Robust Tests for Changing Volatility

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Supplementary Material

This supplementary material provides two appendices for the main paper. Appendix S1 offers the proofs of the Theorems 1-5 and associated Lemmas. Appendix S2 presents some additional simulation results when u_t is not directly observable.

S1 Proofs of the main results

Lemma A1. (i)Suppose Assumptions 1-2 hold, then under $\mathbb{H}_0$ we have $\sqrt{T}(\tilde{g} - g_0) \xrightarrow{d} \frac{\sigma_0 \omega W(1)}{2f_{|\varepsilon|}(c)}$, where $g_0 = c\sigma_0$;

(ii) Suppose Assumptions 1-2 hold, then under $\mathbb{H}_A$ we have $\sqrt{T}\left(\tilde{g} - g^*\right) = O_p\left(1\right)$, where $g^* = c\lim_{T\to\infty}\left(\sum_{s=1}^{T} \frac{f_{|\varepsilon|,s-1}(\tilde{c}_s)}{\sigma_s}\right)^{-1} \sum_{s=1}^{T} f_{|\varepsilon|,s-1}\left(\tilde{c}_s\right)$, $\tilde{c}_s$ is a value between c and $c + \frac{\tilde{g}-g_s}{\sigma_s}$.

Proof: Denote $v = T^{1/2}(g - g_0)$, minimization of (2.4) is equivalent to minimizing the following criterion:

$$Q_T\left(v\right) = \sum_{s=1}^{T} \left|e_s - T^{-1/2}v\right| - \sum_{s=1}^{T} \left|e_s\right|. \qquad \text{(S1.1)}$$

Define

$$z_T = T^{-1/2} \sum_{s=1}^{T} sgn\left(e_s\right); \;\; Z_T = \sum_{s=1}^{T} \int_0^{T^{-1/2}v} \left[I\left(e_s \leq x\right) - I\left(e_s \leq 0\right)\right] dx.$$

By Knight's identity (Knight, 1998), the objective function (S1.1) can be rewritten as

$$Q_T\left(v\right) = -vz_T + 2Z_T. \tag{S1.2}$$

Now we define $B_s = \int_0^{T^{-1/2}v} \left[I\left(e_s \leq x\right) - I\left(e_s \leq 0\right)\right] dx$ and $\mu_s = E(B_s|\mathcal{F}_{s-1})$, where $\mathcal{F}_{s-1}$ is the σ-field generated by $\{\varepsilon_{s-1}, \varepsilon_{s-2}, \cdots\}$. We now rewrite Z_T as

$$Z_T = \sum_{s=1}^{T} \mu_s + \sum_{s=1}^{T} \left(B_s - \mu_s\right). \tag{S1.3}$$

Subsequently, under Assumption 2 we can show that for each $v \in R$,

$$\begin{aligned}\sum_{s=1}^{T} \mu_s = \sum_{s=1}^{T} \int_0^{T^{-1/2}v} & \left[E(I\left(e_s \leq x\right)|\mathcal{F}_{s-1}) - E(I\left(e_s \leq 0\right) |\ \mathcal{F}_{s-1})\right] dx \\ & \to \frac{\frac{1}{T}\sum_{s=1}^{T} f_{|\varepsilon|,s-1}\left(c\right)}{2\sigma_0} v^2. \end{aligned} \tag{S1.4}$$

Next, the integrability of $f^{\gamma}_{|\varepsilon|,s-1}\left(c\right)$ and stationarity of $\left\{f_{|\varepsilon|,s-1}\left(c\right)\right\}$ in Assumption 2 ensure that

$$\sup_{0<r\leq 1} \left| \frac{1}{T^{1-\epsilon}} \sum_{s=1}^{[Tr]} \left[f_{|\varepsilon|,s-1}\left(c\right) - f_{|\varepsilon|}(c)\right] \right| \to 0, \tag{S1.5}$$

for some $\epsilon > 0$ (also see Herce, 1996), which in turn leads to

$$\sum_{s=1}^{T} \mu_s \to \frac{f_{|\varepsilon|}(c)}{2\sigma_0} v^2. \tag{S1.6}$$

As a result we have $Z_T = \frac{f_{|\varepsilon|}(c)}{2\sigma_0} v^2 + o_p(1)$, provided that $\sum_{s=1}^{T}(B_s - \mu_s) = o_p(1)$. Hence it is enough to prove $var\left[\sum_{s=1}^{T}(B_s - \mu_s)\right] = o(1)$. Note that $\{B_s - \mu_s\}$ is a martingale difference sequence, so we have

$$var\left[\sum_{s=1}^{T}(B_s - \mu_s)\right] \leq \sum_{s=1}^{T} E\left(B_s^2\right)$$
$$\leq \frac{f_{|\varepsilon|}(c)}{\sigma_0}\left(T\int_0^{T^{-1/2}v}\int_0^{T^{-1/2}v}\min(x,y)\,dxdy\right) = O\left(T^{-1/2}\right) \tag{S1.7}$$

where $\int_0^{T^{-1/2}v}\int_0^{T^{-1/2}v}\min(x,y)\,dxdy = \frac{1}{3}T^{-3/2}v^3$. By (S1.2)-(S1.7), we obtain

$$Q_T(v) = -vz_T + \frac{f_{|\varepsilon|}(c)}{\sigma_0}v^2 + o_p(1). \tag{S1.8}$$

Because $Q_T(v)$ is convex for each v, Theorem 2 in Kato (2009) implies that

$$\tilde{v} = \frac{\sigma_0 z_T}{2f_{|\varepsilon|}(c)} + o_p(1). \tag{S1.9}$$

Under Assumption 1, using Theorem 2.21(ii) of Fan and Yao(2003) we immediately have

$$z_T = T^{-1/2}\sum_{s=1}^{T} sgn(e_s) \xrightarrow{d} \omega W(1), \tag{S1.10}$$

which, together with (S1.9), gives $\sqrt{T}(\tilde{g} - g_0) \xrightarrow{d} \frac{\sigma_0 \omega W(1)}{2f_{|\varepsilon|}(c)}$.This completes the proof of Lemma A1(i). □

(ii)Under $\mathbb{H}_A$,minimization of (2.4) is equivalent to minimizing the following criterion:

$$Q_T(g) = \sum_{s=1}^{T} |e_s - (g - g_s)| - \sum_{s=1}^{T} |e_s| . \tag{S1.11}$$

By taking similar arguments to those of showing Lemma A1(i), we can show that

$$Q_T(g) = -g \sum_{s=1}^{T} sgn(e_s) + 2 \sum_{s=1}^{T} \int_0^{g-g_s} \left[F_{|\varepsilon|,s-1}\left(c + \frac{x}{\sigma_s}\right) - F_{|\varepsilon|,s-1}(c) \right] dx + o_p(1) . \tag{S1.12}$$

Because $Q_T(g)$ is convex for each g, then its optimal solution is given by

$$-\sum_{s=1}^{T} sgn(e_s) + 2 \sum_{s=1}^{T} \left[F_{|\varepsilon|,s-1}\left(c + \frac{\hat{g} - g_s}{\sigma_s}\right) - F_{|\varepsilon|,s-1}(c) \right] = 0. \tag{S1.13}$$

By using the mean value theorem we have

$$-\sum_{s=1}^{T} sgn(e_s) + 2 \sum_{s=1}^{T} f_{|\varepsilon|,s-1}(\tilde{c}_s) \left(\frac{\tilde{g}}{\sigma_s} - c \right) = 0, \tag{S1.14}$$

where $\tilde{c}_s$ is a value between c and $c + \frac{\tilde{g}-g_s}{\sigma_s}$. As a result, we obtain

$$\tilde{g} = c\left(\sum_{s=1}^{T}\frac{f_{|\varepsilon|,s-1}\left(\tilde{c}_s\right)}{\sigma_s}\right)^{-1}\sum_{s=1}^{T}f_{|\varepsilon|,s-1}\left(\tilde{c}_s\right)+\left(2\sum_{s=1}^{T}\frac{f_{|\varepsilon|,s-1}\left(\tilde{c}_s\right)}{\sigma_s}\right)^{-1}\sum_{s=1}^{T}sgn\left(e_s\right). \tag{S1.15}$$

By (S1.10) we also have $T^{-1/2}\sum_{s=1}^{T}sgn\left(e_s\right)\xrightarrow{d}\omega W(1)$, define $\varphi^* = 2\lim_{T\to\infty}\frac{1}{T}\sum_{s=1}^{T}\frac{f_{|\varepsilon|,s-1}(\tilde{c}_s)}{\sigma_s}$ and $g^* = c\lim_{T\to\infty}\left(\sum_{s=1}^{T}\frac{f_{|\varepsilon|,s-1}(\tilde{c}_s)}{\sigma_s}\right)^{-1}\sum_{s=1}^{T}f_{|\varepsilon|,s-1}\left(\tilde{c}_s\right)$, we immediately have $\sqrt{T}\left(\tilde{g}-g^*\right) = O_p\left(1\right)$. This completes the proof of Lemma A1(ii). □

Lemma A2. (i)Suppose Assumptions 1-2 hold, then under $\mathbb{H}_0$ we have

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]}sgn\left(\tilde{e}_t\right)\xrightarrow{d}\omega\left(W\left(r\right)-rW\left(1\right)\right);$$

(ii)Suppose Assumptions 1-2 hold, then under $\mathbb{H}_A$ we have

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]}sgn\left(\tilde{e}_t\right) = O_p(\sqrt{T}).$$

Proof: Because $sgn\left(\cdot\right)$ is not everywhere differentiable and we cannot directly take a Taylor expansion with $sgn\left(\cdot\right)$, we proceed by treating the function $sgn\left(\cdot\right)$ as a generalized function with a smooth regular sequence $sgn_m\left(\cdot\right)$ defined on an appropriate set of test functions (see Phillips (1995) and Xiao (2012) for more discussion), then $sgn_m\left(\cdot\right)\to sgn\left(\cdot\right)$, and $sgn_m'\left(\cdot\right)\to sgn'\left(\cdot\right)$ as $m\to\infty$, where $sgn_m\left(\cdot\right)$ and its first derivative $sgn_m'\left(\cdot\right)$ are bounded functions for each m. Let $S_{T,m}\left(r\right) =$

$\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn_m(\tilde{e}_t)$,where $\tilde{e}_t = e_t - (\tilde{g} - g_t)$. To obtain asymptotic results of $\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn(\tilde{e}_t)$, it is enough to study the asymptotic behavior of $S_{T,m}(r)$. By using the mean value theorem, we have

$$sgn_m(\tilde{e}_t) = sgn_m(e_t) - (\tilde{g} - g_t)\, sgn'_m(e_t^*), \tag{S1.16}$$

where $e_t^* = e_t - (1-\lambda)(\tilde{g} - g_t)$, $\lambda \in [0,1]$. Then

$$S_{T,m}(r) = \frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn_m(e_t) - \frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} (\tilde{g} - g_t)\, sgn'_m(e_t^*), \tag{S1.17}$$

Firstly, under Assumption 1 it is easy to know

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn_m(e_t) \xrightarrow{d} \omega W(r), \tag{S1.18}$$

as $m \to \infty$ by using Theorem 2.21(ii) of Fan and Yao(2003).

Next, we need to derive the limit of $\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} (\tilde{g} - g_t)\, sgn'_m(e_t^*)$. Let $f_{t,e^*}(x)$ represent the pdf of e_t^* at the point x, we can obtain

$$f_{t,e^*}(x) = \sigma_t^{-1} f_{|\varepsilon|}\left(c + \frac{x + (1-\lambda)(\tilde{g} - g_t)}{\sigma_t}\right). \tag{S1.19}$$

In addition, we also have $E(sgn'(e_t^*)) = 2f_{t,e^*}(0)$ by the formula (27) of Lighthill (1958). Hence

$$E(sgn'(e_t^*)) = 2\sigma_t^{-1} f_{|\varepsilon|}\left(c + \frac{(1-\lambda)(\tilde{g} - g_t)}{\sigma_t}\right). \tag{S1.20}$$

(i) Under $\mathbb{H}_0$, Lemma A1 implies $\sqrt{T}(\tilde{g} - g_0) = O_p(1)$. Using the formula (10) of Phillips(1995), $sgn'_m(\cdot) \to sgn'(\cdot)$ and (S1.20), we immediately obtain

$$\frac{1}{T}\sum_{t=1}^{[Tr]} sgn'_m\left(e_t^*\right) \xrightarrow{p} \frac{2f_{|\varepsilon|}(c)r}{\sigma_0}. \tag{S1.21}$$

Based on Lemma A1 and the result of (S1.21), we have

$$\sqrt{T}\left(\tilde{g}-g_0\right)\frac{1}{T}\sum_{t=1}^{[Tr]} sgn'_m\left(e_t\right) \xrightarrow{d} \omega r W(1). \tag{S1.22}$$

By using the results of (S1.18) and (S1.22) we obtain

$$S_{T,m}\left(r\right) \xrightarrow{d} \omega\left(W\left(r\right)-rW\left(1\right)\right), \tag{S1.23}$$

which means that $\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn\left(\tilde{e}_t\right) \xrightarrow{d} \omega\left(W\left(r\right)-rW\left(1\right)\right)$.

(ii)Similar to the proof of (S1.21), we can show that

$$\begin{aligned}\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]}\left(\tilde{g}-g_t\right)sgn'_m\left(e_t^*\right) &= \sqrt{T}\left(\tilde{g}-g^*\right)\frac{1}{T}\sum_{t=1}^{[Tr]} f_{|\varepsilon|}\left(c+\left(1-\lambda\right)\frac{g^*-g_t}{\sigma_t}\right)\\ &\quad+\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]}\frac{g^*-g_t}{\sigma_t}f_{|\varepsilon|}\left(c+\left(1-\lambda\right)\frac{g^*-g_t}{\sigma_t}\right)+o_p\left(1\right)\\ &= O_p\left(\sqrt{T}\right),\end{aligned} \tag{S1.24}$$

where the first term on R.H.S is $O_p\left(1\right)$ since$\sqrt{T}\left(\tilde{g}-g^*\right)=O_p\left(1\right)$ by Lemma A1(ii), and the second term on R.H.S is $O_p\left(\sqrt{T}\right)$ since $g_t \neq g^*$ under the alternatives.

By using the results of (S1.18) and (S1.24) we have $\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn\left(\tilde{e}_t\right) = O_p(\sqrt{T})$ under $\mathbb{H}_A$. This completes the proof of Lemma A2. □

Lemma A3. (i)Suppose Assumptions 1-3 hold, then under $\mathbb{H}_0$ we have

$\tilde{\omega}^2 \xrightarrow{p} \omega^2$;

(ii) Suppose Assumptions 1-3 hold, then under $\mathbb{H}_A$ we have $\tilde{\omega}^2 = O_p(q_T)$.

Proof: To establish the limiting properties of the LRV estimator $\tilde{\omega}^2$ under $\mathbb{H}_0$ and $\mathbb{H}_A$, we still consider employing $sgn_m(\cdot)$ to approximate $sgn(\cdot)$. Let $\tilde{\gamma}_m(i) = \frac{1}{T}\sum_{t=i+1}^{T} sgn_m(\tilde{e}_t)\, sgn_m(\tilde{e}_{t-i})$ and $\check{\gamma}_m(i) = \frac{1}{T}\sum_{t=i+1}^{T} sgn_m(e_t) sgn_m(e_{t-i})$.
(i) Under $\mathbb{H}_0$, by using (S1.16) we have

$$\begin{aligned}
\max_{0\le i\le q_T} |\tilde{\gamma}_m(i) - \check{\gamma}_m(i)| &\le |\tilde{g} - g_0| \frac{1}{T}\sum_{t=i+1}^{T} \max_{0\le i\le q_T} \left|sgn_m(e_t) sgn'_m(e^*_{t-i})\right| \\
&\le |\tilde{g} - g_0| \frac{1}{T}\sum_{t=i+1}^{T} \max_{0\le i\le q_T} |sgn_m(e_{t-i}) sgn'_m(e^*_t)| \\
&\le |\tilde{g} - g_0| (I_{1T} + I_{2T}) = O_p(T^{-1/2}), \qquad \text{(S1.25)}
\end{aligned}$$

for $0 \le i \le q_T$, where $\left|\sqrt{T}(\tilde{g} - g_0)\right| = O_p(1)$ under $\mathbb{H}_0$, and $I_{1T} = I_{2T} = O_p(1)$ since both $sgn_m(\cdot)$ and $sgn'_m(\cdot)$ are bounded.

Let $\check{\omega}^2 = \sum_{i=-T+1}^{T-1} l(i/q_T)\check{\gamma}(i)$, where $\check{\gamma}(i) = \frac{1}{T}\sum_{t=i+1}^{T} sgn(e_t) sgn(e_{t-i})$ and $\check{\gamma}(i) = \check{\gamma}(-i)$ for $i < 0$. We then have

$$\begin{aligned}
|\tilde{\omega}^2 - \check{\omega}^2| &= \left|\sum_{i=-T+1}^{T-1} l(i/q_T)\left(\tilde{\gamma}(i) - \check{\gamma}(i)\right)\right| \\
&= \max_{0\le i\le q_T} |\tilde{\gamma}(i) - \check{\gamma}(i)| \sum_{i=-T+1}^{T-1} l(i/q_T) = O_p(q_T T^{-1/2}), \qquad \text{(S1.26)}
\end{aligned}$$

where $\max_{0\le i\le q_T}|\tilde{\gamma}(i)-\check{\gamma}(i)|=O_p(T^{-1/2})$ holds because $\max_{0\le i\le q_T}|\tilde{\gamma}_m(i)-\check{\gamma}_m(i)|=O_p(T^{-1/2})$ by (S1.25) when $m\to\infty$, together with $q_T^{-1}\sum_{i=-T+1}^{T-1} l(i/q_T)<\infty$ by Lemma 1 of Jansson(2002). Similar to the proof of Lemma 4 in Cavaliere(2005), we can show $\check{\omega}^2 \xrightarrow{p} \omega^2$ by Assumptions 1 and 3. Hence $\tilde{\omega}^2 \xrightarrow{p} \omega^2$ under $\mathbb{H}_0$.

(ii) Under $\mathbb{H}_A$, we can rewrite (S1.16) as $sgn_m(\tilde{e}_t) \approx sgn_m(e_t) + sgn_m'(e_t^*)(g_t-g^*)$, where $e_t^*=e_t-\lambda(g^*-g_t)$, $\lambda\in[0,1]$ and g^* is defined by Lemma A1(ii). By using the fact that $|g_t-g^*|$, $sgn_m(\cdot)$ and $sgn_m'(\cdot)$ are bounded, we can show that

$$\begin{aligned}\max_{0\le i\le q_T}|\tilde{\gamma}_m(i)-\check{\gamma}_m(i)| &\le \sup_t|g_t-g^*|\left(\frac{1}{T}\sum_{t=i+1}^{T}\max_{0\le i\le q_T}\left|sgn_m(e_t)sgn_m'(e_{t-i}^*)\right|\right)\\ &+\sup_t|g_t-g^*|\left(\frac{1}{T}\sum_{t=i+1}^{T}\max_{0\le i\le q_T}|sgn_m(e_{t-i})sgn_m'(e_t^*)|\right)\\ &=O_p(1). \qquad \text{(S1.27)}\end{aligned}$$

By taking similar arguments to those of showing that $|\tilde{\omega}^2-\check{\omega}^2|=O_p\left(q_TT^{-1/2}\right)$ in (S1.26), we have $|\tilde{\omega}^2-\check{\omega}^2|=O_p(q_T)$ under $\mathbb{H}_A$. In addition, under Assumptions 1 and 3, we still have $\check{\omega}^2 \xrightarrow{p} \omega^2$. As a result, we obtain $\tilde{\omega}^2=O_p(q_T)$. We complete the proof of Lemma A3. □

Proof of Theorem 1: Under $\mathbb{H}_0$, Lemma A2(i) and Lemma A3(i) give Theorem 1. □

Proof of Theorem 2: Under $\mathbb{H}_A$, Lemma A2(ii) and Lemma A3(ii) give Theorem 2. □

Lemma A4. Suppose Assumptions 1-2 and 4 hold, then under $\mathbb{H}_{LA}^1$ we have

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn\left(\tilde{e}_t\right) \xrightarrow{d} \omega\left[W(r)-rW(1)\right]+\frac{2cf_{|\varepsilon|}(c)}{\sigma_0}\left(\int_0^r \pi\left(s\right)ds - r\int_0^1 \pi\left(s\right)ds\right).$$

Proof: Under $\mathbb{H}_{LA}^1$, similar to the proof of Lemma A1 we have

$$Q_T\left(v\right)=\sum_{s=1}^{T}\left|e_s-\left(T^{-1/2}v-cT^{-1/2}\pi_s\right)\right|-\sum_{s=1}^{T}\left|e_s\right|. \quad \text{(S1.28)}$$

where $v=T^{1/2}(g-g_0)$. Define

$$z_T=T^{-1/2}\sum_{s=1}^{T} sgn\left(e_s\right);\ Z_T=\sum_{s=1}^{T}\int_0^{T^{-1/2}v-cT^{-1/2}\pi_s}\left[I\left(e_s\leq x\right)-I\left(e_s\leq 0\right)\right]dx.$$

Similar to the proof of Lemma A1, we can show that

$$Q_T\left(v\right)=-vz_T+f_{|\varepsilon|}\left(c\right)\int_0^1\frac{\left[v-c\pi\left(r\right)\right]^2}{\sigma_0}dr+\frac{c}{\sqrt{T}}\sum_{s=1}^{T}\pi_s sgn\left(e_s\right)+o_p\left(1\right). \quad \text{(S1.29)}$$

Then by minimizing $Q_T\left(v\right)$ with respect to v, we obtain

$$\sqrt{T}(\tilde{g}-g_0)=\frac{\sigma_0 z_T}{2f_{|\varepsilon|}\left(c\right)}+c\int_0^1\pi\left(r\right)dr+o_p\left(1\right). \quad \text{(S1.30)}$$

Based on (S1.30), $\tilde{e}_t=e_t-(\tilde{g}-g_t)$ can be rewritten as

$$\tilde{e}_t=e_t-\frac{\sigma_0}{2f_{|\varepsilon|}\left(c\right)\sqrt{T}}z_T+cT^{-1/2}\left(\pi_t-\int_0^1\pi\left(r\right)dr\right)+o_p\left(1\right). \quad \text{(S1.31)}$$

We still consider employing $sgn_m(\cdot)$ to approximate $sgn(\cdot)$. Hence it is enough to study the asymptotic behavior of $\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn_m\left(\tilde{e}_t\right)$. By using the mean value theorem we have

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn_m\left(\tilde{e}_t\right) \approx \frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn_m\left(e_t\right) + \frac{1}{T}\sum_{t=1}^{[Tr]} sgn_m'\left(e_t\right)\left[c\left(\pi_t - \int_0^1 \pi\left(s\right)ds\right) - \frac{\sigma_0 z_T}{2f_{|\varepsilon|}\left(c\right)}\right]$$
$$\xrightarrow{d} \omega\left[W(r) - rW(1)\right] + \frac{2cf_{|\varepsilon|}(c)}{\sigma_0}\left(\int_0^r \pi\left(s\right)ds - r\int_0^1 \pi\left(s\right)ds\right), \tag{S1.32}$$

where $\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn_m\left(e_t\right) \xrightarrow{d} \omega W(r)$ by (S1.18), $z_T \xrightarrow{d} \omega W(1)$ by (S1.10),$\frac{1}{T}\sum_{t=1}^{[Tr]} sgn_m'\left(e_t\right) \xrightarrow{p} \frac{2rf_{|\varepsilon|}(c)}{\sigma_0}$ by (S1.21), and $\frac{1}{T}\sum_{t=1}^{[Tr]} sgn_m'\left(e_t\right)\pi_t \xrightarrow{p} \frac{2f_{|\varepsilon|}(c)}{\sigma_0}\int_0^r \pi\left(s\right)ds$, whose proof is similar to that of (S1.21). This completes the proof of Lemma A4. □

Lemma A5. Suppose Assumptions 1-2 and 4 hold, then under $\mathbb{H}_{LA}^2$ we have

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn\left(\tilde{e}_t\right) \xrightarrow{d} \omega\left[W(r) - rW(1)\right] + \frac{2cf_{|\varepsilon|}(c)}{\sigma_0}\left(\int_{-\infty}^{\frac{r-\tau^*}{r_T}} \psi\left(s\right)ds - r\int_{-\infty}^{\infty} \psi\left(s\right)ds\right).$$

Proof: Similar to the proof of Lemma A4, we can show that

$$\tilde{e}_t = e_t - \frac{\sigma_0 \omega W(1)}{2f_{|\varepsilon|}\left(c\right)\sqrt{T}} + c\left[d_T\psi\left(\frac{t/T - \tau^*}{r_T}\right) - T^{-1/2}\int_{-\infty}^{\infty}\psi\left(s\right)ds\right] + o_p\left(1\right), \tag{S1.33}$$

and

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} sgn_m\left(\tilde{e}_t\right) \xrightarrow{d} \omega\left[W(r) - rW(1)\right] + \frac{2cf_{|\varepsilon|}(c)}{\sigma_0}\left(\int_{-\infty}^{\frac{r-\tau^*}{r_T}} \psi\left(s\right)ds - r\int_{-\infty}^{\infty} \psi\left(s\right)ds\right), \tag{S1.34}$$

where

$$\int_{-\infty}^{\frac{r-\tau^*}{r_T}} \psi(s)\,ds = \begin{cases} 0, & \text{if } r < \tau^* \\ \int_{-\infty}^{0} \psi(s)\,ds, & \text{if } r = \tau^* \\ \int_{-\infty}^{\infty} \psi(s)\,ds, & \text{if } r > \tau^* \end{cases} .$$

We complete the proof of Lemma A5. □

Proof of Lemma 1: Let $\tau_t = t/T$, for $t = 1, \ldots, T$, and $k_{s,t} = k\left(\frac{\tau_s - \tau_t}{h}\right)$. We consider the following minimization problem,

$$\begin{pmatrix} \hat{g}_t \\ \hat{\varphi}_t \end{pmatrix} = \arg \min_{g,\varphi \in R^2} \sum_{s=1}^{T} k_{s,t} \left| |u_s| - g - (\tau_s - \tau_t)\varphi \right| . \tag{S1.35}$$

Now under Assumption 4, by making the Taylor expansion of $g(\tau_s)$ around the given τ_t, we have

$$g(\tau_s) = g(\tau_t) + g'(\tau_t)(\tau_s - \tau_t) + \frac{1}{2} g''(\bar{\tau}_t)(\tau_s - \tau_t)^2 , \tag{S1.36}$$

where $g'(\cdot)$ and $g''(\cdot)$ are the first and second derivatives of $g(\cdot)$, and $\bar{\tau}_t = \lambda\tau_t + (1-\lambda)\tau_s$, $\lambda \in [0,1]$.

Denote $v_t = (Th)^{1/2}\left[g - g(\tau_t), h(b - g'(\tau_t))\right]'$ and $M_{s,t} = \left[1, \frac{(\tau_s - \tau_t)}{h}\right]'$, Minimization of (S1.35) is equivalent to minimizing the following criterion:

$$H_T(v_t) = \sum_{s=1}^{T} k_{s,t} \left| e_s - (Th)^{-1/2} v_t' M_{s,t} + \frac{1}{2} g''(\bar{\tau}_t)(\tau_s - \tau_t)^2 \right| - \sum_{s=1}^{T} k_{s,t} |e_s| . \tag{S1.37}$$

By Knight's identity (Knight, 1998), (S1.37) can be rewritten as

$$H_T(v_t) = -v_t' H_{1T} + 2H_{2T} + \frac{1}{2} \sum_{s=1}^{T} k_{s,t} g''(\bar{\tau}_t)(\tau_s - \tau_t)^2 sgn(e_s) , \tag{S1.38}$$

where

$$H_{1T} = \frac{1}{\sqrt{Th}} \sum_{s=1}^{T} k_{s,t} M_{s,t} sgn\left(e_s\right),$$
$$H_{2T} = \sum_{s=1}^{T} k_{s,t} \int_0^{(Th)^{-1/2} v_t' M_{s,t} - \frac{1}{2} g''(\bar{\tau}_t)(\tau_s - \tau_t)^2} \left[I\left(e_s \leq z\right) - I\left(e_s \leq 0\right)\right] dz.$$

Now let $d_{s,t} = (Th)^{-1/2} v_t' M_{s,t} - \frac{1}{2} g''\left(\bar{\tau}_t\right)\left(\tau_s - \tau_t\right)^2$, $D_s = k_{s,t} \int_0^{d_{s,t}} \left[I\left(e_s \leq z\right) - I\left(e_s \leq 0\right)\right] dz$ and $\chi_s = E(D_s | \mathcal{F}_{s-1})$, where $\mathcal{F}_{s-1}$ is the σ-field generated by $\{\varepsilon_{s-1}, \varepsilon_{s-2}, \cdots\}$. We rewrite H_{2T} as

$$H_{2T} = \sum_{s=1}^{T} \chi_s + \sum_{s=1}^{T} \left(D_s - \chi_s\right). \tag{S1.39}$$

Subsequently, for each $v_t \in R^2$, we can show that

$$\sum_{s=1}^{T} \chi_s = \frac{1}{2} \sum_{s=1}^{T} \frac{k_{s,t} f_{|\varepsilon|, s-1}\left(c\right)}{\sigma_s} \left[(Th)^{-1/2} v_t' M_{s,t} - \frac{h^2}{2} g''\left(\bar{\tau}_t\right)\left(\frac{s-t}{Th}\right)^2\right]^2. \tag{S1.40}$$

Next, the integrability of $f^{\gamma}_{|\varepsilon|, s-1}\left(c\right)$ and stationarity of $\left\{f_{|\varepsilon|, s-1}\left(c\right)\right\}$ in Assumption 2 ensure that

$$\sup_{0<r\leq 1} \left| \frac{1}{T^{1-\epsilon}} \sum_{s=1}^{[Tr]} \left[f_{|\varepsilon|, s-1}\left(c\right) \lambda\left(s/T\right) - f_{|\varepsilon|}(c) \lambda\left(s/T\right)\right] \right| \to 0, \tag{S1.41}$$

for some $\epsilon > 0$, and for any deterministic bounded function $\lambda\left(\cdot\right)$ on $[0, 1]$, which in turn leads to

$$\sum_{s=1}^{T} \chi_s = \frac{f_{|\varepsilon|}\left(c\right)}{2} \sum_{s=1}^{T} \frac{k_{s,t}}{\sigma_s} \left[(Th)^{-1/2} v_t' M_{s,t} - \frac{h^2}{2} g''\left(\bar{\tau}_t\right)\left(\frac{s-t}{Th}\right)^2\right]^2 + o_p\left(1\right). \tag{S1.42}$$

As a result we have

$$H_{2T} = \frac{f_{|\varepsilon|}(c)}{2} \sum_{s=1}^{T} \frac{k_{s,t}}{\sigma_s} \left[(Th)^{-1/2} v_t' M_{s,t} - \frac{h^2}{2} g''(\bar{\tau}_t) \left(\frac{s-t}{Th} \right)^2 \right]^2 + o_p(1), \tag{S1.43}$$

provided that $\sum_{s=1}^{T} (D_s - \chi_s) = o_p(1)$ .Hence it is enough to prove $var\left[\sum_{s=1}^{T} (D_s - \chi_s)\right] = o(1)$. Note that $\{D_s - \chi_s\}$ is a martingale difference sequence, so we have

$$\begin{aligned} var\left[\sum_{s=1}^{T} (D_s - \chi_s)\right] &\le \sum_{s=1}^{T} E\left(D_s^2\right) \le f_{|\varepsilon|}(c) \sum_{s=1}^{T} \frac{k_{s,t}}{\sigma_s} \left(\int_0^{d_{s,t}} \int_0^{d_{s,t}} \min(x,y)\, dxdy \right) \\ &\le \frac{f_{|\varepsilon|}(c)}{3} \sum_{s=1}^{T} \frac{k_{s,t}}{\sigma_s} \left[(Th)^{-1/2} v_t' M_{s,t} - \frac{1}{2} g''(\bar{\tau}_t)(\tau_s - \tau_t)^2 \right]^3 \\ &= o(1), \end{aligned} \tag{S1.44}$$

where $\int_0^{d_{s,t}} \int_0^{d_{s,t}} \min(x,y)\, dxdy = d_{s,t}^3/3$. By (S1.38)-(S1.44), we obtain

$$H_T(v_t) = -v_t' H_{1T} + f_{|\varepsilon|}(c) \sum_{s=1}^{T} \frac{k_{s,t} \left[(Th)^{-1/2} v_t' M_{s,t} - \frac{h^2}{2} g''(\bar{\tau}_t) \left(\frac{s-t}{Th} \right)^2 \right]^2}{\sigma_s} + o_p(1). \tag{S1.45}$$

As $H_T(v_t)$ is convex for each v_t, Theorem 2 in Kato (2009) implies that

$$\begin{aligned} \left(\frac{1}{Th} \sum_{s=1}^{T} \frac{k_{s,t} M_{s,t} M_{s,t}'}{\sigma_s} \right) \hat{v}_t &= \frac{h^2}{2\sqrt{Th}} \sum_{s=1}^{T} \frac{k_{s,t} g''(\bar{\tau}_t) M_{s,t} \left(\frac{s-t}{Th} \right)^2}{\sigma_s} \\ &\quad + \frac{1}{2 f_{|\varepsilon|}(c) \sqrt{Th}} \sum_{s=1}^{T} k_{s,t} M_{s,t} sgn(e_s) + o_p(1). \end{aligned} \tag{S1.46}$$

Because

$$\frac{1}{Th} \sum_{s=1}^{T} \frac{k_{s,t} M_{s,t} M_{s,t}'}{\sigma_s} \to \begin{pmatrix} \sigma^{-1}(\tau_t) & 0 \\ 0 & \sigma^{-1}(\tau_t) \int_{-1}^{1} u^2 k(u)\, du \end{pmatrix},$$

and

$$\frac{h^2}{\sqrt{Th}}\sum_{s=1}^{T}\frac{k_{s,t}g''\left(\bar{\tau}_t\right)M_{s,t}\left(\frac{s-t}{Th}\right)^2}{\sigma_s}\rightarrow\sqrt{Th}\frac{g''\left(\tau_t\right)h^2}{\sigma\left(\tau_t\right)}\begin{pmatrix}\int_{-1}^{1}u^2k\left(u\right)du\\ \\ 0\end{pmatrix},$$

we then have

$$\sqrt{Th}\left(\hat{g}\left(\tau_t\right)-g\left(\tau_t\right)-\frac{1}{2}h^2c\sigma''\left(\tau_t\right)\mu_2\right)=\frac{\sigma\left(\tau_t\right)}{2f_{|\varepsilon|}\left(c\right)}\frac{1}{\sqrt{Th}}\sum_{s=1}^{T}k_{s,t}sgn\left(e_s\right)+o_p\left(1\right),\tag{S1.47}$$

where $\mu_2=\int_{-1}^{1}u^2k(u)du$. In order to obtain the asymptotic normality of $\frac{1}{\sqrt{Th}}\sum_{s=1}^{T}k_{s,t}sgn\left(e_s\right)$, we can still employ Theorem 2.21(ii) of Fan and Yao (2003), this is because the deterministic and bounded kernel $k_{s,t}$ only acts as a weighting function,and does not alter the strong mixing property in the summation. Under Assumptions 1 and 5,we obtain

$$\frac{1}{\sqrt{Th}}\sum_{s=1}^{T}k_{s,t}sgn\left(e_s\right)\xrightarrow{d}N\left(0,\upsilon_2\xi^2\right),$$

where $\upsilon_2=\int_{-1}^{1}k^2(u)du$ and $\xi^2=\sum_{i=-\infty}^{\infty}E\left[sgn\left(\left|\varepsilon_{t+i}\right|-c\right)sgn\left(\left|\varepsilon_t\right|-c\right)\right]$. As a result, under Assumptions 1-2 and 4-5 we have

$$\sqrt{Th}\left(\hat{g}\left(\tau\right)-g\left(\tau\right)-\frac{1}{2}h^2c\sigma''\left(\tau\right)\mu_2\right)\xrightarrow{d}N\left(0,\frac{\sigma^2\left(\tau\right)\xi^2\upsilon_2}{4f_{|\varepsilon|}^2\left(c\right)}\right).\tag{S1.48}$$

This completes the proof of Lemma 1. □

Proof of Lemma 2: Similar to the proof of Lemma A3, we still employ $sgn_m(\cdot)$ to approximate $sgn(\cdot)$. Let $\hat{\gamma}_m(i)=\frac{1}{T}\sum_{t=i+1}^{T}sgn_m\left(\hat{e}_t\right)sgn_m\left(\hat{e}_{t-i}\right)$ and $\check{\gamma}_m(i)=\frac{1}{T}\sum_{t=i+1}^{T}sgn_m(e_t)sgn_m(e_{t-i})$. By using the mean value theorem we have $sgn_m\left(\hat{e}_t\right) = sgn_m\left(e_t\right) - \left(\hat{g}_t - g_t\right) sgn'_m\left(e^*_t\right)$, where $e^*_t = e_t - \lambda\left(\hat{g}_t - g_t\right), \lambda \in [0,1]$. As a result we obtain

$$\begin{aligned}
\max_i \left|\hat{\gamma}_m(i) - \check{\gamma}_m(i)\right| \leq & \max_i \left| \frac{1}{T} \sum_{t=i+1}^{T} \left(\hat{g}_{t-i} - g_{t-i}\right) sgn_m(e_t) sgn'_m(e^*_{t-i}) \right| \\
& + \max_i \left| \frac{1}{T} \sum_{t=i+1}^{T} \left(\hat{g}_t - g_t\right) sgn_m(e_{t-j}) sgn'_m(e^*_t) \right| \\
& + \max_i \left| \frac{1}{T} \sum_{t=i+1}^{T} \left(\hat{g}_t - g_t\right)\left(\hat{g}_{t-i} - g_{t-i}\right) sgn'_m(e^*_t) sgn'_m\left(e^*_{t-i}\right) \right| \\
= & P_{1T} + P_{2T} + P_{3T}. \qquad \text{(S1.49)}
\end{aligned}$$

For P_{1T} by using Cauchy-Schwarz inequality we have

$$\begin{aligned}
P_{1T} & \leq \max_i \left(\frac{1}{T} \sum_{t=i+1}^{T} \left(\hat{g}_{t-i} - g_{t-i}\right)^2 \right)^{1/2} \left(\frac{1}{T} \sum_{t=i+1}^{T} \left(sgn_m(e_t) sgn'_m(e^*_{t-i}) \right)^2 \right)^{1/2} \\
& \leq \left(\frac{1}{T} \sum_{s=1}^{T} \left(\hat{g}_s - g_s\right)^2 \right)^{1/2} \left(\max_i \frac{1}{T} \sum_{t=i+1}^{T} \left(sgn_m(e_t) sgn'_m(e^*_{t-i}) \right)^2 \right)^{1/2} \\
& = O_p\left(\sqrt{\frac{1}{Th} + h^4} \right) \qquad \text{(S1.50)}
\end{aligned}$$

where we have used $\frac{1}{T}\sum_{s=1}^{T}\left(\hat{g}_s - g_s\right)^2 = O_p\left(\frac{1}{Th} + h^4\right)$ since $E\left[\frac{1}{T}\sum_{s=1}^{T}\left(\hat{g}_s - g_s\right)^2\right] = O\left(\frac{1}{Th} + h^4\right)$ by (S1.48) and $\max_i \frac{1}{T}\sum_{t=i+1}^{T}\left(sgn_m(e_t) sgn'_m(e^*_{t-i})\right)^2 = O_p\left(1\right)$ since $sgn_m(\cdot)$ and $sgn'_m(\cdot)$ are bounded. Hence $P_{1T} = O_p\left(\sqrt{\frac{1}{Th} + h^4}\right)$. In the same manner, we can show $P_{2T} = O_p\left(\sqrt{\frac{1}{Th} + h^4}\right)$ and $P_{3T} = O_p\left(\frac{1}{Th} + h^4\right)$. As a result we obtain $\max_i \left|\hat{\gamma}(i) - \check{\gamma}(i)\right| = O_p\left(\sqrt{\frac{1}{Th} + h^4}\right)$

since $\max_i |\hat{\gamma}_m(i) - \check{\gamma}_m(i)| = O_p\left(\sqrt{\frac{1}{Th} + h^4}\right)$, where $\check{\gamma}(i) = \frac{1}{T}\sum_{t=i+1}^{T} sgn(e_t)sgn(e_{t-i})$. Following the proof of (S1.26), we have $|\hat{\omega}^2 - \breve{\omega}^2| = O_p\left(\sqrt{\frac{q_T^2}{Th} + q_T^2 h^4}\right)$, which is $o_p(1)$ under Assumption 6(ii). In addition, under Assumption 1 and 3 we also have $\breve{\omega}^2 \xrightarrow{p} \omega^2$, whose proof is similar to that of Lemma 4 of Cavaliere(2005). Hence we finally obtain $\hat{\omega}^2 \xrightarrow{p} \omega^2$ under both the null and the alternatives. This completes the proof of Lemma 2. □

Proof of Theorem 3: Under $\mathbb{H}_0$, Lemma A2(i) and Lemma 2 give Theorem 3. □

Proof of Theorem 4: Under $\mathbb{H}_A$, Lemma A2(ii) and Lemma 2 give Theorem 4. □

Proof of Theorem 5: Under $\mathbb{H}_{LA}^1$, Lemma A4 and Lemma 2 give Theorem 5(i). Under $\mathbb{H}_{LA}^2$, Lemma A5 and Lemma 2 give Theorem 5(ii). □

S2 Additional simulation results

In this section we also consider the case in which u_t is not directly observable. Suppose that the time series $\{y_t\}$ follows an AR(1) process: $y_t = 0.1 + 0.5y_{t-1} + u_t$, where u_t is still generated by the model (1). The whole Monte Carlo experiment designs for u_t are the same as those in Section 4 of the paper. To test for changing volatility, we first estimate the $AR(1)$ model by the OLS regression and obtain the estimated residuals $\hat{u}_t$. Then the residual-based tests are constructed by employing $\hat{u}_t$ to replace u_t, and they are denoted as $LM_B^r, CSM_B^r, LM_X^r, CSM_X^r$, $CSM_A^r, QS_A^r, CSM_M^r, QS_M^r, CSM_M^{r,*}$ and $QS_M^{r,*}$. The size results are reported in Table S1, and the power results for DGPP.1-5 are in Tables S2-S6.

Table S1: Empirical rejection probabilities of the tests under the null.

Error	T	LM_B^r	CSM_B^r	LM_X^r	CSM_X^r	CSM_A^r	QS_A^r	CSM_M^r	QS_M^r	$CSM_M^{r,*}$	$QS_M^{r,*}$
$N(0,1)$	250	0.057	0.049	0.111	0.108	0.055	0.072	0.039	0.063	0.061	0.078
	500	0.060	0.058	0.092	0.087	0.066	0.078	0.044	0.053	0.051	0.061
	750	0.055	0.056	0.080	0.071	0.052	0.071	0.053	0.057	0.056	0.062
$t(2)$	250	0.037	0.018	0.064	0.034	0.037	0.064	0.040	0.055	0.063	0.072
	500	0.030	0.017	0.041	0.019	0.036	0.057	0.052	0.058	0.059	0.067
	750	0.037	0.017	0.049	0.022	0.037	0.067	0.055	0.058	0.060	0.063
$st(3,-0.8)$	250	0.041	0.017	0.072	0.039	0.042	0.069	0.051	0.061	0.076	0.080
	500	0.049	0.023	0.064	0.040	0.050	0.065	0.060	0.060	0.073	0.067
	750	0.048	0.023	0.061	0.037	0.048	0.062	0.059	0.058	0.068	0.066
$x^2(1)$	250	0.066	0.028	0.103	0.073	0.066	0.085	0.053	0.063	0.068	0.071
	500	0.052	0.034	0.076	0.054	0.060	0.080	0.057	0.065	0.063	0.067
	750	0.043	0.035	0.063	0.052	0.064	0.079	0.052	0.065	0.058	0.063

Table S2: Empirical rejection probabilities of the tests under DGPP.1.

Error	T	LM_B^r	CSM_B^r	LM_X^r	CSM_X^r	CSM_A^r	QS_A^r	CSM_M^r	QS_M^r	$CSM_M^{r,*}$	$QS_M^{r,*}$
$N(0,1)$	250	0.990	0.998	0.998	1.000	1.000	0.999	0.954	0.957	0.967	0.966
	500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	750	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$t(2)$	250	0.173	0.180	0.259	0.258	0.700	0.741	0.882	0.886	0.903	0.909
	500	0.236	0.260	0.304	0.317	0.854	0.872	0.990	0.991	0.993	0.992
	750	0.318	0.352	0.376	0.394	0.916	0.931	0.998	0.998	0.998	0.998
$st(3,-0.8)$	250	0.275	0.312	0.395	0.422	0.888	0.909	0.993	0.987	0.997	0.990
	500	0.455	0.476	0.526	0.554	0.973	0.968	1.000	1.000	1.000	1.000
	750	0.529	0.556	0.578	0.610	1.000	1.000	1.000	1.000	1.000	1.000
$x^2(1)$	250	0.336	0.405	0.500	0.562	0.929	0.939	1.000	1.000	1.000	1.000
	500	0.646	0.694	0.727	0.776	0.995	0.996	1.000	1.000	1.000	1.000
	750	0.830	0.866	0.874	0.905	1.000	1.000	1.000	1.000	1.000	1.000

Table S3: Empirical rejection probabilities of the tests under DGPP.2.

Error	T	LM_B^r	CSM_B^r	LM_X^r	CSM_X^r	CSM_A^r	QS_A^r	CSM_M^r	QS_M^r	$CSM_M^{r,*}$	$QS_M^{r,*}$
$N(0,1)$	250	0.115	0.143	0.757	0.777	0.618	0.711	0.338	0.279	0.499	0.461
	500	0.856	0.874	0.990	0.992	0.993	0.997	0.807	0.791	0.885	0.867
	750	0.982	0.986	0.998	0.998	1.000	1.000	0.957	0.966	0.982	0.980
$t(2)$	250	0.002	0.003	0.027	0.031	0.108	0.133	0.253	0.208	0.401	0.355
	500	0.011	0.008	0.028	0.029	0.314	0.371	0.694	0.665	0.783	0.765
	750	0.013	0.014	0.035	0.040	0.525	0.577	0.896	0.876	0.944	0.923
$st(3,-0.8)$	250	0.005	0.005	0.039	0.046	0.234	0.288	0.463	0.440	0.652	0.606
	500	0.014	0.021	0.054	0.073	0.617	0.678	0.909	0.922	0.958	0.947
	750	0.051	0.058	0.141	0.166	0.798	0.857	0.998	0.996	0.999	0.999
$x^2(1)$	250	0.005	0.006	0.035	0.050	0.265	0.337	0.711	0.722	0.839	0.829
	500	0.024	0.040	0.107	0.144	0.695	0.771	0.991	0.993	0.995	0.993
	750	0.064	0.092	0.265	0.303	0.926	0.936	1.000	1.000	1.000	1.000

Table S4: Empirical rejection probabilities of the tests under DGPP.3.

Error	T	LM_B^r	CSM_B^r	LM_X^r	CSM_X^r	CSM_A^r	QS_A^r	CSM_M^r	QS_M^r	$CSM_M^{r,*}$	$QS_M^{r,*}$
$N(0,1)$	250	0.921	0.916	0.950	0.953	0.919	0.951	0.588	0.694	0.669	0.731
	500	0.995	0.994	0.996	0.997	0.994	0.997	0.905	0.947	0.928	0.954
	750	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$t(2)$	250	0.139	0.109	0.205	0.160	0.386	0.506	0.474	0.562	0.558	0.615
	500	0.192	0.156	0.247	0.202	0.623	0.721	0.811	0.874	0.846	0.889
	750	0.239	0.194	0.281	0.234	0.734	0.810	0.933	0.969	0.948	0.971
$st(3,-0.8)$	250	0.227	0.206	0.319	0.277	0.611	0.719	0.767	0.830	0.810	0.848
	500	0.355	0.292	0.417	0.357	0.830	0.888	0.974	0.987	0.981	0.989
	750	0.409	0.359	0.460	0.421	0.897	0.932	0.998	1.000	0.998	1.000
$x^2(1)$	250	0.278	0.212	0.379	0.331	0.669	0.758	0.969	0.987	0.982	0.989
	500	0.480	0.445	0.558	0.528	0.889	0.928	0.999	1.000	0.999	1.000
	750	0.646	0.627	0.697	0.689	0.956	0.975	1.000	1.000	1.000	1.000

Table S5: Empirical rejection probabilities of the tests under DGPP.4.

Error	T	LM_B^r	CSM_B^r	LM_X^r	CSM_X^r	CSM_A^r	QS_A^r	CSM_M^r	QS_M^r	$CSM_M^{r,*}$	$QS_M^{r,*}$
$N(0,1)$	250	0.539	0.351	0.801	0.704	0.510	0.553	0.169	0.136	0.303	0.286
	500	0.834	0.655	0.951	0.910	0.812	0.888	0.443	0.486	0.582	0.606
	750	0.952	0.864	0.992	0.982	0.955	0.990	0.717	0.805	0.834	0.869
$t(2)$	250	0.134	0.063	0.193	0.116	0.177	0.186	0.139	0.093	0.220	0.177
	500	0.121	0.055	0.174	0.092	0.264	0.282	0.344	0.336	0.433	0.439
	750	0.165	0.079	0.206	0.120	0.389	0.405	0.538	0.625	0.668	0.712
$st(3,-0.8)$	250	0.161	0.068	0.254	0.157	0.287	0.284	0.259	0.248	0.429	0.434
	500	0.230	0.109	0.307	0.182	0.457	0.486	0.639	0.766	0.792	0.847
	750	0.261	0.141	0.354	0.219	0.626	0.660	0.873	0.949	0.935	0.970
$x^2(1)$	250	0.223	0.110	0.338	0.225	0.318	0.309	0.489	0.688	0.752	0.807
	500	0.317	0.167	0.427	0.272	0.514	0.546	0.954	0.987	0.980	0.993
	750	0.405	0.248	0.527	0.385	0.719	0.768	0.997	1.000	0.997	1.000

Table S6: Empirical rejection probabilities of the tests under DGPP.5.

Error	T	LM_B^r	CSM_B^r	LM_X^r	CSM_X^r	CSM_A^r	QS_A^r	CSM_M^r	QS_M^r	$CSM_M^{r,*}$	$QS_M^{r,*}$
$N(0,1)$	250	0.856	0.894	0.938	0.962	0.943	0.942	0.727	0.744	0.776	0.789
	500	0.990	0.995	0.998	0.998	0.996	0.997	0.952	0.950	0.962	0.957
	750	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$t(2)$	250	0.087	0.091	0.155	0.143	0.370	0.430	0.577	0.604	0.624	0.625
	500	0.115	0.136	0.166	0.161	0.587	0.627	0.894	0.890	0.908	0.907
	750	0.152	0.168	0.203	0.201	0.768	0.783	0.976	0.971	0.976	0.974
$st(3,-0.8)$	250	0.133	0.147	0.220	0.239	0.627	0.660	0.853	0.860	0.888	0.891
	500	0.231	0.273	0.309	0.338	0.835	0.859	0.990	0.993	0.994	0.996
	750	0.300	0.344	0.362	0.400	0.911	0.920	0.999	0.999	0.999	0.999
$x^2(1)$	250	0.179	0.201	0.295	0.321	0.682	0.723	0.993	0.995	0.994	0.995
	500	0.356	0.411	0.458	0.513	0.896	0.906	1.000	1.000	1.000	1.000
	750	0.500	0.565	0.588	0.645	0.966	0.971	1.000	1.000	1.000	1.000

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