Estimation of a distribution with a bias and its applications to competing risks: supplementary material

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1 Proof of Theorem 3.2

We prove the proof for the weak convergence of Z^*_{1n}. The proof for Z^*_{2n} uses similar arguments. First we show the pointwise convergence in distribution. Without loss of generality we assume through out that $S(a) > 0$ for all $a > 0$ in which it suffices to show that weak convergence of $[0, a]$ for all $a > 0$ (see Lindvall, 1973). We have

$$\begin{aligned} Z^*_{1n}(x) &= \sqrt{n}\left[\hat{S}(x) \sup_{0 \le y \le x} \hat{\psi}_1(y) - S_1(x)\right] \\ &= \max\left\{\sqrt{n}\left[\hat{S}(x) \sup_{\eta(x) \le y \le x} \hat{\psi}_1(y) - S_1(x)\right], \sqrt{n}\left[\hat{S}(x) \sup_{y \le \eta(x)} \hat{\psi}_1(y) - S_1(x)\right]\right\}. \end{aligned}$$

Notice that, since $\psi_1(y) = \psi_1(x)$ on $[\eta(x), x]$,

$$\sqrt{n}\left[\hat{S}(x) \sup_{\eta(x) \le y \le x} \hat{\psi}_1(y) - S_1(x)\right] = \sqrt{n}\left[\hat{S}(x) \sup_{\eta(x) \le y \le x} \left\{\hat{\psi}_1(y) - \psi_1(y)\right\} + \psi_1(x)\sqrt{n}[\hat{S}(x) - S(x)]\right].$$

But

$$\sqrt{n}[\hat{\psi}_1(y) - \psi_1(y)] = \frac{1}{S(y)\hat{S}(y)}\left[S(y)Z_{1n}(y) - S_1(y)Z_{3n}(y)\right].$$

Therefore

$$\sqrt{n}[\hat{\psi}_1 - \psi_1] \overset{w}{\Rightarrow} \frac{Z_1 - \psi_1 Z_3}{S} \equiv U_1$$

in $\ell^\infty[0, a]$ for all $a > 0$. Using the continuous mapping theorem, we get

$$\begin{aligned} \sqrt{n}\left[\hat{S}(x) \sup_{\eta(x) \le y \le x} \hat{\psi}_1(y) - S_1(x)\right] &\overset{w}{\Rightarrow} S(x) \sup_{\eta(x) \le y \le x} \left\{\frac{Z_1(y) - \psi_1(y)Z_3(y)}{S(y)}\right\} + \psi_1(x)Z_3(x) \\ &\overset{w}{\Rightarrow} S(x) \sup_{\eta(x) \le y \le x} U_1(y) + \psi_1(x)Z_3(x). \end{aligned}$$

in $\ell^\infty[0, a]$ for all $a > 0$. Next we show that

$$\sqrt{n}\left[\hat{S}(x) \sup_{y \le \eta(x)} \hat{\psi}_1(y) - S_1(x)\right] \overset{d}{\to} S(x)U_1(\eta(x)) + \psi_1(x)Z_3(x).$$

Fix $\delta > 0$ and write

$$\sqrt{n}\left[\hat{S}(x)\sup_{y\leq\eta(x)}\hat{\psi}_1(y) - S_1(x)\right] \;=\; \max(U_{1n}(x), U_{2n}(x))$$

where

$$U_{1n}(x) \;=\; \sqrt{n}\left[\hat{S}(x)\sup_{\eta(x)-\delta\leq y\leq\eta(x)}\hat{\psi}_1(y) - S_1(x)\right] \quad \text{and} \quad U_{2n}(x) = \sqrt{n}\left[\hat{S}(x)\sup_{y\leq\eta(x)-\delta}\hat{\psi}_1(y) - S_1(x)\right].$$

Since $\psi_1(x)$ is nondecreasing,

$$\begin{aligned}\sqrt{n}[\hat{S}(x)\hat{\psi}_1(\eta(x)) - S_1(x)] \leq U_{1n}(x) \;&\leq\; \sqrt{n}\hat{S}(x)\sup_{\eta(x)-\delta\leq y\leq\eta(x)}\left\{(\hat{\psi}_1(y) - \hat{\psi}_1(\eta(x))) - (\psi_1(y) - \psi_1(\eta(x)))\right\}\\ &+\; \sqrt{n}[\hat{S}(x)\hat{\psi}_1(\eta(x)) - S_1(x)].\end{aligned}$$

Since the process $\{\sqrt{n}[\hat{\psi}_1(x) - \psi_1(x)], x \geq 0\}$ converges weakly,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} P[\sqrt{n}\sup_{\eta(s)-\delta\leq y\leq\eta(x)}[(\hat{\psi}_1(y) - \psi_1(y)) - (\hat{\psi}_1(\eta(x)) - \psi_1(\eta(x))] > \epsilon] = 0.$$

This implies that

$$\begin{aligned}U_{1n}(x) \;&=\; \sqrt{n}[\hat{S}(x)\hat{\psi}_1(\eta(x)) - S_1(x))] + o_p(1)\\ &=\; \sqrt{n}\hat{S}(x)\left[\hat{\psi}_1(\eta(x)) - \psi_1(\eta(x))\right] + \psi_1(\eta(x))\sqrt{n}[\hat{S}(x) - S(x)]\\ &\xrightarrow{d}\; S(x)U_1(\eta(x)) + \psi_1(x)Z_3(x).\end{aligned}$$

Consider now $U_{2n}(x)$. We have

$$\begin{aligned}U_{2n}(x) \;&\leq\; \sqrt{n}\hat{S}(x)\sup_{y\leq\eta(x)-\delta}\left\{\hat{\psi}_1(y) - \psi_1(y)\right\} + \sqrt{n}\psi_1(x)[\hat{S}(x) - S(x)]\\ &+\; \sqrt{n}\hat{S}(x)[\psi_1(\eta(x) - \delta) - \psi_1(x)].\end{aligned}$$

Clearly

$$\sqrt{n}\sup_{y\leq\eta(x)-\delta}\left\{\hat{\psi}_1(y) - \psi_1(y)\right\}\hat{S}(x) + \sqrt{n}\psi_1(x)[\hat{S}(t) - S(x)] \stackrel{w}{\Rightarrow} S(x)\sup_{y\leq\eta(x)-\delta}U_1(y) + \psi_1(x)Z_3(x)$$

Since $\psi(x-\delta) - \psi(x) < 0, \sqrt{n}\hat{S}(x)[\psi_1(\eta(x) - \delta) - \psi_1(x)] \to -\infty$, almost surely. Therefore $U_{2n}(x)$ converges to $-\infty$ in probability and we have pointwise convergence in distribution.

Next we show tightness. A careful inspection of the pointwise convergence in distribution shows that the weak convergence of Z^*_{1n} will follow from that of the processes

$$W^*_n(x) = \sqrt{n}[\sup_{0\leq y\leq x}\hat{\psi}_1(y) - \psi_1(x)] \quad \text{and} \quad Z_{3n}(x) = \sqrt{n}[\hat{S}(x) - S(x)].$$

Since Z_{3n} converges weakly, we only need to show that W_n is tight. We need to consider three cases.

1. ψ_1 is constant. When this is the case,

$$\sup_{|x-y|\le\delta} |W_n^*(x) - W_n^*(y)| \quad = \quad \sup_{|x-y|\le\delta} \sqrt{n}|\sup_{0\le s\le x} \hat{\psi}_1(s) - \sup_{0\le s\le y} \hat{\psi}_1 s)|.$$

Since when $x \le y$,

$$\begin{aligned} 0 &\le \sup_{0\le s\le y} \hat{\psi}_1 s) - \sup_{0\le s\le x} \hat{\psi}_1 s) \\ &\le \max[0, \sup_{x\le s\le y} \hat{\psi}_1(s) - \sup_{0\le s\le x} \hat{\psi}_1(s)] \\ &\le \max[0, \sup_{x\le s\le y} \hat{\psi}_1(s) - \hat{\psi}_1(x)] \\ &\le \sup_{x\le s\le y} |\hat{\psi}_1(x) - \hat{\psi}_1(s)| \end{aligned}$$

$$\sup_{|x-y|\le\delta} |W_n^*(x) - W_n^*(y)| \le \sqrt{n} \sup_{|x-y|\le\delta} \sup_{\min(x,y)\le s\le\max(x,y)} |\hat{\psi}_1(x) - \hat{\psi}(s)| = \sqrt{n} \sup_{|x-y|\le\delta} |\hat{\psi}_1(x) - \hat{\psi}_1(y)|$$

and tightness of W_n^* follows from that of $W_n \equiv \sqrt{n}[\hat{\psi}_1 - \psi_1]$.

2. ψ_1 is strictly increasing. Fix $\delta > 0$ and let $\Delta = \inf_{\delta\le x\le a} |\psi_1(x-\delta) - \psi_1(x)| > 0$. Lemma 1 in Rojo and Samaniego (1993) implies that

$$\sup_{0\le x\le a} |\sup_{0\le y\le x} \hat{\psi}_1(y) - \psi_1(x)| = \sup_{0\le x\le a} |\sup_{0\le y\le x} \hat{\psi}_1(y) - \sup_{0\le y\le x} \psi_1(y)| \le \sup_{0\le x\le a} |\hat{\psi}_1(x) - \psi_1(x)| \to 0$$

almost surely. As a result

$$\sup_{0\le x\le a} |(\sup_{0\le y\le x-\delta} \hat{\psi}_1(y) - \sup_{0\le y\le x} \hat{\psi}_1(y)) - (\psi_1(x-\delta) - \psi_1(x))| \to 0$$

This in turn implies that, for $\epsilon = \Delta/2$ and large enough n,

$$\begin{aligned} \sup_{0\le y\le x-\delta} \hat{\psi}_1(y) &\le \sup_{0\le y\le x} \hat{\psi}_1(y) + \psi_1(x-\delta) - \psi_1(x) + \epsilon \\ &\le \sup_{0\le y\le x} \hat{\psi}_1(y) - \epsilon/2 \end{aligned} \tag{1.1}$$

for all $x \in [\delta, a]$ since $\psi_1(x-\delta) - \psi_1(x) \le -\Delta$ over this range. Define

$$\theta_n(x) = \inf\{t, \hat{\psi}_1(t) \ge \sup_{0\le y\le x} \hat{\psi}_1(y) - 1/n\}.$$

Since $\hat{\psi}_1$ is right continuous,

$$\hat{\psi}_1(\theta_n(x)) \ge \sup_{0\le y\le x} \hat{\psi}_1(y) - 1/n.$$

Choose n large enough so that (**??**) is satisfied and $1/n < \epsilon/2$, then for all $x \le a$,

$$\sup_{0\le y\le x-\delta} \hat{\psi}_1(y) \quad \le \quad \sup_{0\le y\le x} \hat{\psi}_1(y) - \epsilon/2 < \sup_{0\le y\le x} \hat{\psi}_1(y) - 1/n.$$

This implies that

$$\sup_{\delta \le x \le a} |\theta_n(x) - x| \le \delta,$$

and since for $x \in [0, \delta], 0 \le \theta_n(x) \le x$,

$$\sup_{0 \le x \le a} |\theta_n(x) - x| \le \delta$$

and we can conclude that

$$\sup_{0 \le x \le a} |\theta_n(x) - x| \to 0 \quad \text{almost surely.} \tag{1.2}$$

Notice that

$$\hat{\psi}_1(x) - \hat{\psi}_1(\theta_n(x)) \le \hat{\psi}_1(x) - \sup_{0 \le y \le x} \hat{\psi}_1(y) + 1/n \le 1/n.$$

Since $\psi_1(\theta_n(x)) \le \psi_1(x)$,

$$n^{-1/2} \ge \sqrt{n}(\hat{\psi}_1(x) - \hat{\psi}_1(\theta_n(x)) \ge \sqrt{n}\left[\hat{\psi}_1(x) - \hat{\psi}_1(\theta_n(x)) - (\psi_1(x) - \psi_1(\theta_n(x))\right].$$

Consequently,

$$\begin{aligned} P\left[\sup_{x \le a} |\sqrt{n}\left[(\hat{\psi}_1(x) - \hat{\psi}_1(\theta_n(x))) - (\psi_1(x) - \psi_1(\theta_n(x)))\right]| > \epsilon\right] \quad &\le \quad P[\sup_{x \le a} |\theta_n(x) - x| > \delta] \\ +P\left[\sup_{|x-y| \le \delta} |\sqrt{n}\left[\hat{\psi}_1(x) - \hat{\psi}_1(y) - (\psi_1(x) - \psi_1(y)\right]| > \epsilon)\right] \to 0 & \end{aligned}$$

by the tightness of W_n. Therefore

$$\sqrt{n} \sup_{0 \le x \le a} |\hat{\psi}_1(x) - \hat{\psi}_1(\theta_n)| \to 0 \quad \text{in probability.} \tag{1.3}$$

By definition of $\theta_n(x)$, for large enough n,

$$\sup_{0 \le s \le x} \hat{\psi}_1(s) \ge \psi_1(\theta_n(x)) \ge \sup_{0 \le s \le x} \hat{\psi}_1(s) - 1/n$$

for all $0 \le x \le a$. Therefore

$$\sup_{0 \le x \le a} |\hat{\psi}_1(\theta_n(x)) - \sup_{0 \le s \le x} \hat{\psi}_1(s)| \le 1/n.$$

Consequently, we have

$$\sqrt{n} \sup_{x \le a} |\hat{\psi}_1(x) - \sup_{0 \le y \le x} \hat{\psi}_1(y)| \le \sqrt{n} \sup_{0 \le x \le a} |\hat{\psi}_1(x) - \hat{\psi}_1(\theta_n)| + n^{-1/2} \to 0$$

in probability using (**??**). This shows that

$$\sup_{0 \le x \le a} |W_n^*(x) - W_n(x)| \to 0$$

in probability. Since W_n converges weakly, W_n^* does also on all intervals of the form $[0, a]$ and we have the desired result in this case.

3. General case. Write

$$\begin{aligned} W_n^*(x) &= \sqrt{n}[\sup_{0 \le y \le x} \hat{\psi}_1(y) - \psi_1(x)] \\ &= \max\left[\sqrt{n}[\sup_{0 \le y \le \eta(x)} \hat{\psi}_1(y) - \psi_1(x)], \sqrt{n}[\sup_{\eta(x) \le y \le x} \hat{\psi}_1(y) - \psi_1(x)]\right] \\ &= \max(W_{1n}^*(x), W_{2n}^*(x)) \end{aligned}$$

where

$$W_{1n}^*(x) = \sqrt{n}[\sup_{0 \le y \le \eta(x)} \hat{\psi}_1(y) - \psi_1(x)] \quad \text{and} \quad W_{2n}^*(x) = \sqrt{n}[\sup_{\eta(x) \le y \le x} \hat{\psi}_1(y) - \psi_1(x)].$$

Notice that

$$W_{1n}^*(x) = \sqrt{n}[\sup_{0 \le y \le \eta(x)} \hat{\psi}_1(y) - \psi_1(x)] = \sqrt{n}[\sup_{0 \le y \le \eta(x)} \hat{\psi}_1(y) - \psi_1(\eta(x))] = \sqrt{n}[\hat{\psi}_1(\eta(x)) - \psi_1(\eta(x))] + o_p(1)$$

uniformly in x from the case of strictly increasing. In addition, we have

$$W_{2n}^*(x) = \sqrt{n} \sup_{\eta(x) \le y \le x} (\hat{\psi}_1(y) - \psi_1(y)) \equiv \sup_{\eta(x) \le s \le x} W_n(s)$$

Since W_n converges weakly to some W. We have

$$\sup_{x \le a} |W_{1n}^*(x) - \inf_{\eta(x) \le y \le x} W(y)| \le \sup_{x \le a} \sup_{\ell(x) \le y \le x} |W_n(s) - W(y)| = \sup_{x \le a} |W_n(x) - W(x)| \to 0$$

almost surely (by embedding the process in a space where can use almost sure convergence). Putting all this together gives $W_n^* \overset{w}{\Rightarrow} W^*$ where $W^*(x) = \sup_{\eta \le y \le x} W(y)$.

2 References

Rojo, J. and Samaniego, F. J. (1993). On estimating a survival curve subject to a uniform stochastic ordering constraint. *Journal of the American Statistical Association* 88 566-572.

Lindvall, T. (1973). Weak convergence of probability measures and random functions in the function space $D[0, \infty)$. *Journal of Applied Probability*, 10 109-121.