yields

$$\frac{\langle X_1 - X_2, Z_1^* - Z_2^* \rangle}{\sqrt{Var\langle X'', X - X' \rangle}} = O_P(1). \tag{039}$$

Now let us substitute (029) and (039) into ((038), we can obtain

$$\frac{\langle X_1 - X_2, Z_1^* - Z_2^* \rangle}{\sqrt{U_n}} = \frac{\langle X_1 - X_2, Z_1^* - Z_2^* \rangle}{\sqrt{Var\langle X'', X - X' \rangle}} + O_P\left(\frac{1}{\sqrt{n}}\right).$$

In a similar way, we have the other formulas in Theorem 5 hold.

This concludes the proof. □

Proof of Theorem 6

Following the same steps in the proof of Theorem 2, and by using Theorem 5, it is not difficult to see that the result holds. This concludes the proof. □

Proof of Theorem 7

To prove the results, we will apply the same argument used in Theorem 3. More specifically, based on the Taylor expansion with Theorem 5, it can be seen that

$$\begin{aligned}
&cos\frac{\langle X_k - X_{k'}, X_j - X_q \rangle}{\sqrt{U_{n,m}}} \qquad (040)\\
&= cos\frac{\langle X_k - X_{k'}, X_j - X_q \rangle}{\sqrt{Var\langle X'', X - X' \rangle}} - sin\frac{\langle X_k - X_{k'}, X_j - X_q \rangle}{\sqrt{Var\langle X'', X - X' \rangle}}\frac{\langle X_k - X_{k'}, X_j - X_q \rangle}{\sqrt{Var\langle X'', X - X' \rangle}}\left(\sqrt{\frac{Var\langle X'', X - X' \rangle}{U_{n,m}}} - 1\right)\\
&- \frac{1}{2}cos\frac{\langle X_k - X_{k'}, X_j - X_q \rangle}{\sqrt{Var\langle X'', X - X' \rangle}}\left(\frac{\langle X_k - X_{k'}, X_j - X_q \rangle}{\sqrt{Var\langle X'', X - X' \rangle}}\right)^2\left(\sqrt{\frac{Var\langle X'', X - X' \rangle}{U_{n,m}}} - 1\right)^2\\
&+ O_P\left(\frac{1}{n\sqrt{n}}\right).
\end{aligned}$$

Similarly

$$
\begin{aligned}
&cos\frac{\langle Y_l - Y_{l'}, X_j - X_q\rangle}{\sqrt{U_{n,m}}} \qquad (041)\\
&= cos\frac{\langle Y_l - Y_{l'}, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} - sin\frac{\langle Y_l - Y_{l'}, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\frac{\langle Y_l - Y_{l'}, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\sqrt{\frac{Var\langle X'', X - X'\rangle}{U_{n,m}}} - 1\right)\\
&- \frac{1}{2}cos\frac{\langle Y_l - Y_{l'}, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\frac{\langle Y_l - Y_{l'}, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)^2\left(\sqrt{\frac{Var\langle X'', X - X'\rangle}{U_{n,m}}} - 1\right)^2\\
&+ O_P\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned}
$$

$$
\begin{aligned}
&cos\frac{\langle X_k - Y_{l'}, X_j - X_q\rangle}{\sqrt{U_{n,m}}} \qquad (042)\\
&= cos\frac{\langle X_k - Y_{l'}, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} - sin\frac{\langle X_k - Y_{l'}, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\frac{\langle X_k - Y_{l'}, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\sqrt{\frac{Var\langle X'', X - X'\rangle}{U_{n,m}}} - 1\right)\\
&- \frac{1}{2}cos\frac{\langle X_k - Y_{l'}, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\frac{\langle X_k - Y_{l'}, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)^2\left(\sqrt{\frac{Var\langle X'', X - X'\rangle}{U_{n,m}}} - 1\right)^2\\
&+ O_P\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&cos\frac{\langle X_{k'} - Y_l, X_j - X_q\rangle}{\sqrt{U_{n,m}}} \qquad (043)\\
&= cos\frac{\langle X_{k'} - Y_l, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} - sin\frac{\langle X_{k'} - Y_l, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\frac{\langle X_{k'} - Y_l, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\sqrt{\frac{Var\langle X'', X - X'\rangle}{U_{n,m}}} - 1\right)\\
&- \frac{1}{2}cos\frac{\langle X_{k'} - Y_l, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\left(\frac{\langle X_{k'} - Y_l, X_j - X_q\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)^2\left(\sqrt{\frac{Var\langle X'', X - X'\rangle}{U_{n,m}}} - 1\right)^2\\
&+ O_P\left(\frac{1}{n\sqrt{n}}\right).
\end{aligned}
$$

Now, based on (04), (040), (041), (042) and (043), we have

$$
\begin{aligned}
&A_{n,m} \qquad (044)\\
&= A_{n,m,1} - \widetilde{A}_{n,m,2}\left(\sqrt{\frac{Var\langle X'', X - X'\rangle}{U_n}} - 1\right) - \widetilde{A}_{n,m,3}\left(\sqrt{\frac{Var\langle X'', X - X'\rangle}{U_n}} - 1\right)^2\\
&+ O_P\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned}
$$

where

$$\widetilde{A}_{n,m,2} = \frac{1}{\binom{n}{4}\binom{m}{2}} \sum_{1\le j<q<k<k'\le n} \sum_{1\le l<l'\le m} \widetilde{H}^s_A(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}),$$

and

$$\widetilde{A}_{n,m,3} = \frac{1}{\binom{n}{4}\binom{m}{2}} \sum_{1\le j<q<k<k'\le n} \sum_{1\le l<l'\le m} \widetilde{h}^s_A(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}).$$

For $\widetilde{H}^s_A(W_j, W_q, W_k, W_{k'}, W_l, W_{l'})$ and $\widetilde{h}^s_A(W_j, W_q, W_k, W_{k'}, W_l, W_{l'})$, denote

$$\widetilde{H}^s_A(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}) = \frac{1}{4!2!} \sum_{\tau\in\pi(j,q,k,k')} \sum_{\gamma\in\pi(l,l')} \widetilde{H}_A(X_{\tau(1)}, X_{\tau(2)}, X_{\tau(3)}, X_{\tau(4)}; Y_{\gamma(1)}, Y_{\gamma(2)}),$$

and

$$\widetilde{h}^s_A(W_j, W_q, W_k, W_{k'}, W_l, W_{l'}) = \frac{1}{4!2!} \sum_{\tau\in\pi(j,q,k,k')} \sum_{\gamma\in\pi(l,l')} \widetilde{h}_A(X_{\tau(1)}, X_{\tau(2)}, X_{\tau(3)}, X_{\tau(4)}; Y_{\gamma(1)}, Y_{\gamma(2)}),$$

where

$$\begin{aligned}
&\widetilde{H}_A(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}) \\
&= sin\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} \frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} - sin\frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} \frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} \\
&\quad - sin\frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} \frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} + sin\frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} \frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}},
\end{aligned}$$

$$\begin{aligned}
&\widetilde{h}_A(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}) \\
&= cos\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} \left(\frac{\langle X_j - X_q, X_k - X_{k'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)^2 - cos\frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} \left(\frac{\langle X_j - X_q, X_k - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)^2 \\
&\quad - cos\frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} \left(\frac{\langle X_j - X_q, X_{k'} - Y_l\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)^2 + cos\frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}} \left(\frac{\langle X_j - X_q, Y_l - Y_{l'}\rangle}{\sqrt{Var\langle X'', X - X'\rangle}}\right)^2.
\end{aligned}$$

Similar to the proof of $A_{n,m}$, we will further show that

$$\begin{aligned} &C_{n,m} \\ &= C_{n,m,1} - \widetilde{C}_{n,m,2}\left(\frac{1}{\sqrt{U_m}} - \frac{1}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\right) - \widetilde{C}_{n,m,3}\left(\frac{1}{\sqrt{U_m}} - \frac{1}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\right)^2 \\ &+ O_P\left(\frac{1}{m\sqrt{m}}\right), \end{aligned} \tag{045}$$

where

$$\widetilde{C}_{n,m,2} = \frac{1}{\binom{n}{2}\binom{m}{4}} \sum_{1\le k<k'\le n} \sum_{1\le j<q<l<l'\le m} \widetilde{H}^s_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}),$$

and

$$\widetilde{C}_{n,m,3} = \frac{1}{\binom{n}{2}\binom{m}{4}} \sum_{1\le k<k'\le n} \sum_{1\le j<q<l<l'\le m} \widetilde{h}^s_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}).$$

For $\widetilde{H}^s_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'})$ and $\widetilde{h}^s_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'})$, we define them as

$$\widetilde{H}^s_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}) = \frac{1}{2!4!} \sum_{\tau\in\pi(k,k')} \sum_{\gamma\in\pi(j,q,l,l')} H_C(X_{\tau(3)}, X_{\tau(4)}; Y_{\gamma(1)}, Y_{\gamma(2)}, Y_{\gamma(3)}, Y_{\gamma(4)}),$$

and

$$\widetilde{h}^s_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}) = \frac{1}{2!4!} \sum_{\tau\in\pi(k,k')} \sum_{\gamma\in\pi(j,q,l,l')} h_C(X_{\tau(3)}, X_{\tau(4)}; Y_{\gamma(1)}, Y_{\gamma(2)}, Y_{\gamma(3)}, Y_{\gamma(4)}),$$

where

$$\begin{aligned} &\widetilde{H}_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}) \\ &= sin\frac{\langle X_k - X_{k'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\frac{\langle X_k - X_{k'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} + sin\frac{\langle Y_l - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\frac{\langle Y_l - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} \\ &- sin\frac{\langle X_k - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\frac{\langle X_k - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}} - sin\frac{\langle X_{k'} - Y_l, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\frac{\langle X_{k'} - Y_l, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}, \end{aligned}$$

and

$$
\begin{aligned}
&\widetilde{h}_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}) \\
&= cos\frac{\langle X_k - X_{k'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\left(\frac{\langle X_k - X_{k'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\right)^2 + cos\frac{\langle Y_l - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\left(\frac{\langle Y_l - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\right)^2 \\
&- cos\frac{\langle X_k - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\left(\frac{\langle X_k - Y_{l'}, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\right)^2 - cos\frac{\langle X_{k'} - Y_l, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\left(\frac{\langle X_{k'} - Y_l, Y_j - Y_q\rangle}{\sqrt{Var\langle Y'', Y - Y'\rangle}}\right)^2.
\end{aligned}
$$

Next, under the null hypothesis, we need to verify the theorem holds. The proof will consist of three steps. In the first step, we establish the limiting null distribution of $\frac{nm}{n+m}(A_{n,m,1} + C_{n,m,1})$ under the high-dimensional distance inference. In the second step, we discuss the asymptotic behavior of $\widetilde{A}_{n,m,2}$, $\widetilde{A}_{n,m,3}$, $\widetilde{C}_{n,m,2}$ and $\widetilde{C}_{n,m,3}$. At last, by using Slutsky's theorem and Theorem 5 to obtain the desired result.

Indeed, based on Theorem 5, similar to the proof of Theorem 3, it can be verified that (023) also holds in high-dimensional cases. The details are omitted. Next, we focus on the limit behavior of $\widetilde{A}_{n,m,2}$, $\widetilde{A}_{n,m,3}$, $\widetilde{C}_{n,m,2}$ and $\widetilde{C}_{n,m,3}$. Under H_0, it is not difficult to see that the kernels

$$
\widetilde{H}^s_A(X_j, X_q, X_k, X_{k'}; Y_l, Y_{l'}), \quad \widetilde{h}^s_A(W_j, W_q, W_k, W_{k'}, W_l, W_{l'}),
$$

and

$$
\widetilde{H}^s_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'}), \quad \widetilde{h}^s_C(X_k, X_{k'}; Y_j, Y_q, Y_l, Y_{l'})
$$

are all degenerate. Hence, using the limit theorem for two-sample degenerate U-statistics, we can obtain that they all follow the mixture of χ^2 distributions.

Now combining the previous results with Slutsky's theorem and Theorem 5, as well as (044) and (045) yields

$$
\frac{nm}{n+m}U_{n,m} \longrightarrow^d \sum_{k=1}^{\infty} 2\lambda_k[(a_k(\theta)Z_{1k} + b_k(\theta)Z_{2k})^2 - (a_k^2(\theta) + b_k^2(\theta))].
$$

This concludes the proof. □

Proof of Theorem 8

Similar to the proof of Theorem 4, under the alternative hypothesis, by applying Slutsky's theorem and Theorem 5, we have the conclusion holds. □

S2 Appendix B: Additional Simulation Results

We conduct additional simulations to assess the finite sample performance of the proposed tests. Let $t_3(\mu, \Sigma)$ stand for the multivariate t distribution with 3 degrees of freedom, location vector μ and shape matrix Σ, $N_p(\mu, \Sigma)$ stands for the multivariate normal distribution with mean vector μ and shape matrix Σ.

Example 1. Suppose $X_k = (X_{k1}, \cdots, X_{kp})$, $Y_l = (Y_{l1}, \cdots, Y_{lp})$ with $k = 1, \cdots, n$, $l = 1, \cdots, m$. We will generate independent identically distributed samples from the models: $X_k \sim t_3(\mu_1, \Sigma_1)$, $Y_l \sim t_3(\mu_2, \Sigma_2)$, where $\mu_1 = \mathbf{0}_{p\times 1}$, $\Sigma_1 = (0.5^{|i-j|})_{p\times p}$, $\mu_2 = \delta\mathbf{1}_{p\times 1}$ and $\Sigma_2 = \sigma^2\Sigma_1$. We consider four scenarios for (δ, σ^2): (0, 1), (0.2, 1), (0.18, 1.3) and (0, 1.6), which corresponds to the null hypothesis H_0, location shift, both location shift and scale difference and scale difference only.

Under the multivariate t distribution with 3 degrees of freedom, the empirical size and power of each test are reported in Table 1. As can be seen from Table 1, when two populations are identically distributed, the sizes of all tests are very close to the significance level $\alpha = 0.05$, indicating that these methods can control the type I error well. When X and Y have different distributions, the results in Table 1 show that no method outperforms the others for all the considered populations throughout our empirical studies. If the location shift is present, i.e., $\delta = 0.2$ or 0.18, the new test performs best, and the ED test follows, which are often substantially higher than BD and MMD tests. For example, the empirical power of ED, BD, MMD and our approach are 0.4325, 0.0750, 0.2075 and 0.5150 under the alternative $\delta = 0.2$, $n = m = 50$

Table 1: Empirical size and power for different test procedures.

Model	Methods	$n = m = 50$		$n = m = 70$		$n = m = 90$	
		$p = 50$	$p = 100$	$p = 50$	$p = 100$	$p = 50$	$p = 100$
$(0, 1)$	CD	0.0675	0.0575	0.0525	0.0600	0.0400	0.0500
	ED	0.0525	0.0675	0.0750	0.0700	0.0600	0.0575
	BD	0.0600	0.0325	0.0350	0.0450	0.0575	0.0675
	MMD	0.0500	0.0500	0.0425	0.0600	0.0575	0.0425
$(0.2, 1)$	CD	0.5150	0.5825	0.6975	0.6700	0.8800	0.9650
	ED	0.4325	0.5825	0.6350	0.6700	0.7850	0.8900
	BD	0.0750	0.0800	0.1100	0.1000	0.1200	0.2000
	MMD	0.2075	0.1950	0.3975	0.3050	0.5375	0.7850
$(0.18, 1.3)$	CD	0.4225	0.5200	0.5950	0.6300	0.7625	0.9000
	ED	0.4000	0.5275	0.6050	0.6400	0.7125	0.8300
	BD	0.3400	0.3350	0.4250	0.3650	0.5625	0.6700
	MMD	0.3975	0.3975	0.5800	0.5025	0.7325	0.9075
$(0, 1.6)$	CD	0.3175	0.3700	0.4525	0.4250	0.5500	0.7500
	ED	0.3250	0.4025	0.5300	0.4625	0.5700	0.7150
	BD	0.6525	0.6875	0.7975	0.7450	0.8975	0.9650
	MMD	0.6150	0.6475	0.8050	0.7475	0.8975	0.9500

and $p = 50$ respectively. However, when only the scale difference is present, i.e., $\sigma^2 = 1.6$, the BD and MMD test procedures have better performance, the ED and our approach deteriorate quickly. For example, the BD test (and the MMD, ED and the new test) has power 0.6875 (and 0.6475, 0.4025 and 0.3700) for the setting with $n = m = 50$ and $p = 100$.

Example 2. In this example, we generate *i.i.d.* samples from the following models: $X_k \sim N_p(\mathbf{0}, \Sigma)$, $Y_l \sim 0.6 \cdot N_p(\mu_3, \Sigma) + 0.4 \cdot N_p(\mu_4, \Sigma)$, where X_k and Y_l are set to be the same as those in Example 1. Besides, three different choices are considered for $\Sigma = (\sigma_{ij})$: (1) $\sigma_{ii} = 1, \sigma_{ij} = 0$ $(i \neq j)$; (2) $\sigma_{ij} = 0.2^{|i-j|}$; (3) $\sigma_{ii} = 1, \sigma_{ij} = 0.2$ $(i \neq j)$, which correspond to Σ_1, Σ_2, and Σ_3 respectively. For Σ_1 and Σ_2, we choose $\mu_3 = (0.3, \cdots, 0.3)^\tau$ and $\mu_4 = (-0.3, \cdots, -0.3)^\tau$. For Σ_3, μ_3 and μ_4 are set to be $(0.5, \cdots, 0.5)^\tau$ and $(-0.5, \cdots, -0.5)^\tau$ respectively.

Table 2 summarizes the simulations results of different choices for Σ, n, m and p. From Table 2, we can see that: (i) The empirical powers of each test procedure improve quickly as p increases. This is perhaps because the deviations from H_0 are accumulating as p diverges. (ii) The CD test is superior to the other three competitive methods under the alternatives Σ_1 and Σ_2. For example, when $n = m = 50$ and $p = 100$, the new test (and the ED, BD and MMD test procedures) has power 0.9850 (and 0.1875, 0.9125 and 0.9050) for the setting with $\Sigma = \Sigma_1$; 0.9600 (and 0.2025, 0.8800 and 0.8100) for the setting with $\Sigma = \Sigma_2$. (iii) It can also be seen that, under the alternative Σ_3, BD test has the best power performance, CD test follows, and ED test is somewhat insensitive.

Example 3. We draw the p-vectors $X_k, k = 1, \cdots, n$, from $N_p(\mu_1, \Sigma_1)$ with $\mu_1 = \mathbf{0}_{p\times 1}$, $\Sigma_1 = (\sigma_{ij})$, $\sigma_{ii} = 1, \sigma_{ij} = 0.7$ $(i \neq j)$, and the p-vectors $Y_l, l = 1, \cdots, m$, from $N_p(\mu_2, \Sigma_2)$ with $\mu_2 = \delta\mathbf{1}_{p\times 1}$, and $\Sigma_2 = \sigma^2\Sigma_1$. We consider three scenarios for (δ, σ^2): (0, 1), (0.5, 1), (0.45, 1.3), which corresponds to the null hypothesis H_0, location shift, both location shift and scale difference, respectively.

Table 2: Empirical power for different test procedures.

Model	Methods	$n = m = 50$		$n = m = 70$		$n = m = 90$	
		$p = 50$	$p = 100$	$p = 50$	$p = 100$	$p = 50$	$p = 100$
Σ_1	CD	0.6325	0.9850	0.9625	1.0000	0.9700	1.0000
	ED	0.1375	0.1875	0.2075	0.2650	0.2200	0.2750
	BD	0.5750	0.9125	0.7425	0.9800	0.8625	0.9900
	MMD	0.4925	0.9050	0.6975	0.9875	0.8450	1.0000
Σ_2	CD	0.5275	0.9600	0.9225	1.0000	0.9300	1.0000
	ED	0.0900	0.2025	0.1850	0.2550	0.2000	0.3625
	BD	0.5525	0.8800	0.7725	0.9400	0.8200	1.0000
	MMD	0.3850	0.8100	0.6150	0.9450	0.7550	0.9500
Σ_3	CD	0.4625	0.5150	0.6900	0.8100	0.8075	0.9200
	ED	0.1625	0.1800	0.2225	0.2350	0.2850	0.4725
	BD	0.6150	0.6575	0.7800	0.8200	0.8750	1.0000
	MMD	0.3675	0.3900	0.5500	0.5900	0.6500	0.8675

Under the multivariate normal distribution, the empirical size and power of each test are reported in Table 3. As can be seen from Table 3, when two populations are identically distributed, the rejection probabilities of all tests are very close to the preselected significance level 0.05. When X and Y have different distributions, the results in Table 3 show that, ED test performs the best, BD and our proposed CD tests follow when $\delta = 0.5$ and $\sigma^2 = 1$, that is, when only the location shift is present. However, if there exists scale difference, i.e., $\sigma^2 = 1.3$, the BD test has the best power performance, our approach deteriorates quickly, indicating that the new test is not so sensitive to the scale difference under high correlations.

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Table 3: Empirical size and power for different test procedures.

Model	Methods	$n = m = 50$		$n = m = 70$		$n = m = 90$	
		$p = 50$	$p = 100$	$p = 50$	$p = 100$	$p = 50$	$p = 100$
(0, 1)	CD	0.0700	0.0575	0.0375	0.0650	0.0675	0.0550
	ED	0.0625	0.0575	0.0400	0.0500	0.0750	0.0300
	BD	0.0700	0.0525	0.0450	0.0550	0.0700	0.0525
	MMD	0.0700	0.0525	0.0425	0.0500	0.0600	0.0675
(0.5, 1)	CD	0.6750	0.6600	0.8500	0.7775	0.9375	0.9500
	ED	0.7500	0.7375	0.9075	0.8525	0.9550	0.9650
	BD	0.6750	0.6825	0.8500	0.7800	0.9100	0.9175
	MMD	0.4475	0.3725	0.6575	0.5350	0.7225	0.7275
(0.45, 1.3)	CD	0.5275	0.5075	0.7200	0.5825	0.8725	0.8775
	ED	0.6000	0.5950	0.8125	0.6650	0.8800	0.8875
	BD	0.6825	0.7000	0.8675	0.7700	0.9200	0.9625
	MMD	0.5725	0.6800	0.8000	0.8250	0.8925	0.9875