Blind Source Separation over Space: an eigenanalysis approach

Bo Zhang, Sixing Hao and Qiwei Yao

University of Science and Technology of China and London School of Economics

Supplementary Material

Appendix A. Some useful lemmas

C_0 which is defined in Condition A1 and A which is defined in Condition A2 are two important notations in our proofs. Without loss of generality, we assume that $C_0 \le A$. It means that

$$\sup_{\beta \ge 1, 1 \le i \le p} \beta^{-1/2} \{E|Z_i(s)|^{\beta}\}^{1/\beta} \le A. \tag{A.1}$$

Thus, any fixed moment of $Z_g(s)$ can be bounded by a constant only depending on A.

Let Z be the $p \times n$ matrix with $(Z_i(s_1), \cdots, Z_i(s_n)) = Z^i$ as its i-th row.

Lemma A.1. *Let conditions A1 and A2 hold, and $p = o(n)$. Then there exists λ_{max} depending only on A such that*

$$\max_{1\le g\le p} \lambda_g \le \lambda_{max} < \infty. \tag{A.2}$$

Proof. For any $g = 1, \cdots, p$, (2.6) implies that

$$\lambda_g = \frac{1}{k}\sum_{h=1}^{k}\sum_{u=1}^{p} E[\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_u(s_j)]^2 \tag{A.3}$$

$$= \frac{1}{k}\sum_{h=1}^{k}\sum_{u\neq g} E[\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_u(s_j)]^2 + \frac{1}{k}\sum_{h=1}^{k} E[\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_g(s_j)]^2.$$

We consider the first part $u \neq g$ for each h,

$$\sum_{u\neq g} E[\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_u(s_j)]^2$$

$$= \sum_{u\neq g}\frac{1}{n^2}\sum_{i,j,\tilde{i},\tilde{j}=1}^{n} f_h(s_i - s_j)f_h(s_{\tilde{i}} - s_{\tilde{j}})E[\tilde{Z}_g(s_i)\tilde{Z}_u(s_j)\tilde{Z}_g(s_{\tilde{i}})\tilde{Z}_u(s_{\tilde{j}})]$$

$$= \sum_{u\neq g}\frac{1}{n^2}\sum_{i,j,\tilde{i},\tilde{j}=1}^{n} f_h(s_i - s_j)f_h(s_{\tilde{i}} - s_{\tilde{j}})E[\tilde{Z}_g(s_i)\tilde{Z}_g(s_{\tilde{i}})]E[\tilde{Z}_u(s_j)\tilde{Z}_u(s_{\tilde{j}})]$$

$$\le \sum_{u\neq g}\frac{1}{n^2}\sum_{i,j,\tilde{i},\tilde{j}=1}^{n} \frac{A}{1+\|s_i - s_j\|^{d+\alpha}}\frac{A}{1+\|s_{\tilde{i}} - s_{\tilde{j}}\|^{d+\alpha}}\frac{A}{1+\|s_i - s_{\tilde{i}}\|^{d+\alpha}}\frac{A}{1+\|s_j - s_{\tilde{j}}\|^{d+\alpha}}.$$

The last inequality is from (3.12) and (3.13). This, together with $p = o(n)$ and $\|s_i - s_j\| \ge \triangle$ for all $n \ge 2$ and $1 \le i \neq j \le n$, implies that

$$\frac{1}{k}\sum_{h=1}^{k}\sum_{u\neq g} E[\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_u(s_j)]^2 = O(A^4 n^{-1} p) = o(1). \tag{A.4}$$

Thus we only need to consider $E[\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_g(s_j)]^2$. Since

$Z^g = (Z_g(s_1), \cdots, Z_g(s_n))$ and

$$(\tilde{Z}_g(s_1), \cdots, \tilde{Z}_g(s_n)) = Z^g[I_n - n^{-1}1_{n\times n}]. \tag{A.5}$$

We can rewrite it as $E(\frac{1}{n}Z^g[I_n - n^{-1}1_{n\times n}]T_h[I_n - n^{-1}1_{n\times n}](Z^g)^\top)^2$, where T_h is a $n \times n$ matrix with the (i,j)th entry $f_h(s_i - s_j)/2 + f_h(s_j - s_i)/2$. Note that $\frac{1}{n}Z^g[I_n - n^{-1}1_{n\times n}]T_h[I_n - n^{-1}1_{n\times n}](Z^g)^\top$ is a quadratic form and $Z_g(s)$ is a sub-Gaussian process. (3.13) implies that $\|T_h\| \le \tilde{C}$, where $\tilde{C}$ only depends on A. These, together with (3.12), imply that there exists a positive constant $\tilde{C}_1$ depending only on A such that

$$\frac{1}{k}\sum_{h=1}^{k} E[\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_g(s_j)]^2 \le \tilde{C}_1.$$

This, together with (A.3)- (A.4), implies that $\lambda_g \le 2\tilde{C}_1$, for any $1 \le g \le p$. We complete the proof. $\square$

Lemma A.2. *Let conditions A1 and A2 hold. For any $n \times n$ non-random symmetric matrix Q with bounded $\|Q\|$, there exists a constant $C > 0$ depending only on A and λ_{max} for which*

$$\max_{1\le g,u\le p} var[\frac{1}{n}\sum_{i,j=1}^{n} Q_{ij}Z_g(s_i)Z_u(s_j)] \le C\|Q\|^2 n^{-1}. \tag{A.6}$$

Here Q_{ij} is the $(i,j)-th$ entry of Q.

Proof. When $g \ne u$, from the independence between $Z_g(s_i)$ and $Z_u(s_j)$ we have

$$
\begin{aligned}
& var[\frac{1}{n}\sum_{i,j=1}^{n} Q_{ij} Z_g(s_i) Z_u(s_j)] \\
= \quad & n^{-2} \sum_{i_1,j_1,i_2,j_2=1}^{n} Q_{i_1 j_1} Q_{i_2 j_2} E[Z_g(s_{i_1}) Z_u(s_{j_1}) Z_g(s_{i_2}) Z_u(s_{j_2})] \\
= \quad & n^{-2} \sum_{i_1,j_1,i_2,j_2=1}^{n} Q_{i_1 j_1} Q_{i_2 j_2} E[Z_g(s_{i_1}) Z_g(s_{i_2})] E[Z_u(s_{j_1}) Z_u(s_{j_2})] \\
\leq \quad & n^{-2} \sum_{i_1,j_1,i_2,j_2=1}^{n} Q_{i_1 j_1} Q_{i_2 j_2} \frac{A}{1+\|s_{i_1}-s_{i_2}\|^{d+\alpha}} \frac{A}{1+\|s_{j_1}-s_{j_2}\|^{d+\alpha}} \\
\leq \quad & C\|Q\|^2 n^{-1}.
\end{aligned}
$$

The first inequality is from (3.12) and (3.13). The second inequality is from $\|s_i - s_j\| \geq \triangle$ for all $n \geq 2$ and $1 \leq i \neq j \leq n$. When $g = u$, we note that $\frac{1}{n}\sum_{i,j=1}^{n} Q_{ij} Z_g(s_i) Z_g(s_j)$ is a quadratic form and $Z_g(s)$ is a sub-Gaussian process. This completes the proof. □

Lemma A.3. *Let conditions A1 and A2 hold, and $p = o(n)$. Then there exists a positive constant C_A depending only on A such that*

$$
\lim_{n\to\infty} P(n^{-1}\|Z\|^2 \leq C_A) = 1. \tag{A.7}
$$

Proof. For any fixed $1 \times n$ unit vector $x = (x_1, \cdots, x_n)$, we denote $xZ^\top$ by $z(x) = \Big(z_1(x), \cdots, z_p(x)\Big)$. Since $Z_1(\cdot), \cdots, Z_p(\cdot)$ are independent, the elements of $z(x)$ are independent. (3.12) implies that $\max_{1\leq j\leq p} E z_j^2(x) \leq$

$\tilde{C}_A$ where $\tilde{C}_A$ only depends on A.

$$xZ^\top Zx^\top = \sum_{j=1}^{p}[z_j^2(x) - Ez_j^2(x)] + \sum_{j=1}^{p} Ez_j^2(x) \le \sum_{j=1}^{p}[z_j^2(x) - Ez_j^2(x)] + p\tilde{C}_A.$$

By the sub-Gaussian property of $Z(s)$, we can conclude that for any fixed $1 \times p$ unit vector x and any $c > 0$ there exists $\tilde{C}_{A,1}$ depending only on A and c such that

$$P\Big(\|xZ^\top\|^2 > \tilde{C}_{A,1}(n+p)\Big) \le c\exp(-5(n+p)). \tag{A.8}$$

As we know, the unit Euclidean sphere S^{n-1} consists of all n-dimensional unit vectors x. Unfortunately the cardinality of S^{n-1} is uncountable cardinal number. We can't use (A.8) to derive an upper bound of $\|Z\|^2$ directly. Thus we introduce a method based on nets to control $\|Z\|^2$. The basic idea is as follows. We define a subset of S^{n-1} as S_ε satisfying $\max_{x\in S^{n-1}} \min_{y\in S_\varepsilon} \|x-y\| \le \varepsilon$. S_ε is a so-called net of S^{n-1} and the cardinality of S_ε is bounded by $(1+2\varepsilon^{-1})^n$. Thus we can control $\max_{y\in S_\varepsilon} \|Zy^\top\|$ in probability by (A.8). Finally, we can control the difference between $\max_{y\in S_\varepsilon} \|Zy^\top\|$ and $\max_{x\in S^{n-1}} \|Zx^\top\|$.

Let S_ε be a subset of S^{n-1}. For any $x \in S^{n-1}$, there exists $\tilde{x} \in S_\varepsilon$ such that $\|\tilde{x} - x\| \le \varepsilon$. This, together with (A.8) and $|S_\varepsilon| \le (1+2\varepsilon^{-1})^n$, implies that

$$P\Big(\max_{\tilde{x}\in S_{1/2}} \|Z\tilde{x}^\top\|^2 > \tilde{C}_{A,1}(n+p)\Big) \le c|S_{1/2}|\exp(-5n-5p) \le c5^n\exp(-5n-5p). \tag{A.9}$$

Then if $\|Zx^\top\| = \|Z\|$, there exists $\tilde{x} \in S_\varepsilon$ such that

$$\|Z\tilde{x}^\top\| \geq \|Zx^\top\| - \|Z(\tilde{x} - x)^\top\| \geq \|Z\| - \varepsilon\|Z\| = (1 - \varepsilon)\|Z\|.$$

Let $\varepsilon = 1/2$,

$$\|Z\|^2 \leq 4 \max_{\tilde{x} \in S_{1/2}} \|Z\tilde{x}^\top\|^2.$$

This, together with (A.9), implies that

$$P\Big(\|Z\|^2 > 4\tilde{C}_{A,1}(n+p)\Big) \leq c|S_{1/2}| \exp(-5n - 5p) \leq c5^n \exp(-5n - 5p). \quad \text{(A.10)}$$

Then (A.7) is implied by (A.10) and $p = o(n)$. □

Definition 1.

$$\hat{N} = \frac{1}{k}\sum_{h=1}^{k}\big\{\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}(s_i)\tilde{Z}(s_j)^\top\big\}\big\{\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}(s_i)\tilde{Z}(s_j)^\top\big\}^\top. \quad \text{(A.11)}$$

Lemma A.4. *Let conditions A1 and A2 hold, and $p = o(n)$. Let M_{gu} be the (g,u)-th entry of $\hat{N} - N$. There exists a positive constant C_1 depending only on A such that*

$$\max_{1 \leq g,u \leq p} EM_{gu}^2 \leq C_1 n^{-1}. \quad \text{(A.12)}$$

Proof. Since N is diagonal, when $g \neq u$,

$$M_{gu} = \frac{1}{k}\sum_{h=1}^{k}\sum_{\tilde{u}=1}^{p}[\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_{\tilde{u}}(s_j)][\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_u(s_i)\tilde{Z}_{\tilde{u}}(s_j)].$$

Divide the term on the RHS of the above equation into three terms: (i)$\tilde{u} = g$, (ii)$\tilde{u} = u$ and (iii) $\tilde{u} \neq g, u$. We control each term as follows. When $\tilde{u} = g$,

$$E\Big([\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_g(s_j)][\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_u(s_i)\tilde{Z}_g(s_j)]\Big) = 0.$$

$$\begin{aligned}
& var\Big([\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_g(s_j)][\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_u(s_i)\tilde{Z}_g(s_j)]\Big) \\
= \quad & E\Big([\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_g(s_j)][\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_u(s_i)\tilde{Z}_g(s_j)]\Big)^2 \\
= \quad & E\Big(n^{-4}\sum_{i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4=1}^{n} f_h(s_{i_1} - s_{j_1})f_h(s_{i_2} - s_{j_2})f_h(s_{i_3} - s_{j_3})f_h(s_{i_4} - s_{j_4}) \\
& \tilde{Z}_g(s_{i_1})\tilde{Z}_g(s_{i_1})\tilde{Z}_g(s_{i_1})\tilde{Z}_g(s_{i_1})\tilde{Z}_g(s_{j_1})\tilde{Z}_g(s_{j_3})\tilde{Z}_u(s_{j_2})\tilde{Z}_u(s_{j_4})\Big) \\
\leq \quad & n^{-4}\sum_{i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4=1}^{n}[\prod_{v=1}^{4}\frac{A}{1+\|s_{i_v} - s_{j_v}\|^{d+\alpha}}]\frac{A}{1+\|s_{j_2} - s_{j_4}\|^{d+\alpha}}EZ_g^6(s) \\
\leq \quad & \tilde{C}_1 n^{-1},
\end{aligned}$$

where $\tilde{C}_1$ only depends on A. The first inequality is from (3.12)-(3.13) and the independence between $Z_g(\cdot)$ and $Z_u(\cdot)$. The second inequality is from (3.11), $C_0 \leq A$ and $\|s_i - s_j\| \geq \triangle$ for all $n \geq 2$ and $1 \leq i \neq j \leq n$. Thus we can control

$$(\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_g(s_j))(\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_u(s_i)\tilde{Z}_g(s_j)).$$

When $\tilde{u} = u$, we can repeat the above method to control

$$(\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_g(s_i)\tilde{Z}_u(s_j))(\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}_u(s_i)\tilde{Z}_u(s_j)).$$

Let's consider the third term

$$\sum_{\tilde{u}\neq g,u}(\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i-s_j)\tilde{Z}_g(s_i)\tilde{Z}_{\tilde{u}}(s_j))(\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i-s_j)\tilde{Z}_u(s_i)\tilde{Z}_{\tilde{u}}(s_j)).$$

We can rewrite it as

$$\begin{aligned}
&\frac{1}{n^2}\sum_{\tilde{u}\neq g,u}\sum_{i,j,\tilde{i},\tilde{j}=1}^{n} f_h(s_i-s_j)f_h(s_{\tilde{i}}-s_{\tilde{j}})\tilde{Z}_g(s_i)\tilde{Z}_u(s_{\tilde{i}})\tilde{Z}_{\tilde{u}}(s_j)\tilde{Z}_{\tilde{u}}(s_{\tilde{j}})\\
=\quad&\frac{1}{n}\sum_{i,\tilde{i}=1}^{n}\Big(\frac{1}{n}\sum_{j,\tilde{j}=1}^{n} f_h(s_i-s_j)f_h(s_{\tilde{i}}-s_{\tilde{j}})\sum_{\tilde{u}\neq g,u}\tilde{Z}_{\tilde{u}}(s_j)\tilde{Z}_{\tilde{u}}(s_{\tilde{j}})\Big)\tilde{Z}_g(s_i)\tilde{Z}_u(s_{\tilde{i}}).
\end{aligned}$$

Let $\tilde{H}$ be a $n\times n$ symmetric matrix with $(i,\tilde{i})$th entry

$$\frac{1}{n}\sum_{j,\tilde{j}=1}^{n} f_h(s_i-s_j)f_h(s_{\tilde{i}}-s_{\tilde{j}})\sum_{\tilde{u}\neq g,u}\tilde{Z}_{\tilde{u}}(s_j)\tilde{Z}_{\tilde{u}}(s_{\tilde{j}}).$$

Recalling (A.5) and (A.6), we define $Q=(I_n-n^{-1}1_{n\times n})\tilde{H}(I_n-n^{-1}1_{n\times n})$. Although Q is random, we can find that Q is independent of $Z_g(s)$ and $Z_u(s)$. It's easy to see

$$E\frac{1}{n}\sum_{i,j=1}^{n} Q_{i,j}Z_g(s_i)Z_u(s_j)=0.$$

$$\begin{aligned}
&var[\frac{1}{n}\sum_{i,j=1}^{n} Q_{i,j}Z_g(s_i)Z_u(s_j)]=E[\frac{1}{n}\sum_{i,j=1}^{n} Q_{i,j}Z_g(s_i)Z_u(s_j)]^2\\
=\quad&\frac{1}{n^2}\sum_{i,j,\tilde{i},\tilde{j}=1}^{n} E(Q_{i,j}Q_{\tilde{i},\tilde{j}})E[Z_g(s_i)Z_g(s_{\tilde{i}})]E[Z_u(s_j)Z_u(s_{\tilde{j}})]\\
\leq\quad&\frac{1}{n^2}\sum_{i,j,\tilde{i},\tilde{j}=1}^{n}(EQ_{i,j}^2)^{1/2}(EQ_{\tilde{i},\tilde{j}}^2)^{1/2}\frac{A}{1+(s_i-s_{\tilde{i}})^{d+\alpha}}\frac{A}{1+(s_j-s_{\tilde{j}})^{d+\alpha}}\\
\leq\quad&\frac{\tilde{C}_2}{n^2}\sum_{i,j=1}^{n} EQ_{i,j}^2=\frac{\tilde{C}_2}{n^2}E\|Q\|_F^2,
\end{aligned}$$

where $\tilde{C}_2$ only depends on A and the first inequality is from (3.12). The second inequality is from $\|s_i - s_j\| \geq \triangle$ for all $n \geq 2$ and $1 \leq i \neq j \leq n$. Recalling the definition of Q, we can rewrite it as

$$Q = \frac{1}{n}(I_n - n^{-1}1_{n\times n})V_h(I_n - n^{-1}1_{n\times n})Z_{-g,-u}^\top Z_{-g,-u}(I_n - n^{-1}1_{n\times n})V_h^\top(I_n - n^{-1}1_{n\times n}),$$

where V_h has the (i,j)th entry $f_h(s_i - s_j)$ and $Z_{-g,-u}$ is a $(p-2)\times n$ matrix without Z^g and Z^u. Then

$$\|Q\|_F^2 \leq \|V_h\|^4 \|\frac{1}{n}Z_{-g,-u}^\top Z_{-g,-u}\|_F^2 \leq \tilde{C}_3 \|\frac{1}{n}Z^\top Z\|_F^2,$$

where $\tilde{C}_3$ only depends on A and the last inequality is from (3.13). Moreover,

$$\begin{aligned}
& E\|\frac{1}{n}Z^\top Z\|_F^2 = E\|\frac{1}{n}ZZ^\top\|_F^2 \\
= \quad & E\sum_{g,u=1}^{p}[n^{-1}\sum_{i=1}^{n} Z_g(s_i)Z_u(s_i)]^2 \\
= \quad & E\sum_{1\leq g\neq u\leq p}[n^{-1}\sum_{i=1}^{n} Z_g(s_i)Z_u(s_i)]^2 + E\sum_{g=1}^{p}[n^{-1}\sum_{i=1}^{n} Z_g^2(s_i)]^2 \\
= \quad & \sum_{1\leq g\neq u\leq p} n^{-2}\sum_{i,j=1}^{n} E[Z_g(s_i)Z_g(s_j)]E[Z_u(s_i)Z_u(s_j)] + \sum_{g=1}^{p} n^{-2}\sum_{i,j=1}^{n} E[Z_g^2(s_i)Z_g^2(s_j)] \\
\leq \quad & \sum_{1\leq g\neq u\leq p} n^{-2}\sum_{i,j=1}^{n}(\frac{A}{1+\|s_i - s_j\|^{d+\alpha}})^2 + \sum_{g=1}^{p} EZ_g^4(s) \\
\leq \quad & \tilde{C}_4 p,
\end{aligned}$$

where $\tilde{C}_4$ only depends on A. The first inequality is from (3.12). The second equation is from (3.11), $C_0 \le A$, $p = o(n)$ and $\|s_i - s_j\| \ge \triangle$ for all $n \ge 2$ and $1 \le i \ne j \le n$. Then we can conclude that

$$E\|Q\|_F^2 \le \tilde{C}_5 p,$$

where $\tilde{C}_5$ only depends on A. From $p = o(n)$,

$$var[\frac{1}{n}\sum_{i,j=1}^{n} Q_{i,j} Z_g(s_i) Z_u(s_j)] \le \frac{\tilde{C}_2\tilde{C}_5}{n^2} p = o(n^{-1}).$$

Thus we control the third term and prove (A.12) for $g \ne u$. When $g = u$, the proof is similar. □

Definition 2. Let J_1 and J_2 be two subsets of $\{1, \cdots, p\}$. Let $\hat{N}_{J_1,J_2}$ be the sub-matrix of $\hat{N}$ consisting of the rows with the indices in J_1 and the columns with the indices in J_2. Write $\hat{N}_{J_1} = \hat{N}_{J_1,J_1}$.

Lemma A.5. *Under the conditions of Lemma A.3 and $J_1 \cap J_2 = \emptyset$, we define the event $B_Z = \{n^{-1}\|Z\|^2 \le C_A\}$. Then there exists a positive constant C_2 depending only on A, c and v such that*

$$P\Big(\|\hat{N}_{J_1,J_2}\|^2 > C_2 n^{-1} v(|J_1|+|J_2|)\Big| B_Z\Big) \le c(5^{|J_1|}+5^{|J_2|})\exp(-5|J_1|v-5|J_2|v). \tag{A.13}$$

Here $v > 0$ can be finite or tending to infinite.

Proof. Since k is finite, it's sufficient to prove (A.13) on

$$n^{-2} Z_{J_1}(I_n - n^{-1}1_{n\times n})V_h(I_n - n^{-1}1_{n\times n})Z^\top Z(I_n - n^{-1}1_{n\times n})V_h^\top(I_n - n^{-1}1_{n\times n})Z_{J_2}^\top,$$

where Z_{J_1} is a sub-matrix of Z with ith row if and only if $i \in J_1$. V_h is a $n \times n$ matrix with the (i,j)th entry $f_h(s_i - s_j)$. We define $\tilde{V}_h = (I_n - n^{-1}1_{n\times n})V_h(I_n - n^{-1}1_{n\times n})$.

$$\begin{aligned}
& Z_{J_1}\tilde{V}_h Z^\top Z \tilde{V}_h^\top Z_{J_2}^\top \\
= \quad & Z_{J_1}\tilde{V}_h Z_{J_1}^\top Z_{J_1}\tilde{V}_h^\top Z_{J_2}^\top + Z_{J_1}\tilde{V}_h Z_{J_2}^\top Z_{J_2}\tilde{V}_h^\top Z_{J_2}^\top \\
+ \quad & Z_{J_1}\tilde{V}_h Z_{J}^\top Z_{J}\tilde{V}_h^\top Z_{J_2}^\top, \qquad \text{(A.14)}
\end{aligned}$$

where J is the complementary set of $J_1 \cup J_2$. At first we deal with $Z_{J_1}\tilde{V}_h Z_{J_1}^\top Z_{J_1}\tilde{V}_h^\top Z_{J_2}^\top$.

$$\begin{aligned}
& \|Z_{J_1}\tilde{V}_h Z_{J_1}^\top Z_{J_1}\tilde{V}_h^\top Z_{J_2}^\top\|^2 \\
= \quad & \|Z_{J_2}\tilde{V}_h Z_{J_1}^\top Z_{J_1}\tilde{V}_h^\top Z_{J_1}^\top Z_{J_1}\tilde{V}_h Z_{J_1}^\top Z_{J_1}\tilde{V}_h^\top Z_{J_2}^\top\| \\
= \quad & \|Z_{J_2} H_{h,J_1} Z_{J_2}^\top\|,
\end{aligned}$$

where

$$H_{h,J_1} = \tilde{V}_h Z_{J_1}^\top Z_{J_1}\tilde{V}_h^\top Z_{J_1}^\top Z_{J_1}\tilde{V}_h Z_{J_1}^\top Z_{J_1}\tilde{V}_h^\top$$

is a $n \times n$ symmetric matrix with rank $|J_1|$ at most. Since $J_1 \cap J_2 = \emptyset$, H_{h,J_1} and $Z_{J_2}^\top$ are independent. Moreover, under the event $B_Z = \{n^{-1}\|Z\|^2 \le C_A\}$,

$$\|H_{h,J_1}\| \le \|\tilde{V}_h\|^4 \|Z_{J_1}^\top Z_{J_1}\|^3 \le \|V_h\|^4 \|Z^\top Z\|^3 \le \|V_h\|^4 n^3 C_A^3.$$

It follows that

$$\lim_{n\to\infty} P(\|H_{h,J_1}\| \le n^3 \tilde{C}_A | B_Z) = 1, \qquad \text{(A.15)}$$

where $\tilde{C}_A$ only depends on A. Now we recall the rank of H_{h,J_1} is not larger than $|J_1|$. For given H_{h,J_1}, we can do eigen-decomposition on it as follows.

$$H_{h,J_1} = U_{h,J_1}\Lambda_{h,J_1}U_{h,J_1}^\top, \tag{A.16}$$

where U_{h,J_1} is a $n \times |J_1|$ matrix and Λ_{h,J_1} is a $|J_1| \times |J_1|$ diagonal matrix. $U_{h,J_1}^\top U_{h,J_1} = I_{|J_1|}$. Then

$$\|Z_{J_1}\tilde{V}_h Z_{J_1}^\top Z_{J_1}\tilde{V}_h^\top Z_{J_2}^\top\|^2 \le \|Z_{J_2}U_{h,J_1}\|^2\|\Lambda_{h,J_1}\|.$$

Since $\|\Lambda_{h,J_1}\|$ can be controlled by (A.15), we only need to consider $\|Z_{J_2}U_{h,J_1}\|^2$. Let $Y = Z_{J_2}U_{h,J_1}$ be a $|J_2| \times |J_1|$ matrix with the (i,j)th entry Y_{ij}. The independence between the rows of Z_{J_2} implies the independence between the rows of Y. For any fixed $1 \times |J_1|$ unit vector $x = (x_1, \cdots, x_{|J_1|})$, we define $xY^\top$ as $Y(x) = (y_1(x), \cdots, y_{|J_2|}(x))$. Then the elements of $Y(x)$ are independent.

$$xY^\top Yx^\top = \sum_{j=1}^{|J_2|}[y_j^2(x) - Ey_j^2(x)] + \sum_{j=1}^{|J_2|} Ey_j^2(x).$$

$Yx^\top = Z_{J_2}U_{h,J_1}x^\top$ and $U_{h,J_1}x^\top$ is an unit vector independent of Z_{J_2}. By the sub-Gaussian property of $Z(s)$, we have

$$xY^\top Yx^\top \le \sum_{j=1}^{|J_2|}[y_j^2(x) - Ey_j^2(x)] + |J_2|\tilde{C}_{A,2},$$

where $\tilde{C}_{A,2}$ only depends on A. Moreover, we can also deal with $\sum_{j=1}^{|J_2|}[y_j^2(x) - Ey_j^2(x)]$ with the sub-Gaussian property of $Z(s)$. Thus, for any fixed $1\times|J_1|$

unit vector x, any $c > 0$ and $v > 0$, there exists $C_{A,3}$ depending only on A, c and v such that

$$P\Big(\|xY^\top\|^2 > C_{A,3}v(|J_1| + |J_2|)\Big| B_Z\Big) \leq c\exp(-5|J_1|v - 5|J_2|v). \quad \text{(A.17)}$$

As we know, the unit Euclidean sphere $S^{|J_1|-1}$ consists of all $|J_1|$-dimensional unit vectors x. Unfortunately, the cardinality of $S^{|J_1|-1}$ are uncountable cardinal number. We can't use (A.17) to conclude the upper bound of $\|Y\|^2$ directly. Thus we use the method based on Nets to control $\|Y\|^2$. Let S_ε be a subset of $S^{|J_1|-1}$. For any $x \in S^{|J_1|-1}$, there exists $\tilde{x} \in S_\varepsilon$ such that $\|\tilde{x} - x\| \leq \varepsilon$. Then if $\|Yx^\top\| = \|Y\|$, there exists $\tilde{x} \in S_\varepsilon$ such that

$$\|Y\tilde{x}^\top\| \geq \|Yx^\top\| - \|Y(\tilde{x} - x)^\top\| \geq \|Y\| - \varepsilon\|Y\| = (1 - \varepsilon)\|Y\|.$$

Let $\varepsilon = 1/2$,

$$\|Y\|^2 \leq 4 \max_{\tilde{x} \in S_{1/2}} \|Y\tilde{x}^\top\|^2.$$

This, together with (A.17) and $|S_\varepsilon| \leq (1 + 2\varepsilon^{-1})^{|J_1|}$, implies that

$$P\Big(\|Y\|^2 > 4C_{A,3}v(|J_1| + |J_2|)\Big| B_Z\Big) \leq c5^{|J_1|}\exp(-5|J_1|v - 5|J_2|v). \quad \text{(A.18)}$$

Recalling (A.15), one can conclude that for any $c > 0$, there exists $C_{A,4}$ only depending on A and c such that

$$P\Big(\|n^{-2}Z_{J_1}\tilde{V}_hZ_{J_1}^\top Z_{J_1}\tilde{V}_h^\top Z_{J_2}^\top\|^2 > 4C_{A,4}vn^{-1}(|J_1| + |J_2|)\Big| B_Z\Big)$$
$$\leq c5^{|J_1|}\exp(-5|J_1|v - 5|J_2|v). \quad \text{(A.19)}$$

Others term in (A.14) can be controlled by the same method. This completes the proof. □

Lemma A.6. *Under conditions A1-A3 and $p = o(n)$,*

$$\|\hat{N}_{J_i} - \Lambda_i\| = O_p(n^{-1/2} q_i^{1/2}), \tag{A.20}$$

where $J_i = \{j \in \mathcal{Z} : p_{i-1} < j \leq p_i\}$, $\Lambda_i = \mathrm{diag}(\lambda_{p_{i-1}+1}, \cdots, \lambda_{p_i})$, and λ_i are specified in Condition A3.

Proof. We divide $\hat{N}_{J_i}$ into two terms: (i) the diagonal term $\hat{N}_{J_i,d}$ and (ii) the off-diagonal term $\hat{N}_{J_i,o}$. Lemma A.4 ensures $\|\hat{N}_{J_i,d} - \Lambda_i\| = O_p(n^{-1/2} q_i^{1/2})$. Thus we only need to show $\|\hat{N}_{J_i,o}\| = O_p(n^{-1/2} q_i^{1/2})$. If q_i is finite, Lemma A.4 can also ensure it. So we only need to consider the case q_i tends to infinity.

We can rewrite $\hat{N}_{J_i,o}$ and control $\|\hat{N}_{J_i,o}\|$ with the following idea.

$$\hat{N}_{J_i,o} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} V_{11} & 0 \\ 0 & V_{22} \end{pmatrix} + \begin{pmatrix} 0 & V_{12} \\ V_{21} & 0 \end{pmatrix} = D_1 + V_{o,1}.$$

Each block is a $q_i/2 \times q_i/2$ matrix. Note that $V_{12} = V_{21}^\top$ and the norm of the second term $V_{o,1}$ (off-diagonal block) can be controlled by $\|V_{12}\|$. Moreover, we can control $\|V_{12}\|$ by Lemmas A.3 and A.5. In details, Lemma A.5 implies that

$$P\Big(\|V_{o,1}\|^2 > C_2 v n^{-1} q_i \Big| B_Z\Big) \leq c(5^{q_i/2} + 5^{q_i/2}) \exp(-5 q_i v). \tag{A.21}$$

For the first term, we can repeat the step on V_{11} and V_{22} to get a new matrix with off-diagonal blocks as follows:

$$V_{o,2} = diag\Big[\begin{pmatrix} 0 & V_{11,12} \\ V_{11,21} & 0 \end{pmatrix}, \begin{pmatrix} 0 & V_{22,12} \\ V_{22,21} & 0 \end{pmatrix}\Big].$$

Lemma A.5 implies that

$$P\Big(\|V_{o,2}\|^2 > C_2 v n^{-1} q_i/2 \Big| B_Z\Big) \leq 2c(5^{q_i/4} + 5^{q_i/4}) \exp(-5q_i v/2). \quad \text{(A.22)}$$

Repeat the steps, we can find that $V_{o,j}$ has 2^{j-1} diagonal blocks and each diagonal block has two $2^{-j}q_i \times 2^{-j}q_i$ off-diagonal blocks. Lemma A.5 implies that

$$P\Big(\|V_{o,j}\|^2 > 2^{1-j} C_2 v n^{-1} q_i \Big| B_Z\Big) \leq 2^{j-1} c(5^{2^{-j}q_i} + 5^{2^{-j}q_i}) \exp(-5q_i v \times 2^{1-j}). \quad \text{(A.23)}$$

We divide it into j_0 matrices: $\hat{N}_{J_i,o} = \sum_{j=1}^{j_0} V_{o,j}$, $2^{j_0-1} \leq q_i$ and $j_0 = O(\log q_i)$. For different j, we choose different v to control (A.23). When $\log q_i = o(2^{1-j}q_i)$, we choose $v = 1$. It follows that

$$P\Big(\|V_{o,j}\|^2 > 2^{1-j} C_2 n^{-1} q_i \Big| B_Z\Big) \leq 2^{j-1} c(5^{2^{-j}q_i} + 5^{2^{-j}q_i}) \exp(-5q_i \times 2^{1-j}) = o(\log^{-1} q_i). \quad \text{(A.24)}$$

Otherwise, we choose $v = q_i^{4/5} \log^{-1} q_i$. It follows that

$$\begin{aligned} & P\Big(\|V_{o,j}\|^2 > C_2 n^{-1} q_i \log^{-2} q_i \Big| B_Z\Big) \quad (A.25) \\ \leq \quad & P\Big(\|V_{o,j}\|^2 > 2^{1-j} C_2 q_i^{4/5} n^{-1} q_i \log^{-1} q_i \Big| B_Z\Big) \\ \leq \quad & 2^{j-1} c(5^{2^{-j} q_i} + 5^{2^{-j} q_i}) \exp(-5 q_i^{9/5} \log^{-1} q_i \times 2^{1-j}) = o(\log^{-1} q_i). \end{aligned}$$

(A.24)-(A.25) and $\|\hat{N}_{J_i,o}\| \leq \sum_{j=1}^{j_0} \|V_{o,j}\|$ imply that

$$P\Big(\|\hat{N}_{J_i,o}\| > 5 C_2^{1/2} n^{-1/2} q_i^{1/2} \Big| B_Z\Big) = o(1). \quad (A.26)$$

Lemma A.3 implies that $\lim_{n\to\infty} P(B_Z) = 1$. This, together with (A.26) and $\|\hat{N}_{J_i,d} - \Lambda_i\| = O_p(n^{-1/2} q_i^{1/2})$, completes the proof. □

Lemma A.7. *Under conditions A1-A2 and $p = o(n)$,*

$$\|\Omega^\top \hat{\Sigma}^{-1} \Omega - I_p\| = O_p(n^{-1/2} p^{1/2}). \quad (A.27)$$

Proof. Since $\tilde{X}(s_j) = \Omega \tilde{Z}(s_j)$,

$$\begin{aligned} \Omega^\top \hat{\Sigma}^{-1} \Omega - I_p = \quad & \Omega^\top [n^{-1} \sum_{1\leq j\leq n} \tilde{X}(s_j) \tilde{X}(s_j)^\top]^{-1} \Omega - I_p \\ = \quad & [n^{-1} \sum_{1\leq j\leq n} \tilde{Z}(s_j) \tilde{Z}(s_j)^\top]^{-1} - I_p. \end{aligned}$$

It suffices to prove

$$\|n^{-1} \sum_{1\leq j\leq n} \tilde{Z}(s_j) \tilde{Z}(s_j)^\top - I_p\| = O_p(n^{-1/2} p^{1/2}).$$

Following the proof of Lemma A.6, one can verify the above equation. □

Appendix B. Proofs of Theorems

Recalling (A.11), write $\hat{N} = \hat{\Gamma}\hat{\Lambda}\hat{\Gamma}^\top$ as its spectral decomposition, i.e.

$$\hat{\Lambda} = \mathrm{diag}(\hat{\lambda}_1, \cdots, \hat{\lambda}_p),$$

where $\hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_p \geq 0$ are the eigenvalues of $\hat{N}$, and the columns of the orthogonal matrix $\hat{\Gamma}$ are the corresponding eigenvectors. Recalling the definition of $\hat{W}$ in (2.9)-(2.10), we can find that

$$\begin{aligned}
\hat{W} = \quad & \frac{1}{k}\sum_{h=1}^{k} \hat{M}(f_h)\hat{M}(f_h)^\top \\
= \quad & \frac{1}{k}\sum_{h=1}^{k} \big\{\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\hat{\Sigma}^{-1/2}\tilde{X}(s_i)\tilde{X}(s_j)^\top\hat{\Sigma}^{-1/2}\big\} \\
& \big\{\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\hat{\Sigma}^{-1/2}\tilde{X}(s_i)\tilde{X}(s_j)^\top\hat{\Sigma}^{-1/2}\big\}^\top \\
= \quad & \frac{1}{k}\hat{\Sigma}^{-1/2}\Omega\sum_{h=1}^{k} \big\{\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}(s_i)\tilde{Z}(s_j)^\top\big\}\Omega^\top\hat{\Sigma}^{-1}\Omega \\
& \big\{\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}(s_i)\tilde{Z}(s_j)^\top\big\}^\top\Omega^\top\hat{\Sigma}^{-1/2} \\
= \quad & \frac{1}{k}\hat{\Sigma}^{-1/2}\Omega\sum_{h=1}^{k} \big\{\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}(s_i)\tilde{Z}(s_j)^\top\big\}(\Omega^\top\hat{\Sigma}^{-1}\Omega - I_p) \\
& \big\{\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i - s_j)\tilde{Z}(s_i)\tilde{Z}(s_j)^\top\big\}^\top\Omega^\top\hat{\Sigma}^{-1/2} + \hat{\Sigma}^{-1/2}\Omega\hat{N}\Omega^\top\hat{\Sigma}^{-1/2}.
\end{aligned}$$

Let $\hat\Sigma^{-1/2}\Omega = \hat V_\Omega \hat\Lambda_\Omega \hat U_\Omega$ where $\hat V_\Omega \hat V_\Omega^\top = \hat U_\Omega \hat U_\Omega^\top = I_p$ and $\hat\Lambda_\Omega$ is a diagonal matrix. Then

$$
\begin{aligned}
\hat W = \quad & \hat V_\Omega \hat U_\Omega \hat\Gamma\hat\Lambda\hat\Gamma^\top \hat U_\Omega^\top \hat V_\Omega^\top + \frac{1}{k}\hat\Sigma^{-1/2}\Omega \sum_{h=1}^{k}\Big\{\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i-s_j)\tilde Z(s_i)\tilde Z(s_j)^\top\Big\} \\
& \hat U_\Omega^\top(\hat\Lambda_\Omega^2 - I_p)\hat U_\Omega \Big\{\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i-s_j)\tilde Z(s_i)\tilde Z(s_j)^\top\Big\}^\top \Omega^\top\hat\Sigma^{-1/2} \\
& + \hat V_\Omega(\hat\Lambda_\Omega - I_p)\hat U_\Omega\hat\Gamma\hat\Lambda\hat\Gamma^\top\hat U_\Omega^\top\hat V_\Omega^\top + \hat V_\Omega\hat\Lambda_\Omega\hat U_\Omega\hat\Gamma\hat\Lambda\hat\Gamma^\top\hat U_\Omega^\top(\hat\Lambda_\Omega - I_p)\hat V_\Omega^\top.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\hat U_\Omega^\top \hat V_\Omega^\top \hat W \hat V_\Omega \hat U_\Omega = \quad & \hat\Gamma\hat\Lambda\hat\Gamma^\top + \frac{1}{k}\hat U_\Omega^\top\hat\Lambda_\Omega\hat U_\Omega \sum_{h=1}^{k}\Big\{\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i-s_j)\tilde Z(s_i)\tilde Z(s_j)^\top\Big\} \\
& \hat U_\Omega^\top(\hat\Lambda_\Omega^2 - I_p)\hat U_\Omega\Big\{\frac{1}{n}\sum_{i,j=1}^{n} f_h(s_i-s_j)\tilde Z(s_i)\tilde Z(s_j)^\top\Big\}^\top \hat U_\Omega^\top\hat\Lambda_\Omega\hat U_\Omega \\
& + \hat U_\Omega^\top(\hat\Lambda_\Omega - I_p)\hat U_\Omega\hat\Gamma\hat\Lambda\hat\Gamma^\top + \hat U_\Omega^\top\hat\Lambda_\Omega\hat U_\Omega\hat\Gamma\hat\Lambda\hat\Gamma^\top\hat U_\Omega^\top(\hat\Lambda_\Omega - I_p)\hat U_\Omega.
\end{aligned}
$$

Then

$$
\|\hat U_\Omega^\top \hat V_\Omega^\top \hat W \hat V_\Omega \hat U_\Omega - \hat\Gamma\hat\Lambda\hat\Gamma^\top\| = O\{\|\hat\Lambda_\Omega - I_p\|\|\hat\Lambda\|(1+\|\hat\Lambda_\Omega\|)^3\}. \qquad \text{(B.28)}
$$

(A.27) implies that $\|\hat\Lambda_\Omega - I_p\| = O_p(n^{-1/2}p^{1/2})$ and $\|\hat\Lambda_\Omega\| = O_p(1)$.

Recalling $\hat\Sigma^{-1/2}\Omega = \hat V_\Omega\hat\Lambda_\Omega\hat U_\Omega$,

$$
\|\hat U_W^\top\hat\Sigma^{-1/2}\Omega - \hat U_W^\top\hat V_\Omega\hat U_\Omega\| \le \|\hat U_W^\top\hat V_\Omega^\top(\hat\Lambda_\Omega - I_p)U_\Omega\| = O_p(n^{-1/2}p^{1/2}). \text{(B.29)}
$$

(B.29) implies that the leading term of $\hat\Gamma_\Omega = \hat U_W^\top\hat\Sigma^{-1/2}\Omega$ is $\hat U_W^\top\hat V_\Omega\hat U_\Omega$. (B.28) implies that $\hat U_W^\top\hat\Sigma^{-1/2}\Omega$ is close to $\hat\Gamma^\top$.

Thus, the asymptotic properties of $\hat{\Gamma}^\top$ is the key point. We will prove the following theorem for $\hat{\Gamma}$ and $\hat{\Lambda}$.

Put $q_i = p_i - p_{i-1}$ for $i = 1, \cdots, m$ (see Condition A3), and

$$\hat{\Gamma} = \begin{pmatrix} \hat{\Gamma}_{11} & \cdots & \hat{\Gamma}_{1m} \\ \dots & \dots & \dots \\ \hat{\Gamma}_{m1} & \cdots & \hat{\Gamma}_{mm} \end{pmatrix}, \qquad \hat{\Lambda} = \mathrm{diag}(\hat{\Lambda}_1, \cdots, \hat{\Lambda}_m), \tag{B.30}$$

where submatrix $\hat{\Gamma}_{ij}$ is of the size $q_i \times q_j$, and $\hat{\Lambda}_i$ is a $q_i \times q_i$ diagonal matrix.

Theorem B.1. *Let Conditions A1-A3 hold. As $n \to \infty$ and $p = o(n)$, it holds that*

$$\|\hat{\Gamma}_{ij}\| = O_p\{n^{-1/2}(q_i + q_j)^{1/2} + n^{-1}p\}, \quad 1 \le i \ne j \le m, \quad \text{and} \tag{B.31}$$

$$\|\hat{\Lambda}_i - \Lambda_i\| = O_p(n^{-1/2}q_i^{1/2} + n^{-1}p), \quad 1 \le i \le m, \tag{B.32}$$

where $\Lambda_i = \mathrm{diag}(\lambda_{p_{i-1}+1}, \cdots, \lambda_{p_i})$, and λ_i are specified in Condition A3.

(B.28), (B.29), (A.27) and Theorem B.1 can conclude Theorem 1. Thus, we now need to prove Theorem B.1.

Proof of Theorem B.1. (3.15) and (A.2) show that m is bounded. Let $J_i = \{j \in \mathcal{Z} : p_{i-1} < j \le p_i\}$. At first we prove (B.32). We only need to prove it when $i = 1$ and other cases can be concluded by a permutation. Define

J_1^c be the complementary set of J_1, then we can rewrite $\det(\lambda I_p - \hat{N}) = 0$ as follows.

$$0 = \det(\lambda I_p - \hat{N}) = \det \begin{pmatrix} \lambda I_{p_1} - \hat{N}_{J_1} & -\hat{N}_{J_1,J_1^c} \\ -\hat{N}_{J_1^c,J_1} & \lambda I_{p-p_1} - \hat{N}_{J_1^c} \end{pmatrix}. \tag{B.33}$$

Lemmas A.3 and A.5 conclude $\|\hat{N}_{J_1^c,J_1}\| = O_p(n^{-1/2}p^{1/2}) = o_p(1)$. Lemmas A.3-A.6 and the condition A3 imply that there exists a positive constant $\tilde{C}_N$ such that

$$\lim_{n\to\infty} P(\|\lambda_l I_{p-p_1} - \hat{N}_{J_1^c}\|_{min} > \tilde{C}_N) = 1 \tag{B.34}$$

for any $1 \le l \le p_1$. Lemma A.6 also implies that

$$\lim_{n\to\infty} P\Big(\lambda_{p_1} - \tilde{C}_N/2 < \|\hat{N}_{J_1}\|_{min} \le \|\hat{N}_{J_1}\| < \lambda_1 + \tilde{C}_N/2\Big) = 1. \tag{B.35}$$

If $\lambda \in (\lambda_{p_1} - \tilde{C}_N/2, \lambda_1 + \tilde{C}_N/2)$ is a solution of (B.33), it is also (with probability 1) a solution of

$$0 = \det\Big(\lambda I_{p_1} - \hat{N}_{J_1} - \hat{N}_{J_1,J_1^c}(\lambda I_{p-p_1} - \hat{N}_{J_1^c})^{-1}\hat{N}_{J_1^c,J_1}\Big). \tag{B.36}$$

Lemma A.5 and (B.34) imply that

$$\|\hat{N}_{J_1,J_1^c}(\lambda I_{p-p_1} - \hat{N}_{J_1^c})^{-1}\hat{N}_{J_1^c,J_1}\| = O_p(n^{-1}p). \tag{B.37}$$

Let $\tilde{\lambda}_1 \ge \cdots \ge \tilde{\lambda}_{p_1}$ be the eigenvalues of $\hat{N}_{J_1}$, (B.36)-(B.37) conclude that

$$\tilde{\lambda}_l - \hat{\lambda}_l = O_p(n^{-1}p) \tag{B.38}$$

for any $1 \leq l \leq p_1$. This, together with (A.20), concludes (B.32).

Now we consider (B.31). We only need to prove it when $j = 1$ and $i > 1$. Other cases can be concluded by a permutation. From $\hat{N} = \hat{\Gamma}\hat{\Lambda}\hat{\Gamma}^\top$ and (B.30), we can find that

$$\begin{pmatrix} \sum_{i=1}^m \hat{N}_{J_1,J_i}\hat{\Gamma}_{i1} \\ \cdots \\ \sum_{i=1}^m \hat{N}_{J_m,J_i}\hat{\Gamma}_{i1} \end{pmatrix} = \hat{N}\begin{pmatrix} \hat{\Gamma}_{11} \\ \cdots \\ \hat{\Gamma}_{m1} \end{pmatrix} = \begin{pmatrix} \hat{\Gamma}_{11}\hat{\Lambda}_1 \\ \cdots \\ \hat{\Gamma}_{m1}\hat{\Lambda}_1 \end{pmatrix}. \tag{B.39}$$

Define $U_{11} = \hat{N}_{J_1,J_1}$, $U_{12} = \hat{N}_{J_1,J_1^c}$, $U_{21} = \hat{N}_{J_1^c,J_1}$ and $U_{22} = \hat{N}_{J_1^c,J_1^c}$. Similarly, define $\tilde{\Gamma}_{21}^\top = (\hat{\Gamma}_{21}^\top, \cdots, \hat{\Gamma}_{m1}^\top)^\top$. Then we can rewrite (B.39) as

$$\begin{pmatrix} U_{11}\hat{\Gamma}_{11} + U_{12}\tilde{\Gamma}_{21} \\ U_{21}\hat{\Gamma}_{11} + U_{22}\tilde{\Gamma}_{21} \end{pmatrix} = \begin{pmatrix} \hat{\Gamma}_{11}\hat{\Lambda}_1 \\ \tilde{\Gamma}_{21}\hat{\Lambda}_1 \end{pmatrix}. \tag{B.40}$$

$$\tilde{\Gamma}_{21}\hat{\Lambda}_1 = \tilde{\Gamma}_{21}(\hat{\Lambda}_1 - \lambda_1 I_{p_1}) + \lambda_1\tilde{\Gamma}_{21}.$$

Then the second line of (B.40) is equivalent to

$$(U_{22} - \lambda_1 I_{p-p_1})\tilde{\Gamma}_{21} = \tilde{\Gamma}_{21}(\hat{\Lambda}_1 - \lambda_1 I_{p_1}) - U_{21}\hat{\Gamma}_{11}.$$

Recalling (B.34), $U_{22} - \lambda_1 I_{p-p_1}$ is invertible with probability 1 as n tends to infinity.

$$\tilde{\Gamma}_{21} = (U_{22} - \lambda_1 I_{p-p_1})^{-1}\tilde{\Gamma}_{21}(\hat{\Lambda}_1 - \lambda_1 I_{p_1}) - (U_{22} - \lambda_1 I_{p-p_1})^{-1}U_{21}\hat{\Gamma}_{11}.$$

(3.14)-(3.15) and Lemmas A.3-A.6 imply that $\|\hat{\Lambda}_1 - \lambda_1 I_{p_1}\| = o_p(1)$ and $\|(U_{22} - \lambda_1 I_{p-p_1})^{-1}\| = O_p(1)$. Then $(\lambda_1 I_{p-p_1} - U_{22})^{-1}U_{21}\hat{\Gamma}_{11}$ is the leading term of $\tilde{\Gamma}_{21}$. Moreover, $\|\hat{\Gamma}_{11}\| = O(1)$. Thus we only need to consider $(\lambda_1 I_{p-p_1} - U_{22})^{-1}U_{21}$. We rewrite $(\lambda_1 I_{p-p_1} - U_{22})^{-1}$ as

$$\begin{pmatrix} \lambda_1 I_{p_2} - \hat{N}_{J_2,J_2} & \cdots & -\hat{N}_{J_2,J_m} \\ \cdots & \cdots & \cdots \\ -\hat{N}_{J_m,J_2} & \cdots & \lambda_1 I_{p_m} - \hat{N}_{J_m,J_m} \end{pmatrix}^{-1} = (\lambda_1 I_{p-p_1} - U_{22})^{-1} = \begin{pmatrix} V_{22} & \cdots & V_{2m} \\ \cdots & \cdots & \cdots \\ V_{m2} & \cdots & V_{mm} \end{pmatrix}.$$

(3.14)-(3.15) and Lemma A.6 ensure $\|(\lambda_1 I_{p_i} - \hat{N}_{J_i,J_i})^{-1}\| = O_p(1)$ for $2 \le i \le m$. Lemma A.5 ensures $\|\hat{N}_{J_i,J_t}\| = O_p(n^{-1/2}p^{1/2}) = o_p(1)$ for $2 \le i \ne t \le m$. Since m is finite, we can find $\|V_{ii}\| = O_p(1)$ and $\|V_{it}\| = O_p(n^{-1/2}p^{1/2})$ for $2 \le i \ne t \le m$. Recall that $\|\hat{N}_{J_i,J_1}\| = O_p(n^{-1/2}(q_1 + q_i)^{1/2})$ for $2 \le i \le m$ and

$$(\lambda_1 I_{p-p_1} - U_{22})^{-1}U_{21} = \begin{pmatrix} V_{22} & \cdots & V_{2m} \\ \cdots & \cdots & \cdots \\ V_{m2} & \cdots & V_{mm} \end{pmatrix} \begin{pmatrix} \hat{N}_{J_2,J_1} \\ \cdots \\ \hat{N}_{J_m,J_1} \end{pmatrix}.$$

It follows that $\|V_{ii}\hat{N}_{J_i,J_1}\| = O_p(n^{-1/2}(q_1 + q_i)^{1/2})$ and $\|\sum_{t \ne i} V_{it}\hat{N}_{J_t,J_1}\| = O_p(n^{-1}p)$. We complete the proof of (B.31). □

Now we prove Theorem 2. By the same idea, we give the following result for $\hat{N}$.

Theorem B.2. *Let conditions A1, A2 and A4 hold. Denote by $\hat{\gamma}_{ij}$ the (i,j)-th entry of matrix $\hat{\Gamma}$ in (B.30). Then as $n, p \to \infty$, it holds that*

$$\hat{\gamma}_{ij} = O_p(n^{-1/2}\upsilon_{\text{gap}}^{-1}|j-i|^{-1}) \quad \text{for } 1 \le i \ne j \le p, \quad \text{and} \tag{B.41}$$

$$\hat{\gamma}_{ii} = 1 + O_p(n^{-1} v_{\text{gap}}^{-2}) \quad \text{for} \ \ i = 1, \cdots, p. \tag{B.42}$$

Moreover,

$$\|\hat{\Lambda} - \Lambda\| = O_p(n^{-1/2} p^{1/2}). \tag{B.43}$$

Proof of Theorem B.2. Following the proof of Lemma A.6, one can verify that $\|\hat{\Lambda} - N\| = O_p(n^{-1/2}p^{1/2})$. This, together with A4, implies (B.43). From $\hat{N}\hat{\Gamma} = \hat{\Gamma}\hat{\Lambda}$, we can find that

$$\hat{\Gamma}\hat{\Lambda} - N\hat{\Gamma} = (\hat{N} - N)\hat{\Gamma}. \tag{B.44}$$

(B.44) implies that

$$\hat{\gamma}_{ij}(\hat{\lambda}_j - \lambda_i) = \sum_{s=1}^{p} M_{is}\hat{\gamma}_{sj}, \tag{B.45}$$

where M_{is} is defined in Lemma A.4. The condition A4 and $\|\hat{\Lambda} - N\| = O_p(n^{-1/2}p^{1/2})$ can control $(\hat{\lambda}_j - \lambda_i)$. Then we can divide the right hand of the above equation into two part.

$$\sum_{s=1}^{p} M_{is}\hat{\gamma}_{sj} = \sum_{s \neq j} M_{is}\hat{\gamma}_{sj} + M_{ij}\hat{\gamma}_{jj}. \tag{B.46}$$

(A.12) implies that $E|M_{ij}\hat{\gamma}_{jj}|^2 \leq E|M_{ij}|^2 \leq C_1 n^{-1}$. Thus we only need to consider the order of $\sum_{s\neq j} M_{is}\hat{\gamma}_{sj}$. Define $v = \max_{1\leq i\leq p}\max_{j\neq i}|\sum_{s\neq j} M_{is}\hat{\gamma}_{sj}|$. Then for any $j \neq i$, (B.45) implies that

$$|\hat{\gamma}_{ij}| \leq (|i-j|v_{gap} - \|\hat{\Lambda} - N\|)^{-1}(v + |M_{ij}|)$$

and

$$
\begin{aligned}
&|\sum_{s\neq j} M_{is}\hat{\gamma}_{sj}| \leq \sum_{s\neq j}|M_{is}||\hat{\gamma}_{sj}| \\
\leq \quad & \sum_{s\neq j}|M_{is}|(|s-j|v_{gap}-\|\hat{\Lambda}-N\|)^{-1}(v+|M_{sj}|) \\
\leq \quad & v\sum_{s\neq j}|M_{is}|(|s-j|v_{gap}-\|\hat{\Lambda}-N\|)^{-1}+\sum_{s\neq j}|M_{is}||M_{sj}|(|s-j|v_{gap}-\|\hat{\Lambda}-N\|)^{-1}.
\end{aligned}
$$

The condition A4, $\|\hat{\Lambda}-N\| = O_p(n^{-1/2}p^{1/2})$ and (A.12) conclude that

$$
\sum_{s\neq j}|M_{is}|(|s-j|v_{gap}-\|\hat{\Lambda}-N\|)^{-1} = O(v_{gap}^{-1}\log p \max_{1\leq i,s\leq p}|M_{is}|) = o_p(1)
$$

and

$$
\sum_{s\neq j}|M_{is}||M_{sj}|(|s-j|v_{gap}-\|\hat{\Lambda}-N\|)^{-1} = o_p(n^{-1/2}).
$$

This, together with the definition of v, implies that $v = o_p(n^{-1/2})$.

$$
|\hat{\gamma}_{ij}| \leq (|i-j|v_{gap}-\|\hat{\Lambda}-N\|)^{-1}[o_p(n^{-1/2})+|M_{ij}|].
$$

This, together with (A.12), concludes (B.41).

$$
\hat{\gamma}_{ii}^2 = 1-\sum_{j\neq i}\hat{\gamma}_{ij}^2 \geq 1-\sum_{j\neq i}(|i-j|v_{gap}-\|\hat{\Lambda}-N\|)^{-2}(v+|M_{ij}|)^2 = 1+O_p(n^{-1}v_{gap}^{-2}).
$$

We complete the proof. □

(B.28) and (A.27) imply that

$$
\|\hat{\Gamma}^\top\hat{U}_\Omega^\top\hat{V}_\Omega^\top\hat{W}\hat{V}_\Omega\hat{U}_\Omega\hat{\Gamma}-\hat{\Lambda}\| = O_p(n^{-1/2}p^{1/2}).
$$

This and Theorem B.2 can conclude the asymptotic properties of $\hat{U}_W^\top\hat{V}_\Omega\hat{U}_\Omega\hat{\Gamma}$.

Then we can prove Theorem 2 by (B.29) and Theorem B.2.

Appendix C. An Additional Example for Numerical Results

In this section, we further present the usefulness of Multiple Ring Kernels by constructing a special example. In this example, Ring Kernel 1 is no longer the best single kernel. We achieve this goal by generating latent fields in a mixing way. To generate data, we split the map of sample locations into 10 rows according to their y coordinates, and all rows have equal width. For each row, let the sample points within be independent from adjacent rows. In order to achieve this, for each of the p latent fields, we generate 3 independent candidate random fields using same set of coordinates and covariance function parameters. The process for generating each candidate random field is the same as described before. The coordinates belong to the $1^{st}, 4^{th}, 7^{th}$ and 10^{th} row would take values from the first candidate random field, those belong to the $2^{nd}, 5^{th}, 8^{th}$ row would take values from the second candidate random field, and the rest of the sample points will take values from the third candidate random field. In this way, the samples from most adjacent rows are independent to each other, and the effectiveness of Ring Kernel 1 is weakened.

We performed simulation using latent random fields constructed from the method above. Dimension of latent field $p = 3$. The sample size, sampling method of coordinates, setting of mixing matrix, and use of matern covariance function is identical to the description of simulation setting in numerical illustration section. The boxplot of $D(\Omega, \hat{\Omega})$ obtained from 1000 replications is presented in figure, and median of $D(\Omega, \hat{\Omega})$ is presented in table.

As the figure shows, kernel 1 is no longer the best-performing single kernel, while multiple kernel remains very close to the best single kernel, and outperforming most other single kernels. Yet as sample size increases, $D(\Omega, \hat{\Omega})$ did not improve, which might due to the artificial nature of this special example. More detailed data is presented in Table C.1.

Kernel	1	2	3	4	5	6	7	8	9	10	Multiple	Multiple Original
n=100	0.2642	0.2218	0.2718	0.2098	0.2583	0.2663	0.2710	0.2448	0.2510	0.2462	0.2123	0.2242
n=500	0.2218	0.1739	0.1583	0.1335	0.2614	0.1974	0.2489	0.1642	0.2582	0.2348	0.1480	0.1045
n=1000	0.2091	0.1712	0.1627	0.1452	0.2500	0.1813	0.2463	0.1544	0.2638	0.2346	0.1535	0.8800
n=2000	0.2190	0.1763	0.1548	0.1474	0.2455	0.1807	0.2703	0.1590	0.2631	0.2442	0.1506	0.0752

Table C.1: Median of $D(\Omega, \hat{\Omega})$ from the proposed method using the 10 single kernels, or multiple kernel(including all 10 ring kernels), and the method of Bachol et el. using the multiple kernel (original) in a simulation with 1000 replications for the mixed random fields. The number of observations n is 100, 500, 1000 or 2000 , and the dimension of random fields is $p = 3$.

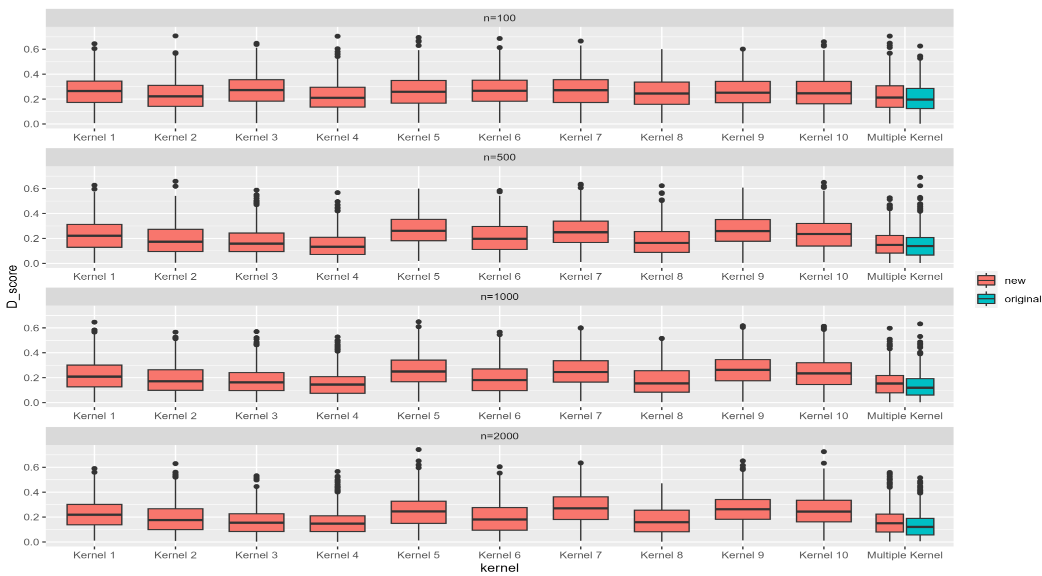

Figure C.1: Boxplots of $D(\Omega, \hat{\Omega})$ for the proposed method using the 10 single kernels, or multiple kernel(including all 10 ring kernels), and the method of Bachol et el. using the multiple kernel (original) in a simulation with 1000 replications for the mix Gaussian random fields. The number of observations n is 100, 500, 1000 or 2000 (from top to bottom), and the dimension of random fields is $p = 3$. Latent field is generated in a mixing approach as described.