MINIMUM ABERRATION FACTORIAL DESIGNS UNDER A MIXED PARAMETRIZATION

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Supplementary Material

In this supplementary material, we present the proofs of theoretical results in the paper and minimum π_B- and π-aberration designs of 8 and 12 runs.

S1 Proofs of the theoretical results

Proof of Part (ii) of Corollary 2. The arguments are similar to those used in Cheng (2014). Let $\xi = (\xi_\phi, \xi^{(1)\mathrm{T}})^\mathrm{T}$ and $X = (1_N, X_1)$. Then the model (2.4) can be written in matrix form as $Y = X\xi + \epsilon$. The information matrix for estimating main effects $\xi^{(1)}$ is given by

$$M_{\xi^{(1)}} = X^\mathrm{T}X - \frac{1}{N}X^\mathrm{T}J_{N\times N}X,$$

where $J_{N\times N}$ is an $N \times N$ matrix of all ones. On the other hand, let $\tilde{X} = (1_N, D)$ and $\beta = (\beta_\phi, \beta_1, \ldots, \beta_m)^\mathrm{T}$. Then under the orthogonal parametrization, the model (2.4) can be written as $Y = \tilde{X}\beta + \epsilon$. The information matrix for estimating the main effects $\beta^{(1)} = (\beta_1, \ldots, \beta_m)^{\mathrm{T}}$ is given by

$$M_{\beta^{(1)}} = \tilde{X}^{\mathrm{T}}\tilde{X} - \frac{1}{N}\tilde{X}^{\mathrm{T}}J_{N\times N}\tilde{X}.$$

Let $\tilde{J} = (0_{N\times 1}, J_{N\times m_1}, 0_{N\times m_2})$, where $0_{N\times 1}$, $J_{N\times m_1}$ and $0_{N\times m_2}$ are, respectively, an $N\times 1$ matrix of all zeros, an $N\times m_1$ matrix of all ones and an $N\times m_2$ matrix of all zeros. Then the matrices X and $\tilde{X}$ are related through $X = \tilde{X} + \tilde{J}$. By some simple algebra, it can be shown that

$$M_{\xi^{(1)}} = (\tilde{X}+\tilde{J})^{\mathrm{T}}(\tilde{X}+\tilde{J}) - \frac{1}{N}(\tilde{X}+\tilde{J})^{\mathrm{T}}J_{N\times N}(\tilde{X}+\tilde{J}) = M_{\beta^{(1)}}.$$

Since an $\mathrm{OA}(N, 2^m, 2)$ is universally optimal for estimating $\beta^{(1)}$ (Cheng, 1980), it follows directly that an $\mathrm{OA}(N, 2^m, 2)$ is also universally optimal for estimating $\xi^{(1)}$. □

Proof of Theorem 1. The proof is similar to that of Theorem 1 in Sun and Tang (2022). Let τ, θ, β and $\tilde{\xi}$ collect all τ_u's, θ_w's, β_w's and ξ_w's in Yates order. Define $L_m = \otimes_{k=1}^m L$ and $H_m = \otimes_{k=1}^m H$ where

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and $\otimes_{k=1}^m$ denotes m-fold Kronecker product. Then in matrix notation we have

$$\tau = H_m\beta, \quad \tau = L_m\theta, \quad \tau = H_{m_2}\otimes L_{m_1}\tilde{\xi}. \tag{S1.1}$$

Then the results in Theorem 1 can be verified directly. For example, $\tilde{\xi} = \otimes_{k=1}^{m_2}(H^{-1}L) \otimes I_{2^{m_1}}\tilde{\theta}$ where $I_{2^{m_1}}$ is the identity matrix of order 2^{m_1}. □

Proof of Lemma 1. Note that the matrix B_2, which contains the rows $2, \ldots, m_1+1$ of the matrix $(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}X_2$, can be written as $B_2 = (B_{2,B\times B}, B_{2,B\times O}, B_{2,O\times O})$, where the three submatrices correspond to the interactions of two B-factors, one B-factor and one O-factor, and two O-factors, respectively. Hence $\pi_2^B = \mathrm{tr}(B_2^{\mathrm{T}}B_2) = \mathrm{tr}(B_{2,B\times B}^{\mathrm{T}}B_{2,B\times B}) + \mathrm{tr}(B_{2,B\times O}^{\mathrm{T}}B_{2,B\times O}) + \mathrm{tr}(B_{2,O\times O}^{\mathrm{T}}B_{2,O\times O})$. Since D is an orthogonal array, it can be easily checked that

$$X^{\mathrm{T}}X = N\begin{bmatrix} 1 & 1_{m_1}^{\mathrm{T}} & 0 \\ 1_{m_1} & I_{m_1}+J_{m_1} & 0 \\ 0 & 0 & I_{m_2}\end{bmatrix}, \quad (X^{\mathrm{T}}X)^{-1} = \frac{1}{N}\begin{bmatrix} m_1+1 & -1_{m_1}^{\mathrm{T}} & 0 \\ -1_{m_1} & I_{m_1} & 0 \\ 0 & 0 & I_{m_2}\end{bmatrix},$$

where J_{m_1} is an $m_1 \times m_1$ matrix of all ones. Through some tedious algebra, one can show that $\mathrm{tr}(B_{2,B\times B}^{\mathrm{T}}B_{2,B\times B}) = 3\sum_{i<j<k} J^2(b_i, b_j, b_k)/N^2 + m_1(m_1-1)$, $\mathrm{tr}(B_{2,B\times O}^{\mathrm{T}}B_{2,B\times O}) = 2\sum_{i<j}\sum_k J^2(b_i, b_j, o_k)/N^2$ and $\mathrm{tr}(B_{2,O\times O}^{\mathrm{T}}B_{2,O\times O}) = \sum_i\sum_{j<k} J^2(b_i, o_j, o_k)/N^2$. This gives the expression of π_2^B in the Lemma. One can also define $O_{2,B\times B}$, $O_{2,B\times O}$ and $O_{2,O\times O}$ similarly and show that $\mathrm{tr}(O_{2,B\times B}^{\mathrm{T}}O_{2,B\times B}) = \sum_{i<j}\sum_k J^2(b_i, b_j, o_k)/N^2$, $\mathrm{tr}(O_{2,B\times O}^{\mathrm{T}}O_{2,B\times O}) = 2\sum_i\sum_{j<k} J^2(b_i, o_j, o_k)/N^2 + m_1m_2$ and $\mathrm{tr}(O_{2,O\times O}^{\mathrm{T}}O_{2,O\times O}) = 3\sum_{i<j<k} J^2(o_i, o_j, o_k)/N^2$, leading to the expression of π_2^O in the Lemma. □

Proof of Theorem 2. First, we do not assume that $D = (b_1, \ldots, b_{m_1}, o_1, \ldots, o_{m_2})$ is regular. Since D is an orthogonal array, there exists a set of $m_3 = N - 1 - m$ orthogonal real columns $E = (e_1, \ldots, e_{m_3})$ such that $e_j^{\mathrm{T}} e_j = N^2$ and e_j's are orthogonal to columns of D and the column 1_N for $j = 1, \ldots, m_3$. Hence, for any $1 \leq i \neq j \leq m_1$, we have that $\sum_{k=1}^{m_1} J^2(b_i, b_j, b_k) + \sum_{k=1}^{m_2} J^2(b_i, b_j, o_k) + \sum_{k=1}^{m_3} J^2(b_i, b_j, e_k) = N^2$. Summing this equation over all (i, j)'s, one can show that

$$\sum_{i<j} \sum_{k} J^2(b_i, b_j, o_k) = -3 \sum_{i<j<k} J^2(b_i, b_j, b_k) - \sum_{i<j} \sum_{k} J^2(b_i, b_j, e_k) + C_1 \tag{S1.2}$$

for some constant C_1. Using similar arguments, we can express $\sum_i \sum_{j<k} J^2(b_i, o_j, o_k)$ and $\sum_{i<j<k} J^2(o_i, o_j, o_l)$ in terms of J-characteristics of columns not involving o_j's. In particular, we have

$$\begin{aligned}\sum_{i} \sum_{j<k} J^2(b_i, o_j, o_k) = 3 \sum_{i<j<k} J^2(b_i, b_j, b_k) + 2 \sum_{i<j} \sum_{k} J^2(b_i, b_j, e_k) \\ + \sum_{i} \sum_{j<k} J^2(b_i, e_j, e_k) + C_2\end{aligned} \tag{S1.3}$$

for some constant C_2 and

$$\begin{aligned}\sum_{i<j<k} J^2(o_i, o_j, o_k) = - \sum_{i<j<k} J^2(b_i, b_j, b_k) - \sum_{i<j} \sum_{k} J^2(b_i, b_j, e_k) \\ - \sum_{i} \sum_{j<k} J^2(b_i, e_j, e_k) - J^2(e_i, e_j, e_k) + C_3\end{aligned} \tag{S1.4}$$

for some constant C_3. Combining (S1.2), (S1.3) and (S1.4), we have $\pi_2^B =$

$\sum_i \sum_{j<k} J^2(b_i, e_j, e_k)/N^2 + C_B$ and $\pi_2^O = -\sum_i \sum_{j<k} J^2(b_i, e_j, e_k)/N^2 - 3\sum_{i<j<k} J^2(e_i, e_j, e_k)/N^2 + C_O$ for some constants C_B and C_O. Therefore, we have proved that sequentially minimizing π_2^B and π_2^O amounts to sequentially minimizing $\sum_i \sum_{j<k} J^2(b_i, e_j, e_k)$ and $-\sum_{i<j<k} J^2(e_i, e_j, e_k)$.

Now suppose that columns of D are selected from a saturated regular design as specified in Theorem 2. Then E can be taken as the complement of D in the saturated regular design. Then we have that $J^2(b_i, e_j, e_k) = N^2$ if b_i, e_j and e_k forms a defining word and $J^2(b_i, e_j, e_k) = 0$ otherwise. It can be verified that when the conditions in part (i) of the theorem are met, b_i must contain an independent column not contained in e_j and e_k, leading to $J^2(b_i, e_j, e_k) = 0$. In addition, the results of Chen and Hedayat (1996) imply that $\sum_{i<j<k} J^2(e_i, e_j, e_k)$ is maximized by design D among all regular OA$(2^h, 2^m, 2)$'s, and in particular, among all OA$(2^h, 2^m, 2)$'s if $m = 2^h - 2^{h_1}$ for some integer h_1. The results of Theorem 2 then follow. □

Proof of Lemma 2. We use similar notations to those in the proof of Lemma 1. For example, $B_{3,B\times B\times B}$ is the submatrix of $(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}X_3$ corresponding to contamination of interaction involving three B-factors on the main effects of B-factors. Then we have

$$\mathrm{tr}(B_{3,B\times B\times B}^{\mathrm{T}} B_{3,B\times B\times B}) = \frac{3}{N^2} \sum_{i<j<k} \{J(b_i, b_j, b_k) + N\}^2$$

$$+\frac{1}{N^2}\sum_{i<j<k}\sum_{l}\left\{J(b_i,b_j,b_k,b_l)+J(b_i,b_j,b_l)+J(b_i,b_k,b_l)+J(b_j,b_k,b_l)\right\}^2,$$

$$\mathrm{tr}(B^{\mathrm{T}}_{3,B\times B\times O}B_{3,B\times B\times O})=\frac{2}{N^2}\sum_{i<j}\sum_{k}J^2(b_i,b_j,o_k)$$

$$+\frac{1}{N^2}\sum_{i<j}\sum_{k}\sum_{l}\left\{J(b_i,b_j,b_k,o_l)+J(b_i,b_k,o_l)+J(b_j,b_k,o_l)\right\}^2,$$

$$\mathrm{tr}(B^{\mathrm{T}}_{3,B\times O\times O}B_{3,B\times O\times O})=\frac{1}{N^2}\sum_{i}\sum_{j<k}J^2(b_i,o_j,o_k)+\frac{1}{N^2}\sum_{i\neq j}\sum_{k<l}\left\{J(b_i,b_j,o_k,o_l)+J(b_i,o_k,o_l)\right\}^2,$$

$$\mathrm{tr}(B^{\mathrm{T}}_{3,O\times O\times O}B_{3,O\times O\times O})=\frac{1}{N^2}\sum_{i}\sum_{j<k<l}J^2(b_i,o_j,o_k,o_l),$$

$$\mathrm{tr}(O^{\mathrm{T}}_{3,B\times B\times B}O_{3,B\times B\times B})=\frac{1}{N^2}\sum_{i<j<k}\sum_{l}\left\{J(b_i,b_j,b_k,o_l)+J(b_i,b_j,o_l)+J(b_i,b_k,o_l)+J(b_j,b_k,o_l)\right\}^2,$$

$$\mathrm{tr}(O^{\mathrm{T}}_{3,B\times B\times O}O_{3,B\times B\times O})=\frac{2}{N^2}\sum_{i<j}\sum_{k<l}\left\{J(b_i,b_j,o_k,o_l)+J(b_i,o_k,o_l)+J(b_j,o_k,o_l)\right\}^2+\frac{1}{2}m_1m_2(m_1-1),$$

$$\mathrm{tr}(O^{\mathrm{T}}_{3,B\times O\times O}O_{3,B\times O\times O})=\frac{3}{N^2}\sum_{i}\sum_{j<k<l}\left\{J(b_i,o_j,o_k,o_l)+J(o_j,o_k,o_l)\right\}^2,$$

$$\mathrm{tr}(O^{\mathrm{T}}_{3,O\times O\times O}O_{3,O\times O\times O})=\frac{4}{N^2}\sum_{i<j<k<l}J^2(o_i,o_j,o_k,o_l).$$

Then π_3 is obtained by taking the sum of all the terms above. Since any two J-characteristics in the same curly bracket cannot be nonzero at the same time, their product will be zero if we expand the square term. By some tedious algebra, we have

$$\pi_3=4A_4+(3m_1-6)A_3+2\pi_2^O+\frac{6}{N}\sum_{i<j<k}J(b_i,b_j,b_k)+\frac{1}{2}m_1(m_1-1)(m_1+m_2-2). \tag{S1.5}$$

In the proof of Theorem 2, we have already obtained that $\pi_2^O=-\sum_i\sum_{j<k}$

$J^2(b_i,e_j,e_k)/N^2 - 3\sum_{i<j<k} J^2(e_i,e_j,e_k)/N^2 + C_O$ for some constant C_O, where $e_1,\ldots,e_{N-1-m}$ are columns of the complement of D. Clearly, if $e_1,\ldots,e_{N-1-m}$ takes the first $2^{h_1}-1$ columns of a saturated regular design, then the first three terms of π_3 in (S1.5) are constant. Then the result of the lemma follows. □

Proof of Theorem 3. Since D is regular, the value of $J(b_i,b_j,b_k)$ is either 0 or $\pm N$ for any $1 \le i < j < k \le m_1$. Thus we have

$$\pi_3 = c_1 N \sum_{i<j<k} J(b_i,b_j,b_k)/N + c_0 \ge -c_1 N \sum_{i<j<k} J^2(b_i,b_j,b_k)/N^2 + c_0$$
$$= -c_1 N A_3(D_B) + c_0, \quad \text{(S1.6)}$$

where $A_3(D_B)$ is the A_3 value of D_B. By results of Chen and Hedayat (1996), we have $A_3(D_B)$ is maximized among all regular designs by the choice of D_B in the construction. In addition, since $J(b_i,b_j,b_k)/N = -J^2(b_i,b_j,b_k)/N^2$, we conclude the lower bound in (S1.6) is achieved. Therefore, $D = (D_B, D_O)$ sequentially minimizes π_2 and π_3 over all regular designs. □

Proof of Lemma 3. The proof can be done by a direct verification. For example, the contamination of k-factor interaction $od_{j_1}\cdots d_{j_{k-1}}$ on the estimation of main effect of d_{j_k}, where o is an O-factor and $d_{j_1},\ldots,d_{j_k}$ are either B-factors or O-factors, will contribute a term $(\sum_{i=1}^N o_i z_{i,j_1} z_{i,j_2}\cdots z_{i,j_{k-1}} d_{i,j_k})^2/N^2$, where $z_{i,j_l} = d_{i,j_l}$ if d_{j_l} is an O-factor and $z_{i,j_l} = d_{i,j_l} + 1$ if d_{j_l} is a B-factor for $l = 1, \ldots, k-1$, in π_k^B or π_k^O depending on whether d_{j_k} is a B-factor or an O-factor. One can see that replacing o_i by $-o_i$ does not affect the value of this term. Therefore, the conclude switching the signs of O-factors in an OA$(N, 2^m, 2)$ does not affect its aberration. □

Proof of Theorem 4. For the design D generated in the theorem, we study the contamination of k-factor interaction $b_{j_1} \cdots b_{j_{k_1}} o_{l_1} \cdots o_{l_{k_2}}$ $(k_1 + k_2 = k)$ on the estimation of the main effect of certain factor d_0. Such a contamination will contribute a term $Q = \{\sum_{i=1}^N (b_{i,j_1}+1) \cdots (b_{i,j_{k_1}}+1) o_{i,l_1} \cdots o_{i,l_{k_2}} d_{i,0}\}^2$ $/N^2$ in π_k. Thus we have

$$Q = \frac{1}{N^2}\left\{J(b_{j_1}, \ldots, b_{j_{k_1}}, o_{l_1}, \ldots, o_{l_{k_2}}, d_0) + \cdots + J(o_{l_1}, \ldots, o_{l_{k_2}}, d_0)\right\}^2.$$

If we expand the square and average E over all possible sign switches of B-factors, the cross-product terms will disappear and we will obtain

$$\tilde{Q} = \frac{1}{N^2}\left\{J^2(b_{j_1}, \ldots, b_{j_{k_1}}, o_{l_1}, \ldots, o_{l_{k_2}}, d_0) + \cdots + J^2(o_{l_1}, \ldots, o_{l_{k_2}}, d_0)\right\}.$$

If we further average $\tilde{Q}$ over all possible choices of B-factors in the orthogonal array, then one can show that the resulting term will be a linear combination of $A_{k+1}, A_k, \ldots, A_{k_2-1}$ with positive coefficients. Then the result of the theorem follows by some tedious algebra. □

Proof of Proposition 1. The result of Theorem 1 implies that there exist an

$(1+m+f)\times(1+m+f)$ upper-triangular matrix P whose diagonal entries are all one, such that $\xi_{\mathcal{F}} = P\beta_{\mathcal{F}}$ and $X_{\mathcal{F}} = Z_{\mathcal{F}}P$. Thus $\det(P) = 1$ and $\det(X_{\mathcal{F}}^T X_{\mathcal{F}}) = \det^2(P)\det(Z_{\mathcal{F}}^T Z_{\mathcal{F}})$. The result of Proposition 1 then follows immediately. □

S2 Minimum π_B- and π-aberration designs

We present minimum π_B- and π-aberration designs of 8 and 12 runs in Tables 2, 3, 4 and 5. All these designs are generated by selecting and sign-switching columns of the saturated designs displayed in Table 1.

Table 1: Two saturated designs of 8 and 12 runs

1	2	3	4	5	6	7
−	−	−	−	−	−	−
+	−	+	−	+	−	+
−	+	+	−	−	+	+
+	+	−	−	+	+	−
−	−	−	+	+	+	+
+	−	+	+	−	+	−
−	+	+	+	+	−	−
+	+	−	+	−	−	+

1	2	3	4	5	6	7	8	9	10	11
−	−	−	−	−	−	−	−	−	−	−
+	−	+	−	−	−	+	+	+	−	+
+	+	−	+	−	−	−	+	+	+	−
−	+	+	−	+	−	−	−	+	+	+
+	−	+	+	−	+	−	−	−	+	+
+	+	−	+	+	−	+	−	−	−	+
+	+	+	−	+	+	−	+	−	−	−
−	+	+	+	−	+	+	−	+	−	−
−	−	+	+	+	−	+	+	−	+	−
−	−	−	+	+	+	−	+	+	−	+
+	−	−	−	+	+	+	−	+	+	−
−	+	−	−	−	+	+	+	−	+	+

Table 2: Minimum π_B and π-aberration designs of 8 runs

m	m_1	m_2	Columns of D_B	Columns of D_O	$(\pi_2^B, \pi_2^O, \pi_3^B, \pi_3^O)$	Criterion
3	1	2	-1	$(2, 4)$	$(0, 2, 0, 0)$	π_B, π
3	2	1	$(-1, -2)$	4	$(2, 2, 0, 1)$	π_B, π
4	1	3	-1	$(2, 4, 7)$	$(0, 3, 1, 3)$	π_B, π
4	2	2	$(-1, -2)$	$(4, 7)$	$(2, 4, 2, 4)$	π_B, π
4	3	1	$(-1, -2, -4)$	7	$(6, 3, 6, 4)$	π_B, π
5	1	4	-2	$(1, 3, 4, 5)$	$(1, 9, 2, 6)$	π_B
5	1	4	-1	$(2, 3, 4, 5)$	$(2, 8, 2, 4)$	π
5	2	3	$(-2, -3)$	$(1, 4, 5)$	$(4, 10, 4, 11)$	π_B
5	2	3	$(-1, -2)$	$(3, 4, 5)$	$(5, 9, 5, 8)$	π
5	3	2	$(-2, -3, -4)$	$(1, 5)$	$(9, 9, 13, 12)$	π_B
5	3	2	$(1, -2, -3)$	$(4, 5)$	$(10, 8, 5, 12)$	π
5	4	1	$(2, -3, -4, -5)$	1	$(16, 6, 28, 10)$	π_B
5	4	1	$(1, -2, -3, -4)$	5	$(17, 5, 21, 9)$	π
6	1	5	-1	$(2, 3, 4, 5, 6)$	$(2, 15, 4, 16)$	π_B, π
6	2	4	$(-1, -2)$	$(3, 4, 5, 6)$	$(6, 16, 10, 22)$	π_B
6	2	4	$(-1, -6)$	$(2, 3, 4, 5)$	$(6, 16, 12, 20)$	π
6	3	3	$(1, -2, -3)$	$(4, 5, 6)$	$(12, 15, 15, 27)$	π_B, π
6	4	2	$(1, -2, -3, -4)$	$(5, 6)$	$(20, 12, 36, 26)$	π_B, π
6	5	1	$(1, -2, -3, -4, -5)$	6	$(30, 7, 62, 18)$	π_B, π
7	1	6	-1	$(2, 3, 4, 5, 6, 7)$	$(3, 24, 7, 36)$	π_B, π
7	2	5	$(-1, -2)$	$(3, 4, 5, 6, 7)$	$(8, 25, 18, 45)$	π_B, π
7	3	4	$(1, -2, -3)$	$(4, 5, 6, 7)$	$(15, 24, 30, 52)$	π_B, π
7	4	3	$(1, -2, -3, -4)$	$(5, 6, 7)$	$(24, 21, 58, 54)$	π_B, π
7	5	2	$(1, -2, -3, -4, -5)$	$(6, 7)$	$(35, 16, 93, 48)$	π_B, π
7	6	1	$(1, 2, 3, -4, -5, -6)$	7	$(48, 9, 138, 31)$	π_B, π

Table 3: Minimum π_B and π-aberration designs of 12 runs for $m = 3, \ldots, 7$ factors

m	m_1	m_2	Columns of D_B	Columns of D_O	$(\pi_2^B, \pi_2^O, \pi_3^B, \pi_3^O)$	Criterion
3	1	2	-1	$(2, 3)$	$(0.11, 2.22, 0.11, 0)$	π_B, π
3	2	1	$(-1, -2)$	3	$(2.22, 2.11, 0.22, 1)$	π_B, π
4	1	3	-1	$(2, 3, 4)$	$(0.33, 4, 0.44, 0)$	π_B, π
4	2	2	$(-1, -2)$	$(3, 4)$	$(2.67, 4.67, 0.67, 2.22)$	π_B, π
4	3	1	$(-1, -2, 3)$	4	$(7, 3.33, 2.33, 3.44)$	π_B, π
5	1	4	-1	$(2, 3, 4, 6)$	$(0.67, 6.67, 1.11, 0.44)$	π_B, π
5	2	3	$(-1, -2)$	$(3, 4, 6)$	$(3.33, 8, 1.56, 3.67)$	π_B
5	2	3	$(-1, -2)$	$(3, 4, 6)$	$(3.33, 8, 1.56, 3.67)$	π
5	3	2	$(-1, -2, 3)$	$(4, 6)$	$(8, 7.33, 3.67, 7.56)$	π_B, π
5	4	1	$(-1, -2, 3, -4)$	6	$(14.67, 4.67, 9.78, 7.78)$	π_B, π
6	1	5	-1	$(2, 3, 4, 5, 6)$	$(1.11, 10.56, 2.22, 7.56)$	π_B, π
6	2	4	$(-1, -2)$	$(3, 4, 5, 6)$	$(4.22, 12.44, 4.89, 12.89)$	π_B, π
6	3	3	$(-1, -2, 3)$	$(4, 5, 6)$	$(9.33, 12.33, 10.11, 18.56)$	π_B
6	3	3	$(1, -2, -3)$	$(4, 5, 10)$	$(9.33, 12.33, 11, 16.33)$	π
6	4	2	$(-1, -2, 3, -4)$	$(5, 6)$	$(16.44, 10.22, 20, 20.44)$	π_B, π
6	5	1	$(-1, -2, 3, -4, 6)$	5	$(25.56, 6.11, 36.67, 14.44)$	π_B, π
7	1	6	-2	$(1, 3, 4, 5, 6, 7)$	$(1.67, 16, 3.89, 17.33)$	π_B, π
7	2	5	$(-1, 3)$	$(2, 4, 5, 6, 7)$	$(5.33, 18.33, 9.11, 29)$	π_B
7	2	5	$(-1, -2)$	$(3, 4, 5, 6, 7)$	$(5.33, 18.33, 9.56, 25)$	π
7	3	4	$(1, -2, -3)$	$(4, 5, 6, 7)$	$(11, 18.67, 18.44, 33.33)$	π_B
7	3	4	$(-1, -3, 5)$	$(2, 4, 6, 7)$	$(11, 18.67, 19.33, 31.56)$	π
7	4	3	$(-1, 3, -4, 6)$	$(2, 5, 7)$	$(18.67, 17, 32.89, 43.33)$	π_B
7	4	3	$(1, 3, 4, 5)$	$(2, 6, 7)$	$(18.67, 17, 33.78, 37.11)$	π
7	5	2	$(-1, -2, 3, -4, 6)$	$(5, 7)$	$(28.33, 13.33, 55.22, 34.67)$	π_B, π
7	6	1	$(-1, -2, 3, -4, -5, 6)$	7	$(40, 7.67, 99.78, 23.44)$	π_B
7	6	1	$(-1, 3, 4, -5, -6, -7)$	2	$(40, 7.67, 100.67, 21.22)$	π

Table 4: Minimum π_B and π-aberration designs of 12 runs for $m = 8, 9$ factors

m	m_1	m_2	Columns of D_B	Columns of D_O	$(\pi_2^B, \pi_2^O, \pi_3^B, \pi_3^O)$	Criterion
8	1	7	1	$(2, 3, 4, 5, 6, 7, 8)$	$(2.33, 23.33, 6.22, 38.22)$	π_B, π
8	2	6	$(-1, 3)$	$(2, 4, 5, 6, 7, 8)$	$(6.67, 26, 14.44, 53.33)$	π_B
8	2	6	$(1, -2)$	$(3, 4, 5, 6, 7, 8)$	$(6.67, 26, 15.33, 48)$	π
8	3	5	$(1, -2, 4)$	$(3, 5, 6, 7, 8)$	$(13, 26.67, 28.11, 60.78)$	π_B
8	3	5	$(1, -2, -7)$	$(3, 4, 5, 6, 8)$	$(13, 26.67, 29, 56.78)$	π
8	4	4	$(1, -2, -3, 8)$	$(4, 5, 6, 7)$	$(21.33, 25.33, 48.89, 64.44)$	π_B, π
8	5	3	$(-1, -2, 3, -4, 6)$	$(5, 7, 8)$	$(31.67, 22, 78.44, 64.67)$	π_B
8	5	3	$(-1, -2, 3, -4, 6)$	$(5, 7, 8)$	$(31.67, 22, 78.44, 64.67)$	π
8	6	2	$(1, -2, -3, -4, -7, 8)$	$(5, 6)$	$(44, 16.67, 127.33, 56.44)$	π_B, π
8	7	1	$(1, -2, 3, 4, 5, -6, -7)$	8	$(58.33, 9.33, 194.11, 35.22)$	π_B, π
9	1	8	-1	$(2, 3, 4, 5, 6, 7, 8, 9)$	$(3.11, 32.89, 9.33, 68.44)$	π_B
9	1	8	-1	$(2, 3, 4, 5, 6, 7, 8, 9)$	$(3.11, 32.89, 9.33, 68.44)$	π
9	2	7	$(1, -2)$	$(3, 4, 5, 6, 7, 8, 9)$	$(8.22, 35.78, 22.89, 84.11)$	π_B
9	2	7	$(-1, -2)$	$(3, 4, 5, 6, 7, 8, 9)$	$(8.22, 35.78, 23.33, 82.78)$	π
9	3	6	$(1, -2, 4)$	$(3, 5, 6, 7, 8, 9)$	$(15.33, 36.67, 40.33, 101.33)$	π_B
9	3	6	$(-1, -2, -4)$	$(3, 5, 6, 7, 8, 9)$	$(15.33, 36.67, 43.89, 96)$	π
9	4	5	$(-1, 2, -3, 5)$	$(4, 6, 7, 8, 9)$	$(24.44, 35.56, 68.44, 107.78)$	π_B, π
9	5	4	$(-1, -2, 3, -4, 6)$	$(5, 7, 8, 9)$	$(35.56, 32.44, 106.89, 108.89)$	π_B, π
9	6	3	$(-1, -2, -3, 4, 7, -8)$	$(5, 6, 9)$	$(48.67, 27.33, 167.78, 107.44)$	π_B
9	6	3	$(-1, -2, 3, -4, 5, 6)$	$(7, 8, 9)$	$(48.67, 27.33, 169.11, 102.11)$	π
9	7	2	$(-1, -2, 3, -4, 5, 6, 7)$	$(8, 9)$	$(63.78, 20.22, 242.33, 85.78)$	π_B, π
9	8	1	$(1, 2, -3, -4, -5, 6, -7, -8)$	9	$(80.89, 11.11, 336, 52)$	π_B, π

Table 5: Minimum π_B and π-aberration designs of 12 runs for $m = 10, 11$ factors

m	m_1	m_2	Columns of D_B	Columns of D_O	$(\pi_2^B, \pi_2^O, \pi_3^B, \pi_3^O)$	Criterion
10	1	9	-1	$(2, 3, 4, 5, 6, 7, 8, 9, 10)$	$(4, 45, 13.33, 112)$	π_B, π
10	2	8	$(-1, -2)$	$(3, 4, 5, 6, 7, 8, 9, 10)$	$(10, 48, 32.89, 132.44)$	π_B
10	2	8	$(-1, -2)$	$(3, 4, 5, 6, 7, 8, 9, 10)$	$(10, 48, 32.89, 132.44)$	π
10	3	7	$(-1, 2, -3)$	$(4, 5, 6, 7, 8, 9, 10)$	$(18, 49, 59, 152.33)$	π_B
10	3	7	$(1, -2, -3)$	$(4, 5, 6, 7, 8, 9, 10)$	$(18, 49, 59.89, 151.44)$	π
10	4	6	$(-1, 2, -3, 5)$	$(4, 6, 7, 8, 9, 10)$	$(28, 48, 92, 169.33)$	π_B
10	4	6	$(-1, -2, 3, -4)$	$(5, 6, 7, 8, 9, 10)$	$(28, 48, 95.56, 165.78)$	π
10	5	5	$(-1, -2, 3, -4, 6)$	$(5, 7, 8, 9, 10)$	$(40, 45, 141.11, 172.22)$	π_B
10	5	5	$(-1, -2, 3, -4, 6)$	$(5, 7, 8, 9, 10)$	$(40, 45, 141.11, 172.22)$	π
10	6	4	$(-1, -2, 3, -4, -5, 6)$	$(7, 8, 9, 10)$	$(54, 40, 215.11, 170.22)$	π_B
10	6	4	$(-1, -2, 3, -4, -5, 6)$	$(7, 8, 9, 10)$	$(54, 40, 215.11, 170.22)$	π
10	7	3	$(1, -2, -3, -4, 5, -6, -7)$	$(8, 9, 10)$	$(70, 33, 302.78, 156.56)$	π_B
10	7	3	$(1, -2, 3, 4, 5, -6, -7)$	$(8, 9, 10)$	$(70, 33, 305.44, 153.89)$	π
10	8	2	$(1, 2, -3, -4, -5, 6, -7, -8)$	$(9, 10)$	$(88, 24, 410.67, 122.67)$	π_B, π
10	9	1	$(-1, 2, -3, 4, 5, 6, -7, -8, -9)$	10	$(108, 13, 540, 73.33)$	π_B, π
11	1	10	-1	$(2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$	$(5, 60, 18.33, 173.33)$	π_B, π
11	2	9	$(-1, -2)$	$(3, 4, 5, 6, 7, 8, 9, 10, 11)$	$(12, 63, 44.67, 201)$	π_B, π
11	3	8	$(1, -2, -3)$	$(4, 5, 6, 7, 8, 9, 10, 11)$	$(21, 64, 80, 226.67)$	π_B, π
11	4	7	$(-1, -2, 3, -4)$	$(5, 6, 7, 8, 9, 10, 11)$	$(32, 63, 125.33, 247.33)$	π_B, π
11	5	6	$(-1, -2, 3, -4, 6)$	$(5, 7, 8, 9, 10, 11)$	$(45, 60, 181.67, 260)$	π_B, π
11	6	5	$(-1, -2, 3, -4, -5, 6)$	$(7, 8, 9, 10, 11)$	$(60, 55, 270, 261.67)$	π_B, π
11	7	4	$(1, -2, -3, -4, 5, -6, -7)$	$(8, 9, 10, 11)$	$(77, 48, 375.33, 249.33)$	π_B
11	7	4	$(1, -2, -3, -4, 5, -6, -7)$	$(8, 9, 10, 11)$	$(77, 48, 375.33, 249.33)$	π
11	8	3	$(1, 2, -3, -4, -5, 6, -7, -8)$	$(9, 10, 11)$	$(96, 39, 498.67, 220)$	π_B, π
11	9	2	$(-1, 2, -3, 4, 5, 6, -7, -8, -9)$	$(10, 11)$	$(117, 28, 649, 170.67)$	π_B, π
11	10	1	$(-1, -2, 3, -4, 5, 6, 7, -8, -9, -10)$	11	$(140, 15, 823.33, 98.33)$	π_B, π

References

Chen, H. and A. S. Hedayat (1996). 2^{n-l} designs with weak minimum aberration. *Ann. Statist. 24*(6), 2536–2548.

Cheng, C. S. (1980). Orthogonal arrays with variable numbers of symbols. *Ann. Statist. 8*(2), 447–453.

Cheng, C.-S. (2014). Optimal biased weighing designs and two-level main-effect plans. *J. Stat. Theory Pract. 8*(1), 83–99.

Sun, C.-Y. and B. Tang (2022). Relationship between orthogonal and baseline parameterizations and its applications to design constructions. *Statist. Sinica 32*(1), 239–250.