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# Unification of Rare and Weak Multiple Testing Models using Moderate Deviations Analysis and Log-Chisquared P-values

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## Supplementary Materials

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## S1 Technical Lemmas

**Lemma 1.** *Let* $\{P_i^{(n)}\}_{i=1}^n$ *be a sequence of probability distributions, each* $P_i^{(n)}$ *has density whose support is contained in* $[0,\infty)$*. Fix* $q>0$*. If,*

$$\lim_{n\to\infty}\max_{i=1,\ldots,n}\frac{\left|\log\left(\frac{dP_i^{(n)}}{d\mathrm{Exp}(2)}(2q\log(n))\right)\right|}{\log(n)}=0, \tag{S1.1}$$

*then*

$$\lim_{n\to\infty}\max_{i=1,\ldots,n}\left|\frac{-\log\Pr\left[P_i^{(n)}\geq 2q\log(n)\right]}{\log(n)}-q\right|. \tag{S1.2}$$

*Proof.* The assumption on the density of $P_i^{(n)}$ ensures that it is absolutely continuous with respect to $\mathrm{Exp}(2)$. Fix $q>0$. We can write (S1.1) as

$$\frac{dP_i^{(n)}}{d\mathrm{Exp}(2)}(2q\log(n))=n^{o(1)},$$

where $o(1)\to 0$ uniformly in $i$ for every fixed $q$. From

$$\frac{d\mathrm{Exp}(2)}{dx}(x)=\frac{e^{-x/2}}{2},$$

we get

$$\begin{aligned}
\Pr\left[P_i^{(n)} \geq 2q\log(n)\right] &= \int_{2q\log(n)}^{\infty} \frac{dP_i^{(n)}}{dx}dx \\
&= \int_{2q\log(n)}^{\infty} \frac{dP_i^{(n)}}{d\mathrm{Exp}(2)}\frac{d\mathrm{Exp}(2)}{dx}dx \\
&= \int_{2q\log(n)}^{\infty} n^{o(1)}e^{-x/2}/2dx \\
&= n^{o(1)}e^{-q\log(n)}/2 = n^{-q+o(1)}
\end{aligned}$$

This implies (S1.2).

□

We require the following lemma from (Cai and Wu, 2014), providing a particular version of Laplace's principle.

**Lemma 2.** *(Cai and Wu, 2014, Lemma 3) Let $(X, \mathcal{F}, \nu)$ be a measure space. Let $F : X \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable. Assume that*

$$\lim_{M\to\infty} \frac{\log F(x, M)}{M} = f(x) \tag{S1.3}$$

*holds uniformly in $x \in X$ for some measureable $f : X \to \mathbb{R}$. If*

$$\int_X \exp(M_0 f(x))d\nu(x) < \infty$$

*for some $M_0 > 0$, then*

$$\lim_{M\to\infty} \frac{1}{M} \log \int_X F(x, M)d\nu(x) = \operatorname*{ess\,sup}_{x\in X} f(x). \tag{S1.4}$$

**Lemma 3.** *Suppose that* $\{Q_i^{(n)}\}_{i=1}^n$ *satisfy* (2.8), $\{E_i^{(n)}\}_{i=1}^n$ *satisfy* (2.11), $Q_i^{(n)}$ *is absolutely continuous with respect to* $E_i^{(n)}$*, and* $E_i^{(n)}$ *is absolutely continuous with respect to the Lebesgue measure on* $[0,\infty)$*. Set*

$$L_i^{(n)}(x) := \frac{dQ_i^{(n)}}{dE_i^{(n)}}(x). \tag{S1.5}$$

*and*

$$\alpha^*(q;r,\sigma) := \max_{y\in[r,q]} \left\{-2\alpha(y;r,\sigma)+y\right\}.$$

*Assume that*

$$\lim_{n\to\infty}\max_{i=1\dots,n} \frac{\left|\log\left(\frac{dQ_i^{(n)}}{d\chi^2(r,\sigma)}(2q\log(n))\right)\right|}{\log(n)} = 0, \quad \forall q\in(r,r+a), \tag{S1.6}$$

*for some* $a>0$ *and* $r>0$*. Then, for any fixed* $q\in(r,r+a)$*,*

$$\lim_{n\to\infty}\max_{i=1\dots,n} \left| \frac{-\log\left(\mathbb{E}_{X\sim Q_i^{(n)}}\left[L_i^{(n)}(X)\mathbf{1}_{\{X\le 2q\log(n)\}}\right]\right)}{\log(n)} - \alpha^*(q;r,\sigma)\right| = 0. \tag{S1.7}$$

*Proof.* Fix $q \in (r, r+a)$. We have

$$\mathbb{E}_{X\sim Q_i^{(n)}}\left[L_i^{(n)}(X)\mathbf{1}_{\{X\le 2q\log(n)\}}\right] = \mathbb{E}_{X\sim E_i^{(n)}}\left[(L_i^{(n)})^2(X)\mathbf{1}_{\{X\le 2q\log(n)\}}\right]$$

$$= \int_0^{2q\log(n)} \left(\frac{dQ_i^{(n)}}{dE_i^{(n)}}(x)\right)^2 E_i^{(n)}(dx)$$

$$= 2\log(n)\int_0^q \left(\frac{dQ_i^{(n)}}{d\,\mathrm{E}_i^{(n)}}(2\log(n)y)\right)^2 E_i^{(n)}(2\log(n)dy) \tag{S1.8}$$

$$= \log(n)\int_0^q \left(\frac{dQ_i^{(n)}}{d\,\mathrm{E}_i^{(n)}}(2\log(n)y)\right)^2 e^{-y\log(n)(1+o(1))}dy \tag{S1.9}$$

$$= \log(n)\int_0^q n^{-2\alpha(y;r,\sigma)+2y+o(1)}\cdot n^{o(1)}\cdot n^{-y}dy = \int_0^q n^{-2\alpha(y;r,\sigma)+y+o(1)}dy, \tag{S1.10}$$

where (S1.8) follows from the change of variables $x = 2y\log(n)$, (S1.9) follows from Lemma 1, and (S1.10) follows from (S1.6) and (**??**). Furthermore, $o(1)$ in (S1.9)-(S1.10) represents a sequence tending to zero uniformly in $i$ and $y \in [0, q]$. We now apply Lemma 2 to (S1.10) with $X = [r, q]$, $M = \log(n)$, $F(x, M) = n^{-2\alpha(x;r,\sigma)+x+o(1)}$, $f(x) = -2\alpha(x; r, \sigma) + x$, and $\nu$ the Lebesgue measure. We obtain:

$$\lim_{n\to\infty}\frac{\log\left(\mathbb{E}_{X\sim Q_i^{(n)}}\left[L_i^{(n)}(X)\mathbf{1}_{\{X>2q\log(n)\}}\right]\right)}{\log(n)} = \max_{y\in[r,q]}\{-2\alpha(y;r,\sigma)+y\}$$

$$= -\alpha^*(q;r,\sigma)$$

uniformly in $i$. Equation (S1.7) follows. □

The following lemma summarizes the truncated likelihood ratio method of (Ingster et al., 2010; Ingster and Suslina, 2012).

**Lemma 4.** *Consider testing*

$$H_0^{(n)} \,:\, (X_1,\ldots,X_n) \sim P_0^{(n)} \tag{S1.11}$$

*versus*

$$H_1^{(n)} \,:\, (X_1,\ldots,X_n) \sim P_1^{(n)} \tag{S1.12}$$

*for* $P_1^{(n)}$ *that is absolutely continuous with respect to* $P_0^{(n)}$*. Denote by* $L_n = \frac{dP_1^{(n)}}{dP_0^{(n)}}$ *the likelihood ratio between* $P_1^{(n)}$ *and* $P_0^{(n)}$*. Suppose that there exists a sequence of sets* $A^{(n)} \subset \mathbb{R}^n$ *such that*

$$1 - \mathbb{E}_{H_0^{(n)}}\left[L_n(X_1,\ldots,X_n)\mathbf{1}_{(X_1,\ldots,X_n)\in A^{(n)}}\right] \leq o(1) \tag{S1.13}$$

*while*

$$\mathbb{E}_{H_0^{(n)}}\left[L_n^2(X_1,\ldots,X_n)\mathbf{1}_{(X_1,\ldots,X_n)\in A^{(n)}}\right] \leq 1 + o(1). \tag{S1.14}$$

*For any sequence of tests* $\psi^{(n)} : \mathbb{R}^n \to \{0,1\}$*,*

$$\liminf_{n\to\infty} \left\{\mathbb{E}_{H_0^{(n)}}\left[\psi^{(n)}(X_1,\ldots,X_n)\right] + \mathbb{E}_{H_1^{(n)}}\left[1-\psi^{(n)}(X_1,\ldots,X_n)\right]\right\} \geq 1.$$

*Proof.* Set

$$\tilde{L}_n := \tilde{L}_n(X_1,\ldots,X_n) := L_n(X_1,\ldots,X_n)\mathbf{1}_{A^{(n)}}(X_1,\ldots,X_n).$$

Conditions (S1.13) and (S1.14) imply

$$\mathbb{E}_{H_0^{(n)}}\left[\tilde{L}_n\right] = \left(\mathbb{E}_{H_0^{(n)}}\left[\tilde{L}_n^2\right] - 1\right) - 2\left(\mathbb{E}_{H_0^{(n)}}\left[\tilde{L}_n\right] - 1\right) \leq o(1),$$

hence $\tilde{L}_n(X) \to 1$ in probability under $H_0^{(n)}$. Next, for some $\psi^{(n)} : \mathbb{R}^n \to \{0,1\}$ and $\epsilon > 0$,

$$\begin{aligned}
&\mathbb{E}_{H_0^{(n)}}\left[\psi^{(n)}\right] + \mathbb{E}_{H_1^{(n)}}\left[1-\psi^{(n)}\right] \\
&= \mathbb{E}_{H_0^{(n)}}\left[\psi^{(n)} + L_n(1-\psi^{(n)})\right] \\
&\geq \mathbb{E}_{H_0^{(n)}}\left[\psi^{(n)} + \tilde{L}_n(1-\psi^{(n)})\right] \\
&\geq \mathbb{E}_{H_0^{(n)}}\left[\psi^{(n)} + \tilde{L}_n(1-\psi^{(n)}) \mid |\tilde{L}_n - 1| < \epsilon\right] \Pr\left[|\tilde{L}_n - 1| < \epsilon\right] \\
&\geq \mathbb{E}_{H_0^{(n)}}\left[\psi^{(n)} + (1-\epsilon)(1-\psi^{(n)})\right] \Pr\left[|\tilde{L}_n - 1| < \epsilon\right] \\
&\geq (1-\epsilon)\Pr\left[|\tilde{L}_n - 1| < \epsilon\right] = (1-\epsilon)(1+o(1)).
\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we have that

$$\liminf_{\psi^{(n)}} \left\{ \mathbb{E}_{H_0^{(n)}}\left[\psi^{(n)}\right] + \mathbb{E}_{H_1^{(n)}}\left[1-\psi^{(n)}\right] \right\} \geq 1.$$

□

**Lemma 5.** *Let $q \in (0,1]$ be fixed. Let $U_1, \ldots, U_n$ be $n$ independent RVs satisfying $\Pr\left[U_i \leq n^{-q}\right] = n^{-q}(1 + a_{n,i}(q))$, and denote by*

$$F_n(t) := \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{U_i \leq t\}}$$

*their empirical CDF. If $\bar{a}_n(q) := n^{-1}\sum_{i=1}^{n} a_{n,i}(q) \leq n^{\frac{q-1}{2}}$, then*

$$\Pr\left[\sqrt{n}\frac{F_n(n^{-q}) - n^{-q}}{\sqrt{n^{-q}(1-n^{-q})}} \geq \log(n)\right] \to 0 \qquad \text{(S1.15)}$$

*Proof.* Denote $t_n = n^{-q}$. We have that $\mathbb{E}\left[F_n(t_n)\right] = t_n(1 + \bar{a}_n(q))$. If $\bar{a}_n(q) \leq 0$ for all $n \geq n_0$ for some $n_0$, then (S1.15) holds. Otherwise, we assume without loss of generality that $r_n := \mathbb{E}\left[F_n(t_n) - t_n\right] = n^{-q}\bar{a}_n(q) > 0$ for all $n$, since the complementary case can be handled by considering only a sub-sequence with that property. Write

$$\Pr\left[\sqrt{n}\frac{F_n(t_n) - t_n}{\sqrt{t_n(1-t_n)}} \geq \log(n)\right] = \Pr\left[F_n(t_n) - t_n \geq (1+\delta)r_n\right],$$

where

$$\delta := -1 + \frac{\sqrt{t_n(1-t_n)}\log(n)}{r_n\sqrt{n}} \geq -1 + \log(n) n^{\frac{q-1}{2}} (\bar{a}_n(q))^{-1}\sqrt{1-1/n}$$
$$\geq -1 + \log(n)(1+o(1)).$$

We have that $\delta \to \infty$. For $X$ the sum of $n$ independent Bernoulli RVs with $\mu = \mathbb{E}[X]$, the Chernoff inequality (Mitzenmacher and Upfal, 2017, Ch 4.) says

$$\Pr\left(X \geq (1+\delta)\mu\right) \leq \left(\frac{e^{-\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu} \leq e^{-\mu\frac{\delta^2}{2+\delta}}, \quad \mu = r_n, \quad \delta \in (0,\infty).$$

We use this inequality with $X = nF_n(t) = \sum_{i=1}^n \mathbf{1}_{\{U_i \leq t\}}$. For $n$ large enough such that $\delta > 2$, we obtain

$$-\log\Pr\left[\sqrt{n}\frac{F_n(t_n) - t_n}{\sqrt{t_n(1-t_n)}} \geq \log(n)\right] \geq \frac{\delta^2 n}{2+\delta} r_n \geq \frac{\delta \cdot n}{2} r_n$$
$$\geq \frac{n}{2}\left(n^{-1/2}\log(n) - r_n\right) = \frac{n^{0.5}}{2}\left(\log(n) - n^{-q+1/2}\bar{a}_n(q)\right) \to \infty.$$

□

**Lemma 6.** *(Donoho and Kipnis, 2022, Lem. 5.7 ) Let $\alpha(\cdot)$ and $\gamma(\cdot)$ be two real-valued functions $\alpha, \gamma : [0,\infty) \to [0,\infty)$. Let $q \in (0,1)$ and $\beta > 0$ be* *fixed. Let $F_n(t)$ be the normalized sum of $n$ independent RVs. Suppose that*

$$\mathbb{E}\left[F_n(n^{-q})\right] = n^{-q+o(1)}(1-n^{-\beta}) + n^{-\beta}n^{-\alpha(q)+o(1)}.$$

*Let $\{a_n\}_{n=1}^{\infty}$ be a positive sequence obeying $a_n n^{-\eta} \to 0$ for any $\eta > 0$. If*

$$\delta(q) + \beta < \gamma(q),$$

*then*

$$\Pr\left[n^{\gamma(q)}(F_n(n^{-q}) - n^{-q}) \le a_n\right] \to 0, \qquad n \to \infty.$$

**Lemma 7.** *Assume that $r < \rho_{\mathsf{Bonf}}(\beta,\sigma)$. Consider $p_1,\ldots,p_n$ as in (2.10). For an interval $I \subset [0,1]$ define*

$$T_I := \min_{i\,:\,p_{(i)}\in I} \frac{p_{(i)}}{i/n}. \tag{S1.16}$$

*For any $0 < a < 1$ and $q < 1$,*

$$\Pr_{H_1^{(n)}}\left[T_{(n^{-q},1]} \le a\right] \to 0. \tag{S1.17}$$

*Proof.* Let $F_n(t) := n^{-1}\sum_{i=1}^{n} \mathbf{1}_{p_i \le t}$ be the empirical CDF of $p_1,\ldots,p_n$. Note that $i/n = F_n(p_{(i)})$, hence

$$\frac{p_{(i)}}{i/n} \le a \Longleftrightarrow F_n(p_{(i)}) \ge p_{(i)}/a. \tag{S1.18}$$

Consequently,

$$\begin{aligned}\Pr_{H_1^{(n)}}\left[T_{(n^{-q},1]} \leq a\right] &\leq \sup_{t>n^{-q}} \Pr_{H_1^{(n)}}\left[F_n(t) \geq t/a\right] \\ &= \sup_{t>n^{-q}} \Pr_{H_1^{(n)}}\left[nF_n(t) \geq nt/a\right] \\ &= \sup_{t>n^{-q}} \Pr_{H_1^{(n)}}\left[nF_n(t) \geq \mathbb{E}_{H_1^{(n)}}\left[nF_n(t)\right](1+\kappa)\right], \qquad \text{(S1.19)}\end{aligned}$$

where

$$\kappa := \kappa(n,a,t) := \frac{t}{a\mathbb{E}\left[F_n(t)\right]} - 1. \qquad \text{(S1.20)}$$

Let $U_i \sim \mathrm{Unif}(0,1)$ and $-2\log(X_i) \sim Q_i^{(n)}$, for $i = 1,\ldots,n$. Using the parameterization $t_n = n^{-q'}$, $q' \leq q < 1$,

$$\begin{aligned}\mathbb{E}_{H_1^{(n)}}\left[F_n(t_n)\right] &= \frac{1}{n}\sum_{i=1}^{n}\Pr_{H_1^{(n)}}\left[p_i \leq n^{-q'}\right] &\text{(S1.21)}\\ &= (1-\epsilon_n)\left[U_i \leq n^{-q'}\right] + \epsilon_n \Pr\left[X_i \leq n^{-q'}\right] &\text{(S1.22)}\\ &= 1 - \epsilon_n + n^{-\alpha(q';r,\sigma)(1+o(1))-\beta}, &\text{(S1.23)}\end{aligned}$$

where the last transition follows from (2.8). Since $\beta + \alpha(q';r,\sigma) \leq \beta + \alpha(1;r,\sigma) < 1$, the last display implies in particular $\mathbb{E}_{H_1^{(n)}}\left[F_n(t_n)\right]/t_n \to 1$. It follows that

$$\sup_{t>n^{-q}} \frac{\mathbb{E}_{H_1^{(n)}}\left[F_n(t_n)\right]}{t_n} = 1 + o(1). \qquad \text{(S1.24)}$$

Since $a < 1$, there exists $\eta > 0$ such that $\kappa \geq 1/a - 1 + \eta > 0$ for all $n \geq n_0(q)$ large enough. Using Chernoff's inequality (Mitzenmacher and

Upfal, 2017, Ch. 4) in (S1.19), we obtain

$$\begin{aligned}\Pr_{H_1^{(n)}}\left[T_{(n^{-q},1]} \leq a\right] &\leq \sup_{t>n^{-q}} \exp\left\{-\frac{n}{a}\frac{\kappa^2}{1+\kappa}\mathbb{E}_{H_1^{(n)}}\left[F_n(t)\right]\right\} \\ &\leq \exp\left\{-\frac{n}{2a}\inf_{t>n^{-q}} E_{H_1^{(n)}}\left[F_n(t)\right]\right\} \\ &= \exp\left\{-\frac{1}{2a}n^{1-\alpha(q;r,\sigma)+o(1)-\beta}\right\} \to 0,\end{aligned}$$

where the last transition follows because $r < \rho_{\mathsf{Bonf}}(\beta,\sigma)$ implies $\beta+\alpha(q;r,\sigma) \leq \beta + \alpha(1;r,\sigma) < 1$. □

**Lemma 8.** *Let $\{a_n\}$,$\{b_n\}$, and $\{\lambda_n\}$ be non-negative sequences such that, as $n \to \infty$, $a_n \to \infty$, $\lambda_n \to \infty$, $a_n/\lambda_n \to 0$, and $a_n/b_n \to c$ for some $c > 1$. For $\lambda' = \lambda_n + \sqrt{\lambda_n b_n}$ and $\Upsilon_{\lambda'} \sim \mathrm{Pois}(\lambda')$,*

$$\lim_{n\to\infty} \frac{\Pr\left[-2\log\bar{\mathsf{P}}(\Upsilon_{\lambda'};\lambda_n) \geq a_n\right]}{(\sqrt{a_n}-\sqrt{b_n})^2} = -\frac{1}{2}. \tag{S1.25}$$

*Proof.* We first develop a moderate deviation estimate for the Poisson survival function. From

$$\begin{aligned}\bar{\mathsf{P}}(x;\lambda) &= e^{-\lambda}\sum_{k=x}^{\infty}\frac{\lambda^k}{k!} \\ &= e^{-\lambda}\frac{\lambda^x}{x!}\left(1+\frac{\lambda^{x+1}}{(x+1)}+\frac{\lambda^{x+2}}{(x+1)(x+2)+\dots}\right),\end{aligned}$$

we get

$$-\log\bar{\mathsf{P}}(x;\lambda) = \lambda - x\log(\lambda) + \log\Gamma(x) + R(x;\lambda),$$

where $\Gamma(x)$ is the Gamma function and

$$R(\lambda;x) := \log\left(1 + \frac{\lambda^{x+1}}{(x+1)} + \frac{\lambda^{x+2}}{(x+1)(x+2)+\dots}\right) \leq -\log\left(1 - \frac{\lambda}{1+x}\right) = O(\lambda/x).$$

Furthermore,

$$\log\Gamma(x) = (x - \frac{1}{2})\log(x) - x + O(1/x)$$

Therefore, for $x > \lambda > 1$, we have that $t \leq -\log \bar{\mathsf{P}}(x;\lambda)$ iff

$$t \leq -(x-\lambda) + x\log(x/\lambda) + O(\lambda/x) = -(x-\lambda)x + x\left(\frac{x-\lambda}{\lambda}\right) + O(\lambda/x),$$

iff

$$0 = x^2 - 2x\lambda + \lambda^2 - t\lambda + o(\lambda/x).$$

Solving the last display for $x > 0$, we obtain $t \leq -\log \bar{\mathsf{P}}(x;\lambda)$ if

$$x \geq \lambda + \sqrt{t\lambda} + O(\sqrt{\lambda/x}) \tag{S1.26}$$

Next, consider the event $A = \{\Upsilon_{\lambda'} \geq \lambda_n\}$. We have

$$\begin{aligned}
\Pr\left[-\log \bar{\mathsf{P}}(\Upsilon_{\lambda'};\lambda_n) \geq a_n | A\right] &\overset{a}{=} \Pr\left[\Upsilon_{\lambda'} \geq \lambda_n + \sqrt{\lambda_n a_n} + O(\sqrt{\lambda_n/\Upsilon_{\lambda'}}) | A\right] \\
&= \Pr\left[\Upsilon_{\lambda'} \geq \lambda_n + \sqrt{a_n\lambda_n} + O(1) | A\right] \\
&= \Pr\left[\Upsilon_{\lambda'} \geq (\lambda_n + \sqrt{a_n\lambda_n})(1+o(1)) | A\right] \\
&\overset{b}{=} \Pr\left[\Upsilon_{\lambda'} \geq (\lambda' + \sqrt{\lambda'}\left(\sqrt{a_n} - \sqrt{b_n}\right)(1+o(1)) | A\right] \\
&= \Pr\left[\Upsilon_{\lambda'} \geq \lambda' + \sqrt{\lambda'}\sqrt{c_n} | A\right],
\end{aligned} \tag{S1.27}$$

where $\{c_n\}$ is a sequence satisfying

$$\frac{\sqrt{c_n}}{\sqrt{a_n}-\sqrt{b_n}} \to 1 \quad \text{as} \quad n \to \infty. \tag{S1.28}$$

In the arguments leading to (S1.27), $(a)$ is due to (S1.26) and $(b)$ is due to

$$\begin{aligned}
\lambda_n + \sqrt{\lambda_n a_n} &= \lambda' + \sqrt{\lambda'}\left(\sqrt{a_n}-\sqrt{b_n}\right)\sqrt{\lambda_n/\lambda'} \\
&= \lambda' + \sqrt{\lambda'}\left(\sqrt{a_n}-\sqrt{b_n}\right)(1+o(1)),
\end{aligned}$$

the last transition because $b_n/\lambda_n \to 0$.

Since $\sqrt{\lambda_n b_n/\lambda'} \to \infty$, the normal approximation $\Upsilon_{\lambda'} \sim \mathcal{N}(\lambda', \lambda')$ implies

$$\Pr[A] \sim \Pr\left[\sqrt{\lambda'}Z + \lambda' \geq \lambda_n\right] = \Pr\left[Z \geq -\sqrt{\lambda_n b_n/\lambda'}\right] \to 1.$$

We obtain

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log \Pr\left[-\log \bar{\mathsf{P}}(\Upsilon_{\lambda'};\lambda_n) \geq a_n\right]}{(\sqrt{a_n}-\sqrt{b_n})^2} &= \lim_{\lambda_n\to\infty} \frac{\log \Pr\left[-\log \bar{\mathsf{P}}(\Upsilon_{\lambda'};\lambda_n) \geq a_n | A\right]}{(\sqrt{a_n}-\sqrt{b_n})^2} \\
&= \lim_{\lambda_n\to\infty} \frac{\log \Pr\left[\Upsilon_{\lambda'} \geq \lambda_n + \sqrt{a_n\lambda_n} | A\right]}{(\sqrt{a_n}-\sqrt{b_n})^2} \\
&= \lim_{n\to\infty} \frac{\log \Pr\left[\Upsilon_{\lambda'} \geq \lambda_n + \sqrt{a_n\lambda_n}\right]}{(\sqrt{a_n}-\sqrt{b_n})^2} \\
&\overset{c}{=} \lim_{n\to\infty} \frac{\log \Pr\left[\Upsilon_{\lambda'} \geq \lambda' + \sqrt{\lambda' c_n}\right]}{(\sqrt{a_n}-\sqrt{b_n})^2} \\
&\overset{d}{=} \lim_{n\to\infty} \frac{\log \Pr\left[\Upsilon_{\lambda'} \geq \lambda' + \sqrt{\lambda' c_n}\right]}{c_n} \\
&\overset{e}{=} -\frac{1}{2}
\end{aligned}$$

where $(c)$ is due to (S1.27), $(d)$ follows from (S1.28), and in $(e)$ we used the following moderate deviation estimate for a Poisson RV from (Arias-Castro and Wang, 2015).

**Lemma 9.** *(Arias-Castro and Wang, 2015, Lemma) Let* $c : (0,\infty) \to (0,\infty)$ *be such that* $c(\lambda) \to \infty$ *and* $c(\lambda)/\lambda \to 0$ *as* $\lambda \to \infty$*. Then*

$$\lim_{\lambda\to\infty} \frac{\log\left(\Upsilon_\lambda \geq \lambda + \sqrt{\lambda c(\lambda)}\right)}{c(\lambda)} = \frac{-1}{2}$$

This completes the proof of Lemma 8. □

# S2 Proofs of Results in Section 2

## S2.1 Proof of Theorem 1

For $r < \rho(\beta,\sigma)$, there exists $\delta > 0$ such that

$$\max_{q\in[0,1]} \left(\frac{1+q}{2} - \alpha(q;r,\sigma)\right) + \delta - \beta < 0. \tag{S2.29}$$

In particular,

$$1 - \beta - \alpha(1;r,\sigma) \leq -\delta, \tag{S2.30}$$

and, by continuity of $q \to \alpha(q;r,\sigma)$, there exists $\eta \in (0,\gamma)$ such that

$$1 - 2\beta - \alpha^*(1+\eta;r,\sigma) < -\delta. \tag{S2.31}$$

Fix $\delta > 0$ satisfying (S2.29), and let $\eta > 0$ satisfy (S2.31). We now use Lemma 4 with

$$P_0^{(n)} = \prod_{i=1}^{n} E_i^{(n)}, \qquad P_1^{(n)} = \prod_{i=1}^{n} \left[(1-\epsilon)E_i^{(n)} + \epsilon Q_i^{(n)}\right],$$

where $\{E_i^{(n)}\}$ satisfy (2.11), and

$$A^{(n)} = \prod_{i=1}^{n} \{X_i \le 2(1+\eta)\log(n)\}.$$

We have

$$\tilde{L}_n = \prod_{i=1}^{n} \bar{L}_i^{(n)}(X_i)\mathbf{1}_{\{X_i \le 2(1+\eta)\log(n)\}}, \tag{S2.32}$$

where

$$\bar{L}_i^{(n)}(x) := (1-\epsilon_n) + \epsilon_n L_i^{(n)} = 1 + \epsilon_n (L_i^{(n)}(x) - 1), \tag{S2.33}$$

and

$$L_i^{(n)}(x) := \frac{dQ_i^{(n)}}{dE_i^{(n)}}(x).$$

Henceforth, all expectations are with respect to $X_i \sim E_i^{(n)}$ unless otherwise specified. For the first moment, since $\mathbb{E}\left[L_i^{(n)}(X_i)\right] = 1$, we have

$$\mathbb{E}\left[\tilde{L}_n\right] = \prod_{i=1}^{n} (1 - a_{n,i}), \tag{S2.34}$$

where,

$$a_{n,i} := \mathbb{E}\left[\bar{L}_i^{(n)}(X_i)\mathbf{1}_{\{X_i > 2(1+\eta)\log(n)\}}\right].$$

Consider

$$\begin{aligned} a_{n,i} &= \Pr_{X_i \sim E_i^{(n)}} \left[ X_i \geq 2(1+\eta)\log(n) \right] + \epsilon_n \mathbb{E}\left[ \left( L_i^{(n)}(X_i) - 1 \right) \mathbf{1}_{\{X_i > 2(1+\eta)\log(n)\}} \right] \\ &\leq \Pr_{X_i \sim E_i^{(n)}} \left[ X_i \geq 2(1+\eta)\log(n) \right] + \epsilon_n \mathbb{E}\left[ L_i^{(n)}(X_i) \mathbf{1}_{\{X_i > 2(1+\eta)\log(n)\}} \right] \\ &= n^{-(1+\eta)+o(1)} + n^{-\beta} n^{-\alpha(1+\eta; r,\sigma)+o(1)}, \end{aligned} \tag{S2.35}$$

where the last transition follows from (2.11) and from (2.8). It follows from (S2.30) that $a_{n,i} = o(1/n)$, hence (S2.34) converges to 1 and the first moment condition of Lemma 4 holds.

As for the second moment, we have

$$\begin{aligned} \mathbb{E}\left[ \tilde{L}_n^2 \right] &= \prod_{i=1}^n \mathbb{E}\left[ \left( (1-\epsilon_n)^2 + 2\epsilon_n(1-\epsilon_n) L_i^{(n)}(X_i) + \epsilon_n^2 (L_i^{(n)}(X_i))^2 \right) \mathbf{1}_{\{X_i \leq 2(1+\eta)\log(n)\}} \right] \\ &= \prod_{i=1}^n \left( (1-\epsilon_n)^2 + 2\epsilon_n(1-\epsilon_n)\mathbb{E}\left[ L_i^{(n)}(X_i) \right] + \epsilon_n^2 \mathbb{E}\left[ (L_i^{(n)}(X_i))^2 \mathbf{1}_{\{X_i \leq 2(1+\eta)\log(n)\}} \right] \right) \\ &\leq \prod_{i=1}^n \left( 1 - \epsilon_n^2 + \epsilon_n^2 \mathbb{E}\left[ (L_i^{(n)}(X_i))^2 \mathbf{1}_{\{X_i \leq 2(1+\eta)\log(n)\}} \right] \right) \leq \prod_{i=1}^n \left( 1 + b_{n,i} \right), \end{aligned} \tag{S2.36}$$

where

$$b_{n,i} := \epsilon_n^2 + \epsilon_n^2 \mathbb{E}\left[ (L_i^{(n)}(X_i))^2 \mathbf{1}_{\{X_i \leq 2(1+\eta)\log(n)\}} \right].$$

By (2.16b),

$$\begin{aligned} \mathbb{E}\left[ (L_i^{(n)}(X_i))^2 \mathbf{1}_{\{X_i \leq 2(1+\eta)\log(n)\}} \right] &= \mathbb{E}_{X_i \sim Q_i^{(n)}} \left[ L_i^{(n)}(X_i) \mathbf{1}_{\{X_i \leq 2(1+\eta)\log(n)\}} \right] \\ &= n^{-\alpha^*(1+\eta\,;\, r,\sigma)+o(1)}. \end{aligned}$$

It follows from (S2.31) that

$$n \cdot b_{n,i} = n^{1-2\beta} + n^{1-2\beta-\alpha^*(1+\eta\,;\,r,\sigma)\}+o(1)} < n^{1-2\beta} + n^{-\delta},$$

which implies $b_{n,i} = o(1/n)$ becasue $\beta > 1/2$. We conclude that (S2.36) converges to 1 hence the second moment condition of Lemma 4 holds and the proof of Theorem 1 is completed. □

## S2.2 Proof of Corollary 1

Condition (2.16a) follows from (2.8). Condition (2.16a) follows Lemma 3.

## S2.3 Proof of Theorem 2

Under (2.11) and (2.17), Lemma 5 implies that

$$\Pr_{H_0^{(n)}}\left[\mathrm{HC}_n^* \geq \log(n)\right] \to 0. \tag{S2.37}$$

Therefore, it is enough to show that $\Pr_{H_1^{(n)}}\left[\mathrm{HC}_n^* \geq \log(n)\right] \to 0$. Set

$$F_n(t) := \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{p_i \leq t}.$$

note that (2.11) and (2.8) imply

$$\mathbb{E}_{H_1^{(n)}}\left[F_n(n^{-q})\right] = n^{-q+o(1)}(1-n^{-\beta}) + n^{-\beta} n^{-\alpha(q;r,\sigma)+o(1)},$$

and that

$$\mathrm{HC}_n^* = \max_{1\leq i \leq n\gamma_0} \sqrt{n}\frac{\frac{i}{n} - p_{(i)}}{\sqrt{p_{(i)}(1-p_{(i)})}} = \sup_{1/n \leq u \leq \gamma_0 n} \sqrt{n}\frac{F_n(u) - u}{\sqrt{u(1-u)}}. \tag{S2.38}$$

Hence, provided $\gamma_0 < 1/2$, we have

$$\mathrm{HC}_n^* \geq \sqrt{n}\frac{F_n(t) - t}{\sqrt{t(1-t)}} \geq \sqrt{\frac{n}{t}}\left(F_n(t) - t\right), \quad \forall t \in [1/n, 1) \tag{S2.39}$$

almost surely. Setting $t_n = n^{-q}$ for $q \leq 1$, we obtain:

$$\begin{aligned}\Pr_{H_1^{(n)}}\left[\mathrm{HC}_n^* \leq \log(n)\right] &\leq \Pr_{H_1^{(n)}}\left(\sqrt{n}\frac{F_n(t_n) - t_n}{\sqrt{t_n(1-t_n)}} \leq \log(n)\right) \\ &\leq \Pr_{H_1^{(n)}}\left[n^{\frac{q+1}{2}}\left(F_n(t_n) - t_n\right) \leq \log(n)\right]. \end{aligned} \tag{S2.40}$$

Apply Lemma 6 with $\delta(q) = \alpha(q; r, \sigma)$, $\gamma(q) = (q+1)/2$, and $a_n = \log(n)$ to conclude that (S2.40) goes to zero as $n \to \infty$. Theorem 2 follows. □

### S2.4 Proof of Theorem 3

The proof is similar to the proof of (Moscovich et al., 2016, Thm. 4.4). In particular, we use:

**Lemma 10.** *(Moscovich et al., 2016, Cor. A1) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence converging to infinity. Let $\mu_n$, $\sigma_n^2$, and $f_n$ denote the mean, variance and density of* $\mathrm{Beta}(\alpha_n, n - \alpha_n + 1)$*, respectively. Let $g(n)$ be any positive function satisfying $g(n) = o(\min\{\alpha_n, n - \alpha_n\})$ as $n \to \infty$. Then,*

$$f_n(\mu_n + \sigma_n \cdot t) \geq \frac{e^{-t^2/2}}{\sqrt{2\pi} \cdot \sigma_n}\left(1 - \frac{t^3}{\sqrt{g(n)}} - \frac{1}{g(n)}\right) \tag{S2.41}$$

Recall that $M_n^- = \min_{i=1,\ldots,n} \pi_i$, where

$$\pi_i = \Pr\left[\mathrm{Beta}(i, n-i+1) \leq p_{(i)}\right], \qquad i = 1, \ldots, n.$$

We use the sequence $t_n = 1/n$ to separate $H_0^{(n)}$ from $H_1^{(n)}$. The limiting distribution of $M_n$ under $H_0$ satisfies (see (Gontscharuk et al., 2015) and (Moscovich et al., 2016, Thm 4.1)),

$$\Pr_{H_0}\left[M_n^- \le \frac{x}{2\log(n)\log\log(n)}\right] \to 1 - e^{-x},$$

from which it follows that

$$\Pr_{H_0}\left[M_n^- \le t_n\right] \to 0. \tag{S2.42}$$

For $X \sim \text{Beta}(i, n-i+1)$, set

$$\mu_i := \mathbb{E}\left[X\right] = \frac{i}{n+1}, \qquad \sigma_i^2 := \text{Var}\left[X\right] = \frac{i(n-1+1)}{(n+1)^2(n+2)},$$

hence, for $x \in \mathbb{R}$,

$$\frac{\mu_i - x}{\sigma_i} = \sqrt{n}\frac{i/n - x}{\sqrt{\frac{i}{n}\left(1-\frac{i}{n}\right)}}(1+o(1)). \tag{S2.43}$$

The proof of Theorem 2 in Section S2.3 implies in particular

$$\Pr_{H_1^{(n)}}\left[\max_{i=1,\dots,n} \sqrt{n}\frac{i/n - p_{(i)}}{\sqrt{\frac{i}{n}\left(1-\frac{i}{n}\right)}} \ge \log(n)\right] \to 1. \tag{S2.44}$$

Together with (S2.43), the last display implies that for any $\delta > 0$ there exists $n_0(\delta)$ and $i^* \in \{1,\dots,n\}$ such that

$$\tau^* := \frac{\mu_{i^*} - p_{(i^*)}}{\sigma_{i^*}} \ge \sqrt{2\log(n)}, \tag{S2.45}$$

with probability at least $1-\delta$. Denote by $f_i$ the density $f_i : [0,1] \to \mathbb{R}^+$ of

Beta$(i, n-i+1)$. We have

$$\begin{aligned}
\pi_{i^*} &= \int_0^{p_{(i^*)}} f_{i^*}(x)dx \\
&= \sigma_{i^*} \int_{-\mu_{i^*}/\sigma_{i^*}}^{\tau^*} f_{i^*}(\mu_{i^*} + \sigma_{i^*} t)dt \\
&\leq \sigma_{i^*} \int_{-\infty}^{\tau^*} f_{i^*}(\mu_{i^*} + \sigma_{i^*} t)dt \\
&\leq \int_{\tau^*}^{\infty} \frac{1+o(1)}{\sqrt{2\pi}} e^{-x^2/2} dx \qquad \text{(S2.46)} \\
&= (1 - \Phi(\tau^*))(1+o(1)) \sim \frac{1}{\tau^*} e^{-\tau^{*2}/2}, \qquad \text{(S2.47)}
\end{aligned}$$

where (S2.46) follows from Lemma 10 and (S2.47) is due to Mills' ratio. Consequently,

$$\pi_{i^*} \leq n^{-1}, \quad \text{for} \quad n \geq n_0(\delta), \qquad \text{(S2.48)}$$

with probability at least $1-\delta$. Hence, for $n \geq n_0(\delta)$,

$$\Pr_{H_1^{(n)}} [M_n \leq t_n] \geq \Pr_{H_1^{(n)}} \left[M_n^- \leq n^{-1}\right] \geq \Pr_{H_1^{(n)}} \left[\pi_{i^*} \leq n^{-1}\right] \geq 1 - \delta. \qquad \text{(S2.49)}$$

Together with (S2.42), the last display implies that the sequence of thresholds $t_n = 1/n$ perfectly separates $H_0^{(n)}$ from $H_1^{(n)}$. □

### S2.5 Proof of Theorem 4

Let

$$\eta := 1 - \alpha(1; r, \sigma) - \beta. \qquad \text{(S2.50)}$$

The condition $r > \rho_{\mathsf{Bonf}}(\beta, \sigma)$ implies $\eta > 0$. By continuity of $q \to \alpha(q; r, \sigma)$, there exits $\delta > 0$ such that

$$1 - \alpha(1 + \delta; r, \sigma) - \beta > \eta/2. \tag{S2.51}$$

For the statistic $p_{(1)} := \min_{i=1\dots,n} p_i$, we show that, along the sequence of thresholds $a_n = n^{-(1+\eta/2)}$, we have $\Pr_{H_0^{(n)}}(p_{(1)} > a_n) \to 1$ while $\Pr_{H_1^{(n)}}(p_{(1)} > a_n) \to 0$. Indeed,

$$\begin{aligned}
\Pr_{H_0^{(n)}}\left[p_{(1)} \le a_n\right] &= 1 - \prod_{i=1}^{n} \Pr_{H_0}\left[p_i > a_n\right] \\
&= 1 - \left(1 - a_n \cdot n^{o(1)}\right)^n \qquad \text{(S2.52)} \\
&= 1 - \left(1 - n^{-(1+\eta/2)+o(1)}\right)^n \to 0,
\end{aligned}$$

where (S2.52) follows from (2.11). On the other hand,

$$\begin{aligned}
\Pr_{H_1^{(n)}}\left[p_{(1)} \le a_n\right] &= 1 - \prod_{i=1}^{n} \Pr_{H_1^{(n)}}\left[p_i > a_n\right] \\
&= 1 - \prod_{i=1}^{n} \left(1 - \Pr_{H_1^{(n)}}\left[p_i \le a_n\right]\right), \qquad \text{(S2.53)}
\end{aligned}$$

hence it is enough to show that $\Pr_{H_1^{(n)}}[p_i \le a_n] > n^{-1+\eta/2+o(1)}$ uniformly in $i$. For $i = 1, \ldots, n$ let $X_i$ be a RV with law $-2\log(X_i) \overset{D}{=} Q_i^{(n)}$. We have:

$$
\begin{aligned}
\Pr_{H_1^{(n)}}[p_i \le a_n] &= (1-\epsilon_n) a_n \cdot n^{o(1)} + \epsilon_n \Pr[X_i \le a_n] \\
&\ge \epsilon_n \Pr[X_i \le a_n] \\
&= n^{-\beta - \alpha(1+\delta; r, \sigma) + o(1)} \qquad \text{(S2.54)} \\
&\ge n^{-1+\eta/2+o(1)}, \qquad \text{(S2.55)}
\end{aligned}
$$

where (S2.54) follows from (2.8), and (S2.55) follows from (S2.51). □

## S2.6 Proof of Theorem 5

The proof is similar to the proof of Theorem 1.4 in (Donoho and Jin, 2004). The main idea is to establish the following claims:

(i) Inference based on FDR thresholding ignores P-values in the range $(n^{-q}, 1]$, for $q < 1$.

(ii) When $r < \rho_{\mathsf{Bonf}}(\beta, \sigma)$, P-values smaller than $n^{-q}$ under $H_1^{(n)}$ are as frequent as under $H_0^{(n)}$.

In order to establish (i) and (ii), define, for an interval $I \subset [0, 1]$,

$$
T_I := \min_{i\,:\,p_{(i)} \in I} \frac{p_{(i)}}{i/n}. \qquad \text{(S2.56)}
$$

For some $q > 0$ and a sequence $\{a_n\}_{n=1}^{\infty}$ of threshold values with $\liminf_{n\to\infty} a_n = 0$,

$$\left| \Pr_{H_0^{(n)}} [\text{FDR rejects}] - \Pr_{H_1^{(n)}} [\text{FDR rejects}] \right| = \left| \Pr_{H_0^{(n)}} \left[T_{[0,1]} < a_n\right] - \Pr_{H_1^{(n)}} \left[T_{[0,1]} < a_n\right] \right|$$

$$\le \Pr_{H_1^{(n)}} \left[T_{(n^{-q},1]} < a_n\right] + \Pr_{H_0^{(n)}} \left[T_{(n^{-q},1]} < a_n\right] \tag{S2.57}$$

$$+ \left| \Pr_{H_1^{(n)}} \left[T_{(0,n^{-q}]} < a_n\right] - \Pr_{H_0^{(n)}} \left[T_{(0,n^{-q}]} < a_n\right] \right|. \tag{S2.58}$$

Note that the terms in (S2.57) are associated with (i) while (S2.58) is associated with (ii).

Lemma 7 implies that the terms in (S2.57) vanish as $n \to \infty$. We now focus on the term (S2.58). Let $I \subset \{1, \ldots, n\}$ be a random set such that $i \in I$ with probability $\epsilon_n = n^{-\beta}$. Considering this randomness, an equivalent way of specifying $H_1^{(n)}$ is

$$-2\log(p_i) \sim \begin{cases} Q_i^{(n)} & i \in I \\ \text{Exp}(2) & i \neq I, \end{cases} \qquad i = 1, \ldots, n. \tag{S2.59}$$

For $i = 1, \ldots, n$, let $X_i$ be a RV satisfying $-2\log(X_i) \overset{D}{=} Q_i^{(n)}$. Choose $r < q < 1$ such that

$$1 - \alpha(q; r, \sigma) - \beta + \delta < 0 \tag{S2.60}$$

for some $\delta > 0$, which is possible since $r < \rho_{\mathsf{Bonf}}(\beta, \sigma) < 1$. Consider the event:

$$E_n^q := \{p_i \leq n^{-q} \text{ for some } i \in I\}.$$

Conditioned on the event $|I| = M$, we have, e.g. by the normal approximation to the Binomial distribution,

$$\begin{aligned}
\Pr\left[E_n^q \mid |I| = M\right] &= \Pr\left[\min_{i=1,\ldots,n} X_i \leq n^{-q} \mid |I| = M\right] \\
&\leq 1 - \left(1 - n^{-\alpha(q;r,\sigma)+o(1)}\right)^M && \text{(S2.61)} \\
&\leq M \cdot n^{-\alpha(q;r,\sigma)+o(1)} && \text{(S2.62)}
\end{aligned}$$

where (S2.61) follows from (2.8) and (S2.62) follows from the inequality $M \cdot \log(1+x) > \log(1+Mx)$, $x \geq -1$. As $M \sim \text{Bin}(n, \epsilon_n)$, we have $\Pr\left[M < n^{1+\delta/2}\epsilon_n\right] = \Pr\left[M < n^{1+\delta/2-\beta}\right] \to 1$. Consequently, for any $\epsilon$,

$$\Pr\left[M \cdot n^{-\alpha(q;r,\sigma)+o(1)} > \epsilon\right] \leq o(1) + \mathbf{1}_{\{n^{1+\delta/2-\beta-\alpha(q;r,\sigma)+o(1)} > \epsilon\}} \to 0$$

where the last transition is due to (S2.60). It follows that $\Pr\left[E_n^q\right] \to 0$. From here, since

$$\Pr_{H_1^{(n)}}\left[T_{[0,n^{-q})} < a_n \mid (E_n^q)^c\right] = \Pr_{H_0^{(n)}}\left[T_{[0,n^{-q})} < a_n\right],$$

we get

$$\Pr_{H_1^{(n)}}\left[T_{(0,n^{-q}]} < a_n\right] = \Pr\left[(E_n^q)^c\right] \Pr_{H_1^{(n)}}\left[T_{[0,n^{-q})} < a_n \mid (E_n^q)^c\right]$$

$$+ \Pr\left[E_n^q\right] \Pr_{H_1^{(n)}}\left[T_{[0,n^{-q})} < a_n \mid E_n^q\right]$$

$$= \Pr_{H_1^{(n)}}\left[T_{[0,n^{-q})} < a_n \mid (E_n^q)^c\right](1+o(1)) + o(1)$$

$$= \Pr_{H_0^{(n)}}\left[T_{[0,n^{-q})} < a_n\right] + o(1),$$

so that (S2.58) vanishes as well and the proof is completed. □

### S2.7 Proof of Theorem 6

We consider first the case where $\{-2\log(p_i)\}$ follow (2.3) and later extend our arguments to the general case of (2.10). Since $F_n \sim \chi^2_{2n}$, under the null in (2.3) we have

$$\mathbb{E}\left[F_n|H_0^{(n)}\right] = 2n, \qquad \mathrm{Var}\left[F_n|H_0^{(n)}\right] = 4n. \tag{S2.63}$$

As $F_n$ is asymptotically normal, it is enough to show that

$$\mathbb{E}\left[F_n|H_1^{(n)}\right] \sim 2n(1+o(1/\sqrt{n})), \quad \text{and} \quad \mathrm{Var}\left[F_n|H_1^{(n)}\right] \sim 4n(1+o(1)). \tag{S2.64}$$

For $X \sim \chi^2(r,\sigma)$, we have

$$\mathbb{E}\left[X\right] = \mu_n(r)^2 + \sigma^2, \qquad \mathbb{E}\left[X^2\right] = \mu_n^4(r) + 4\mu_n^2(r)\sigma^2 + 3\sigma^4, \tag{S2.65}$$

and $\mathrm{Var}\left[X\right] = 2\mu_n^2(r)\sigma^2 + 2\sigma^4$ hence it follows that with $Q_i^{(n)} = \chi^2(r,\sigma)$,

$$2n \leq \mathbb{E}\left[F_n | H_1^{(n)}\right] = 2n(1-\epsilon_n) + n \cdot \epsilon_n \left(2r\log(n) + \sigma^2\right) = 2n(1+o(1/\sqrt{n})),$$

where in the last transition we used that $\beta > 1/2$. Similarly, we have

$$\begin{aligned} 4n \leq \mathrm{Var}\left[F_n^2 | H_1^{(n)}\right] &= 4n(1-\epsilon_n) + n \cdot \epsilon_n \left(4r\log(n)\sigma^2 + 2\sigma^2\right) \\ &= 4n(1+o(1/\sqrt{n})) = 4n(1+o(1)) \end{aligned}$$

hence (S2.64) holds in this case.

For the general case of (2.10) under (2.8), first note that

$$\begin{aligned} \mathbb{E}_{X \sim E_i^{(n)}}\left[X\right] &= \int_0^\infty \Pr\left[X \geq x | X \sim E_i^{(n)}\right] dx \\ &= 2\log(n) \int_0^\infty \Pr\left[X \geq 2y\log(n) | X \sim E_i^{(n)}\right] dy \\ &= 2\log(n) \int_0^\infty e^{-y\log(n)(1+o(1))} dy = 2(1+o(1)), \end{aligned} \tag{S2.66}$$

and

$$\begin{aligned} \mathbb{E}_{X \sim E_i^{(n)}}\left[X^2\right] &= \int_0^\infty x \Pr\left[X \geq x | X \sim E_i^{(n)}\right] dx \\ &= (2\log(n))^2 \int_0^\infty y \Pr\left[X \geq 2y\log(n) | X \sim E_i^{(n)}\right] dy \\ &= (2\log(n))^2 \int_0^\infty y e^{-y\log(n)(1+o(1))} dy = 4(1+o(1)). \end{aligned}$$

It follows that

$$\mathbb{E}\left[F_n | H_0^{(n)}\right] = 2n(1+a_n), \quad \text{and} \quad \mathrm{Var}\left[F_n | H_0^{(n)}\right] = 4n(1+o(1)), \tag{S2.67}$$

where $a_n \to 0$. Next, notice that

$$\begin{aligned}
\mathbb{E}_{X\sim Q_i^{(n)}}[X] &= \int_0^\infty \Pr\left[X \geq x | X \sim Q_i^{(n)}\right] dx \\
&= 2\log(n) \int_0^\infty \Pr\left[X \geq 2y\log(n) | X \sim Q_i^{(n)}\right] dy \\
&= \int_0^\infty n^{-\alpha(y;r,\sigma)+o(1)} dy, \qquad \text{(S2.68)} \\
&= n^{o(1)}, \qquad \text{(S2.69)}
\end{aligned}$$

where (S2.68) follows from (2.8) and (S2.69) follows since

$$\int_0^\infty n^{-\alpha(y;r,\sigma)} dy = \frac{\sigma^2 n^{-\frac{r}{\sigma^2}}}{\log(n)} + 2\sigma\sqrt{\frac{\pi r}{\log(n)}}\Phi\left(\frac{\sqrt{2r\log(n)}}{\sigma}\right) = o(1).$$

From (S2.66) and because $\beta > 1/2$,

$$\begin{aligned}
2n \leq \mathbb{E}\left[F_n | H_1^{(n)}\right] &= 2n(1-\epsilon_n)(1+a_n)) + \epsilon_n n^{1+o(1)} \\
&\leq 2n(1+a_n) + o(1/\sqrt{n})
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}_{X\sim Q_i^{(n)}}\left[X^2\right] &= \int_0^\infty x\Pr\left[X \geq x | X \sim Q_i^{(n)}\right] dx \\
&= (2\log(n))^2 \int_0^\infty y\Pr\left[X \geq 2y\log(n) | X \sim Q_i^{(n)}\right] dy \\
&= (2\log(n))^2 \int_0^\infty y \cdot n^{-\alpha(y;r,\sigma)+o(1)} dy = n^{o(1)},
\end{aligned}$$

where in the last transition we used that

$$\int_0^\infty y \cdot n^{-\alpha(y;r,\sigma)} dy = o(1),$$

as can be deduced from the analytic expression of this integral. We obtain

$$4n \leq \mathbb{E}\left[F_n^2 | H_1^{(n)}\right] = 4n(1-\epsilon_n)(1+o(1)) + n \cdot \epsilon_n \cdot n^{o(1)} \tag{S2.70}$$

$$= 4n(1+o(1)). \tag{S2.71}$$

Evaluations similar to those in (S2.67) and (S2.71) imply that $F_n$ satisfies the conditions of the Lyaponov central limit theorem for sums of independent but perhaps non-identically distributed RVs. Consequently, $F_n$ is asymptotically normal both under $H_0^{(n)}$ and $H_1^{(n)}$. Since we have

$$\frac{\mathbb{E}\left[F_n | H_1^{(n)}\right] - \mathbb{E}\left[F_n | H_0^{(n)}\right]}{\sqrt{\mathrm{Var}\left[F_n | H_0^{(n)}\right]}} \to 0, \quad \text{and} \quad \frac{\mathrm{Var}\left[F_n | H_0^{(n)}\right]}{\mathrm{Var}\left[F_n | H_1^{(n)}\right]} \to 1,$$

we conclude that $F_n$ is asymptotically powerless (e.g., c.f. (Arias-Castro and Wang, 2013, Lem. B.2)). □

# S3 Proofs of Results in Section 3

## S3.1 Proof of Proposition 7

We use Lemma8 with $\lambda_n = \lambda_i$, $a_n = 2q\log(n)$ and $b_n = 2r\log(n)$. We obtain

$$-\log \Pr\left[-2\log \bar{\mathsf{P}}(X_i;\lambda) \geq 2q\log(n)\right] = \log(n)\alpha(q;r,1)(1+o(1)),$$

where $o(1) \to 0$ independently of $\lambda_i$. Proposition 7 follows. □

## S3.2 Proof of Proposition 9

Because $\mathrm{Bin}(m, 1/2)$ is symmetric around $m/2$, for $x \geq 0$ we have

$$\begin{aligned} p_{\mathsf{Bin}}(x) &= \Pr\left[\left|\mathrm{Bin}(m,1/2) - \frac{m}{2}\right| \geq |x - m/2|\right] \\ &= 2\Pr\left[\frac{\mathrm{Bin}(m,1/2) - \frac{m}{2}}{\sqrt{m}/2} \geq \frac{|x-m/2|}{\sqrt{m}/2}\right]. \end{aligned}$$

A Berry-Essen type argument applied to the binomial survival function implies

$$\left| p_{\mathsf{Bin}}(x) - 2\bar{\Phi}\left(\frac{|x-m/2|}{\sqrt{m}/2}\right)\right| \leq \frac{C_1}{\sqrt{t(x)}}, \quad t(x) = \frac{|x-m/2|}{\sqrt{m}/2}.$$

for some constant $C$. Therefore, by Mill's ratio, as $t \to \infty$,

$$-2\log p_{\mathsf{Bin}}(x) = \left(\frac{x-m/2}{\sqrt{m}/2}\right)^2 + O(1).$$

The central limit theorem implies

$$\frac{X - m(1/2+\delta)}{\sqrt{m(1-\delta^2)}} \overset{D}{=} Z + o_p(1), \quad Z \sim \mathcal{N}(0,1),$$

as $m \to \infty$, hence

$$t(X) + o_p(1) \overset{D}{=} \sqrt{1-4\delta^2}Z + 2\sqrt{m}\delta = \sqrt{1-sr}Z + \sqrt{2r\log(n)}$$

converges to infinity in probability as $n \to \infty$. We obtain

$$\begin{aligned}
\log \Pr\left[p_{\mathsf{Bin}}(X) \geq 2q\log(n)\right] &= \log \Pr\left[\left(\frac{X - m/2}{\sqrt{m}/2}\right)^2 + O_P(1) \geq 2q\log(n)\right] \\
&= \log \Pr\left[\frac{X - m/2}{\sqrt{m}/2} \geq \sqrt{2q\log(n)}(1+o(1))\right] \\
&= \log \Pr\left[\sqrt{1-4\delta^2}\frac{X - m(1/2+\delta)}{\sqrt{m(1/4-\delta^2)}} + 2\sqrt{m}\delta \geq \sqrt{2q\log(n)}(1+o(1))\right] \\
&= \log \Pr\left[\sqrt{1-sr}\frac{X - m(1/2+\delta)}{\sqrt{m(1/4-\delta^2)}} + \sqrt{2r\log(n)} \geq \sqrt{2q\log(n)}(1+o(1))\right] \\
&= \log \Pr\left[\frac{X - m(1/2+\delta)}{\sqrt{m(1/4-\delta^2)}} \geq \sqrt{2\log(n)}\frac{\sqrt{q}-\sqrt{r}}{\sqrt{1-rs}}(1+o(1))\right] \\
&= \log \Pr\left[\frac{X - m(1/2+\delta)}{\sqrt{m(1/4-\delta^2)}} \geq \sqrt{2\log(n)\alpha(q;r,1-sr)}(1+o(1))\right].
\end{aligned}$$

From here, (3.42) follows by applying Cramér's (1.1).

### S3.3 Proof of Proposition 8

Our analysis relies on moderate deviation estimate for variance-stabilized Poisson counts as provided in the following lemma from (Donoho and Kipnis, 2022).

**Lemma 11.** *(Donoho and Kipnis, 2022, Lemma 5.3) Let $\Upsilon'_\lambda, \Upsilon_\lambda$ denote two independent Poisson RVs. Let $a(\lambda)$ be a non-negative function. Consider a sequence of pairs $(\lambda, \lambda')$ such that $\lambda \to \infty$, $\lambda' \geq \lambda$, $\lambda'/\lambda \to 1$ as $n \to \infty$.*

*Also suppose* $a(\lambda) - (\sqrt{2\lambda'} - \sqrt{2\lambda}) \to \infty$ *while* $a(\lambda)/\lambda \to 0$. *Then:*

$$\lim_{n\to\infty} \frac{1}{\left(\sqrt{a(\lambda)} - (\sqrt{2\lambda'} - \sqrt{2\lambda})\right)^2} \log\left[\Pr\left(\sqrt{2\Upsilon_{\lambda'}} - \sqrt{2\Upsilon_\lambda} \geq \sqrt{a(\lambda)}\right)\right] = -\frac{1}{2}.$$

Let $W_i := \sqrt{2Y_i} - \sqrt{2X_i}$ and $S_i := -2\log(\pi_i)$. We have

$$\begin{aligned}\Pr\left[S_i > 2q\log(n)\right] &= \Pr\left[2\bar{\Phi}(W_i) < n^{-q}\right] \\ &= \Pr\left[W_i > \bar{\Phi}^{-1}(n^{-q}/2)\right]\end{aligned}$$

By Mill's ratio,

$$\bar{\Phi}^{-1}(n^{-q}/2) = \sqrt{2q\log(n)(1+o(1))}.$$

We now apply Lemma 11 with $\lambda = \lambda_i$, $\lambda' = \lambda_i + \sqrt{2r\log(n)\lambda_i}$ and $\sqrt{a(\lambda)} = \bar{\Phi}^{-1}(n^{-q}/2)$.

Let $\lambda_i' = \lambda_i + \sqrt{\mu_n(r)\lambda_i}$ Note that $\lambda'/\lambda = 1 + \sqrt{2r\log(n)/\lambda_i} \to 1$ uniformly in $i \leq n$ by (3.31), we have

$$\sqrt{2\lambda'} - \sqrt{2\lambda} = \sqrt{r\log(n)}(1+o(1))$$

where here and henceforth $o(1)$ indicates a sequence tending to $\to 0$ uniformly in $i$. Consequently,

$$\begin{aligned}\sqrt{a(\lambda)} - (\sqrt{2\lambda'} - \sqrt{2\lambda}) &= \sqrt{2q\log(n)(1+o(1))} - \sqrt{2\lambda}\,(1+o(1)) \\ &=: 2\log(n)\alpha(q; r/2, 1)(1+o(1)).\end{aligned}$$

Lemma 11 implies

$$\log\left(\Pr\left[W_i > \bar{\Phi}^{-1}(n^{-q}/2)\right]\right) = -\log(n)\alpha(q;r/2,1)(1+o(1)),$$

hence

$$\max_{1\le i\le n}\left|\frac{-\log\Pr\left[S_i > 2q\log(n)\right]}{\log(n)} - \alpha(q;r/2,1)\right| \to 0.$$

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