SUPPLEMENT TO "SIMULTANEOUS INFERENCE FOR THE DISTRIBUTION OF FUNCTIONAL PRINCIPAL COMPONENT SCORES"

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Abstract: This supplement provides additional tables and figures in Sections 5-6 and detailed proofs of the theoretical results with necessarily technical lemmas.

S1. Tables and Figures

Table S.1: Coverage frequencies and width of SCBs for the distribution of ξ_{i1} under standard normal distribution and (rescaled $t(10)$ distribution) based on the smooth estimator $\widehat{F}_{n1}$ and the nonsmooth estimator $\widehat{F}^{\dagger}_{n1}$ over 500 replications.

		Homogenous				Heteroscedastic			
		Smooth		Nonsmooth		Smooth		Nonsmooth	
N	ε	Coverage	Width	Coverage	Width	Coverage	Width	Coverage	Width
100	Normal	.976(.980)	.185(.195)	.968(.954)	.185(.195)	.976(.978)	.186(.195)	.962(.960)	.186(.195)
	Uniform	.976(.962)	.185(.195)	.962(.948)	.185(.195)	.978(.968)	.186(.195)	.976(.952)	.186(.195)
	Laplace	.976(.982)	.185(.194)	.960(.960)	.185(.194)	.978(.978)	.186(.195)	.962(.960)	.186(.195)
200	Normal	.954(.954)	.131(.138)	.936(.940)	.131(.138)	.952(.952)	.131(.137)	.944(.948)	.131(.137)
	Uniform	.956(.948)	.131(.138)	.940(.934)	.131(.138)	.952(.954)	.131(.137)	.940(.928)	.131(.137)
	Laplace	.958(.968)	.131(.138)	.942(.948)	.131(.138)	.960(.970)	.131(.138)	.948(.948)	.131(.138)
400	Normal	.950(.950)	.092(.097)	.936(.938)	.092(.097)	.950(.948)	.092(.097)	.936(.934)	.092(.097)
	Uniform	.948(.948)	.092(.097)	.942(.940)	.092(.097)	.950(.948)	.093(.097)	.946(.942)	.093(.097)
	Laplace	.952(.948)	.092(.097)	.938(.936)	.092(.097)	.956(.946)	.092(.097)	.942(.924)	.092(.097)
600	Normal	.956(.950)	.075(.079)	.956(.946)	.075(.079)	.958(.944)	.075(.079)	.960(.940)	.075(.079)
	Uniform	.962(.946)	.075(.079)	.952(.936)	.075(.079)	.964(.948)	.075(.079)	.954(.942)	.075(.079)
	Laplace	.956(.948)	.075(.079)	.952(.944)	.075(.079)	.958(.952)	.075(.079)	.950(.938)	.075(.079)

Table S.2: Coverage frequencies and width of SCBs for the distribution of ξ_{i2} under standard normal distribution and (rescaled $t(10)$ distribution) based on the smooth estimator $\widehat{F}_{n2}$ and the nonsmooth estimator $\widehat{F}^{\dagger}_{n2}$ over 500 replications.

		Homogenous				Heteroscedastic			
		Smooth		Nonsmooth		Smooth		Nonsmooth	
N	ε	Coverage	Width	Coverage	Width	Coverage	Width	Coverage	Width
100	Normal	.970(.966)	.185(.194)	.952(.956)	.185(.194)	.970(.970)	.185(.194)	.940(.952)	.185(.194)
	Uniform	.964(.962)	.185(.194)	.944(.940)	.185(.194)	.966(.952)	.185(.194)	.948(.944)	.185(.194)
	Laplace	.952(.954)	.185(.195)	.944(.932)	.185(.195)	.954(.954)	.185(.195)	.932(.928)	.185(.195)
200	Normal	.954(.966)	.131(.137)	.944(.948)	.131(.137)	.954(.962)	.131(.137)	.938(.956)	.131(.137)
	Uniform	.962(.970)	.131(.137)	.944(.952)	.131(.137)	.958(.966)	.131(.137)	.944(.944)	.131(.137)
	Laplace	.964(.968)	.131(.137)	.952(.962)	.131(.137)	.956(.968)	.131(.137)	.944(.954)	.131(.137)
400	Normal	.950(.950)	.092(.097)	.942(.942)	.092(.097)	.952(.946)	.092(.097)	.942(.934)	.092(.097)
	Uniform	.956(.956)	.092(.097)	.956(.936)	.092(.097)	.954(.954)	.092(.097)	.950(.942)	.092(.097)
	Laplace	.958(.950)	.092(.097)	.954(.944)	.092(.097)	.960(.956)	.092(.096)	.954(.936)	.092(.096)
600	Normal	.952(.960)	.075(.079)	.950(.956)	.075(.079)	.956(.962)	.075(.079)	.956(.954)	.075(.079)
	Uniform	.950(.960)	.075(.079)	.956(.964)	.075(.079)	.960(.962)	.075(.079)	.952(.962)	.075(.079)
	Laplace	.954(.954)	.075(.079)	.942(.948)	.075(.079)	.956(.956)	.075(.079)	.940(.948)	.075(.079)

Table S.3: Coverage frequencies and width of SCBs for the distribution of ξ_{i3} under standard normal distribution and (rescaled $t(10)$ distribution) based on the smooth estimator $\widehat{F}_{n3}$ and the nonsmooth estimator $\widehat{F}^{\dagger}_{n3}$ over 500 replications.

		Homogenous				Heteroscedastic			
		Smooth		Nonsmooth		Smooth		Nonsmooth	
N	ε	Coverage	Width	Coverage	Width	Coverage	Width	Coverage	Width
100	Normal	.978(.952)	.185(.192)	.956(.944)	.185(.192)	.982(.954)	.185(.192)	.946(.940)	.185(.192)
	Uniform	.962(.966)	.185(.192)	.950(.942)	.185(.192)	.970(.958)	.185(.193)	.948(.932)	.185(.193)
	Laplace	.960(.960)	.185(.192)	.944(.940)	.185(.192)	.964(.962)	.185(.192)	.946(.946)	.185(.192)
200	Normal	.964(.934)	.131(.137)	.950(.912)	.131(.137)	.962(.934)	.131(.137)	.940(.922)	.131(.137)
	Uniform	.958(.940)	.131(.137)	.942(.930)	.131(.137)	.952(.938)	.131(.137)	.940(.928)	.131(.137)
	Laplace	.964(.952)	.131(.137)	.940(.936)	.131(.137)	.972(.954)	.131(.137)	.948(.932)	.131(.137)
400	Normal	.954(.968)	.092(.097)	.940(.954)	.092(.097)	.950(.966)	.092(.096)	.942(.958)	.092(.096)
	Uniform	.954(.956)	.092(.097)	.948(.946)	.092(.097)	.950(.954)	.092(.097)	.944(.954)	.092(.097)
	Laplace	.952(.954)	.092(.096)	.952(.954)	.092(.096)	.960(.952)	.092(.096)	.940(.948)	.092(.096)
600	Normal	.950(.956)	.075(.079)	.942(.948)	.075(.079)	.948(.946)	.075(.079)	.942(.938)	.075(.079)
	Uniform	.952(.946)	.075(.079)	.940(.942)	.075(.079)	.954(.950)	.075(.079)	.940(.940)	.075(.079)
	Laplace	.946(.946)	.075(.079)	.938(.946)	.075(.079)	.950(.950)	.075(.079)	.944(.944)	.075(.079)

Table S.4: Coverage frequencies and width of SCBs for the distribution of ξ_{i4} under standard normal distribution and (rescaled $t(10)$ distribution) based on the smooth estimator $\widehat{F}_{n4}$ and the nonsmooth estimator $\widehat{F}_{n4}^{\dagger}$ over 500 replications.

		Homogenous				Heteroscedastic			
		Smooth		Nonsmooth		Smooth		Nonsmooth	
N	ε	Coverage	Width	Coverage	Width	Coverage	Width	Coverage	Width
100	Normal	.970(.948)	.186(.192)	.958(.936)	.186(.192)	.976(.960)	.186(.192)	.956(.926)	.186(.192)
	Uniform	.968(.958)	.186(.191)	.962(.928)	.186(.191)	.966(.958)	.186(.192)	.960(.932)	.186(.192)
	Laplace	.968(.966)	.186(.191)	.950(.950)	.186(.191)	.970(.968)	.185(.192)	.948(.948)	.185(.192)
200	Normal	.960(.964)	.131(.136)	.940(.944)	.131(.136)	.964(.952)	.131(.136)	.942(.936)	.131(.136)
	Uniform	.964(.952)	.131(.136)	.946(.940)	.131(.136)	.964(.956)	.131(.136)	.938(.948)	.131(.136)
	Laplace	.980(.968)	.131(.136)	.970(.942)	.131(.136)	.976(.968)	.131(.136)	.968(.950)	.131(.136)
400	Normal	.960(.960)	.092(.096)	.948(.960)	.092(.096)	.950(.968)	.092(.097)	.948(.962)	.092(.097)
	Uniform	.962(.950)	.092(.097)	.952(.950)	.092(.097)	.966(.944)	.092(.097)	.958(.936)	.092(.097)
	Laplace	.956(.950)	.092(.097)	.942(.948)	.092(.097)	.956(.948)	.092(.096)	.942(.950)	.092(.096)
600	Normal	.954(.948)	.075(.079)	.946(.948)	.075(.079)	.948(.952)	.075(.079)	.946(.936)	.075(.079)
	Uniform	.950(.944)	.075(.079)	.944(.950)	.075(.079)	.954(.948)	.075(.079)	.944(.942)	.075(.079)
	Laplace	.952(.960)	.075(.079)	.942(.958)	.075(.079)	.950(.950)	.075(.079)	.950(.950)	.075(.079)

Table S.5: Case 1: Empirical levels and powers of the JB-test (JB) in Górecki et al. (2020), the proposed KS-type tests (S_n, $S_n^{\dagger}$) and CVM-type tests (V_n, $V_n^{\dagger}$) under homoscedastic measurement error over 500 replications.

	ξ_{ik}	Normal (H_0)			Uniform (H_1)			Laplace (H_1)		
N	ε_{ij}	$S_n(S_n^{\dagger})$	$V_n(V_n^{\dagger})$	JB	$S_n(S_n^{\dagger})$	$V_n(V_n^{\dagger})$	JB	$S_n(S_n^{\dagger})$	$V_n(V_n^{\dagger})$	JB
100	Normal	0.044(0.052)	0.040(0.042)	0.072	0.476(0.604)	0.944(0.954)	0.468	0.098(0.172)	0.658(0.682)	0.868
	Uniform	0.040(0.058)	0.036(0.042)	0.074	0.470(0.600)	0.950(0.948)	0.448	0.116(0.152)	0.676(0.690)	0.858
	Laplace	0.040(0.056)	0.040(0.040)	0.078	0.482(0.590)	0.950(0.948)	0.470	0.132(0.188)	0.652(0.666)	0.866
200	Normal	0.044(0.056)	0.036(0.042)	0.044	0.952(0.982)	1.000(1.000)	1.000	0.598(0.672)	0.990(0.988)	0.990
	Uniform	0.042(0.050)	0.036(0.042)	0.040	0.960(0.976)	1.000(1.000)	1.000	0.598(0.650)	0.988(0.990)	0.990
	Laplace	0.046(0.054)	0.042(0.040)	0.042	0.954(0.972)	1.000(1.000)	1.000	0.564(0.650)	0.990(0.992)	0.986
400	Normal	0.036(0.046)	0.048(0.050)	0.058	1.000(1.000)	1.000(1.000)	1.000	0.998(0.998)	1.000(1.000)	1.000
	Uniform	0.042(0.048)	0.044(0.048)	0.062	1.000(1.000)	1.000(1.000)	1.000	0.996(0.998)	1.000(1.000)	1.000
	Laplace	0.042(0.044)	0.052(0.048)	0.062	1.000(1.000)	1.000(1.000)	1.000	0.998(0.998)	1.000(1.000)	1.000
600	Normal	0.040(0.036)	0.046(0.054)	0.050	1.000(1.000)	1.000(1.000)	1.000	1.000(1.000)	1.000(1.000)	1.000
	Uniform	0.050(0.052)	0.044(0.054)	0.054	1.000(1.000)	1.000(1.000)	1.000	1.000(1.000)	1.000(1.000)	1.000
	Laplace	0.046(0.040)	0.050(0.048)	0.050	1.000(1.000)	1.000(1.000)	1.000	1.000(1.000)	1.000(1.000)	1.000

Table S.6: Case 1: Empirical levels and powers of the JB-test (JB) in Górecki et al. (2020), the proposed KS-type tests (S_n, $S_n^\dagger$) and CVM-type tests (V_n, $V_n^\dagger$) under heteroscedastic measurement error over 500 replications.

	ξ_{ik}	Normal (H_0)			Uniform (H_1)			Laplace (H_1)		
N	ε_{ij}	$S_n(S_n^\dagger)$	$V_n(V_n^\dagger)$	JB	$S_n(S_n^\dagger)$	$V_n(V_n^\dagger)$	JB	$S_n(S_n^\dagger)$	$V_n(V_n^\dagger)$	JB
100	Normal	0.038(0.054)	0.044(0.042)	0.072	0.468(0.582)	0.946(0.948)	0.470	0.094(0.166)	0.684(0.698)	0.868
	Uniform	0.040(0.060)	0.038(0.044)	0.074	0.492(0.612)	0.948(0.952)	0.456	0.114(0.162)	0.670(0.686)	0.858
	Laplace	0.034(0.050)	0.038(0.038)	0.078	0.490(0.612)	0.950(0.946)	0.470	0.126(0.180)	0.654(0.676)	0.864
200	Normal	0.042(0.052)	0.032(0.040)	0.044	0.952(0.974)	1.000(1.000)	1.000	0.602(0.654)	0.990(0.990)	0.990
	Uniform	0.046(0.050)	0.034(0.036)	0.040	0.964(0.974)	1.000(1.000)	1.000	0.576(0.628)	0.992(0.992)	0.990
	Laplace	0.042(0.052)	0.040(0.034)	0.040	0.960(0.984)	1.000(1.000)	1.000	0.582(0.654)	0.990(0.994)	0.988
400	Normal	0.042(0.048)	0.048(0.052)	0.058	1.000(1.000)	1.000(1.000)	1.000	0.996(0.994)	1.000(1.000)	1.000
	Uniform	0.040(0.050)	0.048(0.046)	0.060	1.000(1.000)	1.000(1.000)	1.000	0.998(1.000)	1.000(1.000)	1.000
	Laplace	0.046(0.048)	0.050(0.046)	0.062	1.000(1.000)	1.000(1.000)	1.000	0.998(1.000)	1.000(1.000)	1.000
600	Normal	0.040(0.044)	0.044(0.050)	0.052	1.000(1.000)	1.000(1.000)	1.000	1.000(1.000)	1.000(1.000)	1.000
	Uniform	0.042(0.050)	0.050(0.054)	0.054	1.000(1.000)	1.000(1.000)	1.000	1.000(1.000)	1.000(1.000)	1.000
	Laplace	0.044(0.038)	0.052(0.054)	0.050	1.000(1.000)	1.000(1.000)	1.000	1.000(1.000)	1.000(1.000)	1.000

Table S.7: Case 2: Empirical levels and powers of the proposed KS-type tests (S_n, $S_n^\dagger$) and CVM-type tests (V_n, $V_n^\dagger$) under homoscedastic measurement error over 500 replications.

	ξ_{ik}	$t(10)$ (H_0)		Uniform (H_1)		Laplace (H_1)	
N	ε_{ij}	$S_n(S_n^\dagger)$	$V_n(V_n^\dagger)$	$S_n(S_n^\dagger)$	$V_n(V_n^\dagger)$	$S_n(S_n^\dagger)$	$V_n(V_n^\dagger)$
100	Normal	0.030(0.058)	0.046(0.046)	0.850(0.916)	1.000(1.000)	0.064(0.094)	0.232(0.240)
	Uniform	0.038(0.044)	0.048(0.052)	0.846(0.902)	1.000(1.000)	0.070(0.082)	0.256(0.264)
	Laplace	0.030(0.042)	0.040(0.046)	0.84(0.918)	1.000(1.000)	0.054(0.084)	0.230(0.252)
200	Normal	0.040(0.036)	0.048(0.046)	1.000(1.000)	1.000(1.000)	0.258(0.314)	0.742(0.762)
	Uniform	0.030(0.040)	0.046(0.052)	1.000(1.000)	1.000(1.000)	0.270(0.326)	0.716(0.742)
	Laplace	0.032(0.044)	0.050(0.046)	1.000(1.000)	1.000(1.000)	0.254(0.298)	0.728(0.748)
400	Normal	0.050(0.050)	0.048(0.050)	1.000(1.000)	1.000(1.000)	0.854(0.846)	1.000(0.998)
	Uniform	0.044(0.046)	0.050(0.050)	1.000(1.000)	1.000(1.000)	0.804(0.832)	0.998(1.000)
	Laplace	0.048(0.052)	0.048(0.054)	1.000(1.000)	1.000(1.000)	0.836(0.850)	0.998(0.998)
600	Normal	0.054(0.064)	0.052(0.052)	1.000(1.000)	1.000(1.000)	0.998(0.994)	1.000(1.000)
	Uniform	0.046(0.054)	0.042(0.052)	1.000(1.000)	1.000(1.000)	0.988(0.992)	1.000(1.000)
	Laplace	0.048(0.064)	0.048(0.046)	1.000(1.000)	1.000(1.000)	0.996(0.990)	1.000(1.000)

Table S.8: Case 2: Empirical levels and powers of the proposed KS-type tests (S_n, $S_n^\dagger$) and CVM-type tests (V_n, $V_n^\dagger$) under heteroscedastic measurement error over 500 replications.

	ξ_{ik}	$t(10)$ (H_0)		Uniform (H_1)		Laplace (H_1)	
N	ε_{ij}	$S_n(S_n^\dagger)$	$V_n(V_n^\dagger)$	$S_n(S_n^\dagger)$	$V_n(V_n^\dagger)$	$S_n(S_n^\dagger)$	$V_n(V_n^\dagger)$
100	Normal	0.038(0.048)	0.040(0.046)	0.854(0.918)	1.000(1.000)	0.068(0.094)	0.238(0.250)
	Uniform	0.038(0.040)	0.048(0.048)	0.87(0.908)	1.000(1.000)	0.064(0.086)	0.238(0.246)
	Laplace	0.028(0.048)	0.044(0.050)	0.836(0.908)	1.000(1.000)	0.064(0.084)	0.240(0.242)
200	Normal	0.036(0.034)	0.040(0.052)	1.000(1.000)	1.000(1.000)	0.262(0.316)	0.736(0.756)
	Uniform	0.024(0.038)	0.040(0.048)	1.000(1.000)	1.000(1.000)	0.270(0.322)	0.734(0.750)
	Laplace	0.034(0.044)	0.044(0.050)	1.000(1.000)	1.000(1.000)	0.250(0.304)	0.730(0.740)
400	Normal	0.042(0.050)	0.052(0.048)	1.000(1.000)	1.000(1.000)	0.842(0.848)	1.000(1.000)
	Uniform	0.046(0.056)	0.048(0.050)	1.000(1.000)	1.000(1.000)	0.828(0.836)	1.000(1.000)
	Laplace	0.044(0.050)	0.050(0.052)	1.000(1.000)	1.000(1.000)	0.848(0.848)	1.000(1.000)
600	Normal	0.062(0.060)	0.050(0.046)	1.000(1.000)	1.000(1.000)	0.990(0.994)	1.000(1.000)
	Uniform	0.044(0.040)	0.046(0.050)	1.000(1.000)	1.000(1.000)	0.994(0.996)	1.000(1.000)
	Laplace	0.056(0.060)	0.052(0.052)	1.000(1.000)	1.000(1.000)	0.998(0.992)	1.000(1.000)

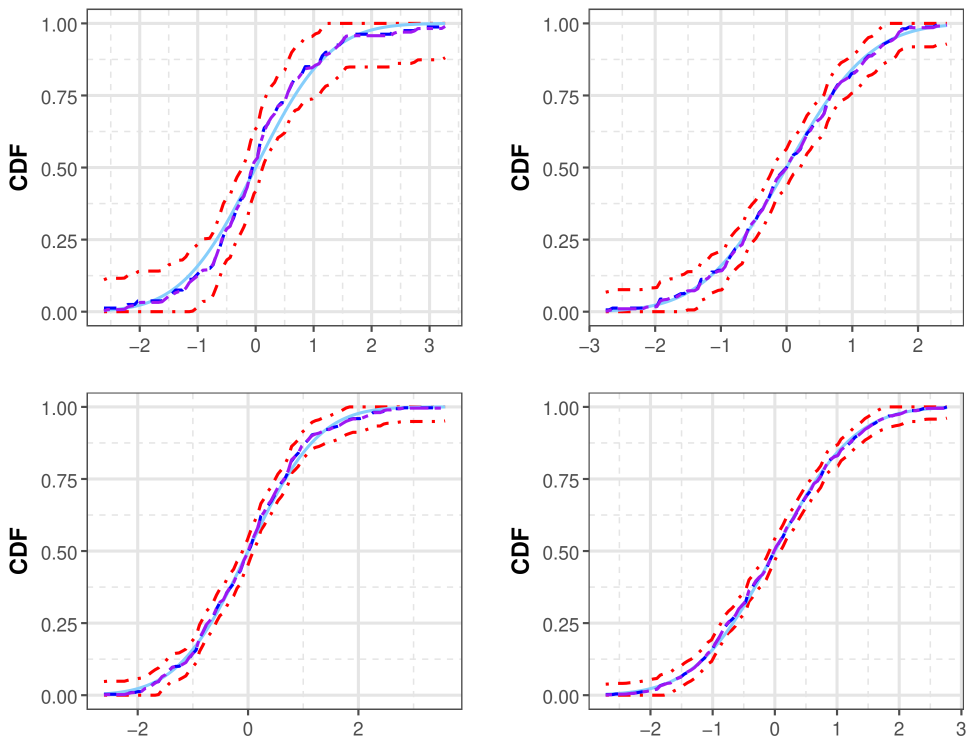

Figure S.1: Smooth estimator (purple, double dashed line), nonsmooth estimator (blue, dashed line), and 95% smooth simultaneous confidence bands (red, dot-dashed line) for true CDF of Normal ξ_{11}(skylightblue, solid line). The sample sizes from left to right are $N = 100, 200, 400, 600$ respectively.

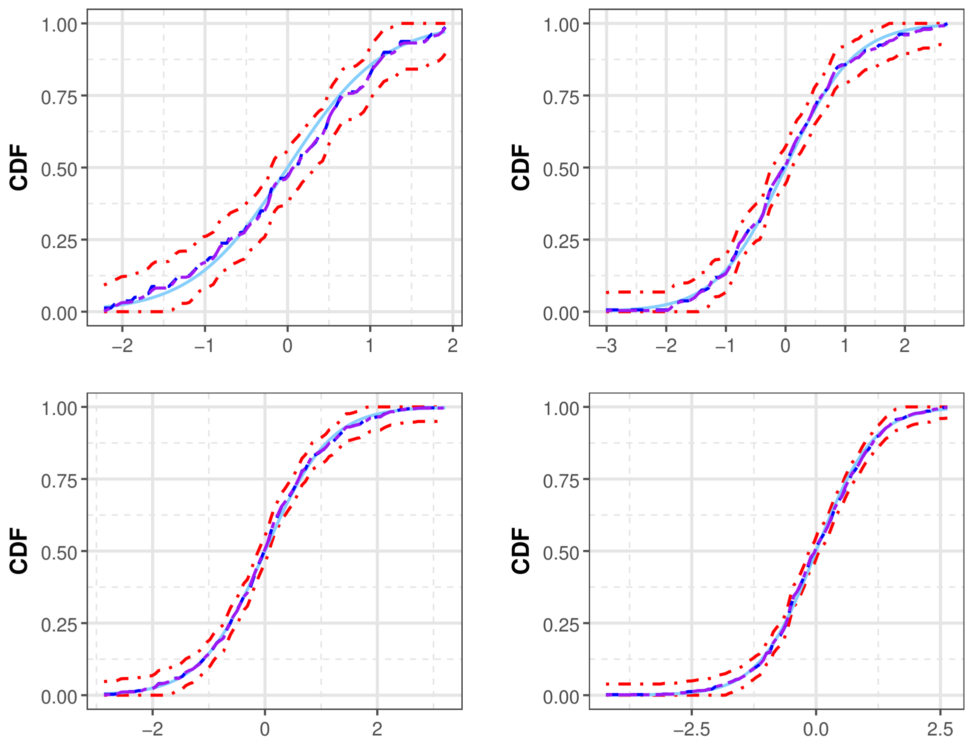

Figure S.2: Smooth estimator (purple, double dashed line), nonsmooth estimator (blue, dashed line), and 95% smooth simultaneous confidence bands (red, dot-dashed line) for true CDF of recaled $t(10)$ ξ_{11}(skylightblue, solid line). The sample sizes from left to right are $N = 100, 200, 400, 600$ respectively.

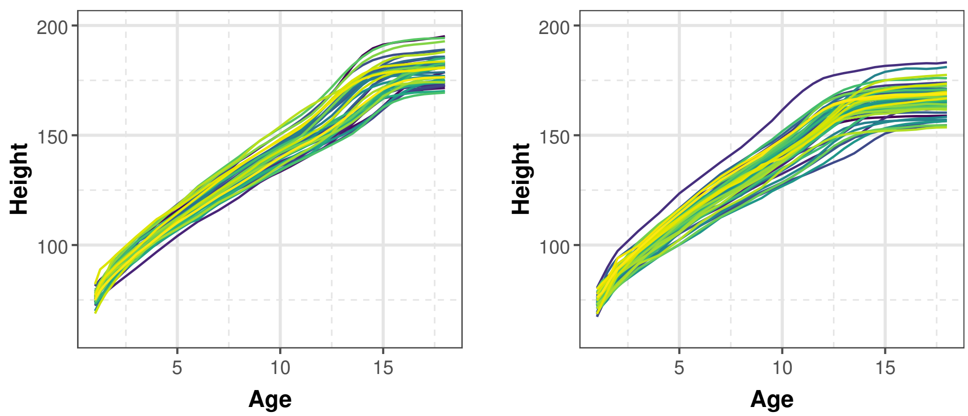

Figure S.3: Left panel: the growth curves of 39 boys. Right panel: the growth curves of 54 girls.

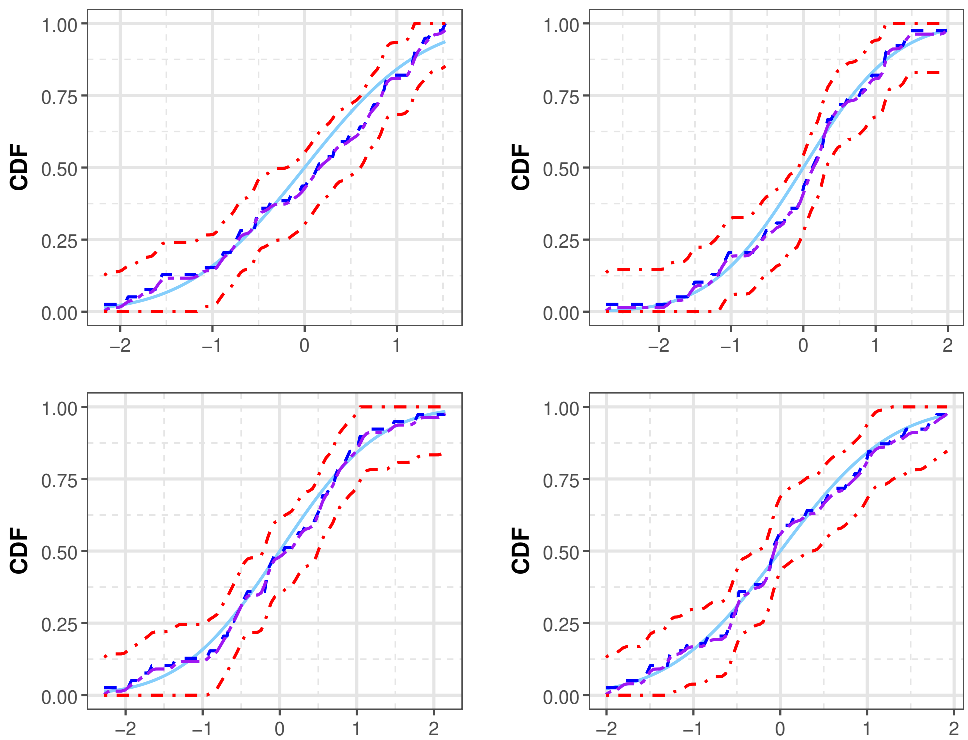

Figure S.4: CDF of standard normal random variable (skylightblue solid line), Smooth estimator (purple double dashed line), nonsmooth estimator (blue dashed line), and 95% smooth simultaneous confidence bands (red dot-dashed line) for CDF of the first four FPC scores (from left to right, $k = 1, 2, 3, 4$) of the growth curves of the boys.

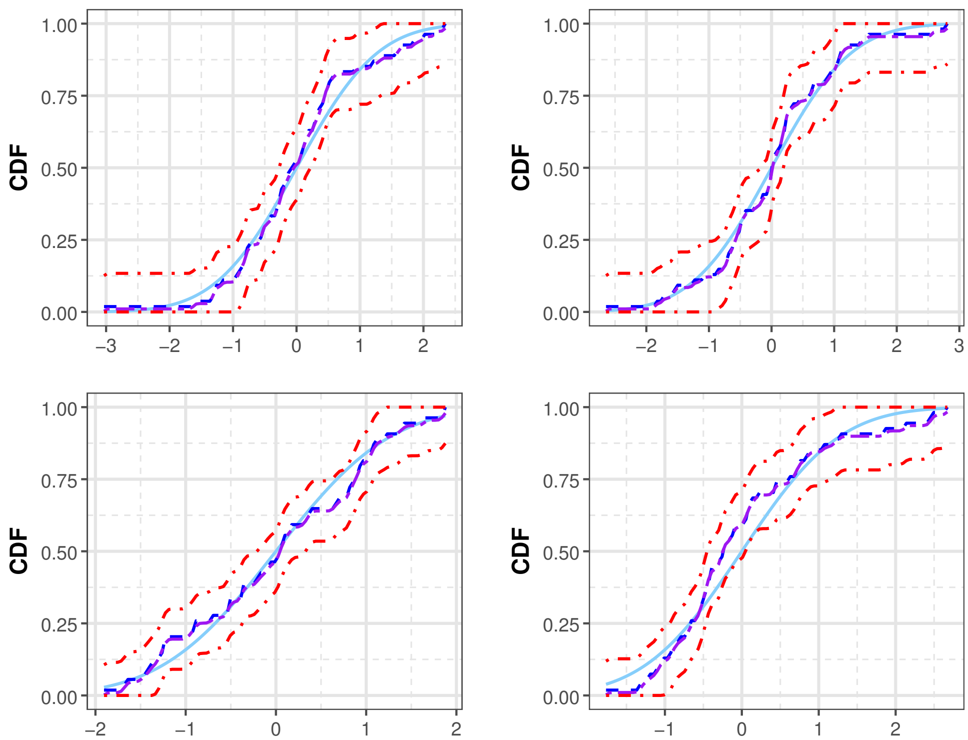

Figure S.5: CDF of standard normal random variable (skylightblue solid line), Smooth estimator (purple double dashed line), nonsmooth estimator (blue dashed line), and 95% smooth simultaneous confidence bands (red dot-dashed line) for CDF of the first four FPC scores (from left to right, $k = 1, 2, 3, 4$) of the growth curves of the girls.

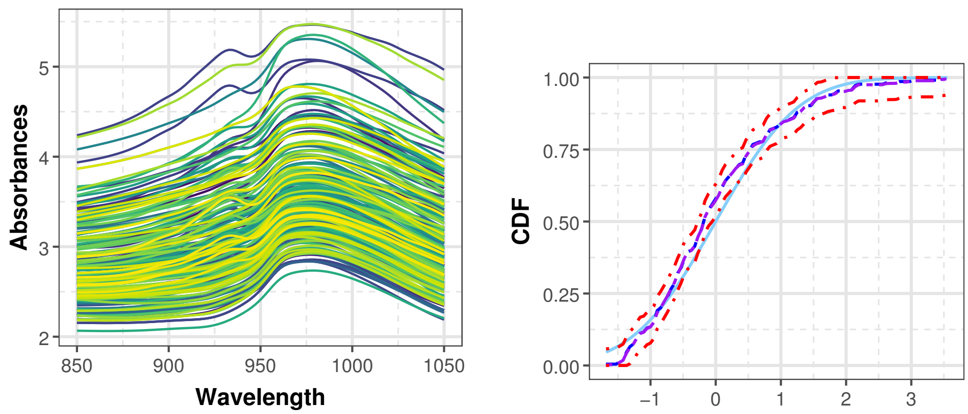

Figure S.6: Left: spectral curves in the *tecator* data set. Right:CDF of standard normal random variable (skylightblue solid line), Smooth estimator (purple twodash line), nonsmooth estimator (blue dash line), and 95% smooth simultaneous confidence bands (red dotdash line) for CDF of the first FPC scores ξ_{i1} of the spectral curves.

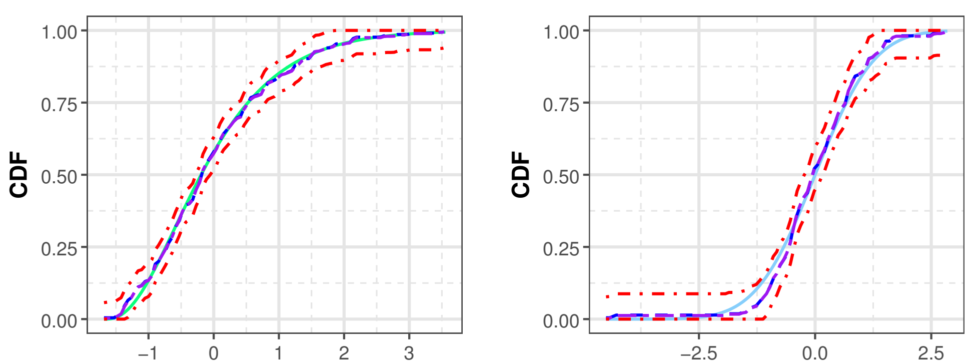

Figure S.7: CDF of centralized gamma distribution with the shape parameter 2.831 and the scale parameter 0.594 (green solid line). CDF of standard normal random variable (skylightblue solid line). Smooth estimator (purple double dashed line), nonsmooth estimator (blue dashed line), and 95% smooth simultaneous confidence bands (red dot-dashed line) for CDF of the first two FPC scores (from left to right, $k = 1, 2$), ξ_{i1} and ξ_{i2}, of the spectral curves.

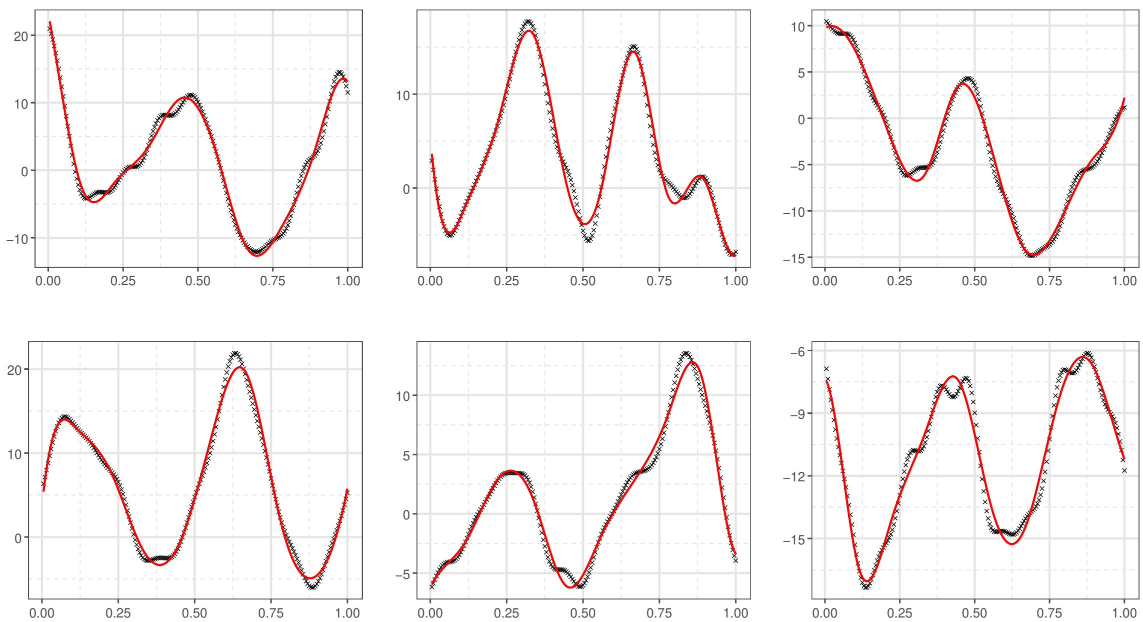

Figure S.8: Randomly selected segments of raw EEG data (black crosses), cubic spline estimated trajectories (red solid lines), in Case 2.

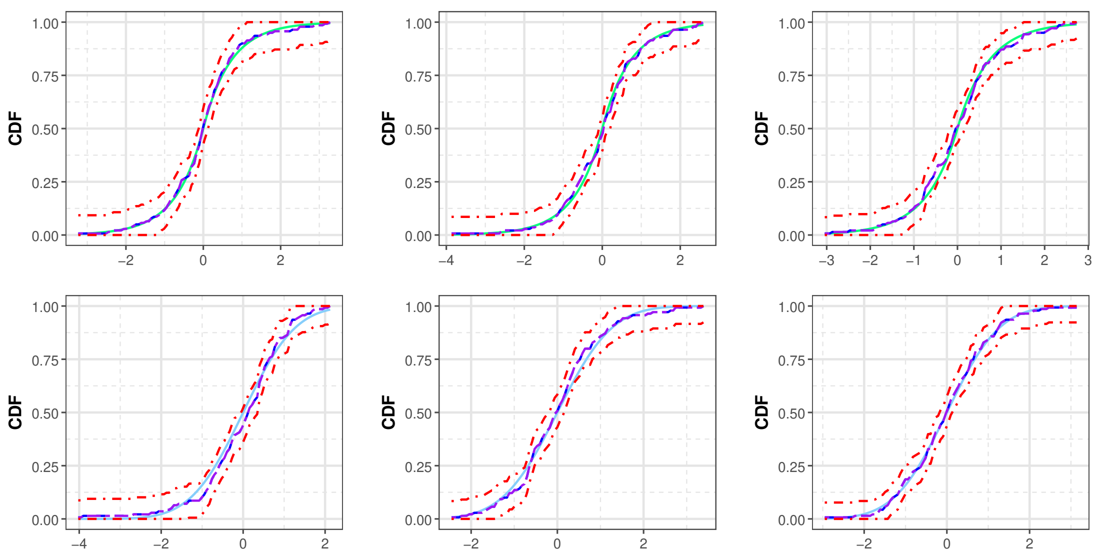

Figure S.9: Smooth estimator (purple double dashed line), nonsmooth estimator (blue), 95% smooth simultaneous confidence bands (red dot-dashed line), CDF of standardized Laplace distribution (green solid line) and CDF of standard normal random variables (skylightblue solid line). From left to right, top to bottom, $k = 1, 2, 3, 4, 5, 6$.

S2. Technical proofs

Denote

$$\zeta_{nk}(x) := \frac{1}{n}\sum_{i=1}^{n}\left\{I\left(\xi_{ik} \le x\right) - F_k(x) + \xi_{ik} f_k(x) + \frac{1}{2} x f_k(x)\left(\xi_{ik}^2 - 1\right)\right\}.$$

Lemma S.1. *Under Assumptions (A2) - (A3) and (A5) - (A7), as $n \to \infty$,*

$$\max_{1\le k\le \kappa_n} \sup_{x\in\mathbb{R}} \sqrt{n}\left|\widetilde{F}_{nk}(x) - F_k(x) - \zeta_{nk}(x)\right| = o_p(1).$$

Proof: Define

$$\begin{aligned}
F_{nk}(x) &= \frac{1}{n}\sum_{i=1}^{n} I\left(\xi_{ik} \le x\right),\\
\breve{F}_{nk}(x) &= \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{x} K_h(u - \xi_{ik})du,\\
\zeta_{0nk}(x) &= \frac{1}{n}\sum_{i=1}^{n}\left\{\xi_{ik} f_k(x) + \frac{1}{2} x f_k(x)\left(\xi_{ik}^2 - 1\right)\right\}.
\end{aligned}$$

The difference $\widetilde{F}_{nk}(x) - \breve{F}_{nk}(x)$ is decomposed by Taylor Expansion as

$$\widetilde{F}_{nk}(x) - \breve{F}_{nk}(x) = T_{k1}(x) + T_{k2}(x)/2 + T_{k3}(x)/6,$$

with

$$T_{k1}(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik})\left(\xi_{ik}-\widetilde{\xi}_{ik}\right),$$
$$T_{k2}(x) = \frac{1}{n}\sum_{i=1}^{n} K'\left(\frac{x-\xi_{ik}}{h}\right)\left(\frac{\xi_{ik}-\widetilde{\xi}_{ik}}{h}\right)^2,$$
$$T_{k2}(x) = \frac{1}{n}\sum_{i=1}^{n} K''\left(\frac{x-\varrho_{ik}}{h}\right)\left(\frac{\xi_{ik}-\widetilde{\xi}_{ik}}{h}\right)^3,$$

in which ϱ_{ik} is a point between ξ_{ik} and $\widetilde{\xi}_{ik}$.

We rewrite $\widetilde{\xi}_{ik}$ as follows

$$\begin{aligned}
\widetilde{\xi}_{ik} &= \widetilde{\lambda}_k^{-1/2}\int_{\mathbb{R}} (\eta_i(x)-\widetilde{m}(x))\,\widetilde{\psi}_k(x)dx \\
&= \widetilde{\lambda}_k^{-1/2}\int_{\mathbb{R}} (\eta_i(x)-m(x))\,\psi_k(x)dx + \widetilde{\lambda}_k^{-1/2}\int_{\mathbb{R}} (m(x)-\widetilde{m}(x))\,\psi_k(x)dx \\
&+ \widetilde{\lambda}_k^{-1/2}\int_{\mathbb{R}} (\eta_i(x)-m(x))\left(\widetilde{\psi}_k(x)-\psi_k(x)\right)dx \\
&+ \widetilde{\lambda}_k^{-1/2}\int_{\mathbb{R}} (m(x)-\widetilde{m}(x))\left(\widetilde{\psi}_k(x)-\psi_k(x)\right)dx \\
&=: \widetilde{\lambda}_k^{-1/2}(E_{ik1}+E_{ik2}+E_{ik3}+E_{ik4}) \\
&= \lambda_k^{-1/2}\sum_{l=1}^{4} E_{ikl} + \left(\widetilde{\lambda}_k^{-1/2}-\lambda_k^{-1/2}\right)\sum_{l=1}^{4} E_{ikl},
\end{aligned} \tag{S2.1}$$

where

$$E_{ik1} = \int_{\mathbb{R}} (\eta_i(x)-m(x))\,\psi_k(x)dx, \quad E_{ik2} = \int_{\mathbb{R}} (m(x)-\widetilde{m}(x))\,\psi_k(x)dx,$$

$$E_{ik3} = \int_{\mathbb{R}} (\eta_i(x) - m(x)) \left(\widetilde{\psi}_k(x) - \psi_k(x)\right) dx,$$
$$E_{ik4} = \int_{\mathbb{R}} (m(x) - \widetilde{m}(x)) \left(\widetilde{\psi}_k(x) - \psi_k(x)\right) dx.$$

Considering the following,

$$\begin{aligned}
&\sqrt{n}\left|\widetilde{F}_{nk}(x) - F_k(x) - \zeta_{nk}(x)\right| \\
=&\sqrt{n}\left|\widetilde{F}_{nk}(x) - \breve{F}_{nk}(x) - \zeta_{0nk}x) + \breve{F}_{nk}(x) - F_{nk}(x)\right| \\
\leq&\sqrt{n}\left|\widetilde{F}_{nk}(x) - \breve{F}_{nk}(x) - \zeta_{0nk}(x)\right| + \sqrt{n}\left|\breve{F}_{nk}(x) - F_{nk}(x)\right|.
\end{aligned}$$

By similar arguments as those given in the proof of Lemmas B.3, B.4 and B.6 - B.9, we could show that

$$\max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \sqrt{n}\left|\widetilde{F}_{nk}(x) - \breve{F}_{nk}(x) - \zeta_{0nk}(x)\right| = \mathcal{o}_p(1).$$

Similar to the proof of Theorem 2.1 in Wang et al. (2013), we obtain

$$\max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \sqrt{n}\left|\breve{F}_{nk}(x) - F_{nk}(x)\right| = \mathcal{o}_p(1),$$

which completes the proof.

Lemma S.2. *Under Assumptions (A3) and (A6), for any $k \in \mathbb{Z}_+$, $\sqrt{n}\zeta_{nk}(x)$ weakly converges to a Gaussian process $\zeta_k(x)$ with mean zero and covariance function $\Sigma_k(\cdot,\cdot)$ as $n \to \infty$,*

$$\sqrt{n}\zeta_{nk}(x) \xrightarrow{d} \zeta_k(x),$$

$$\begin{aligned}\Sigma_k(s,t) = & F_k(s \wedge t) - F_k(s)F_k(t) + f_k(s)f_k(t) + f_k(s)\mathbb{E}\left(\xi_{ik} I\left(\xi_{ik} \le t\right)\right) \\ & + f_k(t)\mathbb{E}\left(\xi_{ik} I\left(\xi_{ik} \le s\right)\right) + \frac{1}{4}\left(\mathbb{E}\xi_{ik}^4 - 1\right) st f_k(s) f_k(t) \\ & + \frac{1}{2} s f_k(s)\mathbb{E}\left((\xi_{ik}^2 - 1) I\left(\xi_{ik} \le t\right)\right) + \frac{1}{2} t f_k(t)\mathbb{E}\left((\xi_{ik}^2 - 1) I\left(\xi_{ik} \le s\right)\right) \\ & + \frac{1}{2} s f_k(s) f_k(t)\mathbb{E}\xi_{ik}^3 + \frac{1}{2} t f_k(t) f_k(s)\mathbb{E}\xi_{ik}^3, \quad s,t \in \mathbb{R}.\end{aligned}$$

Proof: We rewrite

$$\begin{aligned}\sqrt{n}\zeta_{nk}(x) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(I\left(\xi_{ik} \le x\right) - F_k(x) + \xi_{ik} f_k(x) + \frac{1}{2} x f_k(x)\left(\xi_{ik}^2 - 1\right)\right) \\ &=: \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\Xi_{k,x}(\xi_{ik}) - \mathbb{E}\Xi_{k,x}(\xi_{ik})\right),\end{aligned}$$

where

$$\Xi_{k,x}(\xi_{ik}) = I\left(\xi_{ik} \le x\right) + \xi_{ik} f_k(x) + \frac{1}{2} x f_k(x)\left(\xi_{ik}^2 - 1\right).$$

Let $l^\infty(\mathcal{G}_k)$ denote the space of all bounded functions from a set $\mathcal{G}_k$ to $\mathbb{R}$ equipped with the supremum norm $\|f\|_{\mathcal{G}_k} = \sup_{g \in \mathcal{G}_k}|f(g)|$, where $\mathcal{G}_k = \{\Xi_{k,x}(\cdot), x \in \mathbb{R}\}$, for any fixed $k \in \mathbb{Z}_+$. We observe that

$$\sqrt{n}\zeta_{nk}(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\Xi_{k,x}(\xi_{ik}) - \mathbb{E}\Xi_{k,x}(\xi_{ik})\right),$$

is a $\mathcal{G}_k$-index empirical process in $l^\infty(\mathcal{G}_k)$. Thus, it suffices to verify that the class $\mathcal{G}_k$ is Donsker. Following Theorem 2.5.2 and Theorem 2.6.8 in Vaart and Wellner (1996), we need to check that $\mathcal{G}_k$ satisfies the uniform entropy bound, has an envelope function G_k with finite second moment, and is also pointwise separable.

Pointwise separability of $\mathcal{G}_k$ is defined in Chapter 2.3.3 of Vaart and Wellner (1996). Define the sub-class $\mathcal{G}_{k0} = \{\Xi_{k,x}(\cdot), x \in \mathbb{Q}\}$, that is a countable dense subset of $\mathcal{G}_k$. For any sequence $x_m \in \mathbb{Q}$ decreasingly approaching x as $m \to \infty$, and $\Xi_{k,x}(\cdot) \in \mathcal{G}_k$, consider the sequence $\Xi_{k,x_m}(\cdot) \in \mathcal{G}_{k0}$. On the one hand, the sequence $\Xi_{k,x_m}(\cdot)$ satisfies that $\Xi_{k,x_m}(\xi) \to \Xi_{k,x}(\xi)$ pointwisely, as $m \to \infty$, by noticing the fact that $\Xi_{k,x}(\cdot)$ is right continuous for any $x \in \mathbb{R}$. On the other hand, the continuity of the mean and covariance function of the process $\Xi_{k,x}(\xi_{ik})$, $x \in \mathbb{R}$, entails the mean-square continuity by Theorem 7.3.2 in Hsing and Eubank (2015), i.e.,

$$\mathbb{E}\left(\Xi_{k,x_m}(\xi_{ik}) - \Xi_{k,x}(\xi_{ik})\right)^2 \to 0.$$

Hence, pointwise separability of $\mathcal{G}_k$ is verified.

Denote that

$$\mathcal{G}_{k1} = \left\{ \xi \to I\left(\xi \le x\right), x \in \mathbb{R} \right\},$$
$$\mathcal{G}_{k2} = \left\{ \xi \to \xi f_k(x) + \frac{1}{2} x f_k(x) \left(\xi^2 - 1\right), x \in \mathbb{R} \right\}.$$

Clearly, $\mathcal{G}_{k1}$ is a VC-class with $\mathrm{VC}(\mathcal{G}_{k1}) = 2$ by definition. Further notice that $\mathcal{G}_{k2}$ is a finite dimsional vector space with $\dim(\mathcal{G}_{k2}) = 2$. By Lemma 2.6.15 in Vaart and Wellner (1996), we obtain $\mathrm{VC}(\mathcal{G}_{k2}) \le 3$. Thus, one could derive that $\mathcal{G}_k$ satisfies the uniform entropy bound condition with the help of Lemma 7.21 in Sen (2018). Lastly, the finite second moment of envelope function is ensured by the uniform boundness of $x f_k(x)$ and $f_k(x)$, and the condition $\sup_k \mathbb{E}\xi_{ik}^4 < \infty$. The proof is completed.

Proof of Theorem 1 : Theorem 1 follows directly from Lemma S.1 and Lemma S.2.

Lemma S.3. *Under Assumptions (A1) - (A7), as* $n \to \infty$,

$$\max_{1 \le k \le \kappa_n} \sup_{x \in \mathbb{R}} \sqrt{n} \left| \widehat{F}_{nk}(x) - F_k(x) - \zeta_{nk}(x) \right| = o_p(1).$$

Proof: Recall the definition of $F_{nk}(x)$, $\breve{F}_{nk}(x)$ and $\zeta_{0nk}(x)$ in the proof of Lemma

S.1. The difference $\widehat{F}_{nk}(x) - \breve{F}_{nk}(x)$ is decomposed by Mean Value theorem as

$$\widehat{F}_{nk}(x) - \breve{F}_{nk}(x) = T_{k1}(x) + T_{k2}(x)/2 + T_{k3}(x)/6,$$

with

$$\begin{aligned} T_{k1}(x) &= \frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik})\left(\xi_{ik}-\widehat{\xi}_{ik}\right), \\ T_{k2}(x) &= \frac{1}{n}\sum_{i=1}^{n} K'\left(\frac{x-\xi_{ik}}{h}\right)\left(\frac{\xi_{ik}-\widehat{\xi}_{ik}}{h}\right)^2, \\ T_{k2}(x) &= \frac{1}{n}\sum_{i=1}^{n} K''\left(\frac{x-\varrho_{ik}}{h}\right)\left(\frac{\xi_{ik}-\widehat{\xi}_{ik}}{h}\right)^3. \end{aligned}$$

in which ϱ_{ik} is a point between ξ_{ik} and $\widehat{\xi}_{ik}$.

We rewrite $\widehat{\xi}_{ik}$ as follows

$$\begin{aligned}
\widehat{\xi}_{ik} &= \frac{1}{N}\sum_{j=1}^{N}\widehat{\lambda}_k^{-1/2}(Y_{ij} - \widehat{m}\left(\frac{j}{N}\right))\widehat{\psi}_k\left(\frac{j}{N}\right) \\
&= \frac{1}{N}\sum_{j=1}^{N}\widehat{\lambda}_k^{-1/2}\left(\eta_i\left(\frac{j}{N}\right) - m\left(\frac{j}{N}\right)\right)\psi_k\left(\frac{j}{N}\right) + \frac{1}{N}\sum_{j=1}^{N}\widehat{\lambda}_k^{-1/2}\sigma\left(\frac{j}{N}\right)\varepsilon_{ij}\psi_k\left(\frac{j}{N}\right) \\
&+ \frac{1}{N}\sum_{j=1}^{N}\widehat{\lambda}_k^{-1/2}\left(m\left(\frac{j}{N}\right) - \widehat{m}\left(\frac{j}{N}\right)\right)\psi_k\left(\frac{j}{N}\right) \\
&+ \frac{1}{N}\sum_{j=1}^{N}\widehat{\lambda}_k^{-1/2}\left(\eta_i\left(\frac{j}{N}\right) - m\left(\frac{j}{N}\right)\right)\left(\widehat{\psi}_k\left(\frac{j}{N}\right) - \psi_k\left(\frac{j}{N}\right)\right) \\
&+ \frac{1}{N}\sum_{j=1}^{N}\widehat{\lambda}_k^{-1/2}\sigma\left(\frac{j}{N}\right)\varepsilon_{ij}\left(\widehat{\psi}_k\left(\frac{j}{N}\right) - \psi_k\left(\frac{j}{N}\right)\right) \\
&+ \frac{1}{N}\sum_{j=1}^{N}\widehat{\lambda}_k^{-1/2}\left(m\left(\frac{j}{N}\right) - \widehat{m}\left(\frac{j}{N}\right)\right)\left(\widehat{\psi}_k\left(\frac{j}{N}\right) - \psi_k\left(\frac{j}{N}\right)\right) \\
&=: \widehat{\lambda}_k^{-1/2}(D_{ik1} + D_{ik2} + D_{ik3} + D_{ik4} + D_{ik5} + D_{ik6}) \\
&= \lambda_k^{-1/2}\sum_{l=1}^{6}D_{ikl} + \left(\widehat{\lambda}_k^{-1/2} - \lambda_k^{-1/2}\right)\sum_{l=1}^{6}D_{ikl}.
\end{aligned} \tag{S2.2}$$

Considering the fact that

$$\begin{aligned}
&\sqrt{n}\left|\widehat{F}_{nk}(x) - F_k(x) - \zeta_{nk}(x)\right| \\
=&\sqrt{n}\left|\widehat{F}_{nk}(x) - \breve{F}_{nk}(x) - \zeta_{0nk}x) + \breve{F}_{nk}(x) - F_{nk}(x)\right| \\
\leq&\sqrt{n}\left|\widehat{F}_{nk}(x) - \breve{F}_{nk}(x) - \zeta_{0nk}(x)\right| + \sqrt{n}\left|\breve{F}_{nk}(x) - F_{nk}(x)\right|.
\end{aligned} \tag{S2.3}$$

Applying Lemma B.1 - Lemma B.9, we could show that

$$\max_{1\le k\le \kappa_n}\sup_{x\in\mathbb{R}}\sqrt{n}\left|\widehat{F}_{nk}(x)-\breve{F}_{nk}(x)-\zeta_{0nk}(x)\right|=o_p(1).$$

Similar to the proof of Theorem 2.1 in Wang et al. (2013), we obtain

$$\max_{1\le k\le \kappa_n}\sup_{x\in\mathbb{R}}\sqrt{n}\left|\breve{F}_{nk}(x)-F_{nk}(x)\right|=o_p(1),$$

which completes the proof.

Proof of Theorem 2 : Theorem 2 follows directly from Lemma S.1 and Lemma S.3.

Proof of Theorem 3 Theorem 3 is a direct result from Theorem 1, Theorem 2 and the Slutsky theorem.

Proof for Theorem 4 :

For any $1\le k\le \kappa_n$, recall that $l^\infty(\mathcal{G}_k)$ denote the space of all bounded functions from the set $\mathcal{G}_k$ to $\mathbb{R}$ equipped with the supremum norm $\|\varphi\|_{\mathcal{G}_k}=\sup_{g\in\mathcal{G}_k}|\varphi(g)|$, where $\mathcal{G}_k=\{\Xi_{k,x}(\cdot), x\in\mathbb{R}\}$. Then, $\left(l^\infty(\mathcal{G}_k),\|\cdot\|_{\mathcal{G}_k}\right)$ is a Banach space. Besides, we have $\mathcal{G}_k \subset L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_k)$, where $\mathbb{P}_k$ is the probability measure induced by F_k. Recalling the definition of $\mathcal{G}_{k1}$ and $\mathcal{G}_{k2}$ in the proof of Lemma S.2, we know that for any $g \in \mathcal{G}_k$, there exist $g_1 \in \mathcal{G}_{k1}$ and $g_2 \in \mathcal{G}_{k2}$ such that $g = g_1 + g_2$. Besides, the covering number satisfy that

$$\mathcal{N}\left(\epsilon, \mathcal{G}_{k1}, L^2(\mathbb{P}_k)\right) \leq C\epsilon^{-4}, \quad \mathcal{N}\left(\epsilon, \mathcal{G}_{k2}, L^2(\mathbb{P}_k)\right) \leq C\epsilon^{-6},$$

for some universal constant $C > 0$. It means that, for each $g_1 \in \mathcal{G}_{k1}$, there is a $g_{1,l_1} \in \mathcal{G}_{k1}$, $l_1 = l_1(g_1)$ and $1 \leq l_1 \leq \mathcal{N}(\delta, \mathcal{G}_{k1}, L^2(\mathbb{P}_k))$, such that

$$\int \left(g_1(u) - g_{1,l_1}(u)\right)^2 F_k(du) \leq \delta^2. \tag{S2.4}$$

Similarly, for each $g_2 \in \mathcal{G}_{k2}$, there is a $g_{2,l_2} \in \mathcal{G}_{k2}$, $l_2 = l_2(g_2)$ and $1 \leq l_2 \leq \mathcal{N}(\delta, \mathcal{G}_{k2}, L^2(\mathbb{P}_k))$, such that

$$\int \left(g_2(u) - g_{2,l_2}(u)\right)^2 F_k(du) \leq \delta^2. \tag{S2.5}$$

Then, we define the mapping $h_k : \mathbb{R} \to l^\infty(\mathcal{G}_k)$ by setting

$$h_k(u)(g) = g(u) - \int g(u) F_k(du),$$

for any $u \in \mathbb{R}$, and the mapping $\mathcal{P}_{k,m} : l^\infty(\mathcal{G}_k) \to l^\infty(\mathcal{G}_k)$ by setting

$$\mathcal{P}_{k,m}\varphi(g) = \varphi(g_{1,l_1} + g_{2,l_2}),$$

with g_{1,l_1} and g_{2,l_2} defined by (S2.4) and (S2.5) respectively, for any $\varphi(\cdot) \in l^\infty(\mathcal{G}_k)$. Letting $X_{ik} = h_k(\xi_{ik})$, we have

$$\begin{aligned}
&\left(\mathcal{P}_{k,m}X_{ik}\right)(g) = \left(\mathcal{P}_{k,m}h_k(\xi_{ik})\right)(g) \\
&= g_{1,l_1}\left(\xi_{ik}\right) + g_{2,l_2}\left(\xi_{ik}\right) - \mathbb{E}g_{1,l_1}\left(\xi_{ik}\right) - \mathbb{E}g_{2,l_2}\left(\xi_{ik}\right), \quad g \in \mathcal{G}_k.
\end{aligned}$$

Examination of the proof of Theorem 7.1 in Dudley and Philipp (1983) shows that (1.15) therein holds with its ε replaced by m^{-1}, and $\delta^2 \geq C(\varsigma)m^{-2/(1-\varsigma)}$ with ς any small positive constant. Then, for each $m \geq 1$, there is an $n_0 \leq C(\varsigma)m^{(2+2\varsigma)/(1-\varsigma)}$such that for all $n \geq n_0(m)$,

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{g_1(\xi_{ik}) - g_{1,l_1}(\xi_{ik}) - \mathbb{E}g_1(\xi_{ik}) + \mathbb{E}g_{1,l_1}(\xi_{ik})\right\}\right| > \frac{1}{m}\right) < \frac{1}{m}.$$

Besides, when $\delta^2 \leq m^{-2}$, Markov inequality entails that, for any $n \geq 1$, $m \geq 1$

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{g_2(\xi_{ik}) - g_{2,l_1}(\xi_{ik})\right\}\right| > \frac{1}{m}\right) < \frac{1}{m}.$$

Hence, for all $n \geq n_0(m)$ and $m \geq 2$,

$$\mathbb{P}\left(\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_{ik}-\mathcal{P}_{k,m}X_{ik})\right\|_{\mathcal{G}_k} > \frac{1}{m}\right) < \frac{1}{m}.$$

Obviously, the dimension of the linear span of $\mathcal{P}_{k,m}\left(l^\infty(\mathcal{G}_k)\right)$ is bounded by $\mathcal{N}\left(\delta, \mathcal{G}_{k1}, L^2(\mathbb{P}_k)\right) \times \mathcal{N}\left(\delta, \mathcal{G}_{k2}, L^2(\mathbb{P}_k)\right)$. Thus, the dimension of $\mathcal{P}_{k,m}\left(l^\infty(\mathcal{G}_k)\right)$ is a polynomial order of m. Applying Theorem 6.2 in Dudley and Philipp (1983), for any $k \in \mathbb{Z}_+$, there exists a sequence $\{Z_{ik} : i \geq 1\}$ of independent identically distributed Gaussian processes, indexed by $\mathcal{G}_k$ and such that

$$\begin{aligned}
&\mathbb{E}Z_{1k}(g) = 0, \quad g \in \mathcal{G}_k,\\
&\mathbb{E}Z_{1k}(f)Z_{1k}(g) = \int fg dF_k - \int f dF_k \int g dF_k, \quad f, g \in \mathcal{G}_k,
\end{aligned}$$

and as $n \to \infty$,

$$\mathbb{P}\left(\sup_{g\in\mathcal{G}_k}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(g(\xi_{ik}) - \mathbb{E}g(\xi_{ik}) - Z_{ik}(g)\right)\right| \geq \log(n)^{-\vartheta}\right) = \mathcal{O}\left(\log(n)^{-B}\right),$$

for any constant $\vartheta > 1$ and $0 < B < \infty$ not depending on k. Hence, as $n \to \infty$,

$$\mathbb{P}\left(\max_{1\leq k\leq\kappa_n}\sup_{g\in\mathcal{G}_k}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(g(\xi_{ik}) - \mathbb{E}g(\xi_{ik}) - Z_{ik}(g)\right)\right| \geq \kappa_n\log(n)^{-\vartheta}\right) \to 0.$$

Thus, we could show that

$$\max_{1\le k\le \kappa_n} \sup_{g\in\mathcal{G}_k} \left| \frac{1}{n}\sum_{i=1}^{n} \left(g(\xi_{ik}) - \mathbb{E}g(\xi_{ik}) - Z_{ik}(g)\right) \right| = o_p\left(n^{-1/2}\right).$$

Combining the results in Lemma S.3, Applying the Slutsky theorem and Continuous Mapping theorem, the proof is completed.

Proof of Proposition 1 : We first show that

$$\max_{1\le k\le \kappa_n} \sup_{x\in\mathbb{R}} \sqrt{n} \left| \widetilde{F}_{nk}(x) - \widetilde{F}^{\dagger}_{nk}(x) \right| = o_p(1). \tag{S2.6}$$

From (S2.1), we have $\widetilde{\xi}_{ik} = d_{n1}\left(\xi_{ik} + d_{n2}(\chi_i) + d_{n3}\right)$, where

$$d_{n1} = \widetilde{\lambda}_k^{-1/2}\lambda_k^{1/2}, \quad d_{n2}(\chi_i) = \lambda_k^{-1/2}\langle \chi_i, \widetilde{\psi}_k - \psi_k\rangle,$$
$$d_{n3} = \lambda_k^{-1/2}\langle m - \widetilde{m}, \widetilde{\psi}_k - \psi_k\rangle - n^{-1}\sum_{t=1}^{n}\xi_{tk}.$$

Hence, one obtains that

$$\begin{aligned}&\sqrt{n}\left(\widetilde{F}_{nk}(x) - \widetilde{F}^{\dagger}_{nk}(x)\right)\\ &= \sqrt{n}\int_{-\infty}^{\infty}\left(\widetilde{F}^{\dagger}_{nk}(x-hv) - \widetilde{F}^{\dagger}_{nk}(x)\right)K(v)dv\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{n}\int_{-1}^{1}\left(\widetilde{F}^{\dagger}_{nk}(x-hv)-\widetilde{F}^{\dagger}_{nk}(x)\right)K(v)dv \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{-1}^{1}\left(I\left(\widetilde{\xi}_{ik}\le x-hv\right)-I\left(\widetilde{\xi}_{ik}\le x\right)\right)K(v)dv \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{-1}^{1}\left\{I\left(\xi_{ik}+d_{n2}(\chi_i)\le d_{n1}^{-1}(x-hv)-d_{n3}\right)\right. \\
&\left.-I\left(\xi_{ik}+d_{n2}(\chi_i)\le d_{n1}^{-1}x-d_{n3}\right)\right\}K(v)dv.
\end{aligned}
$$

From Theorem 3 in Komlós et al. (1975) , there exists Brownian bridges $\{B_{nk}(x)\}_{k=1}^{\kappa_n}$ and positive constants C_1, C_2 and C_3 such that, for any $1\le k\le\kappa_n$ and any $t\in\mathbb{R}$,

$$
\mathbb{P}\left(\sup_{x\in\mathbb{R}}\left|\sqrt{n}\left(F_{nk}(x)-F_k(x)\right)-B_{nk}(F_k(x))\right|>\frac{C_1\log n+t}{\sqrt{n}}\right)\le C_2\exp\left(-C_3 t\right). \tag{S2.7}
$$

Hence,

$$
\begin{aligned}
&\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{I\left(\xi_{ik}+d_{n2}(\chi_i)\le d_{n1}^{-1}x-d_{n3}\right)-\mathbb{E}_{\chi}F_k(d_{n1}^{-1}x-d_{n3}-d_{n2}(\chi_i))\right\}\right. \\
&\left.-B_{nk}(\mathbb{E}_{\chi}F_k(d_{n1}^{-1}x-d_{n3}-d_{n2}(\chi_i)))\right|=\mathcal{O}_{a.s.}\left(\frac{\log n}{\sqrt{n}}\right).
\end{aligned}
$$

For any Brownian bridge $B(\cdot)$, elementary calculation shows the fact that

$$
\mathbb{E}\left(B(s)-B(t)\right)^2=|s-t|-(s-t)^2\le|s-t|,
$$

which is followed by

$$\left(\mathbb{E}\left(B(s)-B(t)\right)^2\right)^{1/2} \leq |s-t|^{1/2}.$$

According to Theorem 1 in Azmoodeh et al. (2014), this induces that for any $k \in \mathbb{Z}_+$ and any $0 < \mu_1 < 1/2$,

$$|B_{nk}(t) - B_{nk}(s)| \leq C_{nk} \, |t-s|^{\mu_1},$$

where C_{nk} is a non-negative random variable satisfying that $\mathbb{E}\exp(aC_{nk}^b) < \infty$ for any positive constant $a \in \mathbb{R}$ and any $b < 2$. One can show that,

$$\begin{aligned}
&\max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \sup_{v\in[-1,1]} \left|B_{nk}\left(\mathbb{E}_\chi F_k(d_{n1}^{-1}(x-hv) - d_{n2}(\chi_i) - d_{n3}\right) - B_{nk}\left(\mathbb{E}_\chi F_k(d_{n1}^{-1}x - d_{n2}(\chi_i) - d_{n3}\right)\right| \\
&\leq \max_{1\leq k\leq \kappa_n} C_{nk} \sup_k \|f_k\|_\infty^{\mu_1} \left(d_{n1}^{-1}h\right)^{\mu_1} = \mathcal{O}_p\left(h^{\mu_1}\log\kappa_n\right) = o_p(1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \left\{I\left(\widetilde{\xi}_{ik} \leq x - hv\right) - \mathbb{E}_\chi F_k(d_{n1}^{-1}(x-hv) - d_{n3} - d_{n2}(\chi_i))\right\}\right. \\
&\left. - \left\{I\left(\widetilde{\xi}_{ik} \leq x\right) - \mathbb{E}_\chi F_k(d_{n1}^{-1}x - d_{n3} - d_{n2}(\chi_i))\right\}\right| \\
&= \mathcal{O}_{a.s.}\left(n^{-1/2}\log n\right) + \mathcal{O}_p\left(h^{\mu_1}\log\kappa_n\right) = o_p(1).
\end{aligned}$$

Further applying Taylor expansion to $F_k \in \mathcal{H}^{(1,1)}(\mathbb{R})$ and by arguments similar to those in the proof of Theorem 2.1 in Wang et al. (2013), the proof of (S2.6) is completed.

Analogously, one could also show that

$$\max_{1\le k\le \kappa_n} \sup_{x\in\mathbb{R}} \sqrt{n} \left| \widehat{F}_{nk}(x) - \widehat{F}^{\dagger}_{nk}(x) \right| = o_p(1). \tag{S2.8}$$

The proof of (S2.8) can be easily carried out from those of (S2.6) without any essential difficulties. Details are omitted.

Proof of Theorem 5 : Theorem 4, Proposition 1 and the Slutsky theorem yield Theorem 5 directly.

Proof of Proposition 2 : Recalling the covariance estimator is defined as follows,

$$\begin{aligned}
\widehat{\Sigma}(s,t) &= \frac{1}{n}\sum_{i=1}^{n} \widehat{\Xi}_{ik}(s)\widehat{\Xi}_{ik}(t) \\
&= \widehat{F}^{\dagger}_{nk}(s\wedge t) - \widehat{F}^{\dagger}_{nk}(s)\widehat{F}^{\dagger}_{nk}(t) + \widehat{f}_k(s)\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik} I\left(\widehat{\xi}_{ik}\le t\right) \\
&+ \widehat{f}_k(t)\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik} I\left(\widehat{\xi}_{ik}\le s\right) + \widehat{f}_k(s)\widehat{f}_k(t)\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}^2_{ik} \\
&- \widehat{F}^{\dagger}_{nk}(s)\widehat{f}_k(t)\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik} - \widehat{F}^{\dagger}_{nk}(t)\widehat{f}_k(s)\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{i=1}^{n}\left(\widehat{\xi}_{ik}^{2}-1\right)^{2}st\widehat{f}_{k}(s)\widehat{f}_{k}(t)+s\widehat{f}_{k}(s)\frac{1}{2n}\sum_{i=1}^{n}\left(\left(\widehat{\xi}_{ik}^{2}-1\right)I\left(\widehat{\xi}_{ik}\leq t\right)\right)\\
&+t\widehat{f}_{k}(t)\frac{1}{2n}\sum_{i=1}^{n}\left(\left(\widehat{\xi}_{ik}^{2}-1\right)I\left(\widehat{\xi}_{ik}\leq s\right)\right)+s\widehat{f}_{k}(s)\widehat{f}_{k}(t)\frac{1}{2n}\sum_{i=1}^{n}\left(\widehat{\xi}_{ik}^{3}-\widehat{\xi}_{ik}\right)\\
&+t\widehat{f}_{k}(t)\widehat{f}_{k}(s)\frac{1}{2n}\sum_{i=1}^{n}\left(\widehat{\xi}_{ik}^{3}-\widehat{\xi}_{ik}\right)-t\widehat{f}_{k}(t)\widehat{F}_{nk}^{\dagger}(s)\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{\xi}_{ik}^{2}-1\right)\\
&-s\widehat{f}_{k}(s)\widehat{F}_{nk}^{\dagger}(t)\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{\xi}_{ik}^{2}-1\right).
\end{aligned}
$$

$\widehat{\Sigma}(s,t)-\Sigma(s,t)$ could be decomposed as follows,

$$
\begin{aligned}
&\widehat{\Sigma}(s,t)-\Sigma(s,t)\\
&=\left(\widehat{F}_{nk}^{\dagger}(s\wedge t)-F_{k}(s\wedge t)\right)-\left(\widehat{F}_{nk}^{\dagger}(s)\widehat{F}_{nk}^{\dagger}(t)-F_{k}(s)F_{k}(t)\right)\\
&+\left(\widehat{f}_{k}(t)\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik}I\left(\widehat{\xi}_{ik}\leq s\right)-f_{k}(t)\mathbb{E}\left(\xi_{ik}I\left(\xi_{ik}\leq s\right)\right)\right)\\
&+\left(\widehat{f}_{k}(s)\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik}I\left(\widehat{\xi}_{ik}\leq t\right)-f_{k}(s)\mathbb{E}\left(\xi_{ik}I\left(\xi_{ik}\leq t\right)\right)\right)\\
&+\left(\widehat{f}_{k}(s)\widehat{f}_{k}(t)\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik}^{2}-f_{k}(s)f_{k}(t)\right)\\
&-\left(\widehat{F}_{nk}^{\dagger}(s)\widehat{f}_{k}(t)\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik}\right)-\left(\widehat{F}_{nk}^{\dagger}(t)\widehat{f}_{k}(s)\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik}\right)\\
&+\frac{1}{4}st\left(\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{\xi}_{ik}^{2}-1\right)^{2}\widehat{f}_{k}(s)\widehat{f}_{k}(t)-\left(\mathbb{E}\xi_{ik}^{4}-1\right)f_{k}(s)f_{k}(t)\right)\\
&+\frac{1}{2}s\left(\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{\xi}_{ik}^{2}-1\right)I\left(\widehat{\xi}_{ik}\leq t\right)\widehat{f}_{k}(s)-\mathbb{E}\left(\left(\xi_{ik}^{2}-1\right)I\left(\xi_{ik}\leq t\right)\right)f_{k}(s)\right)\\
&+\frac{1}{2}t\left(\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{\xi}_{ik}^{2}-1\right)I\left(\widehat{\xi}_{ik}\leq s\right)\widehat{f}_{k}(t)-\mathbb{E}\left(\left(\xi_{ik}^{2}-1\right)I\left(\xi_{ik}\leq s\right)\right)f_{k}(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}s\left(\widehat{f}_k(s)\widehat{f}_k(t)\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik}^3-f_k(s)f_k(t)\mathbb{E}\xi_{ik}^3\right)\\
&+\frac{1}{2}t\left(\widehat{f}_k(t)\widehat{f}_k(s)\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik}^3-f_k(t)f_k(s)\mathbb{E}\xi_{ik}^3\right)\\
&-t\widehat{f}_k(t)\widehat{F}_{nk}^{\dagger}(s)\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{\xi}_{ik}^2-1\right)-s\widehat{f}_k(s)\widehat{F}_{nk}^{\dagger}(t)\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{\xi}_{ik}^2-1\right)\\
&-\frac{1}{2n}\sum_{i=1}^{n}\widehat{\xi}_{ik}s\widehat{f}_k(s)\widehat{f}_k(t)-\frac{1}{2n}\sum_{i=1}^{n}\widehat{\xi}_{ik}t\widehat{f}_k(s)\widehat{f}_k(t)\\
&=R_1-R_2+R_3+R_4+R_5-R_6-R_7+R_8+R_9+R_{10}\\
&+R_{11}+R_{12}-R_{13}-R_{14}-R_{15}-R_{16}.
\end{aligned}
$$

Firstly, we obtain $\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\widehat{F}_{nk}^{\dagger}(x)-F_k(x)\right|=o_p(1)$ from Theorem 5. Consequently, $\max_{1\le k\le\kappa_n}\sup_{s,t\in\mathbb{R}^2}\left(|R_1|+|R_2|\right)$ is of order $o_p(1)$. Secondly, we desire to demonstrate

$\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\widehat{f}_k(x)-f_k(x)\right|=o_p(1)$. The kernel density estimator

$$
\widehat{f}_k(x)=\frac{1}{n}\sum_{i=1}^{n}L_H\left(\widehat{\xi}_{ik}-x\right).
$$

Define an infeasible density estimator $\widetilde{f}_k(x)=n^{-1}\sum_{i=1}^{n}L_H\left(\xi_{ik}-x\right)$. Notice that

$$
\begin{aligned}
\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\widetilde{f}_k(x)-\mathbb{E}\widetilde{f}_k(x)\right|&=\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\int L_H\left(u-x\right)d\left\{F_{nk}(u)-F_k(u)\right\}\right|\\
&=\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\frac{1}{H^2}\left|\int\left(F_{nk}(u)-F_k(u)\right)L'\left(\frac{u-x}{H}\right)du\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2H^{-1} \left\| L' \right\|_{\infty} \max_{1 \leq k \leq \kappa_n} \left\| F_{nk} - F_k \right\|_{\infty} \\
&= \mathcal{O}_p \left(n^{-1/2} H^{-1} \kappa_n \right)
\end{aligned}
$$

Besides,

$$
\begin{aligned}
&\max_{1 \leq k \leq \kappa_n} \sup_{x \in \mathbb{R}} \left| \widehat{f}_k(x) - \widetilde{f}_k(x) \right| \\
&= \max_{1 \leq k \leq \kappa_n} \sup_{x \in \mathbb{R}} \left| \frac{1}{nH} \sum_{i=1}^{n} \left(L \left(\frac{\widehat{\xi}_{ik} - x}{H} \right) - L \left(\frac{\xi_{ik} - x}{H} \right) \right) \right| \\
&\leq \max_{1 \leq k \leq \kappa_n} \frac{\left\| L' \right\|_{\infty}}{nH^2} \sum_{i=1}^{n} \left| \widehat{\xi}_{ik} - \xi_{ik} \right| \\
&= \mathcal{O}_p \left(n^{-1/2} H^{-2} \kappa_n^{2\tau+1} \right) = o_p(1).
\end{aligned}
$$

by noticing the fact $\max_{1 \leq k \leq \kappa_n} n^{-1} \sum_{i=1}^{n} \left| \widehat{\xi}_{ik} - \xi_{ik} \right| = \mathcal{O}_p \left(n^{-1/2} \kappa_n^{2\tau+1} \right)$ from (S2.2). Additionally,

$$
\begin{aligned}
\max_{1 \leq k \leq \kappa_n} \sup_{x \in \mathbb{R}} \left| \mathbb{E} \widetilde{f}_k(x) - f_k(x) \right| &\leq \max_{1 \leq k \leq \kappa_n} \sup_{x \in \mathbb{R}} \left| \int L_H \left(u - x \right) \left(f_k(u) - f_k(x) \right) du \right| \\
&\leq H \sup_k \left\| F_k \right\|_{1,1} \sup_{x \in \mathbb{R}} \left| \int L_H \left(u - x \right) du \right| \\
&= \mathcal{O}(H).
\end{aligned}
$$

Therefore, we conclude that $\max_{1 \leq k \leq \kappa_n} \sup_{x \in \mathbb{R}} \left| \widehat{f}_k(x) - f_k(x) \right| = o_p(1)$.

Then, we turn to show that $\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|x\widehat{f}_k(x)-xf_k(x)\right|=\mathcal{o}_p(1)$. From the above results, we obtain that $\max_{1\le k\le\kappa_n}\sup_{x\in[-c_n,c_n]}\left|x\widehat{f}_k(x)-xf_k(x)\right|=\mathcal{o}_p(1)$, where $c_n=n^{\epsilon+(1/r_1)}$ for any small $\epsilon>0$. Since $\max_{1\le k\le\kappa_n}\max_{1\le i\le n}|\xi_{ik}|$ is of order $\mathcal{O}_p\left((\kappa_n n)^{1/r_1}\right)$ and $\max_{1\le k\le\kappa_n}\max_{1\le i\le n}\left|\widehat{\xi}_{ik}-\xi_{ik}\right|$ is of order $\mathcal{O}_p\left(n^{1/r_1-1/2}\kappa_n^{2\tau+1}\right)$, we have $\max_{1\le k\le\kappa_n}\max_{1\le i\le n}\left|\widehat{\xi}_{ik}\right|$ is of order $\mathcal{O}_p\left((\kappa_n n)^{1/r_1}\right)$. Hence, with probability approaching 1, we have

$$\max_{1\le k\le\kappa_n}\sup_{x\in(-\infty,-c_n)}\left|x\widehat{f}_k(x)\right|=0,$$

holds for large n. Recalling the fact that $\max_{1\le k\le\kappa_n}\sup_{x\in(-\infty,-c_n)}|xf_k(x)|=\mathcal{o}(1)$, we could show

$$\max_{1\le k\le\kappa_n}\sup_{x\in(-\infty,-c_n)}\left|x\widehat{f}_k(x)-xf_k(x)\right|=\mathcal{o}_p(1).$$

And the proof for the case $x\in(c_n,\infty)$ is similar.

Besides, we derive $\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^n\xi_{ik}I\left(\xi_{ik}\le x\right)-\frac{1}{n}\sum_{i=1}^n\widehat{\xi}_{ik}I\left(\widehat{\xi}_{ik}\le x\right)\right|=\mathcal{o}_p(1)$ as follows. We write

$$\begin{aligned}&\left|\frac{1}{n}\sum_{i=1}^n\xi_{ik}I\left(\xi_{ik}\le x\right)-\frac{1}{n}\sum_{i=1}^n\widehat{\xi}_{ik}I\left(\widehat{\xi}_{ik}\le x\right)\right|\\ &\le\left|\frac{1}{n}\sum_{i=1}^n\xi_{ik}I\left(\xi_{ik}\le x\right)-\frac{1}{n}\sum_{i=1}^n\xi_{ik}I\left(\widehat{\xi}_{ik}\le x\right)\right|\end{aligned}$$

$$+\left|\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}I\left(\widehat{\xi}_{ik}\leq x\right)-\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik}I\left(\widehat{\xi}_{ik}\leq x\right)\right|$$

By Cauchy-Schwarz inequality, one has

$$\begin{aligned}&\left|\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}I\left(\xi_{ik}\leq x\right)-\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}I\left(\widehat{\xi}_{ik}\leq x\right)\right|\\&\leq\sqrt{\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}^{2}}\sqrt{F_{nk}(x)+\widehat{F}_{nk}^{\dagger}(x)-\frac{2}{n}\sum_{i=1}^{n}I\left(\xi_{ik}\leq x,\widehat{\xi}_{ik}\leq x\right)}.\end{aligned}$$

Following the arguments in the proof of Theorem 2 of Gu et al. (2021), we verify that

$$\max_{1\leq k\leq\kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}I\left(\xi_{ik}\leq x,\widehat{\xi}_{ik}\leq x\right)-F_{nk}(x)\right|=\mathcal{O}_p(1).$$

In addition, Borel-Cantelli lemma entails that $\max_{1\leq k\leq\kappa_n}|n^{-1}\sum_{i=1}^{n}\xi_{ik}^2-1|\xrightarrow{a.s.}0$. Hence,

$$\max_{1\leq k\leq\kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}I\left(\xi_{ik}\leq x\right)-\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}I\left(\widehat{\xi}_{ik}\leq x\right)\right|=\mathcal{O}_p(1).$$

Moreover, observe that

$$\max_{1\leq k\leq\kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}I\left(\widehat{\xi}_{ik}\leq x\right)-\frac{1}{n}\sum_{i=1}^{n}\widehat{\xi}_{ik}I\left(\widehat{\xi}_{ik}\leq x\right)\right|$$

$$\leq \max_{1\leq k\leq \kappa_n} \max_{1\leq i\leq n} \left|\widehat{\xi}_{ik} - \xi_{ik}\right| = \mathcal{O}_p\left(n^{1/r_1-1/2}\kappa_n^{2\tau+1}\right) = o_p(1),$$

which completes the proof.

Additionally, Classical empirical process theory implies that, for any $k \in \mathbb{Z}_+$,

$$\mathbb{E}\sup_{x\in\mathbb{R}} \left|\frac{1}{n}\sum_{i=1}^{n} \xi_{ik} I\left(\xi_{ik} \leq x\right) - \mathbb{E}\left(\xi_{ik} I\left(\xi_{ik} \leq x\right)\right)\right| = \mathcal{O}\left(n^{-1/2}\right),$$

which further yields that

$$\max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \left|\frac{1}{n}\sum_{i=1}^{n} \xi_{ik} I\left(\xi_{ik} \leq x\right) - \mathbb{E}\left(\xi_{ik} I\left(\xi_{ik} \leq x\right)\right)\right| = \mathcal{O}_p\left(n^{-1/2}\kappa_n\right) = o_p(1).$$

Lastly, it suffices to show $\max_{1\leq k\leq \kappa_n} \left|\frac{1}{n}\sum_{i=1}^{n} \widehat{\xi}_{ik}^4 - \mathbb{E}\xi_{ik}^4\right| = o_p(1)$. We write

$$\max_{1\leq k\leq \kappa_n} \left|\frac{1}{n}\sum_{i=1}^{n} \widehat{\xi}_{ik}^4 - \mathbb{E}\xi_{ik}^4\right| \leq \max_{1\leq k\leq \kappa_n} \left|\frac{1}{n}\sum_{i=1}^{n} \left(\widehat{\xi}_{ik}^4 - \xi_{ik}^4\right)\right| + \max_{1\leq k\leq \kappa_n} \left|\frac{1}{n}\sum_{i=1}^{n} \xi_{ik}^4 - \mathbb{E}\xi_{ik}^4\right|$$

The first term is of order $o_p(1)$ by considering $\max_{1\leq k\leq \kappa_n} \max_{1\leq i\leq n} \left|\widehat{\xi}_{ik} - \xi_{ik}\right|$ is of order $o_p(1)$. Besides, we bound the send term

$$\mathbb{E}\max_{1\leq k\leq \kappa_n} \left|\frac{1}{n}\sum_{i=1}^{n} \xi_{ik}^4 - \mathbb{E}\xi_{ik}^4\right|^2 \leq \kappa_n \max_{1\leq k\leq \kappa_n} \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} \xi_{ik}^4 - \mathbb{E}\xi_{ik}^4\right)^2 = \mathcal{O}\left(n^{-1}\kappa_n\right),$$

which completes the proof by Markov inequality.

Proof of Proposition 3 : To save space, we only present proofs of the results for S_n and V_n.

To prove (4.27), without loss of generality, we consider sufficiently large n and write the equally spaced grid points $\{z_{kl}\}_{l=1}^{L_k}$ as $\{x_l\}_{l=1}^{L}$, where $x_L = n^{1/8}$ and $L = L_n$ diverges to infinity with a polynomial rate, such that $x_{L_n}/L_n \to 0$ as $n \to \infty$. We could verify that

$$\max_{1\leq k\leq \kappa_n} \sup_{|x|>x_L} |\zeta_k(x)| = \mathcal{O}_p(1),$$

by using Lemma S.3, Theorem 4 and (S2.7). Recalling the definition of Σ_k, we have

$$\sup_{1\leq k\leq \kappa_n} \mathbb{E}\left(\zeta_k(x) - \zeta_k(y)\right)^2 \leq C\left|x-y\right|^{\nu_1}.$$

Theorem 1 of Azmoodeh et al. (2014) implies that for any $\mu_2 \in (0, \nu_1/2)$,

$$\sup_{x\in(x_l, x_{l+1})} |\zeta_k(x) - \zeta_k(x_l)| \leq |C_{kl}||x_{l+1} - x_l|^{\mu_2},$$

where $\sup_{k,l} \|C_{kl}\|_{\psi_1} < \infty$, and $\|\cdot\|_{\psi_1}$ is the sub-exponential norm. Hence, we obtain

$$\max_{1\leq k\leq \kappa_n} \max_{0\leq l<L} \sup_{x\in(x_l,x_{l+1})} |\zeta_k(x) - \zeta_k(x_l)| \leq \max_{1\leq k\leq \kappa_n} \max_{0\leq l<L} |C_{kl}| x_{L_n}^{\mu_2} L_n^{-\mu_2}.$$

Considering that $\left\|\max_{1\leq k\leq\kappa_n}\max_{0\leq l<L}|C_{kl}|\right\|_{\psi_1}=\mathcal{O}\left(\log(L_n\kappa_n)\right)$ and the Orcliz norm is proportional to the standard deviation for Gaussian random variables, we could show that

$$\max_{1\leq k\leq\kappa_n}\max_{0\leq l<L}\sup_{x\in(x_l,x_{l+1})}|\zeta_k(x)-\zeta_k(x_l)|=o_p(1).$$

In addition, a careful check of the proof of Proposition 2 shows that the result could be strengthened to

$$\sup_{1\leq k\leq\kappa_n}\left\|\widehat{\Sigma}_k-\Sigma_k\right\|_\infty=\mathcal{O}_p\left(n^{-c}\right),$$

for some small positive constant c. Applying Lemma 3.1 of Chernozhukov et al. (2013), we have

$$\sup_{t\in\mathbb{R}}\left|\mathbb{P}\left(\max_{1\leq k\leq\kappa_n}\max_{0\leq l<L}\left|\widehat{\zeta}_k(x_l)\right|\leq t|\{Y_{ij}\}_{i=1,j=1}^{n,N}\right)-\mathbb{P}\left(\max_{1\leq k\leq\kappa_n}\max_{0\leq l<L}|\zeta_k(x_l)|\leq t\right)\right|\to 0,\tag{S2.9}$$

where $\left\{\widehat{\zeta}_k(x_l)\right\}_{k=1,l=1}^{\kappa_n,L}$ denotes the generated Gaussian random vectors.

As to (4.28) , we rewrite

$$\left|\sum_{k=1}^{\kappa_n}\lambda_k\sum_{m=1}^{\infty}b_{km}\chi^2_{km}-\sum_{k=1}^{\kappa_n}\widehat{\lambda}_k\sum_{m=1}^{M_k}\widehat{b}_{km}\chi^2_{km}\right|$$

$$\leq \sum_{k=1}^{\kappa_n} \lambda_k \sum_{m=M_k+1}^{\infty} b_{km}\chi^2_{km} + \sum_{k=1}^{\kappa_n} \lambda_k \sum_{m=1}^{M_k} \left| b_{km} - \widehat{b}_{km} \right| \chi^2_{km}$$

$$+ \sum_{k=1}^{\kappa_n} \left| \widehat{\lambda}_k - \lambda_k \right| \sum_{m=1}^{M_k} \left(b_{km} + \left| b_{km} - \widehat{b}_{km} \right| \right) \chi^2_{km}$$

$$\leq \sum_{k=1}^{\kappa_n} \lambda_k \sum_{m=M_k+1}^{\infty} b_{km}\chi^2_{km} + \sum_{k=1}^{\kappa_n} \lambda_k \left\| \widehat{\Sigma}_k - \Sigma_k \right\|_\infty \sum_{m=1}^{M_k} \chi^2_{km}$$

$$+ \left\| \widehat{G} - G \right\|_\infty \sum_{k=1}^{\kappa_n} \sum_{m=1}^{M_k} \left(b_{km} + \left\| \widehat{\Sigma}_k - \Sigma_k \right\|_\infty \right) \chi^2_{km}$$

$$=: O_{n1} + O_{n2} + O_{n3}.$$

where M_k is less than L_k, and diverges to infinity sufficient slowly as $n \to \infty$. The second inequality holds by Lemma 4.2 in Bosq (2000). Considering the fact that $\sup_{1\leq k\leq \kappa_n} \sum_{m=1}^{\infty} b_{km} < \infty$, for any $\epsilon > 0$, one could choose $M_k = M(\epsilon)$ large enough such that $\sup_{1\leq k\leq \kappa_n} \sum_{m=M_k+1}^{\infty} b_{km} < \epsilon$. Hence, O_{n1} is $\mathcal{o}_p(1)$. Combining the result in (S2.9), we get O_{n2} and O_{n3} are of order $\mathcal{o}_p(1)$.

To verify (4.29) , we write

$$S_n \geq \max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \sqrt{n} \left| F_k(x) - F_k^*(x) \right| - \max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \sqrt{n} \left| \widehat{F}_{nk}(x) - F_k(x) \right|.$$

For any x,$y \in \mathbb{R}$, define $u = F_k(x)$, $v = F_k(y)$ and $U_{ik} = F_k(\xi_{ik})$. Then, we let $\widetilde{\zeta}_k(u) = \zeta_k(F_k^{-1}(u))$. Noticing the uniform boundness of $f_k \circ F_k^{-1}$ and $F_k^{-1} \cdot (f_k \circ F_k^{-1})$, one applies Proposition 4.2 of Sen (2018) to obtain that, for any $k \in \mathbb{Z}_+$,

$$\mathbb{E} \sup_{x\in\mathbb{R}} \zeta_k(x) = \mathbb{E} \sup_{u\in(0,1)} \widetilde{\zeta}_k(u) \leq C,$$

where C is some fixed constant not depending on k. Hence,

$$\mathbb{E} \max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} |\zeta_k(x)| \leq C\kappa_n,$$

which is followed by

$$\max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \sqrt{n} \left| \widehat{F}_{nk}(x) - F_k(x) \right| = \mathcal{O}_p\left(\kappa_n\right).$$

Obviously, note that the divergence rate of $\max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \sqrt{n} \left|F_k(x) - F_k^*(x)\right|$ is much faster than $\mathcal{O}_p\left(\kappa_n\right)$ under the alternative.

For (4.30) , note that $V_n \to \infty$ as $n \to \infty$ to complete the proof.

S3. Additional useful Lemmas

Lemma B.1. *Under the assumptions in Lemma S.3,*

$$\max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} K_h(x - \xi_{ik}) \left| \lambda_k^{-1/2} D_{ik1} - \xi_{ik} \right| = o_p\left(n^{-1/2}\right).$$

Proofs: For D_{ik1}, we have

$$\int_0^1 (\eta_i(x) - m(x))\, \psi_k(x) dx - \frac{1}{N} \sum_{j=1}^{N} \left(\eta_i \left(\frac{j}{N} \right) - m \left(\frac{j}{N} \right) \right) \psi_k \left(\frac{j}{N} \right)$$
$$= \sum_{j=1}^{N} \int_{\frac{j-1}{N}}^{\frac{j}{N}} \left[(\eta_i(x) - m(x))\, \psi_k(x) - \left(\eta_i \left(\frac{j}{N} \right) - m \left(\frac{j}{N} \right) \right) \psi_k \left(\frac{j}{N} \right) \right] dx.$$

For any $x \neq y \in [0,1]$ such that $|x - y| \leq N^{-1}$, we write

$$|\eta_i(x) - m(x) - \eta_i(y) + m(y)| \leq \sum_{m=1}^{\infty} |\xi_{im}| \, \|\phi_m\|_{0,\varpi} \, |x-y|^{\varpi} =: M_i \, |x-y|^{\varpi} \, ,$$

Applying Minkowski inequality, we obtain

$$\|M_i\|_{r_1} \leq \sum_{m=1}^{\infty} \|\xi_{im}\|_{r_1} \|\phi_m\|_{0,\varpi} \leq \sup_m \|\xi_{im}\|_{r_1} \sum_{m=1}^{\infty} \|\phi_m\|_{0,\varpi} < \infty,$$

which implies that M_i, $i = 1, 2, \cdots, n$ are nonnegative random variables with finite r_1 moment and $\max_{1 \leq i \leq n} M_i = \mathcal{O}_p(n^{1/r_1})$ by L^{r_1} maximum inequality. Analogously, we derive $\max_{1 \leq i \leq n} \|\chi_i\|_{\infty} = \mathcal{O}_p\left(n^{1/r_1}\right)$. Since ϕ_k and m are uniformly Lipschitz continuos, we show that

$$
\begin{aligned}
&\sup_{x\in[(j-1)/N,j/N]}\left|(\eta_i(x)-m(x))\,\psi_k(x)-\left(\eta_i\left(\frac{j}{N}\right)-m\left(\frac{j}{N}\right)\right)\psi_k\left(\frac{j}{N}\right)\right| \\
&\leq \sup_{x\in[(j-1)/N,j/N]}\left|(\eta_i(x)-m(x))\left(\psi_k(x)-\psi_k\left(\frac{j}{N}\right)\right)\right| \\
&+\sup_{x\in[(j-1)/N,j/N]}\left|\left(\left(\eta_i\left(\frac{j}{N}\right)-m\left(\frac{j}{N}\right)\right)-(\eta_i(x)-m(x))\right)\psi_k\left(\frac{j}{N}\right)\right| \\
&\leq \lambda_k^{-1/2}\left(\|\chi_i\|_\infty\|\phi_k\|_{0,\varpi}N^{-\varpi}+\|\phi_k\|_\infty M_iN^{-\varpi}\right).
\end{aligned}
\tag{S.10}
$$

From (S.10), we have

$$
\max_{1\leq k\leq\kappa_n}\max_{1\leq i\leq n}\left|\lambda_k^{-1/2}D_{ik1}-\xi_{ik}\right|\leq C\lambda_{\kappa_n}^{-1}\left(N^{-\varpi}\max_{1\leq i\leq n}\|\chi_i\|_\infty+N^{-\varpi}\max_{1\leq i\leq n}M_i\right).
$$

Then, $\max_{1\leq k\leq\kappa_n}\max_{1\leq i\leq n}\left|\lambda_k^{-1/2}D_{ik1}-\xi_{ik}\right| = \mathcal{O}_p\left(\kappa_n^\tau n^{1/r_1-\varpi/\theta}\right) = o_p\left(n^{-1/2}\right)$, in which the last equality holds by $r_1>(2\theta)/(2\varpi-\theta)$.

It remains to verify that $\max_{1\leq k\leq\kappa_n}\sup_{x\in\mathbb{R}}n^{-1}\sum_{i=1}^n K_h(x-\xi_{ik})=\mathcal{O}_{a.s.}(1)$. Elementary calculation leads to

$$
\begin{aligned}
&\max_{1\leq k\leq\kappa_n}\sup_{x\in\mathbb{R}}\left|\mathbb{E}K_h\left(x-\xi_{ik}\right)-f_k(x)\right| \\
&\leq \max_{1\leq k\leq\kappa_n}\sup_{x\in\mathbb{R}}\int K(v)\left|f_k(x-vh)-f_k(x)\right|dv \\
&\leq \sup_k\|F_k\|_{1,1}\,h\int K(v)\,|v|\,dv=\mathcal{O}\left(h\right).
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&\max_{1\le k\le \kappa_n}\sup_{x\in\mathbb{R}}\left|\mathbb{E}K_h(x-\xi_{ik})^2-\frac{f_k(x)}{h}\int K^2(v)dv\right| \\
\le &\max_{1\le k\le \kappa_n}\sup_{x\in\mathbb{R}}\frac{1}{h}\int K^2(v)\left|f_k(x-vh)-f_k(x)\right|dv \\
\le &\max_{1\le k\le \kappa_n}\|F_k\|_{1,1}\int K^2(v)\left|v\right|dv=\mathcal{O}\left(1\right),
\end{aligned}
$$

which is followed by, for any $x\in\mathbb{R}$ and $1\le k\le \kappa_n$,

$$
\begin{aligned}
\operatorname{var}\left(K_h\left(x-\xi_{ik}\right)\right) &= \mathbb{E}\left(K_h\left(x-\xi_{ik}\right)\right)^2-\left(\mathbb{E}K_h\left(x-\xi_{ik}\right)\right)^2 \\
&= \frac{f_k(x)}{h}\int K^2(v)dv\left(1+o(1)\right).
\end{aligned}
$$

Besides, the following result holds for any $1\le i\le n$, $k\in\mathbb{Z}_+$, $x\in\mathbb{R}$ and $l=3,4,\cdots$,

$$
\mathbb{E}\left(K_h\left(x-\xi_{ik}\right)-\mathbb{E}K_h\left(x-\xi_{ik}\right)\right)^l\le\left(\frac{2\left\|K\right\|_\infty}{h}\right)^{l-2}\operatorname{var}\left(K_h\left(x-\xi_{ik}\right)\right).
$$

Hence, Bernstein inequality shows that, for some large n, the following concentration property holds with any fixed $x\in\mathbb{R}$ and $k\in\mathbb{Z}_+$,

$$
\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}\left(K_h(x-\xi_{ik})-\mathbb{E}K_h(x-\xi_{ik})\right)\right|>9\sqrt{\frac{\log n}{nh}f_k(x)\int K^2(v)dv}\right)\le 2n^{-8}.
$$

First consider the case $x \in [-a_n, a_n]$, we discrete by equally spaced $-a_n = x_1 < x_2 < \cdots < x_L = a_n$ with $L = n^5$ and $a_n = n^{\iota_1}$. One can choose $\iota_1 = 1$ for example. Then, we obtain

$$\mathbb{P}\left(\max_{1\le k\le \kappa_n}\max_{1\le l\le L}\left|\frac{1}{n}\sum_{i=1}^{n}\left(K_h(x_l-\xi_{ik})-\mathbb{E}K_h(x_l-\xi_{ik})\right)\right| > C\sqrt{\frac{\log n}{nh}}\right) \le 2n^{-3}\kappa_n.$$

Here the constant C does not depend on x, k and n. Applying Borel-Cantelli Lemma, we can show that

$$\max_{1\le k\le \kappa_n}\max_{1\le l\le L}\left|\frac{1}{n}\sum_{i=1}^{n}\left(K_h(x_l-\xi_{ik})-\mathbb{E}K_h(x_l-\xi_{ik})\right)\right| = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\log n}{nh}}\right).$$

Notice that

$$\begin{aligned}
&\max_{1\le k\le \kappa_n}\sup_{x\in[-a_n,a_n]}\left|\frac{1}{n}\sum_{i=1}^{n}\left(K_h(x-\xi_{ik})-\mathbb{E}K_h(x-\xi_{ik})\right)\right| \\
\le &\max_{1\le k\le \kappa_n}\max_{l=1\cdots L}\left|\frac{1}{n}\sum_{i=1}^{n}\left(K_h(x_l-\xi_{ik})-\mathbb{E}K_h(x_l-\xi_{ik})\right)\right| \\
+ &\max_{1\le k\le \kappa_n}\max_{l=1\cdots L-1}\sup_{x\in[x_j,x_{j+1}]}\frac{2}{n}\sum_{i=1}^{n}\left|K_h(x_l-\xi_{ik})-K_h(x-\xi_{ik})\right| \\
\le &\mathcal{O}_{a.s.}\left(\sqrt{\frac{\log n}{nh}}\right) + \mathcal{O}\left(n^{\iota_1-5}h^{-2}\right) = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\log n}{nh}}\right).
\end{aligned}$$

Further observe that

$$
\begin{aligned}
&\max_{1\le k\le\kappa_n}\sup_{x\in(-\infty,-a_n]}\left|\frac{1}{n}\sum_{i=1}^{n}\left(K_h(x-\xi_{ik})-\mathbb{E}K_h(x-\xi_{ik})\right)\right|\\
\le&\max_{1\le k\le\kappa_n}\frac{2}{nh}\sum_{i=1}^{n}I\left(\xi_{ik}\in(-\infty,-a_n+h)\right)\|K\|_\infty\,,
\end{aligned}
$$

where $\max_{1\le k\le\kappa_n}n^{-1}\sum_{i=1}^{n}I\left(\xi_{ik}\in(-\infty,-a_n+h)\right)$ can be bounded by

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}I\left(\xi_{ik}\in(-\infty,-a_n+h)\right)\\
\le&\max_{1\le k\le\kappa_n}\left|\frac{1}{n}\sum_{i=1}^{n}\left(I\left(\xi_{ik}\in(-\infty,-a_n+h)\right)-\mathbb{P}\left(\xi_{ik}\le-a_n+h\right)\right)\right|\\
&+\max_{1\le k\le\kappa_n}\mathbb{P}\left(\xi_{ik}\le-a_n+h\right)\\
\le&\mathcal{O}_p\left(\kappa_n^{1/2}n^{-(r_1\iota_1+1)/2}\right)+n^{-r_1\iota_1}\sup_{1\le k\le\kappa_n}\mathbb{E}\xi_{ik}^{r_1}=\mathcal{O}_p\left(\kappa_n^{1/2}n^{-(r_1\iota_1+1)/2}\right),
\end{aligned}
$$

where the second inequality holds by Markov inequality and L^2 maximum inequality. Therefore, one obtains

$$
\max_{1\le k\le\kappa_n}\sup_{x\in(-\infty,-a_n]}\left|\frac{1}{n}\sum_{i=1}^{n}\left(K_h(x-\xi_{ik})-\mathbb{E}K_h(x-\xi_{ik})\right)\right|=\mathcal{O}_p\left(\kappa_n^{1/2}n^{-(r_1\iota_1+1)/2}h^{-1}\right).
$$

Accordingly, the proof for $x \in [a_n, \infty)$ is similar. We then conclude that

$$\max_{1\le k\le \kappa_n} \sup_{x\in\mathbb{R}} \left| \frac{1}{n}\sum_{i=1}^{n} K_h\left(x-\xi_{ik}\right) - f_k(x) \right| = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\log n}{nh}} + h\right)$$

Therefore, we obtain

$$\begin{aligned}
&\max_{1\le k\le \kappa_n} \sup_{x\in\mathbb{R}} \frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik}) \left|\lambda_k^{-1/2} D_{ik1} - \xi_{ik}\right| \\
&\le \max_{1\le k\le \kappa_n} \sup_{x\in\mathbb{R}} \frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik}) \max_{1\le k\le \kappa_n} \max_{1\le i\le n} \left|\lambda_k^{-1/2} D_{ik1} - \xi_{ik}\right| \\
&= \mathcal{O}_{a.s.}(1)\cdot \mathcal{O}_p\left(\kappa_n^{\tau} n^{1/r_1 - 1/\theta}\right) = o_p\left(n^{-1/2}\right).
\end{aligned}$$

Lemma B.2. *Under the assumptions in Lemma S.3,*

$$\max_{1\le k\le \kappa_n} \sup_{x\in\mathbb{R}} \left| \frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik})\lambda_k^{-1/2} D_{ik2} \right| = o_p\left(n^{-1/2}\right).$$

Proofs: Denote $\widetilde{\varepsilon}_{ijk} = \sigma\left(\frac{j}{N}\right)\phi_k\left(\frac{j}{N}\right)\varepsilon_{ij}$ and $Z_{ijk,\varepsilon}$ is a gaussian random variable such that $\operatorname{var}\left(Z_{ijk,\varepsilon}\right) = \operatorname{var}(\widetilde{\varepsilon}_{ijk})$. By strong approximation theorems for independent but not necessarily identically distributed random variables in Shao (1995) , we have the following result. Denote $\beta_2 = (\theta+2)/r_2$,

$$\max_{1\le k\le \kappa_n}\max_{1\le i\le n}\mathbb{P}\left(\max_{1\le l\le N}\left|\sum_{j=1}^{l}\widetilde{\varepsilon}_{ijk}-\sum_{j=1}^{l}Z_{ijk,\varepsilon}\right|>N^{\beta_2}\right)\le CN^{1-\beta_2 r_2},$$

which further implies that

$$\mathbb{P}\left(\max_{1\le k\le \kappa_n}\max_{1\le i\le n}\left|\frac{1}{N}\sum_{j=1}^{N}\widetilde{\varepsilon}_{ijk}-\frac{1}{N}\sum_{j=1}^{N}Z_{ijk,\varepsilon}\right|>N^{\beta_2-1}\right)\le CN^{\theta+1-\beta_2 r_2}\kappa_n,$$

Hence, we have

$$\max_{1\le k\le \kappa_n}\max_{1\le i\le n}\left|\frac{1}{N}\sum_{j=1}^{N}\widetilde{\varepsilon}_{ijk}-\frac{1}{N}\sum_{j=1}^{N}Z_{ijk,\varepsilon}\right|=\mathcal{O}_p\left(n^{-\frac{1-\beta_2}{\theta}}\right)=o_p(n^{-1/2}).$$

Thus,

$$\max_{1\le k\le \kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\left(\frac{1}{N}\sum_{j=1}^{N}\widetilde{\varepsilon}_{ijk}-\frac{1}{N}\sum_{j=1}^{N}Z_{ijk,\varepsilon}\right)\right|=o_p\left(n^{-1/2}\right)$$

Next, we consider the term $n^{-1}N^{-1}\sum_{i=1}^{n}\sum_{j=1}^{N}K_h(x-\xi_{ik})Z_{ijk,\varepsilon}$. Clearly $\mathbb{E}K_h(x-\xi_{ik})Z_{ijk,\varepsilon}=0$, and

$$\begin{aligned}\mathbb{E}\left(\frac{1}{N}\sum_{j=1}^{N}K_h(x-\xi_{ik})Z_{ijk,\varepsilon}\right)^2&=\frac{f_k(x)}{N^2h}\sum_{j=1}^{N}\sigma^2\left(\frac{j}{N}\right)\phi_k^2\left(\frac{j}{N}\right)\int K^2(v)dv(1+o(1))\\&=\mathcal{O}\left(\frac{1}{Nh}\right).\end{aligned}$$

Bernstein inequality shows that, for any fixed $x \in \mathbb{R}$, $k \in \mathbb{Z}_+$ and large n,

$$\mathbb{P}\left(\left|\frac{1}{nN}\sum_{i=1}^{n}\sum_{j=1}^{N} K_h(x-\xi_{ik})Z_{ijk,\varepsilon}\right| > C\sqrt{\frac{\log n}{nNh}}\right) \leq 2n^{-8}.$$

Applying similar discretization procedures in the proof of Lemma B.1, we derive that

$$\max_{1\leq k\leq \kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{nN}\sum_{i=1}^{n}\sum_{j=1}^{N} K_h(x-\xi_{ik})Z_{ijk,\varepsilon}\right| = \mathcal{O}_{a.s.}\left(\sqrt{\frac{\log n}{nNh}}\right).$$

Hence,

$$\begin{aligned}
&\max_{1\leq k\leq \kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik})\lambda_k^{-1/2}D_{ik2}\right| \\
\leq&\mathcal{O}_{a.s.}\left(\kappa_n^{\tau}\sqrt{\frac{\log n}{nNh}}\right) + \mathcal{O}_p\left(N^{\beta_2-1}\right) \\
=&o_p\left(n^{-1/2}\right).
\end{aligned}$$

Lemma B.3. *Under the assumptions in Lemma S.3,*

$$\max_{1\leq k\leq \kappa_n}\sup_{x\in\mathbb{R}}\left|-\frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik})\lambda_k^{-1/2}D_{ik3} - \frac{1}{n}\sum_{i=1}^{n}\xi_{ik}f_k(x)\right| = o_p\left(n^{-1/2}\right).$$

Proofs: We write

$$
\begin{aligned}
&\left|\frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik})\frac{1}{N}\sum_{j=1}^{n}\lambda_k^{-1}\left(\widehat{m}\left(\frac{j}{N}\right)-m\left(\frac{j}{N}\right)\right)\phi_k\left(\frac{j}{N}\right)-\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}f_k(x)\right| \\
&\leq \left|\frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik})\frac{1}{N}\sum_{j=1}^{N}\lambda_k^{-1}\left(\frac{1}{n}\sum_{l=1}^{n}\sum_{k'=1}^{\infty}\xi_{lk'}\phi_{k'}\left(\frac{j}{N}\right)\right)\phi_k\left(\frac{j}{N}\right)-\frac{1}{n}\sum_{i=1}^{n}\xi_{ik}f_k(x)\right| \\
&+\left|\frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik})\right|\left\|\widehat{m}-\bar{m}\right\|_\infty \lambda_k^{-1}\left\|\phi_k\right\|_\infty
\end{aligned}
$$

When $k'\neq k$, we show that

$$
\begin{aligned}
&\max_{1\leq k\leq \kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik})\lambda_k^{-1}\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{n}\sum_{l=1}^{n}\sum_{k'\neq k}^{\infty}\xi_{lk'}\phi_{k'}\left(\frac{j}{N}\right)\right)\phi_k\left(\frac{j}{N}\right)\right| \\
&\leq \max_{1\leq k\leq \kappa_n}\sup_{x\in\mathbb{R}}\frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik})\lambda_k^{-1}\max_{1\leq k\leq \kappa_n}\left|\sum_{k'\neq k}^{\infty}\frac{1}{N}\sum_{j=1}^{N}\phi_{k'}\left(\frac{j}{N}\right)\phi_k\left(\frac{j}{N}\right)\frac{1}{n}\sum_{l=1}^{n}\xi_{lk'}\right|
\end{aligned}
\tag{S.11}
$$

Observe that

$$
\begin{aligned}
&\max_{1\leq k\leq \kappa_n}\left|\sum_{k'\neq k}^{\infty}\frac{1}{N}\sum_{j=1}^{N}\phi_{k'}\left(\frac{j}{N}\right)\phi_k\left(\frac{j}{N}\right)\frac{1}{n}\sum_{l=1}^{n}\xi_{lk'}\right| \\
&=\max_{1\leq k\leq \kappa_n}\left|\sum_{k'\neq k}^{\infty}\left(\frac{1}{N}\sum_{j=1}^{N}\phi_{k'}\left(\frac{j}{N}\right)\phi_k\left(\frac{j}{N}\right)-\int_0^1\phi_{k'}(x)\phi_k(x)dx\right)\frac{1}{n}\sum_{l=1}^{n}\xi_{lk'}\right| \\
&\leq \max_{1\leq k\leq \kappa_n}\sum_{k'\neq k}^{\infty}\sum_{j=1}^{N}\int_{(j-1)/N}^{j/N}\left|\phi_{k'}\left(\frac{j}{N}\right)\phi_k\left(\frac{j}{N}\right)-\phi_{k'}(x)\phi_k(x)\right|dx\left|\frac{1}{n}\sum_{l=1}^{n}\xi_{lk'}\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq N^{-\varpi} \max_{1\leq k\leq \kappa_n} \sum_{k'\neq k}^{\infty} \left(\|\phi_k\|_\infty \|\phi_{k'}\|_{0,\varpi} + \|\phi_{k'}\|_\infty \|\phi_k\|_{0,\varpi} \right) \left| \frac{1}{n}\sum_{l=1}^{n} \xi_{lk'} \right| \\
&\leq C N^{-\varpi} \sum_{k'=1}^{\infty} \left(\|\phi_{k'}\|_{0,\varpi} + \|\phi_{k'}\|_\infty \right) \left| \frac{1}{n}\sum_{l=1}^{n} \xi_{lk'} \right| .
\end{aligned}
$$

Minkowski inequality implies that

$$
\begin{aligned}
\mathbb{E}\left(\sum_{k'=1}^{\infty} \|\phi_{k'}\|_\infty \left| \frac{1}{n}\sum_{l=1}^{n} \xi_{lk'} \right| \right)^{r_1} &\leq \left(\sum_{k'=1}^{\infty} \|\phi_{k'}\|_\infty \right)^{r_1} \sup_{k'} \mathbb{E} \left| \frac{1}{n}\sum_{l=1}^{n} \xi_{lk'} \right|^{r_1} \\
&\leq \left(\sum_{k'=1}^{\infty} \|\phi_{k'}\|_\infty \right)^{r_1} \sup_{k'} \mathbb{E} \left| \frac{1}{n}\sum_{l=1}^{n} \xi_{lk'} \right|^{r_1} \\
&= \mathcal{O}\left(n^{-r_1/2} \right) .
\end{aligned}
$$

Similarly,

$$
\left\| \left(\sum_{k'=1}^{\infty} \|\phi_{k'}\|_{0,\varpi} \left| \frac{1}{n}\sum_{l=1}^{n} \xi_{lk'} \right| \right) \right\|_{r_1} = \mathcal{O}\left(n^{-1/2} \right) .
$$

Hence, the right hand side of (S.11) is of order $\mathcal{O}_p\left(\kappa_n^{\tau} n^{-1/2-\varpi/\theta} \right)$.

When $k' = k$,

$$
\begin{aligned}
&\max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \left| \frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik}) \lambda_k^{-1} \frac{1}{n}\sum_{l=1}^{n} \xi_{lk} \frac{1}{N}\sum_{j=1}^{N} \phi_k^2\left(\frac{j}{N}\right) - f_k(x) \frac{1}{n}\sum_{l=1}^{n} \xi_{lk} \right| \\
\leq &\max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \left| \frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik}) \lambda_k^{-1} \frac{1}{n}\sum_{l=1}^{n} \xi_{lk} \right| \left| \frac{1}{N}\sum_{j=1}^{N} \phi_k^2\left(\frac{j}{N}\right) - \lambda_k \right| \\
+ &\max_{1\leq k\leq \kappa_n} \sup_{x\in\mathbb{R}} \left| \frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik}) - f_k(x) \right| \left| \frac{1}{n}\sum_{l=1}^{n} \xi_{lk} \right|
\end{aligned}
$$

$$
\begin{aligned}
&=\mathcal{O}_p\left(\kappa_n^{\tau+1/r_1} n^{-1/2} N^{-\varpi}\right)+\mathcal{O}_p\left(h\kappa_n^{1/r_1} n^{-1/2}\right) \\
&=\scriptstyle{\mathcal{O}}_p\left(n^{-1/2}\right).
\end{aligned}
$$

Clearly,

$$
\max_{1\leq k\leq\kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\right|\|\widehat{m}-\bar{m}\|_\infty=\mathcal{O}_{a.s.}(1)\cdot o_p(n^{-1/2}).
$$

Therefore,

$$
\max_{1\leq k\leq\kappa_n}\sup_{x\in\mathbb{R}}\left|-\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\lambda_k^{-1/2}D_{ik3}-f_k(x)\frac{1}{n}\sum_{l=1}^{n}\xi_{lk}\right|=o_p\left(n^{-1/2}\right),
$$

which completes the proof.

Lemma B.4. *Under the assumptions in Lemma S.3,*

$$
\max_{1\leq k\leq\kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\lambda_k^{-1/2}D_{ik4}\right|=o_p\left(n^{-1/2}\right).
$$

Proofs: We write

$$
\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\lambda_k^{-1/2}\frac{1}{N}\sum_{j=1}^{N}\left(\eta_i\left(\frac{j}{N}\right)-m\left(\frac{j}{N}\right)\right)\left(\widehat{\psi}_k\left(\frac{j}{N}\right)-\psi_k\left(\frac{j}{N}\right)\right)\right|
$$

$$
\begin{aligned}
&= \left| \frac{1}{n} \sum_{i=1}^{n} K_h(x - \xi_{ik}) \lambda_k^{-1/2} \frac{1}{N} \sum_{j=1}^{N} \sum_{k'=1}^{\infty} \xi_{ik'} \phi_{k'} \left(\frac{j}{N} \right) \left(\widehat{\psi}_k \left(\frac{j}{N} \right) - \psi_k \left(\frac{j}{N} \right) \right) \right| \\
&\leq \lambda_k^{-1/2} \left| \frac{1}{n} \sum_{i=1}^{n} K_h(x - \xi_{ik}) \xi_{ik} \frac{1}{N} \sum_{j=1}^{N} \phi_k \left(\frac{j}{N} \right) \left(\widehat{\psi}_k \left(\frac{j}{N} \right) - \psi_k \left(\frac{j}{N} \right) \right) \right| \\
&+ \lambda_k^{-1/2} \frac{1}{N} \sum_{j=1}^{N} \left| \sum_{k' \neq k}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} K_h(x - \xi_{ik}) \xi_{ik'} \phi_{k'} \left(\frac{j}{N} \right) \right) \left(\widehat{\psi}_k \left(\frac{j}{N} \right) - \psi_k \left(\frac{j}{N} \right) \right) \right| \\
&= A_{nk1}(x) + A_{nk2}(x).
\end{aligned}
$$

We decompose $A_{nk1}(x)$ as follows,

$$
\begin{aligned}
A_{nk1}(x) &\leq \lambda_k^{-1/2} \left| \frac{1}{n} \sum_{i=1}^{n} K_h(x - \xi_{ik}) \xi_{ik} \frac{1}{N} \sum_{j=1}^{N} \phi_k \left(\frac{j}{N} \right) \left(\widehat{\psi}_k \left(\frac{j}{N} \right) - \widetilde{\psi}_k \left(\frac{j}{N} \right) \right) \right| \\
&+ \lambda_k^{-1/2} \left| \frac{1}{n} \sum_{i=1}^{n} K_h(x - \xi_{ik}) \xi_{ik} \frac{1}{N} \sum_{j=1}^{N} \phi_k \left(\frac{j}{N} \right) \left(\widetilde{\psi}_k \left(\frac{j}{N} \right) - \psi_k \left(\frac{j}{N} \right) \right) \right| \\
&=: A_{nk11}(x) + A_{nk12}(x).
\end{aligned}
$$

Consider the class of measurable functions $\{f_x(u) = K_h(x-u)u, x \in \mathbb{R}\}$, with envelope function $F(\xi) = \xi h^{-1} \|K\|_\infty$ and finite VC dimension. Empirical process theory yields that for any $k \in \mathbb{Z}_+$,

$$
\mathbb{E} \sup_{x \in \mathbb{R}} \left| \frac{1}{n} \sum_{i=1}^{n} K_h(x - \xi_{ik}) \xi_{ik} - \mathbb{E} \left(K_h(x - \xi_{ik}) \xi_{ik} \right) \right| \leq C n^{-1/2} \|F\|_2 = \mathcal{O} \left(n^{-1/2} h^{-1} \right).
$$

Hence,

$$\mathbb{E} \max_{1\le k\le \kappa_n} \sup_{x\in\mathbb{R}} \left| \frac{1}{n}\sum_{i=1}^{n} K_h(x-\xi_{ik})\xi_{ik} - \mathbb{E}\left(K_h(x-\xi_{ik})\xi_{ik}\right) \right| = \mathcal{O}\left(\kappa_n n^{-1/2} h^{-1}\right) = o(1).$$

Notice that

$$\begin{aligned}
&\left|\mathbb{E}\left(K_h(x-\xi_{ik})\xi_{ik}\right) - x f_k(x)\right| \\
\le & x\int K(v)\left|f_k(x-vh)-f_k(x)\right| dv + h\int vK(v)\left|f_k(x-vh)-f_k(x)\right| dv \\
\le & xh\left\|F_k\right\|_{1,1}\int |v|K(v)dv + h^2\left\|F_k\right\|_{1,1}\int |v|^2 K(v)dv.
\end{aligned}$$

Hence, $\max_{1\le k\le\kappa_n}\sup_{x\in[-b_n,b_n]}\left|\mathbb{E}\left(K_h(x-\xi_{ik})\xi_{ik}\right) - x f_k(x)\right| = o(1)$, where $b_n = n^{\iota_2}$ with $\iota_2 = 1/9$ and $b_n h = o(1)$. Further, for sufficient large n,

$$\begin{aligned}
\max_{1\le k\le\kappa_n}\sup_{x\in[b_n,\infty)} \mathbb{E}\left(K_h(x-\xi_{ik})\xi_{ik}\right) &\le \frac{\|K\|_\infty}{h}\max_{1\le k\le\kappa_n}\mathbb{E}\left(\xi_{ik} I\left(\xi_{ik} > n^{\iota_2}+h\right)\right) \\
&\le Cn^{-\iota_2(r_1-1)}h^{-1}\max_{1\le k\le\kappa_n}\mathbb{E}\left(\xi_{ik}^{r_1} I\left(\xi_{ik} > 2n^{\iota_2}\right)\right) \\
&= o\left(n^{-\iota_2(r_1-1)}h^{-1}\right).
\end{aligned}$$

Taking similar procedures to the case $x\in(-\infty,-b_n]$, we obtain

$$\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\mathbb{E}\left(K_h(x-\xi_{ik})\xi_{ik}\right) - x f_k(x)\right| = o\left(n^{-\iota_2(r_1-1)}h^{-1}\right) + \mathcal{O}\left(n^{\iota_2}h\right).$$

Then, we derive that

$$\max_{1\le k\le \kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\xi_{ik}\right| = \mathcal{O}_p(1).$$

From Cai and Hu (2024), we obtain the fact that $\max_{1\le k\le\kappa_n}\left\|\widehat{\psi}_k-\widetilde{\psi}_k\right\|_\infty = \mathcal{O}_p\left(\kappa_n^{3\tau/2+1}\left\|\widehat{G}-\widetilde{G}\right\|_\infty\right)$, which is then followed by $\sup_{x\in\mathbb{R}} A_{nk11}(x) = o_p\left(n^{-1/2}\right)$.

We could obtain the following fact from Cai and Hu (2024),

$$\max_{1\le k\le \kappa_n}\sup_{x\in\mathbb{R}}\sqrt{n}\left|\left(\widetilde{\psi}_k(x)-\psi_k(x)\right)-\sum_{l\neq k}^{\infty}\frac{\sqrt{\lambda_l\lambda_k}}{\lambda_l-\lambda_k}\left(\frac{1}{n}\sum_{m=1}^{n}\xi_{mk}\xi_{ml}\right)\psi_l(x)\right| = o_p(1),$$

Besides, we have

$$\begin{aligned}
&\max_{1\le k\le\kappa_n}\sqrt{n}\lambda_k^{-1}\left|\frac{1}{N}\sum_{j=1}^{N}\phi_k\left(\frac{j}{N}\right)\sum_{l\neq k}^{\infty}\frac{\sqrt{\lambda_l\lambda_k}}{\lambda_l-\lambda_k}\left(\frac{1}{n}\sum_{m=1}^{n}\xi_{mk}\xi_{ml}\right)\phi_l\left(\frac{j}{N}\right)\right| \\
&\le \max_{1\le k\le\kappa_n}\sqrt{n}\lambda_k^{-1}\sum_{l\neq k}^{\infty}\frac{\sqrt{\lambda_l\lambda_k}}{|\lambda_l-\lambda_k|}\left|\frac{1}{n}\sum_{m=1}^{n}\xi_{mk}\xi_{ml}\right|\left|\frac{1}{N}\sum_{j=1}^{N}\phi_k\left(\frac{j}{N}\right)\phi_l\left(\frac{j}{N}\right)-\int_0^1\phi_k(x)\phi_l(x)dx\right| \\
&\le \max_{1\le k\le\kappa_n}C\lambda_k^{-1/2}\delta_k^{-1}\sum_{l\neq k}^{\infty}\left|\frac{1}{\sqrt{n}}\sum_{m=1}^{n}\xi_{mk}\xi_{ml}\right|\left(\|\phi_l\|_\infty+\|\phi_l\|_{0,\varpi}\right)N^{-\varpi} \\
&= \mathcal{O}_p\left(\kappa_n^{3\tau/2+1+1/r_1}N^{-\varpi}\right),
\end{aligned}$$

where the last equality holds by Minkowski inequality and L^{r_1} maximum inquality. Thus, we yield that $\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}A_{nk12}(x) = o_p\left(n^{-1/2}\right)$.

As to $A_{nk2}(x)$, we have

$$\max_{1\le k\le \kappa_n} \sup_{x\in\mathbb{R}} A_{nk2}(x) \le \max_{1\le k\le \kappa_n} \lambda_k^{-1/2} \sum_{k'\neq k}^{\infty} \left\{ \|\phi_{k'}\|_\infty \sup_{x\in\mathbb{R}} \left| \frac{1}{n}\sum_{i=1}^{n} \xi_{ik'} K_h(x-\xi_{ik}) \right| \right\} \left\| \widehat{\psi}_k - \psi_k \right\|_\infty .$$

Making use of the independence between ξ_{ik} and $\xi_{ik'}$, $k \neq k'$, classical kernel smoothing theory yields that, for any $k' \in \mathbb{Z}_+$,

$$\mathbb{E} \max_{1\le k\le \kappa_n} \sup_{x\in\mathbb{R}} \left| \frac{1}{n}\sum_{i=1}^{n} \xi_{ik'} K_h(x-\xi_{ik}) \right| = \mathcal{O}\left(\sqrt{\frac{\log n}{nh}} \right).$$

Then,

$$\begin{aligned} &\mathbb{E} \max_{1\le k\le \kappa_n} \sum_{k'\neq k}^{\infty} \left\{ \|\phi_{k'}\|_\infty \sup_{x\in\mathbb{R}} \left| \frac{1}{n}\sum_{i=1}^{n} \xi_{ik'} K_h(x-\xi_{ik}) \right| \right\} \\ \le& \sum_{k'=1}^{\infty} \left\{ \|\phi_{k'}\|_\infty \mathbb{E} \max_{1\le k\le \kappa_n} \sup_{x\in\mathbb{R}} \left| \frac{1}{n}\sum_{i=1}^{n} \xi_{ik'} K_h(x-\xi_{ik}) \right| \right\} \\ =& \mathcal{O}\left(\sqrt{\frac{\log n}{nh}} \right), \end{aligned}$$

which further implies that $\max_{1\le k\le \kappa_n} \sup_{x\in\mathbb{R}} A_{nk2}(x) = \mathcal{O}_p\left(\kappa_n^{2\tau+1} n^{-1} h^{-1/2} \log^{1/2} n \right) = \scriptstyle{\mathcal{O}}_p\left(n^{-1/2}\right)$.

Lemma B.5. *Under the assumptions in Lemma S.3,*

$$\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\lambda_k^{-1/2}D_{ik5}\right|=o_p\left(n^{-1/2}\right).$$

Proofs: We show that

$$\begin{aligned}
&\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\lambda_k^{-1/2}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\frac{1}{N}\sum_{j=1}^{N}\sigma\left(\frac{j}{N}\right)\varepsilon_{ij}\left(\widehat{\psi}_k\left(\frac{j}{N}\right)-\psi_k\left(\frac{j}{N}\right)\right)\right|\\
&\le\max_{1\le k\le\kappa_n}\lambda_k^{-1/2}\frac{1}{N}\sum_{j=1}^{N}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\sigma\left(\frac{j}{N}\right)\varepsilon_{ij}\right|\left|\widehat{\psi}_k\left(\frac{j}{N}\right)-\psi_k\left(\frac{j}{N}\right)\right|\\
&\le\max_{1\le k\le\kappa_n}\lambda_k^{-1/2}\frac{1}{N}\sum_{j=1}^{N}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\sigma\left(\frac{j}{N}\right)\varepsilon_{ij}\right|\left\|\widehat{\psi}_k-\psi_k\right\|_\infty.
\end{aligned}$$

Classic kernel smoothing theory entails that, for any $1\le j\le N$,

$$\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\sigma\left(\frac{j}{N}\right)\varepsilon_{ij}\right|=\mathcal{O}_{a.s.}\left(\sqrt{\frac{\log n}{nh}}\right).$$

Further notice that fact $\max_{1\le k\le\kappa_n}\left\|\widehat{\psi}_k-\psi_k\right\|_\infty=\mathcal{O}_p\left(\kappa_n^{3\tau/2+1}n^{-1/2}\right)$ to complete the proof.

Lemma B.6. *Under the assumptions in Lemma S.3,*

$$\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\lambda_k^{-1/2}D_{ik6}\right|=o_p\left(n^{-1/2}\right).$$

Proofs: Consider that $\max_{1\le k\le \kappa_n}\left\|\widehat{\psi}_k-\psi_k\right\|_\infty=\mathcal{O}_p\left(\kappa_n^{3\tau/2+1}n^{-1/2}\right)$ and $\|\widehat{m}-m\|_\infty=\mathcal{O}_p\left(n^{-1/2}\right)$ to complete the proof.

Lemma B.7. *Under the assumptions in Lemma S.3,*

$$\sup_{x\in\mathbb{R}}\left|-\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\left(\widehat{\lambda}_k^{-1/2}-\lambda_k^{-1/2}\right)D_{ik1}-xf_k(x)\frac{1}{2n}\sum_{i=1}^{n}\left(\xi_{ik}^2-1\right)\right|=o_p\left(n^{-1/2}\right).$$

Proofs: Since we have the following fact from Cai and Hu (2024)

$$\max_{1\le k\le\kappa_n}\left|\sqrt{n}\left(\widehat{\lambda}_k-\lambda_k\right)-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\lambda_k\left(\xi_{ik}^2-1\right)\right|=\mathcal{O}_p\left(\kappa_n^{2(\tau+1)}n^{-1/2}\right),$$

which further yields that,

$$\max_{1\le k\le\kappa_n}\left|\sqrt{n}\left(1-\frac{\lambda_k^{1/2}}{\widehat{\lambda}_k^{1/2}}\right)-\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\left(\xi_{ik}^2-1\right)\right|=o_p(1).$$

It is noted that

$$\begin{aligned}
&\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\left(\widehat{\lambda}_k^{-1/2}-\lambda_k^{-1/2}\right)D_{ik1}-\mathbb{E}K_h(x-\xi_{ik})\xi_{ik}\left(\frac{\lambda_k^{1/2}}{\widehat{\lambda}_k^{1/2}}-1\right)\right|\\
&\le\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\lambda_k^{-1}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\left(\frac{\lambda_k^{1/2}}{\widehat{\lambda}_k^{1/2}}-1\right)\left(\frac{1}{N}\sum_{j=1}^{N}\sum_{k'\neq k}^{\infty}\xi_{ik'}\phi_{k'}\left(\frac{j}{N}\right)\phi_k\left(\frac{j}{N}\right)\right)\right|\\
&+\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\lambda_k^{-1}\left|\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\xi_{ik}\left(\frac{\lambda_k^{1/2}}{\widehat{\lambda}_k^{1/2}}-1\right)\left(\frac{1}{N}\sum_{j=1}^{N}\phi_k\left(\frac{j}{N}\right)\phi_k\left(\frac{j}{N}\right)-1\right)\right|\\
&+\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\lambda_k^{-1}\left|\left(\frac{1}{n}\sum_{i=1}^{n}K_h(x-\xi_{ik})\xi_{ik}-\mathbb{E}K_h(x-\xi_{ik})\xi_{ik}\right)\left(\frac{\lambda_k^{1/2}}{\widehat{\lambda}_k^{1/2}}-1\right)\right|,
\end{aligned}\tag{S.12}$$

in which

$$\begin{aligned}
&\max_{1\le k\le\kappa_n}\left|\frac{1}{N}\sum_{j=1}^{N}\sum_{k'\neq k}^{\infty}\xi_{ik'}\phi_{k'}\left(\frac{j}{N}\right)\phi_k\left(\frac{j}{N}\right)\right|\\
\le&\max_{1\le k\le\kappa_n}\sum_{k'\neq k}^{\infty}|\xi_{ik'}|\left|\frac{1}{N}\sum_{j=1}^{N}\phi_{k'}\left(\frac{j}{N}\right)\phi_k\left(\frac{j}{N}\right)-\int_0^1\phi_{k'}(x)\phi_k(x)dx\right|\\
\le&CN^{-\varpi}\sum_{k'=1}^{\infty}|\xi_{ik'}|\left(\|\phi_{k'}\|_\infty+\|\phi_{k'}\|_{0,\varpi}\right)\\
=&\mathcal{O}_p\left(N^{-\varpi}\right),
\end{aligned}$$

and

$$\begin{aligned}
\max_{1\le k\le\kappa_n}\left|1-\frac{\lambda_k^{1/2}}{\widehat{\lambda}_k^{1/2}}\right|&=\max_{1\le k\le\kappa_n}\left|\frac{1}{2n}\sum_{i=1}^{n}\left(\xi_{ik}^2-1\right)\right|+o_p\left(n^{-1/2}\right)\\
&=\mathcal{O}_p\left(\kappa_n^{2/r_1}n^{-1/2}\right).
\end{aligned}\tag{S.13}$$

Hence, the left hand side of (S.12) is of order $\mathcal{O}_p\left(\kappa_n^{2/r_1+\tau}n^{-1/2}N^{-\varpi}+\kappa_n^{2/r_1+\tau+1}n^{-1}h^{-1}\right)$. Recalling the fact that

$$\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\mathbb{E}\left(K_h(x-\xi_{ik})\xi_{ik}\right)-xf_k(x)\right|=o\left(n^{-\iota_2(r_1-1)}h^{-1}\right)+\mathcal{O}\left(n^{\iota_2}h\right).$$

we conclude that

$$\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\mathbb{E}K_h(x-\xi_{ik})\xi_{ik}\left(1-\frac{\lambda_k^{1/2}}{\widehat{\lambda}_k^{1/2}}\right)-xf_k(x)\frac{1}{2n}\sum_{i=1}^n\left(\xi_{ik}^2-1\right)\right|=o_p\left(n^{-1/2}\right).$$

Lemma B.8. *Under the assumptions in Lemma S.3,*

$$\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}T_{k2}(x)=o_p\left(n^{-1/2}\right).$$

Proofs: Note that

$$\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}T_{k2}(x)\le\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\frac{1}{nh^2}\sum_{i=1}^n\left|K'\left(\frac{x-\xi_{ik}}{h}\right)\right|\max_{1\le k\le\kappa_n}\max_{1\le i\le n}\left(\xi_{ik}-\widehat{\xi}_{ik}\right)^2.$$

Taking similar procedures as in the proof of Lemma B.2, we could obtain

$$\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|\frac{1}{nh^2}\sum_{i=1}^n\left|K'\left(\frac{x-\xi_{ik}}{h}\right)\right|-\frac{f_k(x)}{h}\int\left|K'(v)\right|dv\right|=\mathcal{O}_{a.s.}\left(\sqrt{\frac{\log n}{nh^3}}+h\right).$$

Recalling the decomposition of $\widehat{\xi}_{ik}$ in (S2.2), tedious but elementary calculations show that $\max_{1\le k\le\kappa_n}\max_{1\le i\le n}\left(\xi_{ik}-\widehat{\xi}_{ik}\right)^2=\mathcal{O}_p\left(n^{-1+2/r_1}\kappa_n^C\right)$,

Lemma B.9. *Under the assumptions in Lemma S.3,*

$$\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}T_{k3}(x)=\mathcal{o}_p\left(n^{-1/2}\right).$$

Proofs: Note that

$$\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}T_{k3}(x)\le\max_{1\le k\le\kappa_n}\sup_{x\in\mathbb{R}}\left|K''\left(\frac{x-\varrho_{ik}}{h}\right)\right|\max_{1\le k\le\kappa_n}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\xi_{ik}-\widehat{\xi}_{ik}}{h}\right)^3.$$

Recalling the decomposition of $\widehat{\xi}_{ik}$ in (S2.2), tedious but elementary calculations show that

$\max_{1\le k\le\kappa_n}n^{-1}\sum_{i=1}^{n}\left(\xi_{ik}-\widehat{\xi}_{ik}\right)^3h^{-3}=\mathcal{o}_p\left(n^{-1/2}\right)$.

Some modification for theoretical derivation in Zhong and Yang (2022): Assume that $\mathbb{E}\left\|\chi_i\right\|_{q,\mu}^{r_1}<\infty$ rather than $\sum_{k=1}^{\infty}\left\|\phi_k\right\|_{q,\mu}<\infty$. For any fixed $\omega\in\Omega$, $\chi_i(\omega,\cdot)\in\mathcal{H}^{(q,\mu)}[0,1]$. Then, there exists a spline function $\boldsymbol{B}^\top(\cdot)\boldsymbol{\alpha}(\omega)$ such that $\left\|\chi_i(\omega,\cdot)-\boldsymbol{B}^\top(\cdot)\boldsymbol{\alpha}(\omega)\right\|_\infty\le C_{q,\mu}\left\|\chi_i(\omega,\cdot)\right\|_{q,\mu}N_s^{-p^*}$. Recall the definition of $\tilde{\chi}_i$ in Zhong and Yang (2022), we obtain that $\left\|\tilde{\chi}_i(\omega,\cdot)-\boldsymbol{B}^\top(\cdot)\boldsymbol{\alpha}(\omega)\right\|_\infty\lesssim\left\|\chi_i(\omega,\cdot)\right\|_{q,\mu}N_s^{-p^*}$.

Thus, $\|\tilde{\chi}_i(\omega,\cdot)-\chi_i(\omega,\cdot)\|_\infty \lesssim \|\chi_i(\omega,\cdot)\|_{q,\mu} N_s^{-p^*}$. Let

$$A=\left\{\omega : \max_{1\le i\le n} \|\chi_i(\omega,\cdot)\|_{q,\mu} \le (n\log n)^{2/r_1}\right\}.$$

Markov inequality shows that $\mathbb{P}(A^c)\to 0$ as $n\to\infty$. Thus, we still have

$$\max_{1\le i\le n} \|\tilde{\chi}_i(\cdot)-\chi_i(\cdot)\|_\infty = \mathcal{O}_p\left((n\log n)^{2/r_1} N_s^{-p^*}\right).$$

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