Supplementary Materials for

Combining p-values using heavy tailed distributions and their asymptotic results with applications to genomic data

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Supplementary Material

In this supplementary materials, we provide all proofs of theorems, lemmas and corollaries. Additional results for numerical studies are also presented. We introduce some notations. Denote $f(x) \sim g(x)$ as $x \to c$ if $\lim_{x\to c} f(x)/g(x) = 1$. The statement $a \lesssim b$ means that $a \leq \gamma \cdot b$, where $\gamma > 0$ is a fixed constant independent of dimension d. Let $\phi(x)$, $\Phi(x)$ and $\bar{\Phi}(x)$ be the density function, the cumulative distribution function and the survival function of standard normal distribution, respectively. For $f(x) > 0$ and $g(x) > 0$, we define $f(x) \ll g(x)$ if $\limsup_{x\to\infty} f(x)/g(x) = C$ for a constant $C \in (1, \infty)$.

S1 Proofs of Theorem 1, 2 and 3 and Lemma 1

S1.1 Proof of Theorem 1

1. Under the null hypothesis, with $0 \leq \rho_{ij} = \rho < 1$, we have

$$\begin{aligned}
P(T_{\text{Stouffer}} \geq z_\alpha) &= P\left(\frac{1}{\sqrt{d + d(d-1)\rho}} \sum_{i=1}^{d} \bar{\Phi}^{-1}(p_i) \geq \frac{\sqrt{d}}{\sqrt{d + d(d-1)\rho}} z_\alpha\right) \\
&= P\left(\frac{1}{\sqrt{d + d(d-1)\rho}} \sum_{i=1}^{d} X_i \geq \frac{\sqrt{d}}{\sqrt{d + d(d-1)\rho}} z_\alpha\right) \\
&= 1 - \Phi\left(\frac{\sqrt{d} z_\alpha}{\sqrt{d + d(d-1)\rho}}\right) \\
&= 1 - \Phi\left(\frac{z_\alpha}{\sqrt{1 + (d-1)\rho}}\right).
\end{aligned}$$

Hence, for fixed $\rho > 0$, we have

$$\lim_{d \to \infty} P(T_{\text{Stouffer}} \geq z_\alpha) = \frac{1}{2},$$

and if $\rho = o(1/d)$, we obtain

$$\lim_{d \to \infty} P(T_{\text{Stouffer}} \geq z_\alpha) = \alpha.$$

2. To show the case of T_{Fisher}, we first introduce a lemma in Alouini et al. (2001).

Lemma S1 (Corollary 1 in Alouini et al. (2001))**.** *Let* $\{Y_i\}_{i=1}^d$ *be a set of* d *correlated gamma variates with parameters* φ *and* ψ*, i.e., for* $i = 1, \ldots, d$, $Y_i \sim$ *Gamma*(φ, ψ) *and let* $\xi = \mathsf{Corr}(Y_i, Y_j)$*. Then the pdf* *of* $Y = \sum_{i=1}^{d} Y_i$ *can be expressed as follows :*

$$p_{T_{Fisher}}(t) = \left[\prod_{i=1}^{d}\left(\frac{\lambda_1}{\lambda_i}\right)^{\varphi}\right]\sum_{k=0}^{\infty}\frac{\delta_k t^{d\varphi+k-1}e^{-t/\lambda_1}}{\lambda_1^{d\varphi+k}\Gamma(d\varphi+k)}I(t>0), \qquad \text{(S1.1)}$$

where $\Gamma(\cdot)$ *is the gamma function,* $\lambda_1 = \min_{1\le i\le d}\{\lambda_i\}$*,* $\{\lambda\}_{i=1}^{d}$ *are the eigenvalues of the matrix* $A = DC$ *where* D *is a* $d\times d$ *diagonal matrix whose diagonal elements are* ψ *and* C *is a* $d\times d$ *matrix whose diagonal entries are 1 and off-diagonal entries are* $\sqrt{\xi}$ *and the coefficients* δ_k *can be obtained recursively:*

$$\begin{aligned}\delta_0 &= 1 && \text{(S1.2)}\\ \delta_{k+1} &= \frac{\varphi}{k+1}\sum_{i=1}^{k+1}\left[\sum_{j=1}^{d}\left(1-\frac{\lambda_1}{\lambda_j}\right)^{i}\right]\delta_{k+1-i}, \quad k\ge 0. && \text{(S1.3)}\end{aligned}$$

For $i = 1, \ldots, d$, let $Y_i = -2\log(p_i) - 2$, then we have $Y_i \sim \chi^2(2d) \overset{d}{=}$ Gamma$(1, 2)$ and $T_{\text{Fisher}} = \sum_{i=1}^{d} Y_i$ by definition. For exchangeable p-values, let $\xi = \mathsf{Corr}(Y_i, Y_j)$. Then, for $\varphi = 1$ and $\psi = 2$, the density function $p_{T_{\text{Fisher}}}(t)$ of T_{Fisher} is given as follows:

$$p_{T_{\text{Fisher}}}(t) = \left(\frac{\lambda_1}{\lambda_d}\right)\sum_{k=0}^{\infty}\frac{\left(1-\frac{\lambda_1}{\lambda_d}\right)^{k} t^{d+k-1}e^{-t/\lambda_1}}{\lambda_1^{d+k}\Gamma(d+k)}I(t>0), \qquad \text{(S1.4)}$$

where $\lambda_1 = 2(1-\sqrt{\xi})$ and $\lambda_d = 2(1+(d-1)\sqrt{\xi})$. When $\xi = 0$, since $\lambda_1 = \lambda_2 = \cdots = \lambda_d = 2$, we have

$$p_{T_{\text{Fisher}}}(t) = \frac{t^{d-1}e^{-t/2}}{2^d\Gamma(d)}I(t>0),$$

which is the density function of Gamma$(d,2) \stackrel{d}{=} \chi^2(2d)$. Assuming $\xi > 0$, we decompose $p_{T_{\text{Fisher}}}$ into two parts:

$$\begin{aligned}
p_{T_{\text{Fisher}}}(t) &= \frac{1}{\lambda_1}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}\left(\frac{\lambda_1}{\lambda_d}\right)\sum_{k=0}^{\infty}\frac{\left(1-\frac{\lambda_1}{\lambda_d}\right)^{d+k-1}t^{d+k-1}e^{-t/\lambda_1}}{\lambda_1^{d+k-1}\Gamma(d+k)}, \\
&= \frac{1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}e^{-t/\lambda_1}\sum_{k=0}^{\infty}\frac{\left(\frac{t}{\lambda_1}-\frac{t}{\lambda_d}\right)^{d+k-1}}{(d+k-1)!} \\
&= \frac{1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}e^{-t/\lambda_1}\left[\sum_{s=0}^{\infty}\frac{\left(\frac{t}{\lambda_1}-\frac{t}{\lambda_d}\right)^{s}}{s!}-\sum_{s=0}^{d-2}\frac{\left(\frac{t}{\lambda_1}-\frac{t}{\lambda_d}\right)^{s}}{s!}\right] \\
&= \frac{1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}e^{-t/\lambda_1}\left[e^{t\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_d}\right)}-\sum_{s=0}^{d-2}\frac{\left(\frac{t}{\lambda_1}-\frac{t}{\lambda_d}\right)^{s}}{s!}\right] \\
&= \frac{1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}e^{-t/\lambda_d}-\frac{1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}e^{-t/\lambda_1}\sum_{s=0}^{d-2}\frac{\left(\frac{t}{\lambda_1}-\frac{t}{\lambda_d}\right)^{s}}{s!} \\
&= A(t)-B(t).
\end{aligned}$$

To obtain the tail probability of T_{Fisher} such as $P(T_{\text{Fisher}} \geq \chi^2_\alpha(2d)) = \int_{\chi^2_\alpha(2d)}^{\infty} p_{T_{\text{Fisher}}}(t)dt = \int_{\chi^2_\alpha(2d)}^{\infty} A(t)dt - \int_{\chi^2_\alpha(2d)}^{\infty} B(t)dt$, we first consider the first term which is

$$\begin{aligned}
\int_{\chi^2_\alpha(2d)}^{\infty} A(t)dt &= \frac{1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}\int_{\chi^2_\alpha(2d)}^{\infty} e^{-t/\lambda_d}dt \\
&= \left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d} e^{-\frac{\chi^2_\alpha(2d)}{\lambda_d}} \\
&= \left(1-\frac{\lambda_1}{\lambda_d}\right)\left(1-\frac{\lambda_1}{\lambda_d}\right)^{-d} e^{-\frac{\chi^2_\alpha(2d)}{\lambda_d}}.
\end{aligned}$$

Since the upper α quantile of chi-square distribution $\chi^2_\alpha(2d) = 2d(1+$

$O(1/\sqrt{d})$), we have $e^{-\frac{\chi_\alpha^2(2d)}{\lambda_d}} = e^{-\frac{1}{\sqrt{\xi}} - O(\frac{1}{\sqrt{d\xi}})}$ which leads to

$$\int_{\chi_\alpha^2(2d)}^\infty A(t)dt = \left(1 - \frac{\lambda_1}{\lambda_d}\right)\left(1 - \frac{\lambda_1}{\lambda_d}\right)^{-d} e^{-\frac{1}{\sqrt{\xi}} - O(\frac{1}{\sqrt{d\xi}})}. \quad \text{(S1.5)}$$

We now consider the integration of $B(t)$. Using the uniform convergence for this series as in Moschopoulos (1985),

$$\begin{aligned}
&\int_{\chi_\alpha^2(2d)}^\infty B(t)dt \\
=\ & \frac{\lambda_1}{\lambda_d}\left(1 - \frac{\lambda_1}{\lambda_d}\right)^{1-d} \sum_{s=0}^{d-2} \left(1 - \frac{\lambda_1}{\lambda_d}\right)^s \frac{1}{\lambda_1^{s+1}\Gamma(s+1)} \int_{\chi_\alpha^2(2d)}^\infty t^s e^{-t/\lambda_1} dt \\
=\ & \frac{\lambda_1}{\lambda_d}\left(1 - \frac{\lambda_1}{\lambda_d}\right)^{1-d} \sum_{s=0}^{d-2} \left(1 - \frac{\lambda_1}{\lambda_d}\right)^s \frac{1}{\lambda_1^{s+1}\Gamma(s+1)} \left(\frac{\lambda_1}{2}\right)^{s+1} \int_{2\chi_\alpha^2(2d)/\lambda_1}^\infty y^s e^{-y/2} dy.
\end{aligned}$$

Integrating by parts iteratively implies that, for $s < d$,

$$\begin{aligned}
&\int_{2\chi_\alpha^2(2d)/\lambda_1}^\infty y^{d-1} e^{-y/2} dy \\
=\ & 2\left(\frac{2\chi_\alpha^2(2d)}{\lambda_1}\right)^{d-1} e^{-\frac{\chi_\alpha^2(2d)}{\lambda_1}} + 2(d-1)\int_{2\chi_\alpha^2(2d)/\lambda_1}^\infty y^{d-2} e^{-y/2} dy \\
=\ & 2\left(\frac{2\chi_\alpha^2(2d)}{\lambda_1}\right)^{d-1} e^{-\frac{\chi_\alpha^2(2d)}{\lambda_1}} + 2^2(d-1)\left(\frac{2\chi_\alpha^2(2d)}{\lambda_1}\right)^{d-2} e^{-\frac{\chi_\alpha^2(2d)}{\lambda_1}} \\
& \quad + 2^2(d-1)(d-2)\int_{2\chi_\alpha^2(2d)/\lambda_1}^\infty y^{d-3} e^{-y/2} dy \\
& \vdots \\
=\ & \left[\sum_{w=1}^{d-s-1} 2^w \frac{(d-1)!}{(d-w)!}\left(\frac{2\chi_\alpha^2(2d)}{\lambda_1}\right)^{d-w} e^{-\chi_\alpha^2(2d)/\lambda_1}\right] \\
& + 2^{d-s-1}\frac{(d-1)!}{s!}\int_{2\chi_\alpha^2(2d)/\lambda_1}^\infty y^s e^{-y/2} dy.
\end{aligned}$$

From the definition of $\chi_\alpha^2(2d)$, define $0 < \gamma_{\xi,d} < 1$ which depends on ξ

and d such that

$$\gamma_{\xi,d} := \frac{\int_{2\chi^2_\alpha(2d)/\lambda_1}^{\infty} y^{d-1}e^{-y/2}dy}{2^d\Gamma(d)} < \frac{\int_{\chi^2_\alpha(2d)}^{\infty} y^{d-1}e^{-y/2}dy}{2^d\Gamma(d)} = \alpha.$$

Then we have

$$\begin{aligned}
\gamma_{\xi,d} &= \frac{\int_{2\chi^2_\alpha(2d)/\lambda_1}^{\infty} y^{d-1}e^{-y/2}dy}{2^d\Gamma(d)} \\
&= \frac{1}{2^d\Gamma(d)}\left\{\left[\sum_{w=1}^{d-s-1} 2^w \frac{(d-1)!}{(d-w)!}\left(\frac{2\chi^2_\alpha(2d)}{\lambda_1}\right)^{d-w} e^{-\chi^2_\alpha(2d)/\lambda_1}\right] + 2^{d-s-1}\frac{(d-1)!}{s!}\int_{2\chi^2_\alpha(2d)/\lambda_1}^{\infty} y^s e^{-y/2}dy\right\},
\end{aligned}$$

so that $\int_{2\chi^2_\alpha(2d)/\lambda_1}^{\infty} y^s e^{-y/2}dy$ can be decomposed as follows:

$$\begin{aligned}
&\int_{2\chi^2_\alpha(2d)/\lambda_1}^{\infty} y^s e^{-y/2}dy \\
&= \frac{\gamma_{\xi,d}\cdot 2^d\Gamma(d)}{2^{d-s-1}(d-1)!/s!} - \frac{\sum_{w=1}^{d-s-1} 2^w \frac{(d-1)!}{(d-w)!}\left(\frac{2\chi^2_\alpha(2d)}{\lambda_1}\right)^{d-w} e^{-\chi^2_\alpha(2d)/\lambda_1}}{2^{d-s-1}(d-1)!/s!} \\
&= \underbrace{\gamma_{\xi,d}\cdot 2^{s+1}s!}_{=B_1} - \underbrace{\sum_{w=1}^{d-s-1}\frac{1}{(d-w)!}\left(\frac{\chi^2_\alpha(2d)}{\lambda_1}\right)^{d-w} e^{-\chi^2_\alpha(2d)/\lambda_1}2^{s+1}s!}_{=B_2} \\
&= B_1 - B_2.
\end{aligned}$$

Thus, we have

$$\int_{\chi^2_\alpha(2d)}^{\infty} B(t)dt = \frac{\lambda_1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}\sum_{s=0}^{d-2}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{s}\frac{1}{\lambda_1^{s+1}\Gamma(s+1)}\left(\frac{\lambda_1}{2}\right)^{s+1}\cdot(B_1-B_2).$$

First, consider B_1.

$$\begin{aligned}
&\frac{\lambda_1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}\sum_{s=0}^{d-2}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{s}\frac{1}{\lambda_1^{s+1}\Gamma(s+1)}\left(\frac{\lambda_1}{2}\right)^{s+1}\cdot B_1\\
&=\frac{\lambda_1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}\sum_{s=0}^{d-2}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{s}\frac{1}{\lambda_1^{s+1}s!}\left(\frac{\lambda_1}{2}\right)^{s+1}\gamma_{\xi,d}\cdot 2^{s+1}s!\\
&=\gamma_{\xi,d}\cdot\frac{\lambda_1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}\sum_{s=0}^{d-2}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{s}\\
&=\gamma_{\xi,d}\cdot\frac{\lambda_1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}\frac{1-\left(1-\frac{\lambda_1}{\lambda_d}\right)^{d-1}}{1-\left(1-\frac{\lambda_1}{\lambda_d}\right)}\\
&=\gamma_{\xi,d}\cdot\left[\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}-1\right]. \qquad \text{(S1.6)}
\end{aligned}$$

Next, consider B_2.

$$\begin{aligned}
&\frac{\lambda_1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}\sum_{s=0}^{d-2}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{s}\frac{1}{\lambda_1^{s+1}\Gamma(s+1)}\left(\frac{\lambda_1}{2}\right)^{s+1}\cdot B_2\\
&=\frac{\lambda_1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}\sum_{s=0}^{d-2}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{s}\frac{1}{\lambda_1^{s+1}\Gamma(s+1)}\left(\frac{\lambda_1}{2}\right)^{s+1}\sum_{w=1}^{d-s-1}\frac{1}{(d-w)!}\left(\frac{\chi^2_\alpha(2d)}{\lambda_1}\right)^{d-w}e^{-\frac{\chi^2_\alpha(2d)}{\lambda_1}}2^{s+1}s!\\
&=\frac{\lambda_1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}e^{-\frac{\chi^2_\alpha(2d)}{\lambda_1}}\sum_{s=0}^{d-2}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{s}\cdot\sum_{w=1}^{d-s-1}\frac{1}{(d-w)!}\left(\frac{\chi^2_\alpha(2d)}{\lambda_1}\right)^{d-w}\\
&=\frac{\lambda_1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}e^{-\frac{\chi^2_\alpha(2d)}{\lambda_1}}\sum_{s=0}^{d-2}\left(\frac{\chi^2_\alpha(2d)}{\lambda_1}\right)^{s+1}\frac{1}{(s+1)!}\sum_{j=0}^{s}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{j}\\
&=\frac{\lambda_1}{\lambda_d}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}e^{-\frac{\chi^2_\alpha(2d)}{\lambda_1}}\sum_{s=0}^{d-2}\left(\frac{\chi^2_\alpha(2d)}{\lambda_1}\right)^{s+1}\frac{1}{(s+1)!}\frac{1-\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1+s}}{1-\left(1-\frac{\lambda_1}{\lambda_d}\right)}\\
&=\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}e^{-\frac{\chi^2_\alpha(2d)}{\lambda_1}}\sum_{s=0}^{d-2}\left(\frac{\chi^2_\alpha(2d)}{\lambda_1}\right)^{s+1}\frac{1}{(s+1)!}\left[1-\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1+s}\right]\\
&=\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}e^{-\frac{\chi^2_\alpha(2d)}{\lambda_1}}\left[\sum_{s=0}^{d-2}\left(\frac{\chi^2_\alpha(2d)}{\lambda_1}\right)^{s+1}\frac{1}{(s+1)!}-\sum_{s=0}^{d-2}\left(\frac{\chi^2_\alpha(2d)}{\lambda_1}\right)^{s+1}\frac{1}{(s+1)!}\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1+s}\right]\\
&=\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}\left[\sum_{s=0}^{d-2}\frac{\left(\frac{\chi^2_\alpha(2d)}{\lambda_1}\right)^{s+1}e^{-\frac{\chi^2_\alpha(2d)}{\lambda_1}}}{(s+1)!}-\sum_{s=0}^{d-2}\frac{\left(\frac{\chi^2_\alpha(2d)}{\lambda_1}\left(1-\frac{\lambda_1}{\lambda_d}\right)\right)^{s+1}e^{-\frac{\chi^2_\alpha(2d)}{\lambda_1}}}{(s+1)!}\right]\\
&=:\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}\left(B_{2,1}-B_{2,2}\right).
\end{aligned}$$

Let $f_{G(\varphi,\psi)}(x)$ be a density function of Gamma distribution with parameter φ and ψ. Then

$$\begin{aligned}
B_{2,1} &= \lambda_1 \sum_{s=0}^{d-2} \frac{\left(\chi^2_\alpha(2d)\right)^{s+1} e^{-\frac{\chi^2_\alpha(2d)}{\lambda_1}}}{\lambda_1^{s+2}\Gamma(s+2)} \\
&= \lambda_1 \sum_{s=0}^{d-2} f_{G(s+2,\lambda_1)}(\chi^2_\alpha(2d)) \\
B_{2,2} &= \sum_{s=0}^{d-2} \frac{\left(\chi^2_\alpha(2d)\left(1-\frac{\lambda_1}{\lambda_d}\right)\right)^{s+1} e^{-\frac{\chi^2_\alpha(2d)\left(1-\frac{\lambda_1}{\lambda_d}\right)}{\lambda_1}} \cdot e^{-\frac{\chi^2_\alpha(2d)\frac{\lambda_1}{\lambda_d}}{\lambda_1}}}{\lambda_1^{s+1}(s+1)!} \\
&= \lambda_1 e^{-\frac{\chi^2_\alpha(2d)}{\lambda_d}} \sum_{s=0}^{d-2} f_{G(s+2,\lambda_1)}\left(\chi^2_\alpha(2d)\left(1-\frac{\lambda_1}{\lambda_d}\right)\right).
\end{aligned}$$

In $B_{2,1}$,

$$f_{G(s+2,\lambda_1)}(\chi^2_\alpha(2d)) = f_{G(s+2,\lambda_1)}(\chi^2_\alpha(2d)) - f_{G(s+2,\lambda_1)}(0) = \int_0^{\chi^2_\alpha(2d)} \frac{\partial f_{G(s+2,\lambda_1)}(x)}{\partial x} dx,$$

where

$$\begin{aligned}
\frac{\partial f_{G(s+2,\lambda_1)}(x)}{\partial x} &= \frac{\partial}{\partial x}\left\{\frac{x^{s+1}e^{-\frac{x}{\lambda_1}}}{\lambda_1^{s+2}\Gamma(s+2)}\right\} \\
&= \frac{1}{\lambda_1^{s+2}(s+1)!}\left[(s+1)x^s e^{-\frac{x}{\lambda_1}} - \frac{1}{\lambda_1}x^{s+1}e^{-\frac{x}{\lambda_1}}\right] \\
&= \frac{x^s e^{-\frac{x}{\lambda_1}}}{\lambda_1^{s+2}s!} - \frac{1}{\lambda_1}\frac{x^{s+1}e^{-\frac{x}{\lambda_1}}}{\lambda_1^{s+2}(s+1)!} \\
&= \frac{1}{\lambda_1}f_{G(s+1,\lambda_1)}(x) - \frac{1}{\lambda_1}f_{G(s+2,\lambda_1)}(x).
\end{aligned}$$

Then we have

$$f_{G(s+2,\lambda_1)}(\chi^2_\alpha(2d)) = \int_0^{\chi^2_\alpha(2d)} \frac{1}{\lambda_1} f_{G(s+1,\lambda_1)}(x)dx - \int_0^{\chi^2_\alpha(2d)} \frac{1}{\lambda_1} f_{G(s+2,\lambda_1)}(x)dx,$$

so that

$$\lambda_1 \sum_{s=0}^{d-2} f_{G(s+2,\lambda_1)}(\chi^2_\alpha(2d)) = \int_0^{\chi^2_\alpha(2d)} f_{G(1,\lambda_1)}(x)dx - \int_0^{\chi^2_\alpha(2d)} f_{G(d,\lambda_1)}(x)dx. \tag{S1.7}$$

Using similar steps in $B_{2,1}$ implies that for $B_{2,2}$,

$$\begin{aligned}
&\lambda_1 \cdot e^{-\frac{\chi^2_\alpha(2d)}{\lambda_d}} \sum_{s=0}^{d-2} f_{G(s+2,\lambda_1)}\left(\chi^2_\alpha(2d)\left(1-\frac{\lambda_1}{\lambda_d}\right)\right)\\
&= e^{-\frac{\chi^2_\alpha(2d)}{\lambda_d}}\left[\int_0^{\chi^2_\alpha(2d)\left(1-\frac{\lambda_1}{\lambda_d}\right)} f_{G(1,\lambda_1)}(x)dx - \int_0^{\chi^2_\alpha(2d)\left(1-\frac{\lambda_1}{\lambda_d}\right)} f_{G(d,\lambda_1)}(x)dx\right]\\
&= e^{-\frac{1}{\sqrt{\xi}}-O(\frac{1}{\sqrt{d\xi}})}\left[\int_0^{\chi^2_\alpha(2d)\left(1-\frac{\lambda_1}{\lambda_d}\right)} f_{G(1,\lambda_1)}(x)dx - \int_0^{\chi^2_\alpha(2d)\left(1-\frac{\lambda_1}{\lambda_d}\right)} f_{G(d,\lambda_1)}(x)dx\right]
\end{aligned} \tag{S1.8}$$

From (S1.5), (S1.6), (S1.7) and (S1.8),

$$\begin{aligned}
P(T_{\text{Fisher}} \geq \chi^2_\alpha(2d)) = &\left(1-\frac{\lambda_1}{\lambda_d}\right)\left(1-\frac{\lambda_1}{\lambda_d}\right)^{-d} e^{-\frac{1}{\sqrt{\xi}}-O(\frac{1}{\sqrt{d\xi}})} - \gamma_{\xi,d}\cdot\left[\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}-1\right]\\
&+\left(1-\frac{\lambda_1}{\lambda_d}\right)^{1-d}\left\{\int_0^{\chi^2_\alpha(2d)} f_{G(1,\lambda_1)}(x)dx - \int_0^{\chi^2_\alpha(2d)} f_{G(d,\lambda_1)}(x)dx\right.\\
&\left.-e^{-\frac{1}{\sqrt{\xi}}-O(\frac{1}{\sqrt{d\xi}})}\left[\int_0^{\chi^2_\alpha(2d)\left(1-\frac{\lambda_1}{\lambda_d}\right)} f_{G(1,\lambda_1)}(x)dx - \int_0^{\chi^2_\alpha(2d)\left(1-\frac{\lambda_1}{\lambda_d}\right)} f_{G(d,\lambda_1)}(x)dx\right]\right\}.
\end{aligned} \tag{S1.9}$$

Since as $d \to \infty$,

$$\left(1-\frac{\lambda_1}{\lambda_d}\right)^{-d} \stackrel{d\to\infty}{\longrightarrow} e^{-\left(1-\frac{1}{\sqrt{\xi}}\right)},$$

we have the following asymptotic result; For fixed $\xi > 0$,

$$\lim_{d\to\infty} P(T_{\text{Fisher}} \geq \chi^2_\alpha(2d)) = \frac{1}{e} + \gamma_\xi\left(1-\frac{1}{e}\right),$$

where $\gamma_\xi := \lim_{d\to\infty} \gamma_{\xi,d} = 0$ since $\gamma_{\xi,d} = P\left(\frac{G(d,2)-2d}{\sqrt{4d}} > \frac{\frac{\chi^2_\alpha(2d)}{1-\sqrt{\xi}}-2d}{\sqrt{4d}}\right)$ and

$$\left|\gamma_{\xi,d} - 1 + \Phi\left(\frac{\frac{\chi^2_\alpha(2d)}{1-\sqrt{\xi}} - 2d}{\sqrt{4d}}\right)\right| \le \sup_t |F_d(t) - \Phi(t)| \le O\left(\frac{1}{\sqrt{d}}\right)$$

as $d \to \infty$ due to the Berry-Essen Theorem where $F_d(t) = P\left(\frac{G(d,2)-2d}{\sqrt{4d}} \le t\right)$. Since $\frac{\frac{\chi^2_\alpha(2d)}{1-\sqrt{\xi}}-2d}{\sqrt{4d}} \to \infty$, we have $1 - \Phi\left(\frac{\frac{\chi^2_\alpha(2d)}{1-\sqrt{\xi}}-2d}{\sqrt{4d}}\right) \to 0$. Therefore, we have $\gamma_{\xi,d} \to 0$. Therefore we have, for fixed $\xi > 0$,

$$\lim_{d\to\infty} P(T_{\text{Fisher}} \ge \chi^2_\alpha(2d)) = \frac{1}{e}.$$

Now, assume that $\xi = o(1/d)$, that is, $d\xi \to 0$. To derive the type I error probability, consider the following three cases:

(i) $d\sqrt{\xi} \to \infty$,

(ii) $d\sqrt{\xi} \to c$ for some constant $c > 0$,

(iii) $d\sqrt{\xi} \to 0$.

For the first case of (i), since we have

$$\left(1 - \frac{\lambda_1}{\lambda_d}\right)^{-d} = \left(1 - \frac{1-\sqrt{\xi}}{1+(d-1)\sqrt{\xi}}\right)^{-d} = \exp\left(\frac{d(1-\sqrt{\xi})}{1+(d-1)\sqrt{\xi}}\right) \overset{d\to\infty}{\longrightarrow} 0$$

and

$$\begin{aligned}
\gamma_{\xi,d} &\sim 1-\Phi\left(\frac{z_\alpha\sqrt{4d}+z_\alpha^2/2+2d\sqrt{\xi}}{(1-\sqrt{\xi})\sqrt{4d}}\right)\\
&= 1-\Phi\left(\frac{z_\alpha}{1-\sqrt{\xi}}+\frac{z_\alpha^2}{4(1-\sqrt{\xi})\sqrt{d}}+\frac{\sqrt{d\xi}}{1-\sqrt{\xi}}\right)\\
&\to 1-\Phi(z_\alpha)=\alpha
\end{aligned}$$

from (S1.9), we obtain

$$\lim_{d\to\infty} P(T_{\text{Fisher}} \geq \chi_\alpha^2(2d)) = \alpha.$$

The case of (ii), $d\sqrt{\xi}\to c$, implies that

$$\left(1-\frac{\lambda_1}{\lambda_d}\right)^{-d} \overset{d\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \gamma_{\xi,d}\to\alpha$$

leading to

$$\lim_{d\to\infty} P(T_{\text{Fisher}} \geq \chi_\alpha^2(2d)) = \alpha.$$

Now, consider (iii), i.e., $d\sqrt{\xi}\to 0$. (S1.4) can be expressed as a sum of two parts:

$$p_{T_{\text{Fisher}}}(t) = \left(\frac{\lambda_1}{\lambda_d}\right)\frac{t^{d-1}e^{-t/\lambda_1}}{\lambda_1^d\Gamma(d)}I(t>0) + \left(\frac{\lambda_1}{\lambda_d}\right)\sum_{k=1}^{\infty}\frac{\left(1-\frac{\lambda_1}{\lambda_d}\right)^k t^{d+k-1}e^{-t/\lambda_1}}{\lambda_1^{d+k}\Gamma(d+k)}I(t>0).$$

Hence, by the uniform convergence theorem, we have

$$\begin{aligned}
&P(T_{\text{Fisher}} \geq \chi_\alpha^2(2d))\\
= &\underbrace{\left(\frac{\lambda_1}{\lambda_d}\right)\gamma_{\xi,d}}_{(I)} + \underbrace{\left(\frac{\lambda_1}{\lambda_d}\right)\sum_{k=1}^{\infty}\left(1-\frac{\lambda_1}{\lambda_d}\right)^k P(G(d+k,\lambda_1)>\chi_\alpha^2(2d))}_{(II)},
\end{aligned} \tag{S1.10}$$

where $G(d+k,\lambda_1)$ denotes a random variable following the Gamma distribution with parameters $d+k$ and λ_1. For (I) in (S1.10), it is easy to show that $\gamma_{\xi,d} \to \alpha$ as $d \to \infty$. For (II) in (S1.10), since $d\sqrt{\xi} \to 0$ implies $\lambda_d \to \lambda_1$, we have

$$\begin{aligned}
(II) &\leq \left(\frac{\lambda_1}{\lambda_d}\right) \sum_{k=1}^{\infty} \left(1 - \frac{\lambda_1}{\lambda_d}\right)^k \\
&= \left(\frac{\lambda_1}{\lambda_d}\right) \lim_{N\to\infty} \sum_{k=1}^{N} \left(1 - \frac{\lambda_1}{\lambda_d}\right)^k \\
&= \lim_{N\to\infty} \left(1 - \frac{\lambda_1}{\lambda_d}\right) \left[1 - \left(1 - \frac{\lambda_1}{\lambda_d}\right)^N\right] \\
&\leq 1 - \frac{\lambda_1}{\lambda_d} \\
&\to 0.
\end{aligned}$$

Therefore, we obtain $\lim_{d\to\infty} P(T_{\text{Fisher}} \geq \chi^2_\alpha(2d)) = \alpha$. From results of the cases (i), (ii) and (iii), it can be concluded that, if $\xi = o(1/d)$, we have $\lim_{d\to\infty} P(T_{\text{Fisher}} \geq \chi^2_\alpha(2d)) = \alpha$.

S1.2 Proof of Theorem 2

1. Under the null hypothesis,

$$\begin{aligned} P(T_{\text{Stouffer}} \geq z_\alpha) &= P\left(\frac{1}{\sqrt{d+\sum_{i\neq j}\rho_{ij}}}\sum_{i=1}^{d}\bar{\Phi}^{-1}(p_i) \geq \frac{\sqrt{d}}{\sqrt{d+\sum_{i\neq j}\rho_{ij}}} z_\alpha\right) \\ &= P\left(\frac{1}{\sqrt{d+\sum_{i\neq j}\rho_{ij}}}\sum_{i=1}^{d} X_i \geq \frac{\sqrt{d}}{\sqrt{d+\sum_{i\neq j}\rho_{ij}}} z_\alpha\right) \\ &= 1-\Phi\left(\frac{\sqrt{d}z_\alpha}{\sqrt{d+\sum_{i\neq j}\rho_{ij}}}\right) \\ &= 1-\Phi\left(\frac{z_\alpha}{\sqrt{1+\frac{1}{d}\sum_{i\neq j}\rho_{ij}}}\right) \geq \alpha. \end{aligned}$$

If $\sum_{i\neq j}\rho_{ij} = o(d)$,

$$\lim_{d} P(T_{\text{Stouffer}} \geq z_\alpha) = \alpha.$$

2. With $\varphi = 1$ and $\psi = 2$, (S1.1) can be expressed as follows:

$$P(T_{\text{Fisher}} = t) = \frac{\lambda_1^d}{\prod_{i=1}^d \lambda_i}\frac{t^{d-1}e^{-t/\lambda_1}}{\lambda_1^d\Gamma(d)} I(t>0) + \frac{\lambda_1^d}{\prod_{i=1}^d \lambda_i}\sum_{k=1}^{\infty}\frac{\delta_k t^{d+k-1}e^{-t/\lambda_1}}{\lambda_1^{d+k}\Gamma(d+k)} I(t>0).$$

Hence by the uniform convergence theorem, we have

$$\begin{aligned} P(T_{\text{Fisher}} \geq \chi^2_\alpha(2d)) =& \frac{\lambda_1^d}{\prod_{i=1}^d \lambda_i} P\left(G(d,\lambda_1) \geq \chi^2_\alpha(2d)\right) \\ &+ \frac{\lambda_1^d}{\prod_{i=1}^d \lambda_i}\sum_{k=1}^{\infty}\delta_k P\left(G(d+k,\lambda_1) \geq \chi^2_\alpha(2d)\right). \end{aligned} \tag{S1.11}$$

By the arithmetic and geometric means, the maximum of $\prod_{j=1}^{d} \lambda_j$ is attained at $\lambda_1 = \cdots = \lambda_d$. Since $\sum_{j=1}^{d} \lambda_j = 2d$, the maximum is attained at $\lambda_1 = \cdots = \lambda_d = 2$ which are eigenvalues of a $2 \cdot I$ where I is a $d \times d$ identity matrix. Also, when $\lambda_1 = \cdots = \lambda_d = 2$, we have $\delta_{k+1} = 0$ for $k \geq 0$ from (S1.3). Therefore the infimum of $P(T_{\text{Fisher}} \geq \chi^2_\alpha(2d))$ is attained at $\Sigma = I$, so that we have

$$\inf_{\Sigma \in \mathcal{F}_{d,\rho}} P_\Sigma(T_{\text{Fisher}} \geq \chi^2_\alpha(2d)) = P_I(T_{\text{Fisher}} \geq \chi^2_\alpha(2d)) = \alpha.$$

Furthermore, from (S1.11) and by the definition of δ_k for $k \geq 1$, we have the following property such that, if $\lambda_d/\lambda_1 \to 1$ then

$$P(T_{\text{Fisher}} \geq \chi^2_\alpha(2d)) \to \alpha$$

since $1 \leq \lambda_2/\lambda_1 \leq \cdots \leq \lambda_d/\lambda_1$. Note that λ_d/λ_1 is the condition number of the matrix $A = DC$ defined in (S1.1). Denote the condition number of A by $\kappa(A)$ and the condition number of C by $\kappa(C)$ then $\kappa(A) = \kappa(C)$, since $D = 2I$.

Now consider $\kappa(C)$. From Merikoski et al. (1997), we have

$$\kappa(C) \leq \left(\frac{1 + \sqrt{1 - \left(\frac{d}{\|C\|_F^2}\right)^d \mathsf{det}(C)^2}}{1 - \sqrt{1 - \left(\frac{d}{\|C\|_F^2}\right)^d \mathsf{det}(C)^2}} \right)^{1/2},$$

where $\|C\|_F$ is the Frobenius norm and $\|C\|_F^2 = d + \sum_{i \neq j} \xi_{ij}$. And by the definition of the condition number, we have $1 \leq \kappa(C)$. Hence, if

$\sqrt{1-\left(\frac{d}{\|C\|_F^2}\right)^d \mathsf{det}(C)^2} \to 0$, we have $\kappa(C) \to 1$. Indeed,

$$\left(\frac{d}{\|C\|_F^2}\right)^d \mathsf{det}(C)^2 = \frac{\mathsf{det}(C)^2}{\left(1+\frac{1}{d}\sum_{i\neq j}\xi_{ij}\right)^d}$$

and by the arithmetic and geometric means, that is, $\mathsf{det}(C) = \prod_{i=1}^d \lambda_i \leq 1^d$ we have that

$$\left(\frac{d}{\|C\|_F^2}\right)^d \mathsf{det}(C)^2 \leq 1.$$

Also, by Grone et al. (1984), if $s^2 = \|C\|_F^2/d - 1$,

$$\mathsf{det}(C) \geq \left(1 - s\sqrt{d-1}\right)\left(1 + s/\sqrt{d-1}\right)^{d-1}.$$

Hence, we have

$$\begin{aligned}\left(\frac{d}{\|C\|_F^2}\right)^d \mathsf{det}(C)^2 &\geq \left(1 - s\sqrt{d-1}\right)^2\left(1 + s/\sqrt{d-1}\right)^{2(d-1)}\left(\frac{1}{1+s^2}\right)^d \\ &\geq \left(1 - s\sqrt{d-1}\right)^2\left(\frac{1}{1+s^2}\right)^d \\ &\sim \left(1 - s\sqrt{d-1}\right)^2 e^{-ds^2}.\end{aligned}$$

From this result, if $ds^2 = \sum_{i\neq j}\xi_{ij} = o(1/d)$, we have $\kappa(A) = 1+o(1/d)$ which implies that $\lambda_1/\lambda_i = 1 - o(1/d)$ for all $i = 1, \ldots, d$ so that

$$\prod_{i=1}^d \left(\frac{\lambda_1}{\lambda_i}\right) = (1 - o(1/d))^d \sim e^{-o(1)}.$$

And since

$$\delta_k \leq \left[\sum_{j=1}^d \left(1 - \frac{\lambda_1}{\lambda_j}\right)\right]^k,$$

we have that, in (S1.11),

$$\sum_{k=1}^{\infty} \delta_k P\left(G(d+k, \lambda_1) \geq \chi^2_\alpha(2d)\right) \leq \sum_{k=1}^{\infty} \delta_k \leq \sum_{k=1}^{\infty} \left[\sum_{j=1}^{d} \left(1 - \frac{\lambda_1}{\lambda_j}\right)\right]^k = o(1).$$

Therefore, we have $\lim_{d\to\infty} P_\Sigma(T_{\text{Fisher}} \geq \chi^2_\alpha(2d)) = P_I(T_{\text{Fisher}} \geq \chi^2_\alpha(2d)) = \alpha$.

3. Regarding $T_{\text{minP}} = \max_{1\leq i\leq d} X_i$, we first use the result in Slepian (1962)

$$P_{\Sigma_1}(X_1 \leq c, X_2 \leq c, \ldots, X_d \leq c) \geq P_{\Sigma_2}(X_1 \leq c, X_2 \leq c, \ldots, X_d \leq c) \tag{S1.12}$$

for $\Sigma_1 = (\rho_{ij})$ and $\Sigma_2 = (\kappa_{ij})$ with $\rho_{ij} \geq \kappa_{ij}$ for all i and j. This implies that $P_{\Sigma_1}(\max_{1\leq i\leq d} X_i \leq c) \geq P_{\Sigma_2}(\max_{1\leq i\leq d} \leq c)$, equivalently

$$P_{\Sigma_1}(\max_{1\leq i\leq d} X_i \geq c) \leq P_{\Sigma_2}(\max_{1\leq i\leq d} \geq c). \tag{S1.13}$$

Therefore, for any $\Sigma \in \mathcal{F}_{\rho,d}$, we have $P_\Sigma(\max_{1\leq i\leq d} X_i \geq c) \leq P_I(\max_{1\leq i\leq d} \geq c)$ leading to

$$P_\Sigma(\max_{1\leq i\leq d} X_i \geq x_\alpha) \quad \leq \quad P_I(\max_{1\leq i\leq d} X_i \geq x_\alpha) = \alpha$$

where the last equality is due to the fact that x_α is the upper α quantile of $\max_{1\leq i\leq d} X_i$ under independence of X_i for $1 \leq i \leq d$.

To show the second result that $P_\Sigma\left(\max_i X_i \geq x_\alpha\right) \to 0$ as $d \to \infty$ for

Σ with $\rho_{ij} \geq \epsilon > 0$, we first derive the followings:

$$
\begin{aligned}
P_{\Sigma}(\max_i X_i \geq x_\alpha) &= 1 - P_{\Sigma}(\max_i X_i \leq x_\alpha) \\
&\leq 1 - P_{\Sigma_\epsilon}(\max_i X_i \leq x_\alpha) \\
&= 1 - \int_{-\infty}^{\infty} \Phi\left(\frac{x_\alpha - \epsilon^{1/2} y}{\sqrt{1-\epsilon}}\right)^d \phi(y) dy \\
&= \int_{-\infty}^{\infty} \left[1 - \Phi\left(\frac{x_\alpha - \epsilon^{1/2} y}{\sqrt{1-\epsilon}}\right)^d\right] \phi(y) dy \\
&\leq \int_{-\infty}^{y_0} \left[1 - \Phi\left(\frac{x_\alpha - \epsilon^{1/2} y}{\sqrt{1-\epsilon}}\right)^d\right] \phi(y) dy + \int_{y_0}^{\infty} \phi(y) dy \\
&\leq \int_{-\infty}^{y_0} d\left[1 - \Phi\left(\frac{x_\alpha - \epsilon^{1/2} y}{\sqrt{1-\epsilon}}\right)\right] \phi(y) dy + \delta \\
&\qquad \text{using } (1 - z^d) \leq d(1-z) \text{ for } z \in (0,1) \\
&\leq d\left[1 - \Phi\left(\frac{x_\alpha - \epsilon^{1/2} y_0}{\sqrt{1-\epsilon}}\right)\right] + \delta \\
&\leq d\frac{\sqrt{1-\epsilon}}{x_\alpha} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x_\alpha - \epsilon^{1/2} y_0)^2}{2(1-\epsilon)}\right) + \delta.
\end{aligned}
$$

Since $x_\alpha = \Phi^{-1}((1-\alpha)^{1/d})$ and $x_\alpha = \sqrt{2\log\frac{d}{\alpha}}(1 + o(1))$, we have

$$
\begin{aligned}
&\lim_{d\to\infty} P_{\Sigma}(\max_i X_i \geq x_\alpha) \\
&= \lim_{d\to\infty} O\left(\exp\left(\log d - \frac{1}{1-\epsilon}\log\frac{d}{\alpha} + \frac{\epsilon^{1/2} y_0}{1-\epsilon}\sqrt{2\log\frac{d}{\alpha}}\right)\right) + \delta \\
&= \delta. \qquad \text{(S1.14)}
\end{aligned}
$$

This is true for arbitrary $\delta > 0$. This implies that the limit (S1.14) is zero by letting $\delta \to 0$. Therefore, for Σ with $\rho_{ij} \geq \epsilon > 0$, we have $P_{\Sigma}(\max_i X_i \geq x_\alpha) \to 0$ as $d \to \infty$ for any given $\alpha > 0$.

S1.3 Proof of Lemma 1

We take $\xi_d = \sqrt{1-\rho}\left(\sqrt{2\log d} - \frac{\log\log d}{\sqrt{2\log d}}\right)$ and show that $P_\Sigma(\max X_{1\le i\le d} \le \xi_d) \to 0$.

We define Σ_ρ as an equi-correlated correlation matrix with the correlation $\rho \ge 0$. Based on (S1.13) for Σ_ρ and $\Sigma = (\rho_{ij})$ with $0 \le \rho_{ij} \le \rho < 1$ for all i and j, we have

$$\begin{aligned}
P_\Sigma(\max_{1\le i\le d} X_i \le \xi_d) &\le P_{\Sigma_\rho}(\max_{1\le i\le d} X_i \le \xi_d) \\
&= \int_{-\infty}^{\infty} \prod_{i=1}^{d} \Phi\left(\frac{\xi_d - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)\phi(y)dy \\
&\le \int_{-\infty}^{y_0} \prod_{i=1}^{d} \Phi\left(\frac{\xi_d - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)\phi(y)dy + \epsilon \\
&\le \Phi\left(\frac{\xi_d}{\sqrt{1-\rho}}\right)^d + \epsilon
\end{aligned}$$

for a sufficiently large $y_0 > 0$ so that $\int_{y_0}^{\infty}\phi(y)dy \le \epsilon$. Since

$$1 - \Phi\left(\frac{\xi_d}{\sqrt{1-\rho}}\right) \sim \frac{\sqrt{1-\rho}}{\sqrt{2\pi}\sqrt{2\log d}}\exp\left(-\log d + \log\log d - \frac{(\log\log d)^2}{4\log d}\right),$$

we have

$$d\left(1 - \Phi\left(\frac{\xi_d}{\sqrt{1-\rho}}\right)\right) \sim \frac{\sqrt{1-\rho}}{2\sqrt{\pi}}\sqrt{\log d} \to \infty$$

leading to

$$P_\Sigma(\max X_i \le \xi_d) \le \exp\left(-d\left(1 - \Phi\left(\frac{\xi_d}{\sqrt{1-\rho}}\right)\right)\right) + \epsilon \to \epsilon$$

as $d \to \infty$. Since ϵ is arbitrary, we let $\epsilon \to 0$ then we obtain $\lim_{d\to\infty} P_\Sigma\left(\max X_i \leq \xi_d\right) = 0$ for any $\Sigma \in \mathcal{F}_{\rho,d}$, equivalently $\lim_{d\to\infty} P_\Sigma\left(\max X_i > \xi_d\right) = 1$ which proves this lemma.

S1.4 Proof of Theorem 3

Before we present the proof of Theorem 3, we provide the following lemma which is used in the proof of Theorem 3.

Lemma S2. *For* $h^{-1}(t) = \Phi^{-1}\left[2\bar{\Phi}\left(\frac{1}{\sqrt{t}}\right)\right]$, *we have*

$$(1-O(1/(\log t)^2))(1+O(1/\log t))\sqrt{\log t} \leq h^{-1}(t) \leq (1+O(1/\log t))\sqrt{\log t}.$$

as $t \to \infty$.

Proof. By Mill's ratio, that is, for $x > 0$,

$$\frac{x\phi(x)}{1+x^2} \leq \bar{\Phi}(x) \leq \frac{\phi(x)}{x},$$

we have the following lower and upper bounds of Φ^{-1},

$$(1-O(1/(\log t)^2))\sqrt{-2\log(1-t)} \leq \Phi^{-1}(t) \leq \sqrt{-2\log(1-t)},$$

or equivalently, for small t,

$$(1-O(1/(\log t)^2))\sqrt{-2\log t} \leq \Phi^{-1}(1-t) \leq \sqrt{-2\log t}. \tag{S1.15}$$

By Taylor expansion at 0, we have

$$2\Phi\left(\frac{1}{\sqrt{t}}\right) - 1 = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}} - O(t^{-3/2})$$

leading to

$$\begin{aligned} h^{-1}(t) &= \Phi^{-1}\left[1-\left(2\Phi\left(\frac{1}{\sqrt{t}}\right)-1\right)\right] \\ &= \Phi^{-1}\left[1-\left(\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}-O(t^{-3/2})\right)\right]. \end{aligned} \tag{S1.16}$$

Using (S1.15) and (S1.16), we have the following upper and lower bounds of $h^{-1}(t)$ which is

$$(1-O(1/(\log t)^2))\sqrt{-2\log\left(\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}-O(t^{-3/2})\right)} \le h^{-1}(t) \le \sqrt{-2\log\left(\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}-O(t^{-3/2})\right)}.$$

Since $\sqrt{-2\log\left(\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}-O(t^{-3/2})\right)} = (1+O(1/\log t))\sqrt{\log t}$, it can be shown that

$$(1-O(1/(\log t)^2))(1+O(1/\log t))\sqrt{\log t} \le h^{-1}(t) \le (1+O(1/\log t))\sqrt{\log t}. \tag{S1.17}$$

$\square$

Proof of Theorem 3 : Now, we prove the Theorem 3. First, we consider splitting $P(T_{\text{Lévy}} \geq l_\alpha)$ into two parts as follows:

$$
\begin{aligned}
& P(T_{\text{Lévy}} \geq \ell_\alpha) \\
= \; & P\left(\frac{1}{d^2} \sum_{i=1}^{d} \left[\Phi^{-1}((1+p_i)/2)\right]^{-2} \geq \ell_\alpha \right) \\
= \; & \underbrace{P\left(\frac{1}{d^2} \sum_{i=1}^{d} \left[\Phi^{-1}((1+p_i)/2)\right]^{-2} \geq \ell_\alpha, \quad \bigcup_{j=1}^{d} \left\{ \left[\Phi^{-1}((1+p_i)/2)\right]^{-2} > d^2(1+\delta_\alpha)\ell_\alpha \right\} \right)}_{=P(A)} \\
& + \underbrace{P\left(\frac{1}{d^2} \sum_{i=1}^{d} \left[\Phi^{-1}((1+p_i)/2)\right]^{-2} \geq \ell_\alpha, \quad \bigcap_{j=1}^{d} \left\{ \left[\Phi^{-1}((1+p_i)/2)\right]^{-2} \leq d^2(1+\delta_\alpha)\ell_\alpha \right\} \right)}_{=P(B)} \\
\equiv \; & P(A) + P(B)
\end{aligned}
$$

where $\delta_\alpha > 0$ is a constant depending on only ℓ_α with $\delta_\alpha \to 0$. Denote transformation functions with respect to p-value and corresponding X by h_p and h_X, respectively. Specifically, we have

$$
h_p(p) = \left[\Phi^{-1}\left(\frac{1+p}{2}\right)\right]^{-2}, \quad \text{and} \quad h_X(X) = \left[\Phi^{-1}\left(\frac{2-\Phi(X)}{2}\right)\right]^{-2}.
$$

By the definition of one-sided p-value $p = 1 - \Phi(X)$ for X and the corresponding p-value p, we have $h_X(x) = h_p(p)$. Then we obtain

$$
A = \bigcup_{j=1}^{d} A_j = \bigcup_{j=1}^{d} \left\{ h_p(p_j) > d^2(1+\delta_\alpha)\ell_\alpha, \quad \sum_{i=1}^{d} h_p(p_i) \geq d^2 \ell_\alpha \right\}
$$

and

$$
B = \bigcap_{j=1}^{d} B_j = \bigcap_{j=1}^{d} \left\{ h_p(p_j) \leq d^2(1+\delta_\alpha)\ell_\alpha, \quad \sum_{i=1}^{d} h_p(p_i) \geq d^2 \ell_\alpha \right\}.
$$

Our goal is to show $\frac{P(T_{\text{Lévy}} \geq l_\alpha)}{\alpha} = \frac{P(A)}{\alpha} + \frac{P(B)}{\alpha} \to 1$, equivalently

$$\frac{P(A)}{\alpha} \to 1, \quad \text{and} \quad \frac{P(B)}{\alpha} \to 0$$

as $d \to \infty$ and $\alpha \to 0$.

1. **The proof of $\frac{P(B)}{\alpha} \to 0$.**

   We consider the bound of $P(B)$ consisting of two parts :

$$\begin{aligned}
P(B) &\leq \sum_{j=1}^{d} P\left(d\ell_\alpha \leq h_p\left(p_j\right) \leq d^2(1+\delta_\alpha)\ell_\alpha, \quad \sum_{i=1}^{d} h_p\left(p_i\right) \geq d^2\ell_\alpha\right) \\
&\leq \underbrace{\sum_{j=1}^{d} P\left(d\ell_\alpha \leq h_p\left(p_j\right) \leq d^2(1-\delta_\alpha)\ell_\alpha, \quad \sum_{i=1}^{d} h_p\left(p_i\right) \geq d^2\ell_\alpha\right)}_{=I_1} \\
&\quad + \underbrace{\sum_{j=1}^{d} P\left(d^2(1-\delta_\alpha)\ell_\alpha \leq h_p\left(p_j\right) \leq d^2(1+\delta_\alpha)\ell_\alpha\right)}_{=I_2} \\
&\equiv I_1 + I_2.
\end{aligned}$$

   Regarding I_1, we have

$$\begin{aligned}
I_1 &\leq \sum_{j=1}^{d} P\left(d\ell_\alpha \leq h_p\left(p_j\right) \leq d^2(1-\delta_\alpha)\ell_\alpha, \quad \sum_{i:i\neq j} h_p\left(p_i\right) \geq \delta_\alpha d^2\ell_\alpha\right) \\
&\leq \sum_{j=1}^{d} P\left(d\ell_\alpha \leq h_p\left(p_j\right) \leq d^2(1-\delta_\alpha)\ell_\alpha, \quad \bigcup_{i:i\neq j} \left\{h_p\left(p_i\right) \geq \delta_\alpha d\ell_\alpha\right\}\right) \\
&\leq \sum_{j=1}^{d} \sum_{i:i\neq j} \underbrace{P\left(d\ell_\alpha \leq h_p\left(p_j\right) \leq d^2(1-\delta_\alpha)\ell_\alpha, \quad h_p\left(p_i\right) \geq \delta_\alpha d\ell_\alpha\right)}_{=J_{ij}}.
\end{aligned}$$

   It can be shown that, using the bivariate normality of $X_i = \rho_{ij}X_j + \sqrt{1-\rho_{ij}^2}Z_{ij}$ where Z_{ij} is a standard normal random variable and independent of X_j, we derive an upper bound of J_{ij} as follows :

$$\begin{aligned}
J_{ij} &= P\left(h_p\left(p_j\right) \geq d\ell_\alpha,\ h_p\left(p_j\right) \leq d^2(1-\delta_\alpha)\ell_\alpha,\quad h_p\left(p_i\right) \geq \delta_\alpha d\ell_\alpha\right) \\
&= P\left(h_p\left(p_j\right) \geq d\ell_\alpha,\ h_X\left(X_j\right) \leq d^2(1-\delta_\alpha)\ell_\alpha,\quad h_X\left(X_i\right) \geq \delta_\alpha d\ell_\alpha\right) \\
&= P\left(h_p\left(p_j\right) \geq d\ell_\alpha,\ X_j \leq h_X^{-1}\left(d^2(1-\delta_\alpha)\ell_\alpha\right),\quad X_i \geq h_X^{-1}\left(\delta_\alpha d\ell_\alpha\right)\right) \\
&= P\left(h_p\left(p_j\right) \geq d\ell_\alpha,\ X_j \leq h_X^{-1}\left(d^2(1-\delta_\alpha)\ell_\alpha\right),\quad \rho_{ij}X_j + \sqrt{1-\rho_{ij}^2}Z_{ij} \geq h_X^{-1}\left(\delta_\alpha d\ell_\alpha\right)\right) \\
&\leq P\left(h_p\left(p_j\right) \geq d\ell_\alpha, Z_{ij} \geq \frac{1}{\sqrt{1-\rho_{ij}^2}}\left[h_X^{-1}\left(\delta_\alpha d\ell_\alpha\right) - \rho_{ij}h_X^{-1}\left(d^2(1-\delta_\alpha)\ell_\alpha\right)\right]\right) \\
&= P\left(h_p\left(p_j\right) \geq d\ell_\alpha\right) \cdot P\left(Z_{ij} \geq \frac{1}{\sqrt{1-\rho_{ij}^2}}\left[h_X^{-1}\left(\delta_\alpha d\ell_\alpha\right) - \rho_{ij}h_X^{-1}\left(d^2(1-\delta_\alpha)\ell_\alpha\right)\right]\right).
\end{aligned} \tag{S1.18}$$

Let $\delta_\alpha = \ell_\alpha^{-\epsilon}$ and $d = \ell_\alpha^{a_L}$. Then, using Lemma S2, we obtain

$$\begin{aligned}
h_X^{-1}\left(\delta_\alpha d\ell_\alpha\right) &\geq (1-O(1/\log(\delta_\alpha d\ell_\alpha))(1+O(1/\log(\delta_\alpha d\ell_\alpha)))\sqrt{(1+a_L-\epsilon)\log\ell_\alpha}, \\
h_X^{-1}\left(d^2(1-\delta_\alpha)\ell_\alpha\right) &\leq (1+O(1/\log(d^2\ell_\alpha)))\sqrt{(1+2a_L)\log\ell_\alpha},
\end{aligned}$$

where all $O(\cdot)$ are positives. Using these bounds, the second probability in (S1.18) is

$$
\begin{aligned}
& P\left(Z_{ij} \geq \frac{1}{\sqrt{1-\rho_{ij}^2}}\left[h_X^{-1}\left(\delta_\alpha d\ell_\alpha\right) - \rho_{ij} h_X^{-1}\left(d^2(1-\delta_\alpha)\ell_\alpha\right)\right]\right) \\
& \leq P\left(Z_{ij} \geq \frac{\left[(1-O(1/\log(\delta_\alpha d\ell_\alpha))\sqrt{1+a_L-\epsilon} - \rho_{ij}(1+O(1/\log(d^2\ell_\alpha)))\sqrt{1+2a_L}\right]}{\sqrt{1-\rho_{ij}^2}}\sqrt{\log \ell_\alpha}\right) \\
& = \bar{\Phi}\left(\frac{\left[(1-o(1))\sqrt{1+a_L-\epsilon} - \rho_{ij}(1+o(1))\sqrt{1+2a_L}\right]}{\sqrt{1-\rho_{ij}^2}}\sqrt{\log \ell_\alpha}\right) \\
& \overset{(i)}{\leq} \bar{\Phi}\left(\underbrace{\frac{\left[(1-o(1))\sqrt{1+a_L-\epsilon} - \rho(1+o(1)))\sqrt{1+2a_L}\right]}{\sqrt{1-\rho^2}}}_{=I}\sqrt{\log \ell_\alpha}\right) \\
& =: \bar{\Phi}(I \cdot \sqrt{\log \ell_\alpha}) \\
& \sim \frac{\ell_\alpha^{-I^2/2}}{I\sqrt{\log \ell_\alpha}},
\end{aligned}
\tag{S1.19}
$$

where the inequality (i) is due to the fact that $I(x) = \frac{\left[\sqrt{1+a_L-\epsilon} - x\sqrt{1+2a_L}\right]}{\sqrt{1-x^2}}$ is a decreasing function in $0 \leq x \leq \rho < 1$ since $\frac{dI(x)}{dx} = \frac{\rho\sqrt{1+a_L-\epsilon} - \sqrt{1+2a_L}}{(1-x^2)^{3/2}} < 0$. We decompose I in (S1.19) into two parts, I_A and I_B such that

$$
\begin{aligned}
I &= \frac{\left[\sqrt{1+a_L-\epsilon} - \rho\sqrt{1+2a_L}\right]}{\sqrt{1-\rho^2}} \\
&\quad + \frac{\left[-o(1)\sqrt{1+a_L-\epsilon} - \rho \cdot o(1) \cdot \sqrt{1+2a_L}\right]}{\sqrt{1-\rho^2}} \\
&=: I_A + I_B
\end{aligned}
$$

Then we have

$$
\frac{\ell_\alpha^{-I^2/2}}{I\sqrt{\log \ell_\alpha}} \sim \frac{c \cdot \ell_\alpha^{-I_A^2/2}}{I_A\sqrt{\log \ell_\alpha}},
$$

where $c > 0$ is a constant defined later since $I_B = O(1/\log(\delta_\alpha d\ell_\alpha))$ by definition and

$$\begin{aligned}
\frac{\ell_\alpha^{-I^2/2}}{I\sqrt{\log \ell_\alpha}} \bigg/ \frac{\ell_\alpha^{-I_A^2/2}}{I_A\sqrt{\log \ell_\alpha}} &= \frac{I_A}{I_A+I_B}\ell_\alpha^{-\frac{1}{2}I_B^2 - I_A\cdot I_B} \\
&= \frac{I_A}{I_A+I_B}\exp\left(O\left(\frac{\log \ell_\alpha}{\log \ell_\alpha^{a_L+1-\epsilon}}\right)\right) \\
&\to \exp\left(O\left(\frac{1}{a_L+1-\epsilon}\right)\right) =: c.
\end{aligned}$$

Thus, the second probability in (S1.18) can be expressed as follows :

$$\begin{aligned}
& P\left(Z_{ij} \geq \frac{1}{\sqrt{1-\rho_{ij}^2}}\left[h_X^{-1}\left(\delta_\alpha d\ell_\alpha\right) - \rho_{ij}h_X^{-1}\left(d^2(1-\delta_\alpha)\ell_\alpha\right)\right]\right) \\
\leq\ & \bar{\Phi}(I\cdot\sqrt{\log \ell_\alpha}) \sim \frac{c\cdot \ell_\alpha^{-I_A^2/2}}{I_A\sqrt{\log \ell_\alpha}}. \qquad \text{(S1.20)}
\end{aligned}$$

Furthermore, the first probability in (S1.18) is

$$P\left(h_p\left(p_j\right) \geq d\ell_\alpha\right) = P\left(h_p\left(p_j\right) \geq \ell_\alpha^{1+a_L}\right) \sim \ell_\alpha^{-\frac{1+a_L}{2}}. \qquad \text{(S1.21)}$$

Combining (S1.20) and (S1.21), (S1.18) can be bounded as follows:

$$J_{ij} \leq \text{(S1.18)} \lesssim \frac{\ell_\alpha^{-\frac{1+a_L+I_A^2}{2}}}{\sqrt{\log \ell_\alpha}}.$$

Therefore we have the following bound

$$I_1 \lesssim d^2\frac{\ell_\alpha^{-\frac{1+a_L+I_A^2}{2}}}{\sqrt{\log \ell_\alpha}} \lesssim \ell_\alpha^{2a_L-\frac{1+a_L+I_A^2}{2}}.$$

In order to show $I_1 = o(\ell_\alpha^{-1/2})$, it suffices to show $2a_L - \frac{1+a_L+I_A^2}{2} < -\frac{1}{2}$ which is the exponent of ℓ_α. Since we handle with only a tail probability, we can consider a case of $I_A > 0$. Hence we prove $2a_L - \frac{1+a_L+I_A^2}{2} < -\frac{1}{2}$ by showing $I_A > \sqrt{3a_L}$.

We show that the given condition on a_L satisfies $I_A > \sqrt{3a_L}$. We first define

$$\begin{aligned}
\kappa_1(\rho) &= 25\rho^4 - 28\rho^2 + 4, \\
\kappa_2(\rho) &= 5\rho^4 - (3+\epsilon)\rho^2 - 2(1-\epsilon), \\
\kappa_3(\rho) &= (\rho^2 - (1-\epsilon))^2.
\end{aligned}$$

By fundamental calculations, we have the following inequality

$$\kappa_1(\rho)a_L^2 + 2\kappa_2(\rho)a_L + \kappa_3(\rho) > 0. \tag{S1.22}$$

from $I_A^2 > 3a_L$.

From (S1.22), we consider three cases such as (i) : $\kappa_1(\rho) > 0$, (ii) : $\kappa_1(\rho) < 0$ and $\kappa(\rho) = 0$.

(i) For $\kappa_1(\rho) > 0$, that is, $0 < \rho < \frac{\sqrt{12}-\sqrt{2}}{5}$ or $\frac{\sqrt{12}+\sqrt{2}}{5} < \rho < 1$, we have

$$0 < a_L < U_{a_L}, \quad \text{or} \quad L_{a_L} < a_L \tag{S1.23}$$

where

$$\begin{aligned}
U_{a_L} &= \frac{-\kappa_2(\rho) + \sqrt{(\kappa_2(\rho))^2 - \kappa_1(\rho)\kappa_3(\rho)}}{\kappa_1(\rho)} \\
&= \frac{-(5\rho^4 - (3+\epsilon)\rho^2 - 2(1-\epsilon)) - 2\sqrt{3}\rho\sqrt{[(4-5\epsilon)\rho^2 - 2\epsilon^2 + 16\epsilon - 4](1+\rho^2) - 8\rho^2}}{25\rho^4 - 28\rho^2 + 4}, \\
L_{a_L} &= \frac{-\kappa_2(\rho) - \sqrt{(\kappa_2(\rho))^2 - \kappa_1(\rho)\kappa_3(\rho)}}{\kappa_1(\rho)} \\
&= \frac{5\rho^4 - (3+\epsilon)\rho^2 - 2(1-\epsilon) - 2\sqrt{3}\rho\sqrt{[(4-5\epsilon)\rho^2 - 2\epsilon^2 + 16\epsilon - 4](1+\rho^2) - 8\rho^2}}{-(25\rho^4 - 28\rho^2 + 4)}.
\end{aligned}$$

Note that the condition in (S1.23), $L_{a_L} < a_L$ can not imply $I_A > \sqrt{3a_L}$ so that for $\kappa_1(\rho) > 0$, the condition to hold $I_A > \sqrt{3a_L}$ is the following inequality:

$$0 < a_L < U_{a_L} \tag{S1.24}$$

(*ii*) For $\kappa_1(\rho) < 0$, that is, $\frac{\sqrt{12}-\sqrt{2}}{5} < \rho < \frac{\sqrt{12}+\sqrt{2}}{5}$, we have

$$L_{a_L} < a_L < U_{a_L}. \tag{S1.25}$$

For $\frac{\sqrt{12}-\sqrt{2}}{5} < \rho < \frac{\sqrt{12}+\sqrt{2}}{5}$ and sufficiently small $\epsilon > 0$, we obtain $L_{a_L} < 0$ so that (S1.25) can be expressed by

$$0 < a_L < U_{a_L}. \tag{S1.26}$$

(*iii*) We consider the case of $\kappa_1(\rho) = 0$, that is, $\rho^2 = \frac{14-4\sqrt{6}}{25}$ or $\rho^2 = \frac{14+4\sqrt{6}}{25}$. When $\kappa_1(\rho) = 0$, we have $a_L < -\frac{\kappa_3(\rho)}{\kappa_2(\rho)}$ from (S1.22) due to $\kappa_2(\rho) < 0$ for any given $\rho \in (0, 1)$ and sufficiently small ϵ. In fact, it can be calculated $U_{a_L} = -\frac{\kappa_3(\rho)}{\kappa_2(\rho)}$ if $\kappa_1(\rho) = 0$. Therefore, when $\kappa_1(\rho) = 0$, we have

$$0 < a_L < U_{a_L}. \tag{S1.27}$$

In particular, when $\rho^2 = \frac{14-4\sqrt{6}}{25}$, (S1.22) can be reduced as follows :

$$2\left(-\frac{(14-4\sqrt{6})(1+4\sqrt{6})}{125} - 2 - \epsilon\left(\frac{14-4\sqrt{6}}{25} - 2\right)\right)a_L + \left(-\frac{11+4\sqrt{6}}{25} + \epsilon\right)^2 > 0,$$

so that we have

$$a_L < \frac{\left(\frac{11+4\sqrt{6}}{25} - \epsilon\right)^2}{2\left(\frac{(14-4\sqrt{6})(1+4\sqrt{6})}{125} + 2 + \epsilon\left(\frac{14-4\sqrt{6}}{25} - 2\right)\right)}.$$

Similarly, when $\rho^2 = \frac{14+4\sqrt{6}}{25}$, we have,

$$a_L < \frac{\left(\frac{11-4\sqrt{6}}{25} - \epsilon\right)^2}{2\left(\frac{(14+4\sqrt{6})(1-4\sqrt{6})}{125} + 2 + \epsilon\left(\frac{14+4\sqrt{6}}{25} - 2\right)\right)}.$$

Combining the results from (S1.24), (S1.26) and (S1.27), we have the common condition

$$0 < a_L < U_{a_L} \tag{S1.28}$$

for three cases.

U_{a_L} in (S1.27) includes $\epsilon > 0$, but we eliminate ϵ by letting it converge to zero. More specifically, in the definition of $\delta_\alpha = d^{-\epsilon}$, the power $\epsilon > 0$

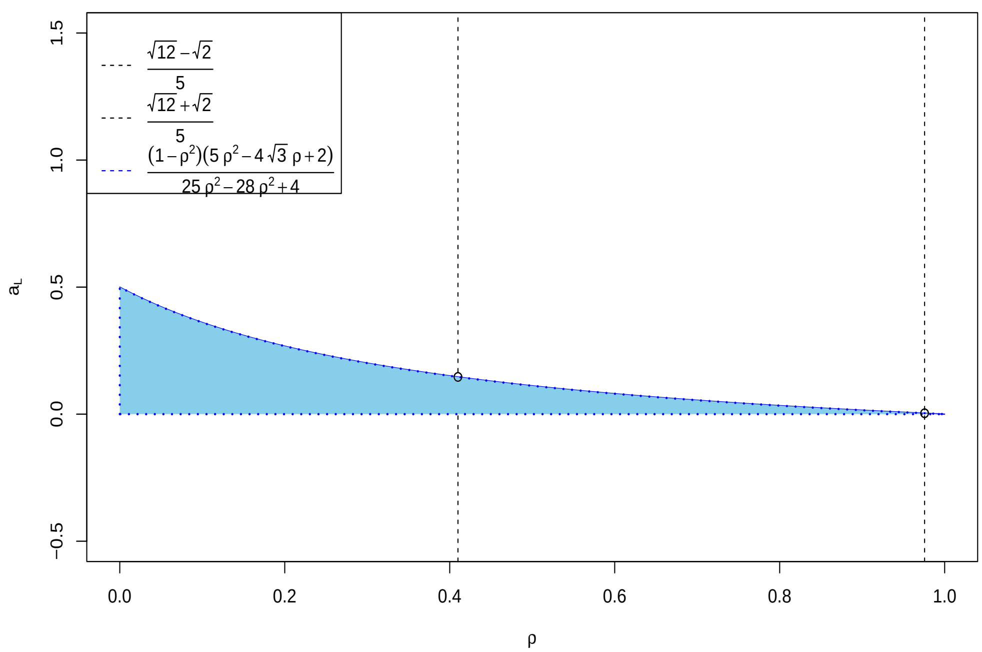

Figure S1: Region of a_L to satisfy $2a_L - \frac{1+a_L+I_A^2}{2} < -\frac{1}{2}$. Black dotted lines indicate $\kappa_1(\rho) = 0$ and blue dotted lines indicate U_{a_L}. The blue region represents (S1.29). Two points indicate upper bounds when $\kappa_1(\rho) = 0$, (S1.30) and (S1.31).

is defined arbitrarily and independent of a_L. Hence, as $\epsilon \to 0^+$, (S1.28) can be expressed as follows : for $0 < \rho < 1$ and $\rho^2 \neq (14 - 4\sqrt{6})/25$ and $\rho^2 \neq (14 + 4\sqrt{6})/25$, we have

$$0 < a_L < \frac{(1-\rho^2)(5\rho^2 - 4\sqrt{3}\rho + 2)}{25\rho^4 - 28\rho^2 + 4} = \frac{1-\rho^2}{5\rho^2 + 4\sqrt{3}\rho + 2}. \qquad \text{(S1.29)}$$

Also, for $\rho^2 = (14 - 4\sqrt{6})/25$ or $\rho^2 = (14 + 4\sqrt{6})/25$, we have

$$0 < a_L < \frac{(11 + 4\sqrt{6})^2}{5\left(2(14 - 4\sqrt{6})(1 + 4\sqrt{6}) + 500\right)}, \tag{S1.30}$$

and

$$0 < a_L < \frac{(11 - 4\sqrt{6})^2}{5\left(2(14 + 4\sqrt{6})(1 - 4\sqrt{6}) + 500\right)}, \tag{S1.31}$$

respectively.

Figure S1 represents regions of a_L satisfying $2a_L - \frac{1+a_L+I_A^2}{2} < -\frac{1}{2}$ for arbitrary small $\epsilon > 0$. To satisfy $I_A > \sqrt{3a_L}$, a_L should hold the blue region in Figure S1.

Now, we can show $\frac{P(B)}{\alpha} \leq \frac{I_1}{\alpha} + \frac{I_2}{\alpha} \to 0$ as $\alpha \to 0$ as follows. For a_L under (S1.29), it can be shown

$$\frac{\sqrt{1 + a_L} - \rho\sqrt{1 + 2a_L}}{\sqrt{1 - \rho^2}} > \sqrt{3a_L} \tag{S1.32}$$

which leads to

$$\frac{I_1}{\alpha} = o\left(\frac{\ell_\alpha^{-1/2}}{\alpha}\right) = o(1) \tag{S1.33}$$

due to $\alpha \sim \sqrt{\frac{2}{\pi}}\ell_\alpha^{-1/2}$.

For I_2, using $\alpha \sim \sqrt{\frac{2}{\pi}}\ell_\alpha^{-1/2}$ and $P(h_p(p) > c) \sim \sqrt{\frac{2}{\pi}}c^{-1/2}$ as $c \to \infty$, we have

$$\begin{aligned}\frac{I_2}{\alpha} =& \frac{1}{\alpha}\sum_{j=1}^{d} P\left(d^2(1-\delta_\alpha)\ell_\alpha \le h_p\left(p_j\right) \le d^2(1+\delta_\alpha)\ell_\alpha\right)\\ &= \frac{1}{\alpha}\sum_{j=1}^{d}\left[P\left(h_p\left(p_j\right) \ge d^2(1-\delta_\alpha)\ell_\alpha\right) - P\left(h_p\left(p_j\right) \ge d^2(1+\delta_\alpha)\ell_\alpha\right)\right]\\ &\sim \left[\frac{1}{\sqrt{1-\delta_\alpha}} - \frac{1}{\sqrt{1+\delta_\alpha}}\right] \to 0 \qquad \text{(S1.34)}\end{aligned}$$

since δ_α is defined to decrease to 0 as $\alpha \to 0$.

Using (S1.33) and (S1.34), we have

$$\frac{P(B)}{\alpha} \le \frac{I_1}{\alpha} + \frac{I_2}{\alpha} \to 0. \qquad \text{(S1.35)}$$

2. **Proof of $\frac{P(A)}{\alpha} \to 1$.**

By Boole's inequality, we have

$$\left|P(A) - \sum_{j=1}^{d} P\left(A_j\right)\right| \le \underbrace{\sum_{1\le i<j\le d} P(A_i \cap A_j)}_{=\mathcal{C}_1}$$

where

$$\begin{aligned}\sum_{j=1}^{d} P\left(A_j\right) \quad &= \quad \underbrace{\sum_{j=1}^{d} P\left(h_p\left(p_j\right) > d^2(1+\delta_\alpha)\ell_\alpha\right)}_{\mathcal{C}_2}\\ &\quad \underbrace{-\sum_{j=1}^{d} P\left(h_p\left(p_j\right) > d^2(1+\delta_\alpha)\ell_\alpha, \quad \sum_{i=1}^{d} h_p\left(p_i\right) \le d^2\ell_\alpha\right)}_{\mathcal{C}_3}\\ &= \quad \mathcal{C}_2 - \mathcal{C}_3.\end{aligned}$$

Since

$$\begin{aligned}\mathcal{C}_2 &= \sum_{j=1}^{d} P\left(h_p\left(p_j\right) > d^2(1+\delta_\alpha)\ell_\alpha\right) \\ &\sim d\sqrt{\frac{2}{\pi}}\frac{1}{d\sqrt{(1+\delta_\alpha)l_\alpha}} = \sqrt{\frac{2}{\pi}}{l_\alpha}^{-1/2} \sim \alpha, \end{aligned} \tag{S1.36}$$

and

$$\left\{h_p\left(p_j\right) > d^2(1+\delta_\alpha)\ell_\alpha, \quad \sum_{i=1}^{d} h_p\left(p_i\right) \le d^2\ell_\alpha\right\} = \emptyset$$

which implies $\mathcal{C}_3 = 0$, it suffices to show

$$\frac{\mathcal{C}_1}{\alpha} = \frac{1}{\alpha}\sum_{1\le i<j\le d} P(A_i \cap A_j) \to 0. \tag{S1.37}$$

We split $\mathcal{C}_1$ into two parts, namely $\mathcal{C}_{1,1}$ and $\mathcal{C}_{1,2}$ as follows :

$$\begin{aligned}\mathcal{C}_1 &= \sum_{1\le i<j\le d} P(A_i \cap A_j) \\ &= \sum_{1\le i<j\le d} P\left(h_p\left(p_i\right) > d^2(1+\delta_\alpha)\ell_\alpha, \quad h_p\left(p_j\right) > d^2(1+\delta_\alpha)\ell_\alpha, \quad \sum_{k=1}^{d} h_p\left(p_k\right) \ge d^2\ell_\alpha\right) \\ &= \sum_{1\le i<j\le d} P\left(h_p\left(p_i\right) > d^2(1+\delta_\alpha)\ell_\alpha, \quad h_p\left(p_j\right) > d^2(1+\delta_\alpha)\ell_\alpha\right) \\ &\quad - \sum_{1\le i<j\le d} P\left(h_p\left(p_i\right) > d^2(1+\delta_\alpha)\ell_\alpha, \quad h_p\left(p_j\right) > d^2(1+\delta_\alpha)\ell_\alpha, \quad \sum_{k=1}^{d} h_p\left(p_k\right) \le d^2\ell_\alpha\right) \\ &= \mathcal{C}_{1,1} - \mathcal{C}_{1,2}.\end{aligned}$$

From the bivariate normality and by an upper bound of multivariate Gaussian tail probability (Hashorva and Hüsler, 2003) with the fact that $(1-o(1))\sqrt{\log t} \le h^{-1}(t) \le (1+o(1))\sqrt{\log t}$ in (S2), we have

$$
\begin{aligned}
\frac{\mathcal{C}_{1,1}}{\alpha} &= \frac{1}{\alpha}\cdot \sum_{1\le i<j\le d} P\left(h_p\left(p_i\right) > d^2(1+\delta_\alpha)\ell_\alpha,\quad h_p\left(p_j\right) > d^2(1+\delta_\alpha)\ell_\alpha\right) \\
&= \frac{1}{\alpha}\cdot \sum_{1\le i<j\le d} P\left(X_i > h_X^{-1}\left(d^2(1+\delta_\alpha)\ell_\alpha\right),\quad X_j > h_X^{-1}\left(d^2(1+\delta_\alpha)\ell_\alpha\right)\right) \\
&\lesssim \frac{(1-\rho^2)\sqrt{1-\rho^2}}{2\pi(1-\rho)^2}\cdot d^2 \cdot \ell_\alpha^{\frac{1}{2}} \cdot \left[d^2(1+\delta_\alpha)\ell_\alpha\right]^{-\frac{1}{1+\rho}} \cdot (1+o(1)) \\
&= \frac{(1-\rho^2)\sqrt{1-\rho^2}}{2\pi(1-\rho)^2}\cdot (1+\delta_\alpha)^{-\frac{1}{1+\rho}} \cdot \ell_\alpha^{\frac{2\rho a_L - 1}{1+\rho}+\frac{1}{2}} \cdot (1+o(1)) \\
&\lesssim \ell_\alpha^{\frac{\rho(4a_L+1)-1}{2(1+\rho)}}.
\end{aligned}
$$

Hence, if $4a_L < 1/\rho - 1$, we have

$$
\begin{aligned}
\frac{\mathcal{C}_{1,1}}{\alpha} &= \frac{1}{\alpha} \sum_{1\le i<j\le d} P\left(h_p\left(p_i\right) > d^2(1+\delta_\alpha)\ell_\alpha,\quad h_p\left(p_j\right) > d^2(1+\delta_\alpha)\ell_\alpha\right) \\
&\to 0. \qquad \text{(S1.38)}
\end{aligned}
$$

Using result that $\mathcal{C}_{1,1}/\alpha \to 0$, we have

$$
\begin{aligned}
\frac{\mathcal{C}_{1,2}}{\alpha} &= \frac{1}{\alpha}\cdot \sum_{1\le i<j\le d} P\left(h_p\left(p_i\right) > d^2(1+\delta_\alpha)\ell_\alpha,\quad h_p\left(p_j\right) > d^2(1+\delta_\alpha)\ell_\alpha,\quad \sum_{k=1}^{d} h_p\left(p_k\right) \le d^2\ell_\alpha\right) \\
&\le \frac{1}{\alpha}\cdot \sum_{1\le i<j\le d} P\left(h_p\left(p_i\right) > d^2(1+\delta_\alpha)\ell_\alpha,\quad h_p\left(p_j\right) > d^2(1+\delta_\alpha)\ell_\alpha\right) \\
&= \frac{\mathcal{C}_{1,1}}{\alpha} \to 0. \qquad \text{(S1.39)}
\end{aligned}
$$

Combining (S1.38) and (S1.39), we obtain

$$
\frac{\mathcal{C}_1}{\alpha} = \frac{1}{\alpha}\cdot \sum_{1\le i<j\le d} P(A_i \cap A_j) \to 0. \qquad \text{(S1.40)}
$$

Hence, we conclude

$$\frac{P(A)}{\alpha} \to 1. \tag{S1.41}$$

With (S1.35) and (S1.41), we finally have

$$\frac{P(T_{\text{Lévy}} \geq \ell_\alpha)}{\alpha} = \frac{P(A)}{\alpha} + \frac{P(B)}{\alpha} \to 1. \tag{S1.42}$$

Next, we consider T_{Cauchy}. The Cauchy combination method can be decomposed into two parts similar to the case of $T_{\text{Lévy}}$: Let $h_{C,p}(p) = \tan\left((1/2 - p)\pi\right)$ and $h_{C,X}(X) = \tan\left((\Phi(X) - 1/2)\pi\right)$. Define

$$\begin{aligned} A_C &= \bigcup_{j=1}^{d} A_{C,j} = \bigcup_{j=1}^{d} \left\{ h_{C,p}\left(p_j\right) > d(1+\delta_\alpha)c_\alpha, \quad \sum_{i=1}^{d} h_{C,p}\left(p_i\right) \geq dc_\alpha \right\} \\ B_C &= \bigcap_{j=1}^{d} B_{C,j} = \bigcap_{j=1}^{d} \left\{ h_{C,p}\left(p_j\right) \leq d(1+\delta_\alpha)c_\alpha, \quad \sum_{i=1}^{d} h_{C,p}\left(p_i\right) \geq dc_\alpha \right\}, \end{aligned}$$

where h_C is a Cauchy transformation of a p-value, $\delta_\alpha = d^{-\epsilon}$ for $\epsilon > 0$ and c_α is the α-quantile of the standard Cauchy distribution. Then we have

$$P(T_{\text{Cauchy}} \geq c_\alpha) = P(A_C) + P(B_C).$$

To show that $P(B_C)/\alpha \to 0$, split $P(B_C)$ into two parts, $I_{C,1}$ and $I_{C,2}$ as follows :

$$\begin{aligned}
P(B_C) &= \sum_{j=1}^{d} P\left(c_\alpha < h_{C,p}(p_j) \le d(1+\delta_\alpha^C)c_\alpha, \sum_{i=1}^{d} h_{C,p}(p_i) > dc_\alpha\right) \\
&\le \sum_{j=1}^{d} P\left(c_\alpha < h_{C,p}(p_j) \le d(1-\delta_\alpha^C)c_\alpha, \sum_{i=1}^{d} h_{C,p}(p_i) > dc_\alpha\right) \\
&\quad + \sum_{j=1}^{d} P\left(d(1-\delta_\alpha^C)c_\alpha < h_{C,p}(p_j) \le d(1+\delta_\alpha^C)c_\alpha\right) \\
&= I_{C,1} + I_{C,2}.
\end{aligned}$$

Using similar steps in proving I_2 in (S1.34), it can be shown that

$$\frac{I_{C,2}}{\alpha} \to 0.$$

Similarly, we also use the steps in the proof of I_1 for $T_{\text{Lévy}}$, it can be also shown that for $d = c_\alpha^{a_C}$,

$$\begin{aligned}
I_{C,1} &\le \sum_{j=1}^{d} P\left(c_\alpha < h_{C,p}(p_j) \le d(1-\delta_\alpha^C)c_\alpha, \sum_{i\ne j} h_{C,p}(p_i) > d\delta_\alpha^C c_\alpha\right) \\
&\le \sum_{j=1}^{d} \sum_{i:i\ne j} \underbrace{P\left(c_\alpha < h_{C,p}(p_j) \le d(1-\delta_\alpha^C)c_\alpha, \quad h_{C,p}(p_i) > \delta_\alpha^C c_\alpha\right)}_{=J_{C,ij}}.
\end{aligned}$$

Based on the bivariate normality, for a standard normal random variable Z_{ij} which is independent of X_i, we derive a bound of $J_{C,ij}$ which is

$$\begin{aligned}
J_{C,ij} &\le P\left(c_\alpha < h_{C,p}(p_j), X_j \le h_{C,X}^{-1}\left(d(1-\delta_\alpha^C)c_\alpha\right), \rho_{ij}X_j + \sqrt{1-\rho_{ij}^2}Z_{ij} > h_{C,X}^{-1}\left(\delta_\alpha^C c_\alpha\right)\right) \\
&= P\left(h_{C,p}(p_j) > c_\alpha\right) \cdot P\left(Z_{ij} \ge \frac{1}{\sqrt{1-\rho_{ij}^2}}\left[h_{C,X}^{-1}\left(c_\alpha^{1-\epsilon}\right) - \rho_{ij} h_{C,X}^{-1}\left(c_\alpha^{1+a_C}\right)\right]\right).
\end{aligned}$$

For $x \geq \xi > 0$ or equivalently $p = 1 - \Phi(x) \leq 1 - \Phi(\xi) < 1/2$, it follows from Mill's ratio and Taylor expansion of cosine function that

$$\begin{aligned}
h_{C,p}(p) &= \frac{\cos(p\pi)}{\sin(p\pi)} \geq \frac{\cos(p\pi)}{p\pi} = \frac{\cos(p\pi)}{(1-\Phi(x))\pi} \\
&\geq \frac{\cos(p\pi)}{\pi} \cdot \frac{x}{\phi(x)} \\
&\geq (1 - (p\pi)^2/2) \cdot \sqrt{\frac{2}{\pi}} \cdot x e^{x^2/2} \\
&\geq (1 - \pi^2/8) \cdot \sqrt{\frac{2}{\pi}} \cdot \xi \cdot e^{x^2/2} \\
&=: c_1 \cdot e^{x^2/2},
\end{aligned}$$

where $c_1 = (1 - \pi^2/8) \cdot \sqrt{\frac{2}{\pi}} \cdot \xi$ and

$$\begin{aligned}
h_{C,p}(p) &= \frac{\cos(p\pi)}{\sin(p\pi)} \leq \frac{1}{p\pi} = \frac{1}{(1-\Phi(x))\pi} \\
&\leq \frac{1}{\pi} \cdot \frac{x}{\phi(x)} \cdot \left(1 + \frac{1}{x^2}\right) \\
&\leq \sqrt{\frac{2}{\pi}} \cdot \left(1 + \frac{1}{\xi^2}\right) \cdot x \cdot e^{x^2/2} \\
&=: c_2 \cdot x e^{x^2/2}
\end{aligned}$$

where $c_2 = \sqrt{\frac{2}{\pi}} \cdot \left(1 + \frac{1}{\xi^2}\right)$. Then, we have

$$c_1 \cdot e^{x^2/2} \leq h_{C,X}(x) \leq c_2 \cdot x e^{x^2/2}$$

which leads to

$$(1 - o(1))\sqrt{2 \log \frac{t}{c_2}} \leq h_{C,X}^{-1}(t) \leq \sqrt{2 \log \frac{t}{c_1}} \tag{S1.43}$$

since $\log h_{C,X}(x) \leq \log c_2 + \log x + x^2/2$. Here $c_1 < 0$ and $c_2 > 0$ so that

(S1.43) can be expressed as $t \to \infty$,

$$(1 - o(1))\sqrt{2\log t} \leq h_{C,X}^{-1} \leq (1 + o(1))\sqrt{2\log t}. \tag{S1.44}$$

Using (S1.44),

$$\begin{aligned}
&P\left(Z_{ij} \geq \frac{1}{\sqrt{1-\rho_{ij}^2}}\left[h_{C,X}^{-1}\left(c_\alpha^{1-\epsilon}\right) - \rho_{ij} h_{C,X}^{-1}\left(c_\alpha^{1+a_C}\right)\right]\right) \\
&\leq P\left(Z_{ij} \geq \frac{1}{\sqrt{1-\rho_{ij}^2}}\left[(1-o(1))\sqrt{2(1-\epsilon)\log c_\alpha} - \rho_{ij}(1+o(1))\sqrt{2(1+a_C)\log c_\alpha}\right]\right).
\end{aligned}$$

Hence, using the above inequality, we have

$$\begin{aligned}
&J_{C,ij} \\
&\leq P\left(h_C(p_j) > c_\alpha\right) \cdot P\left(Z_{ij} \geq \frac{1}{\sqrt{1-\rho_{ij}^2}}\left[(1-o(1))\sqrt{2(1-\epsilon)\log c_\alpha} - \rho_{ij}(1+o(1))\sqrt{2(1+a_C)\log c_\alpha}\right]\right) \\
&\overset{(i)}{\leq} P\left(h_C(p_j) > c_\alpha\right) \cdot P\left(Z_{ij} \geq \frac{1}{\sqrt{1-\rho^2}}\left[(1-o(1))\sqrt{2(1-\epsilon)\log c_\alpha} - \rho(1+o(1))\sqrt{(1+a_C)\log c_\alpha}\right]\right) \\
&\sim P\left(h_C(p_j) > c_\alpha\right) \cdot P\left(Z_{ij} \geq \underbrace{\frac{\sqrt{2(1-\epsilon)} - \rho\sqrt{2(1+a_C)}}{\sqrt{1-\rho^2}}}_{I_C}\sqrt{\log c_\alpha}\right) \\
&\sim c_\alpha^{-1} \cdot \frac{c \cdot c_\alpha^{-I_C^2/2}}{I_C\sqrt{\log c_\alpha}},
\end{aligned}$$

where the inequality (i) holds since $I_C(x) = \frac{\sqrt{2(1-\epsilon)} - x\sqrt{2(1+a_C)}}{\sqrt{1-x^2}}$ is decreasing in $0 \leq x \leq \rho$ and $c > 0$. Therefore, we have the following bound which is

$$I_{C,1} \lesssim d^2 c_\alpha^{-1} \cdot \frac{c_\alpha^{-I_C^2/2}}{I_c\sqrt{\log c_\alpha}} \lesssim c_\alpha.$$

To show that $I_{C,1}$ in the above actually has the bound $o(c_\alpha^{-1})$, it suffices to show that $I_C > \sqrt{4a_C}$, or equivalently

$$\frac{\sqrt{2(1-\epsilon)} - \rho\sqrt{2(1+a_C)}}{\sqrt{1-\rho^2}} > \sqrt{4a_C}.$$

Hence, for given $0 < \rho < 1$ and since $\epsilon > 0$ is defined arbitrarily and independent of a_C, as $\epsilon \to 0^+$, we can see

$$\frac{\sqrt{2} - \rho\sqrt{2(1+a_C)}}{\sqrt{1-\rho^2}} > \sqrt{4a_C}.$$

Also to achieve $I_C > \sqrt{4a_C}$, a_C should hold the following inequalities

$$0 < a_C < \frac{1-\rho^2}{(\sqrt{3}\rho + \sqrt{2})^2}. \tag{S1.45}$$

Regions of (S1.45), which is the blue region in Figure S2. Under (S1.45), we have $\frac{I_{C,1}}{\alpha} \to 0$, and it can be concluded that

$$P(B_C)/\alpha \to 0.$$

For A_C, under (S1.45), similar steps in T_{Cauchy} with Boole's inequality imply that,

$$\frac{P(A_C)}{\alpha} \to 1,$$

and finally we have

$$\frac{P(T_{\text{Cauchy}} \geq c_\alpha)}{\alpha} = \frac{P(A_C)}{\alpha} + \frac{P(B_C)}{\alpha} \to 1.$$

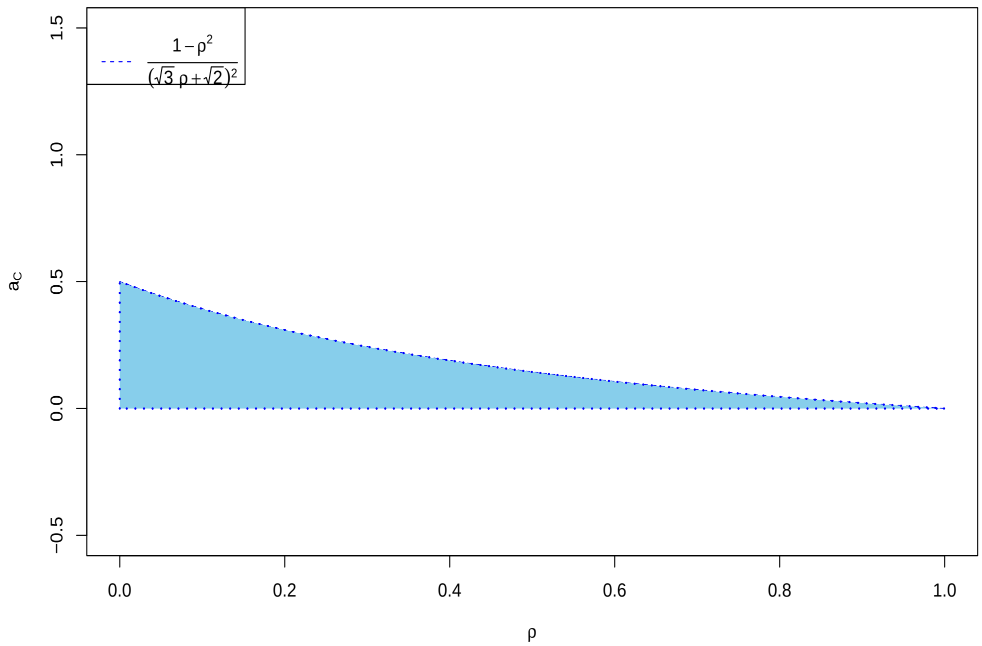

Figure S2: Region of a_C to satisfy $I_C > \sqrt{4a_C}$. The blue region indicates (S1.45).

S2 Proofs of Lemma 2 and Corollary 1

S2.1 Proof of Lemma 2

Let $C(0,1)$ and $L(0,1)$ be random variables of standard Lévy and Cauchy distributions, respectively and c_α and ℓ_α be α-upper quantiles of the standard Cauchy and Lévy distributions, respectively. For given sufficiently small $\alpha \in (0,1)$, by definitions, we have

$$P\left(C(0,1) > c_\alpha\right) = \alpha, \quad P\left(L(0,1) > \ell_\alpha\right) = \alpha.$$

Approximations of tail probabilities of Cauchy and Lévy distributions show that

$$\alpha = P\left(C(0,1) > c_\alpha\right) \sim \frac{1}{\pi c_\alpha}, \quad \alpha = P\left(L(0,1) > \ell_\alpha\right) \sim \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{\ell_\alpha}} \tag{S2.46}$$

since both c_α and l_α diverges due to $\alpha \to 0$. Hence, for any $\alpha \to 0$, we have the following relationship $2\pi c_\alpha^2 \sim \ell_\alpha$ which proves this lemma.

S2.2 Proof of Corollary 1

1. Since $\mathcal{I}_C \subset \mathcal{I}_L$, it is obvious that we have $\mathcal{G}_C \subset \mathcal{G}_L$.

2. For $\alpha_L \sim \alpha_C$, we have $(\frac{1}{d_L})^{1/(2a_L)} \sim (\frac{1}{d_C})^{1/a_C}$ leading to $d_C \sim d_L^{a_C/(2a_L)}$. Therefore, we have $d_C/d_L = d_L^{a_C/(2a_L)-1} \to 0$ since $d_L \to \infty$ and $a_C/(2a_L)-1 < 0$. Similarly, for $d_L \sim d_C$, we have $\alpha_C/\alpha_L \sim d_L^{\frac{1}{2a_L}-\frac{1}{a_C}} \to 0$ since $\frac{1}{2a_L} - \frac{1}{a_C} < 0$.

S3 Proofs of Theorem 4 and Corollary 2

S3.1 Proof of Theorem 4

Let c_α be an upper α-quantile of the standard Cauchy distribution and $\epsilon > 0$ be a constant.

$$
\begin{aligned}
&\text{Type II error}\\
&= P_{H_1}\left(\frac{1}{d}\sum_{i=1}^{d} h(p_i) \le c_\alpha\right)\\
&= P_{H_1}\left(\sum_{i\in\mathcal{N}} h(p_i) + \sum_{i\in\mathcal{S}} h(p_i) \le dc_\alpha\right)\\
&\overset{(i)}{=} P_{H_1}\left(\sum_{i\in\mathcal{N}} h(p_i) + \sum_{i\in\mathcal{S}} \left[\frac{1}{\pi p_i} + O_p(p_i^2)\right] \le dc_\alpha\right)\\
&\overset{(ii)}{\le} P_{H_1}\Bigg(\sum_{i\in\mathcal{N}} h(p_i) + \sum_{i\in\mathcal{S}} \left[\frac{1}{\pi p_i} + O_p(p_i^2)\right] \le dc_\alpha,\\
&\qquad\qquad \max_{i\in\mathcal{S}} |Z_i| \ge \sqrt{1-\rho}\sqrt{2\log d} + \sqrt{2\tau\log d} + o(1)\Bigg)\\
&\quad + P_{H_1}\left(\max_{i\in\mathcal{S}} |Z_i| < \sqrt{1-\rho}\sqrt{2\log d} + \sqrt{2\tau\log d} + o_P(1)\right)\\
&\overset{(iii)}{\lesssim} P_{H_1}\left(\sum_{i\in\mathcal{N}} h(p_i) + \frac{1}{\pi}|\mathcal{S}|d^{1-\rho+\tau+\sqrt{\tau}\sqrt{1-\rho}} + o_p(1) \le dc_\alpha\right) + o(1)\\
&\lesssim P_{H_1}\left(\frac{1}{|\mathcal{N}|}\sum_{i\in\mathcal{N}} h(p_i) \le \frac{d}{|\mathcal{N}|}\left(c_\alpha - d^{\beta-\rho+\tau+\sqrt{\tau}\sqrt{1-\rho}}\right)\right) + o(1)\\
&= P_{H_1}\left(\frac{1}{|\mathcal{N}|}\sum_{i\in\mathcal{N}} h(p_i) \le \frac{d}{|\mathcal{N}|}\left(d^{1/a_C} - d^{\beta-\rho+\tau+\sqrt{\tau}\sqrt{1-\rho}}\right)\right) + o(1).
\end{aligned}
$$

In (i), we use $h(p_i) = \frac{1}{\pi p_i} + O_p\left(p_i^2\right)$.

In (ii), from Lemma 1, $\max_{i\in\mathcal{S}} |Z_i| \ge \sqrt{1-\rho}\sqrt{2\log d} + \sqrt{2\tau\log d} + o_P(1)$

implies that

$$\frac{1}{p_i} \sim \left(\sqrt{1-\rho}\sqrt{2\log d} + \sqrt{2\tau \log d}\right) \exp\left((1-\rho)\log d + \tau \log d + \sqrt{1-\rho}\sqrt{\tau}\log d\right).$$

In (iii), $O_p(p_i^2)$ can be expressed as $o_p(1)$ if $\beta < 2\tau$ since $|\mathcal{S}| \cdot p_i^2 \sim d^{\beta - 2\tau}$.

Sufficient conditions under which the type II error is asymptotically zero are $\beta < 2\tau$ and $1/a_C < \beta + \tau - 1$, or equivalently $a_C > (\beta + \tau - 1)^{-1}$. Combining this to the condition of type I error shows that

$$\frac{1}{\beta + \tau - 1} < a_C < \frac{1-\rho^2}{(\sqrt{3}\rho + \sqrt{2})^2}. \tag{S3.47}$$

Similarly, sufficient conditions of the Lévy combination test statistic can be shown that

$$\frac{1}{\beta + 2\tau - 2} < a_L < \frac{(1-\rho^2)(5\rho^2 - 4\sqrt{3}\rho + 2)}{25\rho^4 - 28\rho^2 + 4}.$$

S3.2 Proof of Corollary 2

For Cauchy combination method, since $\alpha = P(C(0,1) > c_\alpha) \sim 1/(\pi c_\alpha)$ for sufficiently large c_α, or small α,

$$d^{1/a_C} = c_\alpha \sim \frac{1}{\pi \alpha}.$$

Hence we have the following relationship :

$$\frac{1}{a_C} \log d \sim \log(1/\pi\alpha) \sim \log(1/\alpha) \ \Leftrightarrow \ a_C \sim \frac{\log d}{\log(1/\pi\alpha)}.$$

The sufficient condition (S3.47) can be expressed by notations of α.

$$
\begin{aligned}
\text{(S3.47)} \quad &\Leftrightarrow \quad \frac{1}{\beta+\tau-1} \ll \frac{\log d}{\log(1/\alpha)} \ll \frac{1-\rho^2}{(\sqrt{3}\rho+\sqrt{2})^2} \\
&\Leftrightarrow \quad \frac{1}{\beta+\tau-1} \cdot \frac{1}{\log d} \ll \frac{1}{\log(1/\alpha)} \ll \frac{1-\rho^2}{(\sqrt{3}\rho+\sqrt{2})^2} \cdot \frac{1}{\log d} \\
&\Leftrightarrow \quad \frac{1}{\beta+\tau-1} \cdot \frac{1}{\log d} \ll -\frac{1}{\log \alpha} \ll \frac{1-\rho^2}{(\sqrt{3}\rho+\sqrt{2})^2} \cdot \frac{1}{\log d} \\
&\Leftrightarrow \quad -\frac{1}{\beta+\tau-1} \cdot \frac{1}{\log d} \gg \frac{1}{\log \alpha} \gg -\frac{1-\rho^2}{(\sqrt{3}\rho+\sqrt{2})^2} \cdot \frac{1}{\log d} \\
&\Leftrightarrow \quad -(\beta+\tau-1) \cdot \log d \ll \log \alpha \ll -\frac{(\sqrt{3}\rho+\sqrt{2})^2}{1-\rho^2} \cdot \log d \\
&\Leftrightarrow \quad d^{1-\beta-\tau} \ll \alpha \ll d^{-\frac{(\sqrt{3}\rho+\sqrt{2})^2}{1-\rho^2}}.
\end{aligned}
$$

Since $d^{1/a_L} = \ell_\alpha \sim c_\alpha^2 = d^{2/a_C}$ so that $a_L \sim a_C/2$, we have the following condition for Lévy case.

$$
d^{1-\beta/2-\tau} \ll \alpha \ll d^{-\frac{25\rho^4-28\rho^2+4}{2(1-\rho^2)(5\rho^2-4\sqrt{3}\rho+2)}} \tag{S3.48}
$$

where $f(x) \ll g(x)$ if $\limsup_{x\to\infty} f(x)/g(x) = C$ for a constant $C \in (1, \infty)$.

S4 Proofs of Theorem 5, Theorem 6, Lemma 1 and Theorem 7

S4.1 Proof of Theorem 5

First, we consider splitting $\mathsf{E}_{\Sigma} T_{gHMP(\eta)}$ into two parts as follows:

$$
\begin{aligned}
& \mathsf{E}_{\Sigma} T_{gHMP(\eta)} \\
= \; & P\left(p_{global}^{gHMP(\eta)} < \alpha\right) \\
= \; & P\left(\frac{d}{\left(\sum_{j=1}^{d} \frac{1}{p_j^{1/\eta}}\right)^{\eta}} < \alpha\right) \\
= \; & P\left(\frac{1}{d}\left(\sum_{j=1}^{d} \frac{1}{p_j^{1/\eta}}\right)^{\eta} > \frac{1}{\alpha}\right) \\
= \; & P\left(\frac{1}{d^{1/\eta}} \sum_{j=1}^{d} \frac{1}{p_j^{1/\eta}} > \frac{1}{\alpha^{1/\eta}}\right) \\
= \; & \underbrace{P\left(\frac{1}{d^{1/\eta}} \sum_{j=1}^{d} \frac{1}{p_j^{1/\eta}} > \frac{1}{\alpha^{1/\eta}}, \quad \bigcup_{i=1}^{d} \left\{\frac{1}{p_i^{1/\eta}} > (1+\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta}\right\}\right)}_{=P(A)} \\
& + \underbrace{P\left(\frac{1}{d^{1/\eta}} \sum_{j=1}^{d} \frac{1}{p_j^{1/\eta}} > \frac{1}{\alpha^{1/\eta}}, \quad \bigcap_{i=1}^{d} \left\{\frac{1}{p_i^{1/\eta}} \leq (1+\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta}\right\}\right)}_{=P(B)} \\
\equiv \; & P(A) + P(B)
\end{aligned}
$$

where $\delta_\alpha > 0$ is a constant depending on only ℓ_α with $\delta_\alpha \to 0$. Then our goal is to show

$$\frac{P(A)}{\alpha} \to 1, \quad \text{and} \quad \frac{P(B)}{\alpha} \to 0$$

as $d \to \infty$ and $\alpha \to 0$.

The proof of $\frac{P(B)}{\alpha} \to 0$.

First, consider the event B.

$$\begin{aligned}
B &\subseteq \bigcap_{j=1}^{d} \left\{ \frac{1}{p_j^{1/\eta}} \le (1+\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta}, \quad \sum_{i=1}^{d} \frac{1}{p_i^{1/\eta}} > \left(\frac{d}{\alpha}\right)^{1/\eta} \right\} \\
&\subseteq \bigcap_{j=1}^{d} \left\{ \frac{1}{p_j^{1/\eta}} \le (1+\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta}, \quad \bigcup_{i=1}^{d} \left\{ \frac{1}{p_i^{1/\eta}} > \frac{1}{d}\left(\frac{d}{\alpha}\right)^{1/\eta} \right\}, \quad \sum_{i=1}^{d} \frac{1}{p_i^{1/\eta}} > \left(\frac{d}{\alpha}\right)^{1/\eta} \right\} \\
&\subseteq \bigcup_{j=1}^{d} \left\{ \frac{1}{d}\left(\frac{d}{\alpha}\right)^{1/\eta} \le \frac{1}{p_j^{1/\eta}} \le (1+\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta}, \quad \sum_{i=1}^{d} \frac{1}{p_i^{1/\eta}} > \left(\frac{d}{\alpha}\right)^{1/\eta} \right\}.
\end{aligned}$$

We consider the bound of $P(B)$ consisting of two parts :

$$\begin{aligned}
P(B) &\le \sum_{j=1}^{d} P\left(\frac{1}{d}\left(\frac{d}{\alpha}\right)^{1/\eta} \le \frac{1}{p_j^{1/\eta}} \le (1+\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta}, \quad \sum_{i=1}^{d} \frac{1}{p_i^{1/\eta}} > \left(\frac{d}{\alpha}\right)^{1/\eta} \right) \\
&\le \underbrace{\sum_{j=1}^{d} P\left(\frac{1}{d}\left(\frac{d}{\alpha}\right)^{1/\eta} \le \frac{1}{p_j^{1/\eta}} \le (1-\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta}, \quad \sum_{i=1}^{d} \frac{1}{p_i^{1/\eta}} > \left(\frac{d}{\alpha}\right)^{1/\eta} \right)}_{=I_1} \\
&\quad + \underbrace{\sum_{j=1}^{d} P\left((1-\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta} \le \frac{1}{p_j^{1/\eta}} \le (1+\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta} \right)}_{=I_2} \\
&\equiv I_1 + I_2.
\end{aligned}$$

For I_2,

$$
\begin{aligned}
I_2 &\leq \sum_{j=1}^{d}\left[P\left(\frac{1}{p_j^{1/\eta}} \geq (1-\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta}\right) - P\left(\frac{1}{p_j^{1/\eta}} \geq (1+\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta}\right)\right] \\
&= \sum_{j=1}^{d}\left[P\left(p_j \leq \frac{1}{(1-\delta_\alpha)^\eta\left(\frac{d}{\alpha}\right)}\right) - P\left(p_j \leq \frac{1}{(1+\delta_\alpha)^\eta\left(\frac{d}{\alpha}\right)}\right)\right] \\
&= \sum_{j=1}^{d}\left[\frac{1}{(1-\delta_\alpha)^\eta} - \frac{1}{(1+\delta_\alpha)^\eta}\right]\left(\frac{\alpha}{d}\right).
\end{aligned}
$$

Therefore, since $\delta_\alpha \to 0$,

$$
\frac{I_2}{\alpha} = \frac{1}{(1-\delta_\alpha)^\eta} - \frac{1}{(1+\delta_\alpha)^\eta} \to 0.
$$

Next, regarding I_1, by the definition of p-value, $p_j = 1 - \Phi(X_j)$, we have

$$
\begin{aligned}
& I_1 \\
= \; & \sum_{j=1}^{d} P\left(\frac{1}{d}\left(\frac{d}{\alpha}\right)^{1/\eta} \leq \frac{1}{p_j^{1/\eta}} \leq (1-\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta}, \quad \sum_{i=1}^{d} \frac{1}{p_i^{1/\eta}} > \left(\frac{d}{\alpha}\right)^{1/\eta}\right) \\
\leq \; & \sum_{j=1}^{d} P\left(\frac{1}{d}\left(\frac{d}{\alpha}\right)^{1/\eta} \leq \frac{1}{p_j^{1/\eta}} \leq (1-\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta}, \quad \bigcup_{i=1}^{d}\left\{\frac{1}{p_i^{1/\eta}} > \frac{1}{d}\left(\frac{d}{\alpha}\right)^{1/\eta}\right\}\right) \\
\leq \; & \sum_{j=1}^{d} P\left(\frac{1}{d}\left(\frac{d}{\alpha}\right)^{1/\eta} \leq \frac{1}{p_j^{1/\eta}} \leq (1-\delta_\alpha)\left(\frac{d}{\alpha}\right)^{1/\eta}, \quad \bigcup_{i:i\neq j}\left\{\frac{1}{p_i^{1/\eta}} > \frac{1}{d}\left(\frac{d}{\alpha}\right)^{1/\eta}\right\}\right) \\
\leq \; & \sum_{j=1}^{d}\sum_{i:i\neq j} P\left(\frac{1}{(1-\delta_\alpha)^\eta}\left(\frac{\alpha}{d}\right) \leq p_j \leq d^\eta\left(\frac{\alpha}{d}\right), \quad \frac{1}{p_i^{1/\eta}} > \frac{1}{d}\left(\frac{d}{\alpha}\right)^{1/\eta}\right) \\
= \; & \sum_{j=1}^{d}\sum_{i:i\neq j} P\left(\frac{1}{(1-\delta_\alpha)^\eta}\left(\frac{\alpha}{d}\right) \leq p_j \leq d^\eta\left(\frac{\alpha}{d}\right), \quad p_i < d^\eta\left(\frac{\alpha}{d}\right)\right) \\
= \; & \sum_{j=1}^{d}\sum_{i:i\neq j} P\left(\frac{1}{(1-\delta_\alpha)^\eta}\left(\frac{\alpha}{d}\right) \leq 1-\Phi(X_j) \leq d^\eta\left(\frac{\alpha}{d}\right), \quad 1-\Phi(X_i) < d^\eta\left(\frac{\alpha}{d}\right)\right) \\
= \; & \sum_{j=1}^{d}\sum_{i:i\neq j} \underbrace{P\left(1-d^\eta\left(\frac{\alpha}{d}\right) \leq \Phi(X_j) \leq 1-\frac{1}{(1-\delta_\alpha)^\eta}\left(\frac{\alpha}{d}\right), \quad \Phi(X_i) > 1-d^\eta\left(\frac{\alpha}{d}\right)\right)}_{=J_{ij}}.
\end{aligned}
$$

It can be shown that, using the bivariate normality of $X_i = \rho_{ij}X_j + \sqrt{1-\rho_{ij}^2}Z_{ij}$ where Z_{ij} is a standard normal random variable and independent of X_j, we derive an upper bound of J_{ij} as follows :

$$
\begin{aligned}
& J_{ij} \\
= \;& P\left(\Phi^{-1}\left(1-d^{\eta}\left(\frac{\alpha}{d}\right)\right) \leq X_j \leq \Phi^{-1}\left(1-\frac{1}{(1-\delta_\alpha)^{\eta}}\left(\frac{\alpha}{d}\right)\right), X_i > \Phi^{-1}\left(1-d^{\eta}\left(\frac{\alpha}{d}\right)\right)\right) \\
= \;& P\left(\Phi^{-1}\left(1-d^{\eta}\left(\frac{\alpha}{d}\right)\right) \leq X_j \leq \Phi^{-1}\left(1-\frac{1}{(1-\delta_\alpha)^{\eta}}\left(\frac{\alpha}{d}\right)\right)\right) \\
& \times P\left(\rho_{ij}X_j + \sqrt{1-\rho_{ij}^2}Z_{ij} > \Phi^{-1}\left(1-d^{\eta}\left(\frac{\alpha}{d}\right)\right)\right) \\
\leq \;& \underbrace{P\left(p_j \leq d^{\eta}\left(\frac{\alpha}{d}\right)\right)}_{J_{ij,1}} \\
& \times \underbrace{P\left(Z_{ij} > \frac{1}{\sqrt{1-\rho_{ij}^2}}\left[\Phi^{-1}\left(1-d^{\eta}\left(\frac{\alpha}{d}\right)\right) - \rho_{ij}\Phi^{-1}\left(1-\frac{1}{(1-\delta_\alpha)^{\eta}}\left(\frac{\alpha}{d}\right)\right)\right]\right)}_{J_{ij,2}}.
\end{aligned}
$$

With $\epsilon := \epsilon(\alpha) \to 0^+$, let $\delta_\alpha = 1-\alpha^{\epsilon/\eta} \to 0$ and $d = \alpha^{-a_G}$. Then, since p-values follow the uniform distribution,

$$
J_{ij,1} = d^{\eta}\left(\frac{\alpha}{d}\right) = \alpha^{(1-\eta)a_G+1}. \tag{S4.49}
$$

For $0 < \rho_{ij} < \rho < 1$,

$$
\begin{aligned}
J_{ij,2} \;\leq\;& P\left(Z_{ij} > \frac{1}{\sqrt{1-\rho^2}}\left[\Phi^{-1}\left(1-d^{\eta}\left(\frac{\alpha}{d}\right)\right) - \rho\cdot\Phi^{-1}\left(1-\frac{1}{(1-\delta_\alpha)^{\eta}}\left(\frac{\alpha}{d}\right)\right)\right]\right) \\
=\;& \bar{\Phi}\left(\frac{1}{\sqrt{1-\rho^2}}\left[\Phi^{-1}\left(1-\alpha^{(1-\eta)a_G+1}\right) - \rho\cdot\Phi^{-1}\left(1-\alpha^{1+a_G-\epsilon}\right)\right]\right).
\end{aligned}
$$

Lemma S3 (Blair et al. (1976))**.**

$$
1-\frac{1}{y\sqrt{\pi}}e^{-y^2} \leq \operatorname{erf}(y) \leq 1-\frac{1}{y\sqrt{\pi}}e^{-y^2} + \frac{1}{y^3 2\sqrt{\pi}}e^{-y^2}.
$$

Using Lemma S3, it can be shown that ,

$$(1 - o(1))\sqrt{-2\log(1-x)} \le \Phi^{-1}(x) \le \sqrt{-2\log(1-x)}.$$

Hence, we have

$$\begin{aligned}
&J_{ij,2} \\
&\le \bar{\Phi}\left(\frac{1}{\sqrt{1-\rho^2}}\left[\sqrt{2((1-\eta)a_G+1)\log(1/\alpha)} - \rho(1-o(1))\sqrt{2(1+a_G-\epsilon)\log(1/\alpha)}\right]\right) \\
&= \bar{\Phi}\left(\underbrace{\frac{\sqrt{2((1-\eta)a_G+1)} - \rho(1-o(1))\sqrt{2(1+a_G-\epsilon)}}{\sqrt{1-\rho^2}}}_{L}\sqrt{\log(1/\alpha)}\right) \\
&\sim \frac{(1/\alpha)^{-L^2/2}}{L\sqrt{\log(1/\alpha)}}
\end{aligned}$$

For a constant c, it can be shown that

$$\frac{(1/\alpha)^{-L^2/2}}{L\sqrt{\log(1/\alpha)}} \sim \frac{c\cdot(1/\alpha)^{-I^2/2}}{I\sqrt{\log(1/\alpha)}},$$

where

$$I = \sqrt{\frac{2}{1-\rho^2}}\left[\sqrt{((1-\eta)a_G+1)} - \rho\sqrt{2(1+a_G-\epsilon)}\right].$$

Hence, we have the following asymptotic equivalence:

$$J_{ij,2} \sim \frac{(1/\alpha)^{-I^2/2}}{I\sqrt{\log(1/\alpha)}}. \tag{S4.50}$$

Combining (S4.49) and (S4.50),

$$J_{ij} \lesssim \frac{(1/\alpha)^{-\frac{2(1-\eta)a_G+2+I^2}{2}}}{\sqrt{\log(1/\alpha)}}.$$

Therefore we have the following bound

$$I_1 \lesssim d^2 \frac{(1/\alpha)^{-\frac{2(1-\eta)a_G+2+I^2}{2}}}{\sqrt{\log(1/\alpha)}} \lesssim (1/\alpha)^{2a_G - \frac{2(1-\eta)a_G+2+I^2}{2}}.$$

In order to show $I_1/\alpha \to 0$, it suffices to show

$$2a_G - \frac{2(1-\eta)a_G + 2 + I^2}{2} + 1 = (1+\eta)a_G - \frac{I^2}{2} < 0.$$

Since we handle with only a tail probability, we can consider a case of $I > 0$.

Let $\epsilon \to 0$. We first define

$$\begin{aligned}
\kappa_1(\rho) &= (\eta+2)^2\rho^4 - 4(\eta^2+\eta+1)\rho^2 + 4\eta^2, \\
\kappa_2(\rho) &= (1-\epsilon)(\eta+2)\rho^4 + (\eta - 2(1-\epsilon))\rho^2 - 2\eta, \\
\kappa_3(\rho) &= (1-\rho^2(1-\epsilon))^2.
\end{aligned}$$

By fundamental calculations, we have the following inequality

$$\kappa_1(\rho)a_L^2 + 2\kappa_2(\rho)a_L + \kappa_3(\rho) > 0.$$

As the case of Lévy transformation method, as $\epsilon \to 0$, we have the condition

$$0 < a_G < U_{a_G}, \tag{S4.51}$$

where

$$U_{a_G} = \frac{(1-\rho^2)\left[(2+\eta)\rho^2 + 2\eta - 2\sqrt{(2\eta+1)(\eta+1)}\right]}{(\eta+2)^2\rho^4 - 4(\eta^2+\eta+1)\rho^2 + 4\eta^2}.$$

Hence, a_G under (S4.51), it can be shown

$$\frac{P(B)}{\alpha} \le \frac{I_1}{\alpha} + \frac{I_2}{\alpha} \to 0.$$

As similar steps of Theorem 3, it can be shown that $\frac{P(A)}{\alpha} \to 1$ and we have

$$\frac{\mathsf{E}_{\Sigma} T_{gHMP(\eta)}}{\alpha} = 1.$$

S4.2 Proof of Theorem 6

Following steps of proof in Theorem 4, we have

$$\begin{aligned}
&\text{Type II error} \\
&= P_{H_1}\left(\frac{d}{\left(\sum_{j=1}^{d} \frac{1}{p_j^{1/\eta}}\right)^{\eta}} \le \alpha \right) \\
&= P_{H_1}\left(\sum_{i\in\mathcal{N}} \frac{1}{p_j^{1/\eta}} + \sum_{i\in\mathcal{S}} \frac{1}{p_j^{1/\eta}} \le \left(\frac{d}{\alpha}\right)^{1/\eta} \right) \\
&\le P_{H_1}\Bigg(\sum_{i\in\mathcal{N}} \frac{1}{p_j^{1/\eta}} + \sum_{i\in\mathcal{S}} \frac{1}{p_j^{1/\eta}} \le \left(\frac{d}{\alpha}\right)^{1/\eta}, \\
&\qquad\qquad \max_{i\in\mathcal{S}} |Z_i| \ge \sqrt{1-\rho}\sqrt{2\log d} + \sqrt{2\tau \log d} + o(1) \Bigg) \\
&\quad + P_{H_1}\left(\max_{i\in\mathcal{S}} |Z_i| < \sqrt{1-\rho}\sqrt{2\log d} + \sqrt{2\tau\log d} + o_P(1) \right) \\
&\overset{(i)}{\lesssim} P_{H_1}\left(\sum_{i\in\mathcal{N}} \frac{1}{p_j^{1/\eta}} + |\mathcal{S}| d^{(1-\rho+\tau+\sqrt{\tau}\sqrt{1-\rho})/\eta} + o_p(1) \le \left(\frac{d}{\alpha}\right)^{1/\eta} \right) + o(1) \\
&\lesssim P_{H_1}\left(\frac{1}{|\mathcal{N}|} \sum_{i\in\mathcal{N}} \frac{1}{p_j^{1/\eta}} \le \frac{1}{|\mathcal{N}|} \left(\left(\frac{d}{\alpha}\right)^{1/\eta} - d^{\beta+(1-\rho+\tau+\sqrt{\tau}\sqrt{1-\rho})/\eta} \right) \right) + o(1) \\
&\overset{(ii)}{=} P_{H_1}\left(\frac{1}{|\mathcal{N}|} \sum_{i\in\mathcal{N}} \frac{1}{p_j^{1/\eta}} \le \frac{1}{|\mathcal{N}|} \left(d^{(1+1/a_G)/\eta} - d^{\beta+(1-\rho+\tau+\sqrt{\tau}\sqrt{1-\rho})/\eta} \right) \right) + o(1) \\
&= P_{H_1}\left(\frac{1}{|\mathcal{N}|} \sum_{i\in\mathcal{N}} \frac{1}{p_j^{1/\eta}} \le \frac{d}{|\mathcal{N}|} \left(d^{(1+1/a_G)/\eta-1} - d^{\beta-1+(1-\rho+\tau+\sqrt{\tau}\sqrt{1-\rho})/\eta} \right) \right) + o(1).
\end{aligned}$$

In (i), we used Lemma 1, and

$$\frac{1}{p_i^{1/\eta}} \sim \left(\sqrt{1-\rho}\sqrt{2\log d} + \sqrt{2\tau\log d}\right)^{1/\eta} \exp\left(\frac{(1-\rho)\log d + \tau\log d + \sqrt{1-\rho}\sqrt{\tau}\log d}{\eta}\right).$$

In (ii), $d = \alpha^{-a_G}$ so that $\alpha = d^{-1/a_G}$. Therefore, sufficient conditions under which the type II error is asymptotically zero are $1/a_G < \eta\beta + \tau - 1$, or equivalently $a_G > (\eta\beta + \tau - 1)^{-1}$.

S4.3 Proof of Theorem 7

By definitions of combined p-values, we have

$$\left(\frac{p_{global}^{gHMP}}{p_{global}^{S(\eta,\psi)}}\right)^{-1/\eta} \sim \frac{\sum_{i=1}^d p_i^{-1/\eta}}{c_{\eta,\psi}^{-1/\eta}\sum_{i=1}^d h_{\eta,\psi}(p_i)}.$$

For given $\epsilon > 0$, we define $\mathcal{E} = \{1 \le i \le d : p_i < \epsilon\}$ and let $|\mathcal{E}|(\le d)$ be the cardinality of $\mathcal{E}$. Then, we have

$$\begin{aligned}
\frac{\sum_{i=1}^d p_i^{-1/\eta}}{c_{\eta,\psi}^{-1/\eta}\sum_{i=1}^d h_{\eta,\psi}(p_i)} &= c_{\eta,\psi}^{1/\eta} \cdot \frac{\sum_{i\in\mathcal{E}} p_i^{-1/\eta} + \sum_{i\in\mathcal{E}^c} p_i^{-1/\eta}}{\sum_{i\in\mathcal{E}} h_{\eta,\psi}(p_i) + \sum_{i\in\mathcal{E}^c} h_{\eta,\psi}(p_i)} \\
&= c_{\eta,\psi}^{1/\eta} \cdot \frac{\left[\sum_{i\in\mathcal{E}} p_i^{-1/\eta}\right] \cdot \left[1 + \frac{\sum_{i\in\mathcal{E}^c} p_i^{-1/\eta}}{\sum_{i\in\mathcal{E}} p_i^{-1/\eta}}\right]}{\left[\sum_{i\in\mathcal{E}} h_{\eta,\psi}(p_i)\right] \cdot \left[1 + \frac{\sum_{i\in\mathcal{E}^c} h_{\eta,\psi}(p_i)}{\sum_{i\in\mathcal{E}} h_{\eta,\psi}(p_i)}\right]} \\
&= \frac{\left[\sum_{i\in\mathcal{E}} p_i^{-1/\eta}\right] \cdot \left[1 + \frac{\sum_{i\in\mathcal{E}^c} p_i^{-1/\eta}}{\sum_{i\in\mathcal{E}} p_i^{-1/\eta}}\right]}{\left[\sum_{i\in\mathcal{E}} p_i^{-1/\eta}\left(1 + o_P(1)\right)\right] \cdot \left[1 + \frac{\sum_{i\in\mathcal{E}^c} h_{\eta,\psi}(p_i)}{\sum_{i\in\mathcal{E}} h_{\eta,\psi}(p_i)}\right]}.
\end{aligned} \tag{S4.52}$$

The equality in (S4.52) holds from the fact that $h_{\eta,\psi}(p_i) = F^{-1}(1 - p_i|\eta,\psi) \sim c_\eta^{1/\eta} \cdot p_i^{-1/\eta}$ where $F(\cdot|\eta,\psi)$ is a cumulative distribution function of heavy-tailed stable distribution with stability parameter η and skewness parameter ψ. Hence, from (S4.52), it suffices to show that

$$\frac{\sum_{i\in\mathcal{E}^c} p_i^{-1/\eta}}{\sum_{i\in\mathcal{E}} p_i^{-1/\eta}} = o_P(1), \quad \frac{\sum_{i\in\mathcal{E}^c} h_{\eta,\psi}(p_i)}{\sum_{i\in\mathcal{E}} h_{\eta,\psi}(p_i)} = o_P(1). \tag{S4.53}$$

By definition, for $i \in \mathcal{E}^c$, we have $p_i^{-1/\eta} \le \epsilon^{-1/\eta}$. For the first condition (i), it follows from Lemma 6 in Cai, Liu, and Xia (2014) that is $\max_{1\le i\le d}\{Z_i\} \ge \sqrt{2\log d} + o_P(1)$, so that we have

$$\min_{1\le i\le d}\{p_i\} = \bar{\Phi}(\max_{1\le i\le d}\{X_i\}) \le \bar{\Phi}(\sqrt{2\log d} + o_P(1))$$

and by Mill's ratio,

$$\begin{aligned} \frac{1}{\min_{1\le i\le d}\{p_i\}} &= \frac{1}{\bar{\Phi}(\max_{1\le i\le d}\{X_i\})} \ge \left(\sqrt{2\log d} + o_P(1)\right) e^{\log d + o_P(1)} \\ &= d\sqrt{2\log d}(1 + o_P(1)). \end{aligned}$$

Let $\mathcal{E}^- = \mathcal{E}\backslash\{1 \le i \le d : p_i = \min_{1\le j\le d}\{p_j\}\}$, then

$$\begin{aligned} \frac{\sum_{i\in\mathcal{E}^c} p_i^{-1/\eta}}{\sum_{i\in\mathcal{E}} p_i^{-1/\eta}} &\le \frac{|\mathcal{E}^c| \cdot \epsilon^{-1/\eta}}{\sum_{i\in\mathcal{E}} p_i^{-1/\eta}} \\ &= \frac{|\mathcal{E}^c| \cdot \epsilon^{-1/\eta}}{(\min_{1\le i\le d}\{p_i\})^{-1/\eta} + \sum_{i\in\mathcal{E}^-} p_i^{-1/\eta}} \\ &\le \frac{d \cdot \epsilon^{-1/\eta}}{d^{1/\eta}\,(2\log d)^{1/(2\eta)}\,(1 + o_P(1)) + \sum_{i\in\mathcal{E}^-} p_i^{-1/\eta}} \\ &\le \frac{1}{\epsilon^{1/\eta} d^{1/\eta - 1}\,(2\log d)^{1/(2\eta)}\,(1 + o_P(1))} \qquad \text{(S4.54)} \\ &\to 0, \end{aligned}$$

as $d \to \infty$. Hence, we have $\frac{\sum_{i\in\mathcal{E}^c} p_i^{-1/\eta}}{\sum_{i\in\mathcal{E}} p_i^{-1/\eta}} = o_P(1)$. Note that for $i \in \mathcal{E}^c$, since

$h_{\eta,\psi}(p_i) \le h_{\eta,\psi}(\epsilon)$, similarly we have the following inequalities.

$$\begin{aligned}\frac{\sum_{i\in\mathcal{E}^c} h_{\eta,\psi}(p_i)}{\sum_{i\in\mathcal{E}} h_{\eta,\psi}(p_i)} &\le \frac{|\mathcal{E}^c|\cdot h_{\eta,\psi}(\epsilon)}{\sum_{i\in\mathcal{E}} h_{\eta,\psi}(p_i)} \\ &\le \frac{d\cdot h_{\eta,\psi}(\epsilon)}{h_{\eta,\psi}\left(\left(2d^2\log d\right)^{-1/2}(1+o_P(1))\right)}.\end{aligned}$$

Since $h_{\eta,\psi}(x) \sim c_\eta^{1/\eta}\cdot x^{-1/\eta}$, the right hand side is also $o_P(1)$.

For the second condition (ii), we obtain from Lemma 1

$$\begin{aligned}\frac{1}{\min_{1\le i\le d}\{p_i\}} &= \frac{1}{\bar{\Phi}\left(\max_{1\le i\le d}\{X_i\}\right)} \\ &\ge \left(\sqrt{2\log d}+o_P(1)\right)e^{(1-\rho)\log d+o_P(1)} \\ &= d^{1-\rho}\sqrt{2\log d}\,(1+o_P(1)).\end{aligned}$$

leading to

$$\begin{aligned}\frac{\sum_{i\in\mathcal{E}^c} p_i^{-1/\eta}}{\sum_{i\in\mathcal{E}} p_i^{-1/\eta}} &\le \frac{d\cdot\epsilon^{-1/\eta}}{d^{\frac{1-\rho}{\eta}}\left(2(1-\rho)\log d\right)^{1/(2\eta)}(1+o_P(1))} \\ &= \frac{1}{\epsilon^{1/\eta}d^{\frac{1-\rho}{\eta}-1}\left(2(1-\rho)\log d\right)^{1/(2\eta)}(1+o_P(1))} \\ &= \exp\left(-\frac{1}{\eta}\log\epsilon+\left(1-\frac{1-\rho}{\eta}\right)\log d-\frac{1}{2\eta}\log\log d+o(1)\right) \\ &= \exp\left(-\frac{1}{\eta}\log\epsilon+\frac{1}{2\eta}\left[2(\eta-1+\rho)\log d-\log\log d\right]+o(1)\right).\end{aligned} \tag{S4.55}$$

Hence, for $\eta < 1$, (S4.55)$\to 0$ in probability if $\rho < 1-\eta$. When $\eta = 1$, (S4.55)$\to 0$ in probability if $\rho = o\left(\frac{\log\log d}{\log d}\right)$. Using similar steps, it can be shown $\frac{\sum_{i\in\mathcal{E}^c} h_{\eta,\psi}(p_i)}{\sum_{i\in\mathcal{E}} h_{\eta,\psi}(p_i)} = o_P(1)$, since $h_{\eta,\psi}(x)\sim c_\eta^{1/\eta}\cdot x^{-1/\eta}$.

S5 Additional numerical results

In this section, we provide additional numerical results in section 5 and 6. We consider threes types of correlation matrices, that is, matrices with exchangeable, polynomially decaying and exponentially decaying correlation coefficients for simulation studies.

S5.1 The upper bounds of a_C and $2a_L$

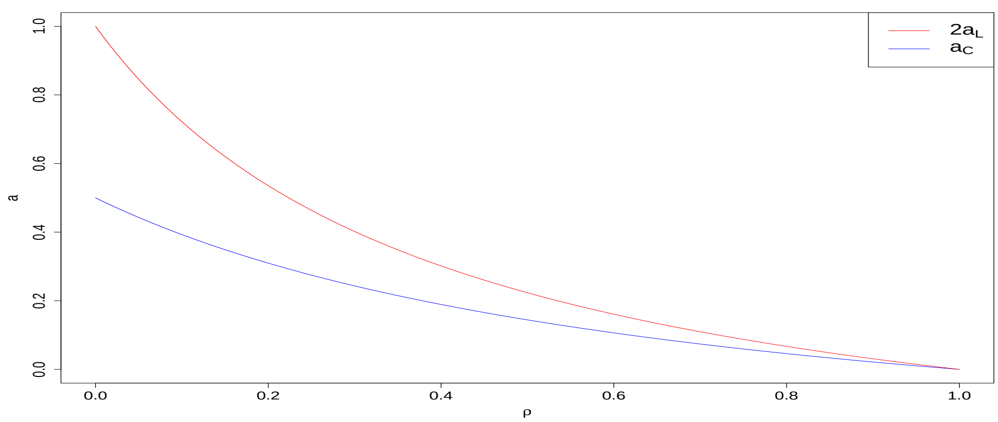

Figure S3: The upper bounds of a_C and $2a_L$ for $0 \leq \rho < 1$.

S5.2 Numerical Results of Asymptotic Equivalence between Heavy Tailed combination test and Generalized Harmonic mean method under various dependency structures

Figures S4 and S5 show the asymptotic equivalences between $p_{\text{global}}^{S(1,0)}$ and $p_{\text{global}}^{gHMP(1)}$ and between $p_{\text{global}}^{S(1/2,1)}$ and $p_{\text{global}}^{gHMP(1/2)}$ in section 5.1, respectively. x-

axes in Figures S4 and S5 indicate values of the global p-values of the Cauchy and Lévy methods, respectively and y-axes represent values of $p_{\text{global}}^{gHMP(1)}$ and $p_{\text{global}}^{gHMP(1/2)}$, respectively. We consider exchangeable correlation coefficient varying 0 to 0.9 (top panels), exponentially decaying correlation coefficients varying 0.4 to 0.9 (middle panels) and polynomially decaying correlation coefficients varying 0.1 to 0.9. We use $d = 200$ and generate 500 each global p-values.

S5.3 Additional numerical results of Type I errors

In this subsection, we present numerical results of Type I errors of T_{Stouffer}, T_{Fisher}, T_{Cauchy}, $T_{\text{Lévy}}$ and T_{minP} at $\alpha = 0.01$. We also present numerical results when dimension d is extremely large. Figure S6 shows boxplots of Type I error of each method at significant level 0.01. Compared to the case $\alpha = 0.05$, all methods are more robust to increasing correlation coefficients. Figure S7 shows Type I error of each method for exchangeable correlation coefficients when $d = 2000, 3000$. Lastly, we present numerical result of Type I error of $T_{\text{gHMP(1)}}$ with T_{Cauchy}, $T_{\text{Lévy}}$ and T_{minP} in Figure S8.

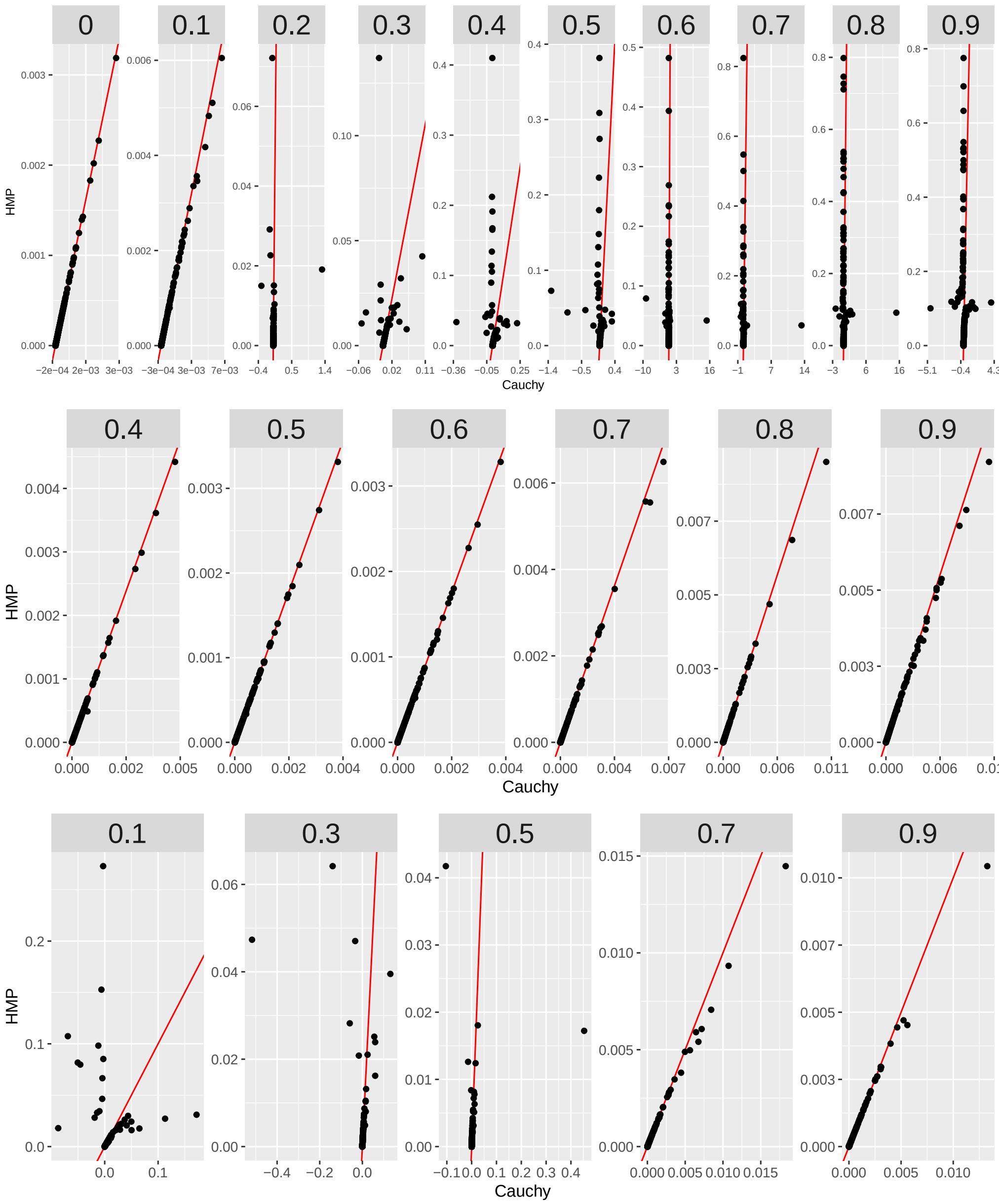

Figure S4: Dot plots of $p_{\text{global}}^{S(1,0)}$ and $p_{\text{global}}^{gHMP(1)}$ with various dependency structures (Top: Exchangeable, $\rho = 0, 0.1, \cdots, 0.9$, Middle: AR(1), $\rho = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$, Bottom: Polynomially decaying, $r = 0.1, 0.3, 0.5, 0.7, 0.9$). Combined p-value of Harmonic mean method is Y-axis and those of Cauchy combination is X-axis. Red line indicates $y = x$.

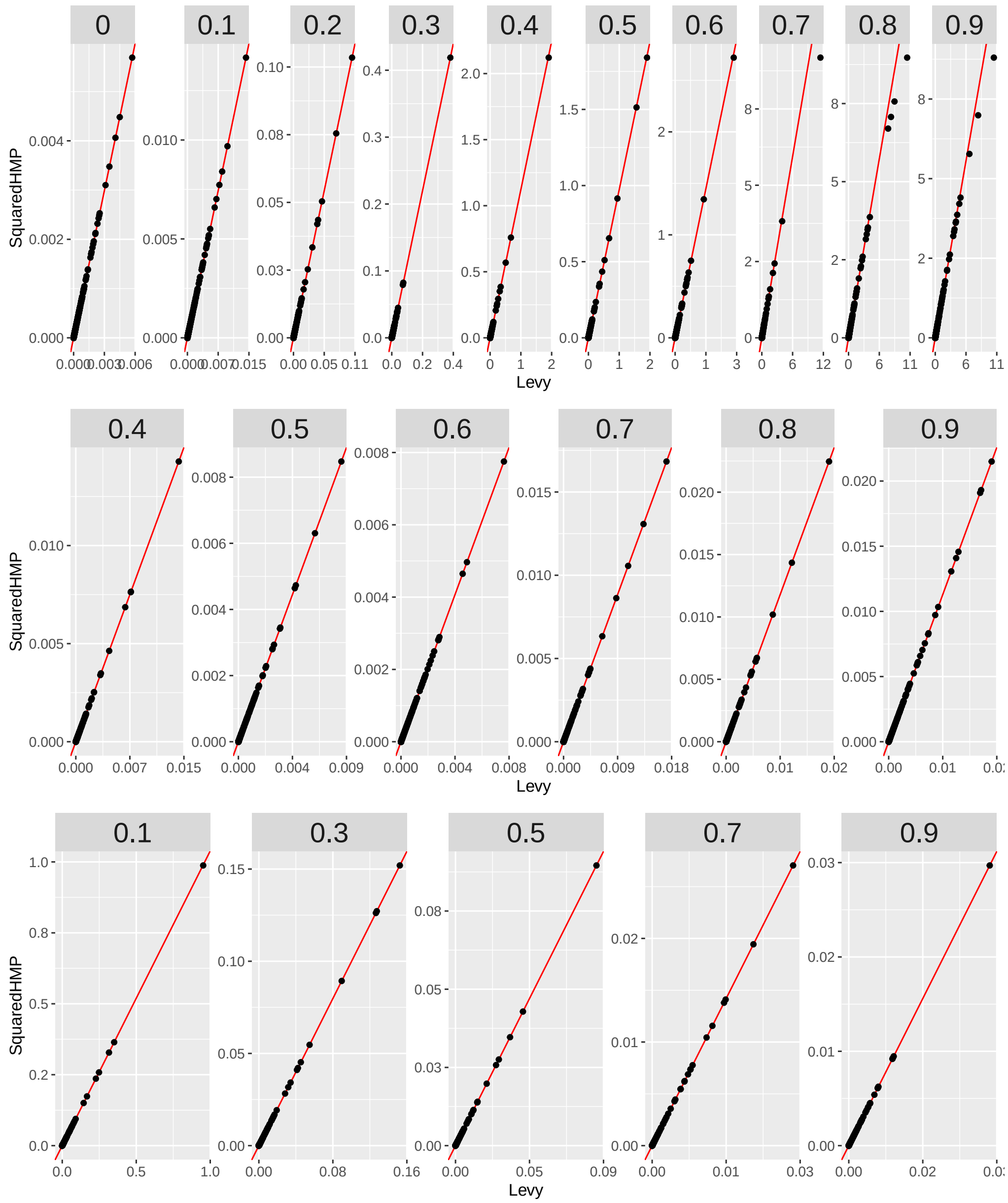

Figure S5: Dot plots of $p_{\text{global}}^{S(1/2,1)}$ and $p_{\text{global}}^{gHMP(1/2)}$ with various dependency structures (Top: Exchangeable, $\rho = 0, 0.1, \cdots, 0.9$, Middle: AR(1), $\rho = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$, Bottom: Polynomially decaying, $r = 0.1, 0.3, 0.5, 0.7, 0.9$). Combined p-value of generalized Harmonic mean method with $\eta = 1/2$ is Y-axis and those of Lévy combination is X-axis. Red line indicates $y = x$.

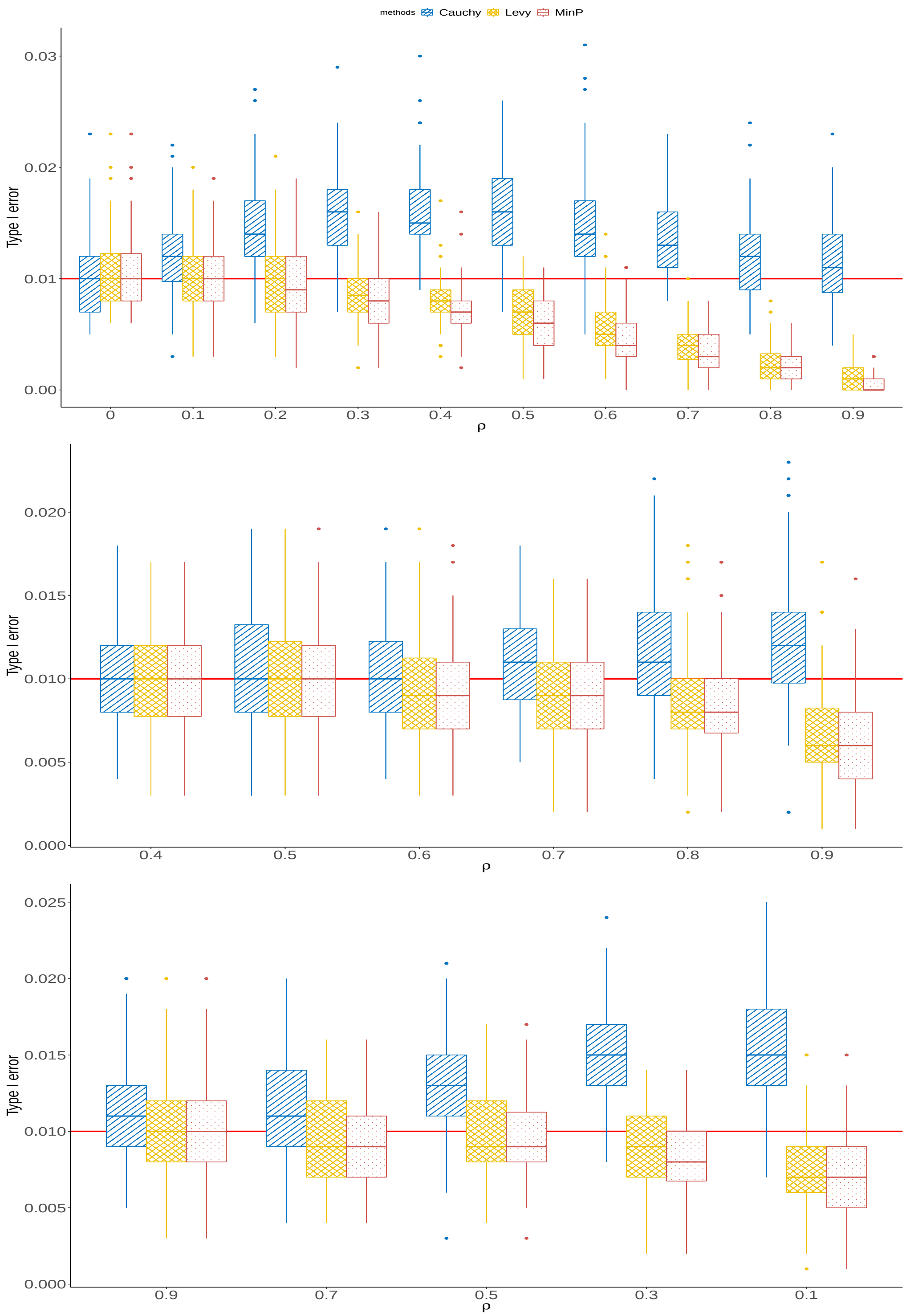

Figure S6: Box plots of Type I error of T_{Cauchy}, $T_{\text{Lévy}}$ and T_{minP} at 0.01 with exchangeable, polynomially and exponentially decaying dependency structures. Blue, yellow and red boxes indicate Type I error of T_{Cauchy}, $T_{\text{Lévy}}$ and T_{minP}, respectively.

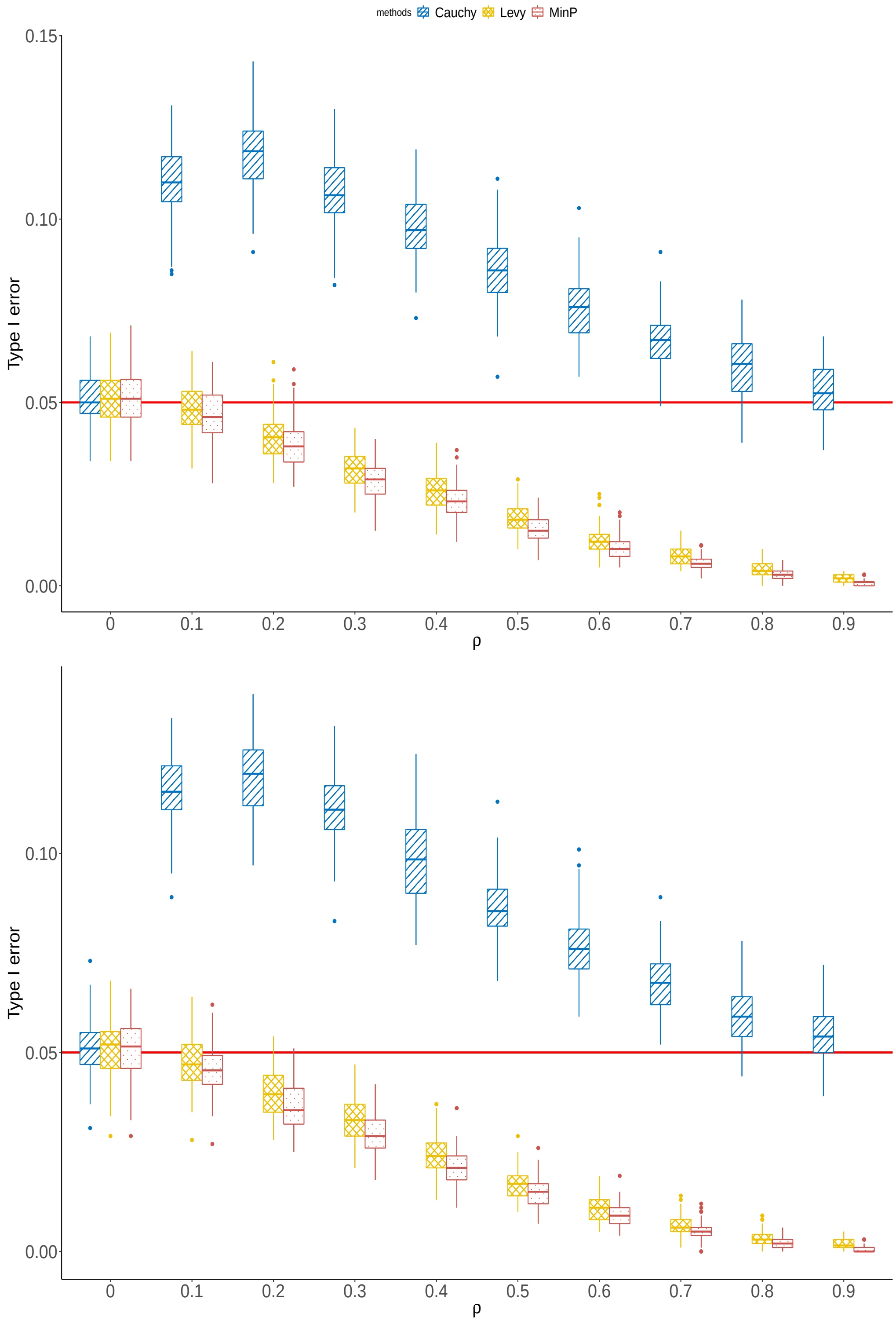

Figure S7: Box plots of Type I error of T_{Cauchy}, $T_{\mathrm{Lévy}}$ and T_{minP} at 0.05 for $d = 2000, 3000$ with exchangeable dependency. Blue, yellow and grey boxes indicate Type I error of T_{Cauchy}, $T_{\mathrm{Lévy}}$ and T_{minP}, respectively.

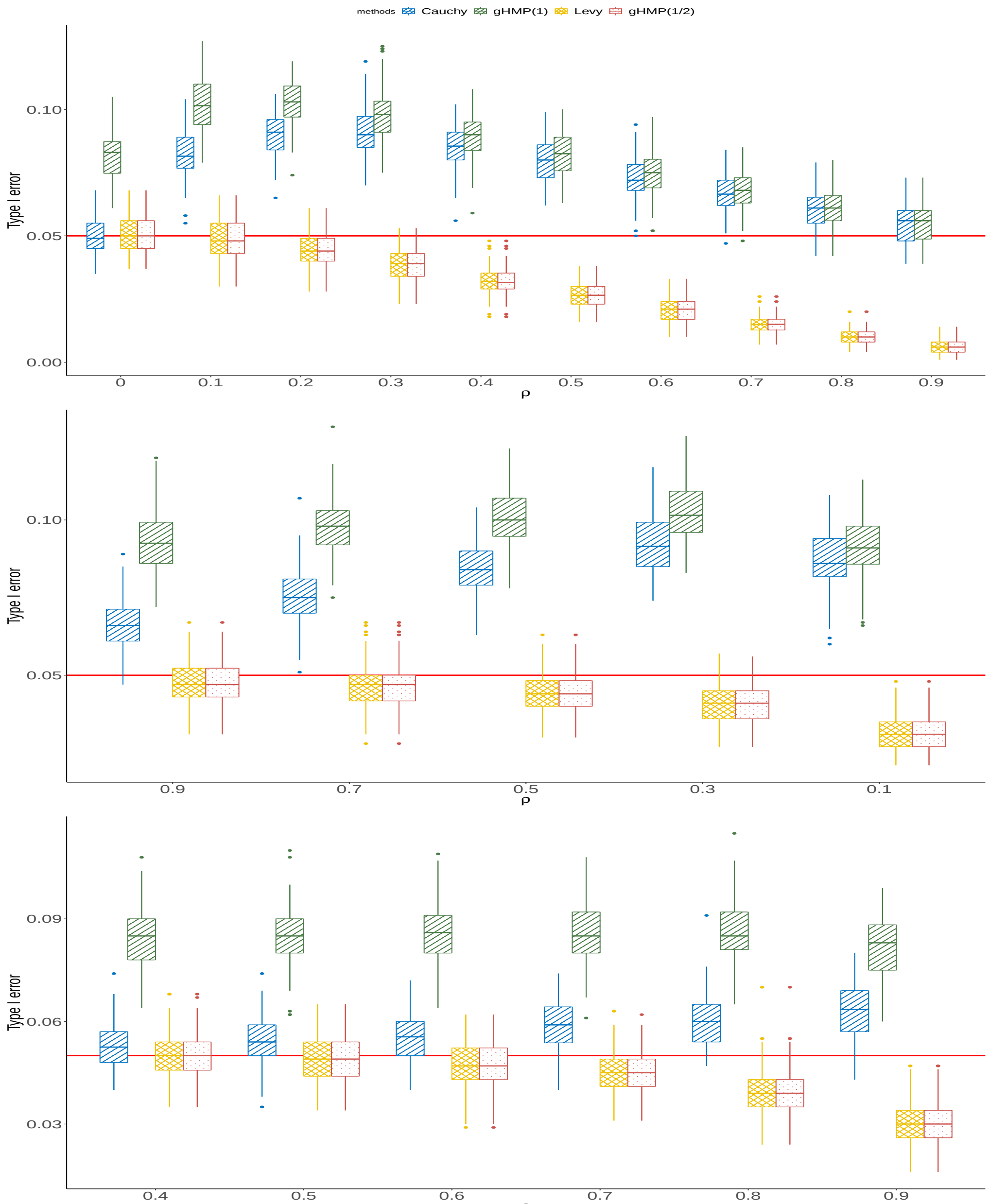

Figure S8: Box plots of Type I error of T_{Cauchy}, $T_{\text{gHMP(1)}}$, $T_{\text{Lévy}}$ and $T_{\text{gHMP(1/2)}}$ at 0.05 with exchangeable, polynomially and exponentially decaying dependency structures

S5.4 Additional numerical results of power comparison

Figure S9 and Figure S10 show powers of T_{Stouffer}, T_{Fisher}, T_{Cauchy}, $T_{\text{Lévy}}$ and T_{minP} under polynomially and exponentially decaying dependency structures, respectively, at significance level $\alpha = 0.05$. Simulation settings other than the dependency structures are the same in Figure 3.

Figure S11, Figure S12 and Figure S13 represent size-adjusted powers of T_{Stouffer}, T_{Fisher}, T_{Cauchy}, $T_{\text{Lévy}}$ and T_{minP} in finite samples under exchangeable case, polynomially and exponentially decaying dependency structures, respectively, at significance level $\alpha = 0.05$. For the size-adjusted power, as in Liu and Xie (2020), null p-values are permuted to generate $(1-\alpha)$-quantile for the size-adjusting. Results of the size-adjusted power show that T_{minP} has the highest power and $T_{\text{Lévy}}$ and T_{Cauchy} have higher powers compared to the T_{Stouffer} and T_{Fisher}. Indeed, the heavier the tail, the higher the power when the power is size-adjusted. As shown in numerical results of Ling and Rho (2022), this order of powers is the reverse order of the results obtained in the case of raw power. However, especially in real data analysis, there is many cases that generating the quantile of null p-values and obtaining size-adjusted power of test statistics are unfeasible since we do not know the exact null distribution of the test statistics due to the unknown dependent structure of p-values.

Figure S9: Power comparison for polynomially decaying case. The columns from left to right correspond to the dimension $d = 20, 40, 60, 100, 300, 500$. The rows from top to bottom correspond to the signal proportion $1\%, 5\%, 10\%, 15\%$.

S5.5 Real Data Analysis

Tables of global p-values

Table S1 and Table S2 represent p-values generated from T_{Cauchy}, $T_{\text{Lévy}}$ and T_{minP}. Table S1 contains only 15 genes rejected by T_{Cauchy} and $T_{\text{Lévy}}$ simultaneously. On the other hand, Table S2 contains 15 genes rejected by T_{Cauchy} but cannot be rejected by $T_{\text{Lévy}}$ and T_{minP}.

The simulated null p-values

In real data, although it is not feasible to check whether test statistics can control a given Type I error, we provide some partial answer to this question under some conditions. To check whether test statistics can control a give Type I error, we provide an algorithm that generates simulated null p-values. We use “zero-assumption” introduced in Efron (2010). To generate null p-values, we first find null p-values from null/signal mixed p-values using the zero-assumption. p-values larger than a fixed constant can be considered as null p-values so that they are assumed to follow the uniform distribution marginally. A small portion of p-values are resampled and rescaled to generate simulated null p-values. Then the simulated p-values of each gene are compared to the uniform distribution. In Efron (2010), Efron (2010) used central 50% statistics to analyzing null distribution under the

GENE	Stouffer	Fisher	Cauchy	Lévy	MinP
DLEU1	0.0172	0.0228	0.0309	0.0456	0.0539
RNF220	0.2616	0.2848	0.0490	0.0458	0.0448
AC011997.1	0.0000	0.0000	0.0309	0.0459	0.0458
FRMD6	0.3398	0.1397	0.0458	0.0461	0.0454
AC079950.1	0.1379	0.0249	0.0328	0.0471	0.0643
LINC02607	0.0971	0.0370	0.0353	0.0472	0.0493
ADCY3	0.0001	0.0001	0.0264	0.0473	0.0569
ATP10A	0.0179	0.0008	0.0340	0.0479	0.0482
RIPOR2	0.0248	0.0208	0.0388	0.0479	0.0471
ATXN7	0.0010	0.0001	0.0328	0.0483	0.0589
ADK	0.0018	0.0003	0.0363	0.0488	0.0486
AC087280.2	0.0003	0.0003	0.0265	0.0488	0.0666
RBMS3-AS3	0.3671	0.0864	0.0439	0.0490	0.0495
MIAT	0.0000	0.0000	0.0220	0.0492	0.0690
PER3	0.0000	0.0003	0.0303	0.0495	0.0584

Table S1: Global p-values of 15 genes which are rejected by the Cauchy method and other robust methods at significant level 0.05.

GENE	Stouffer	Fisher	Cauchy	Lévy	MinP
AC073332.1	0.0258	0.0080	0.0385	0.0500	0.0506
LNCARSR	0.1503	0.0780	0.0467	0.0502	0.0503
DSCAM	0.0027	0.0006	0.0282	0.0506	0.0671
ANTXR1	0.0314	0.0061	0.0327	0.0507	0.0572
PPM1H	0.4229	0.0793	0.0341	0.0509	0.0548
TMEM132E	0.1267	0.0446	0.0401	0.0512	0.0533
RASGEF1B	0.0015	0.0012	0.0393	0.0513	0.0507
Z94721.2	0.0000	0.0000	0.0223	0.0516	0.0755
LINC02571	0.0009	0.0010	0.0309	0.0523	0.0648
C12orf45	0.0330	0.0191	0.0437	0.0526	0.0530
SYT9	0.0001	0.0002	0.0435	0.0527	0.0523
LINC00540	0.5760	0.3873	0.0485	0.0529	0.0555
LRRK2	0.0000	0.0000	0.0259	0.0532	0.0642
FRMPD2	0.0006	0.0001	0.0332	0.0536	0.0668
CAPZB	0.0000	0.0000	0.0256	0.0542	0.0626

Table S2: Global p-values of 15 genes held different results between the Cauchy method and other methods at significant level 0.05.

Figure S10: Power comparison for exponentially decaying case. The columns from left to right correspond to the dimension $d = 20, 40, 60, 100, 300, 500$. The rows from top to bottom correspond to the signal proportion $1\%, 5\%, 10\%, 15\%$.

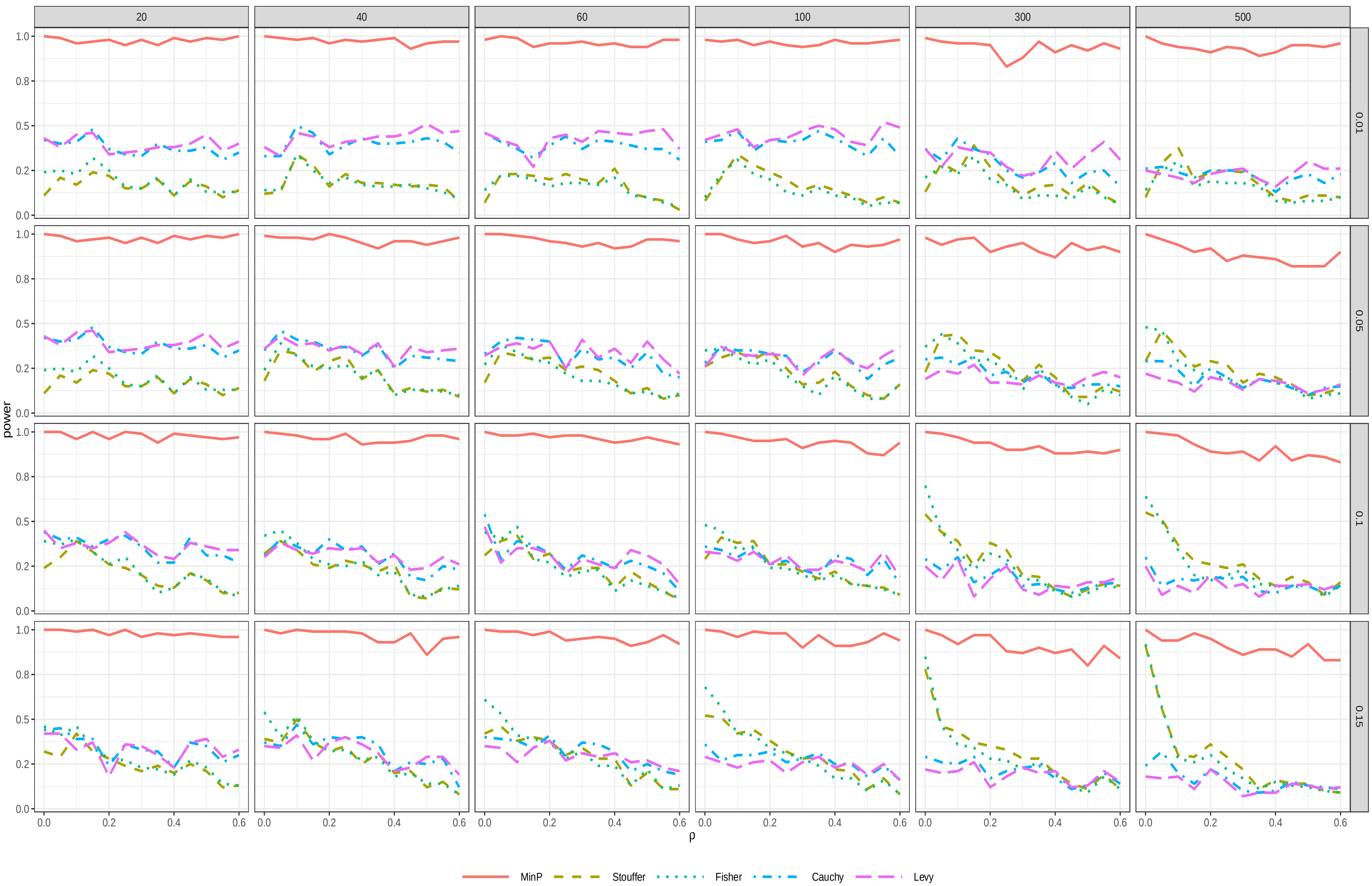

Figure S11: Size-adjusted power comparison for exchangeable case. The columns from left to right correspond to the dimension $d = 20, 40, 60, 100, 300, 500$. The rows from top to bottom correspond to the signal proportion $1\%, 5\%, 10\%, 15\%$.

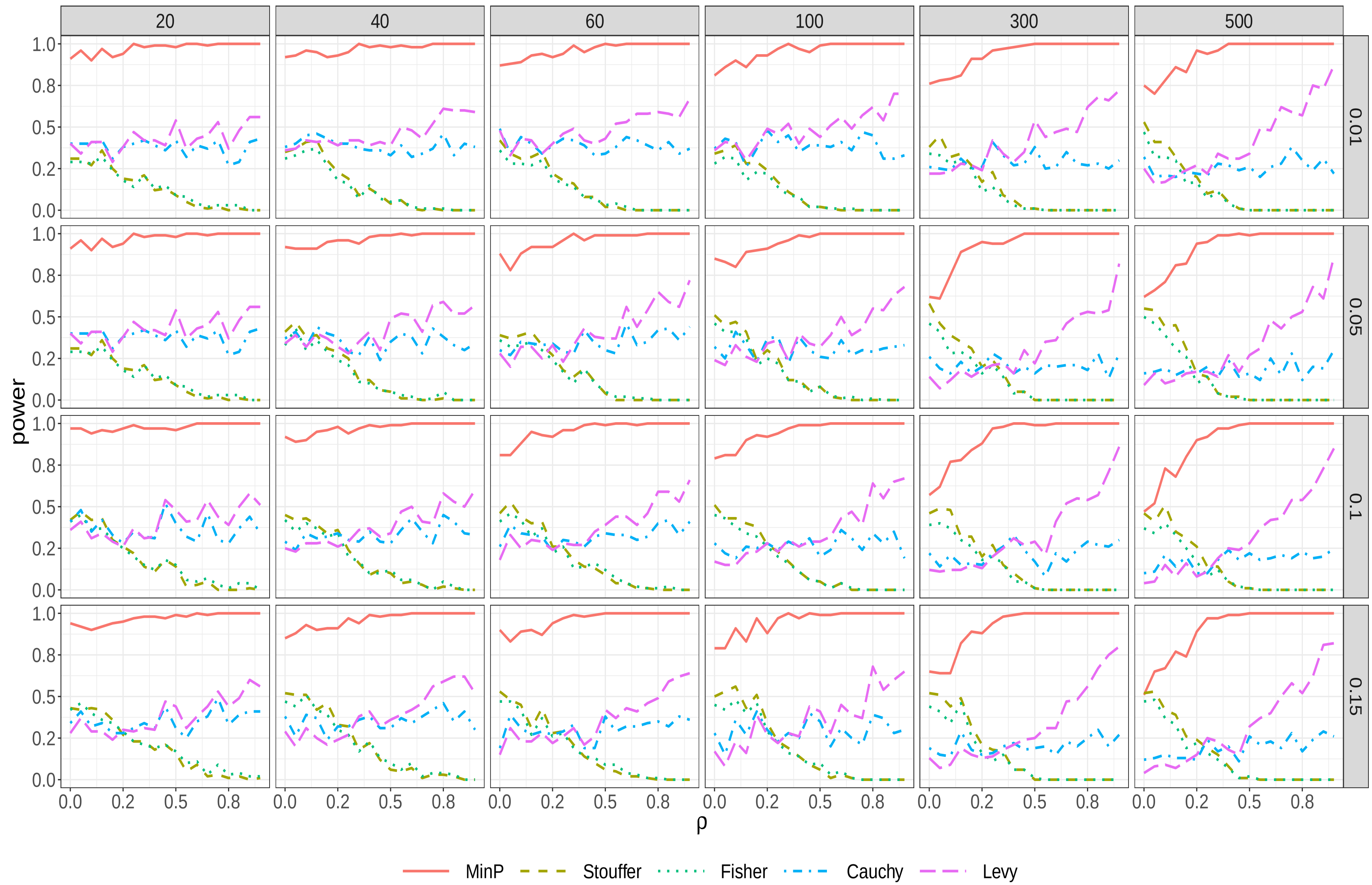

Figure S12: Size-adjusted power comparison for polynomially decaying case. The columns from left to right correspond to the dimension $d = 20, 40, 60, 100, 300, 500$. The rows from top to bottom correspond to the signal proportion $1\%, 5\%, 10\%, 15\%$.

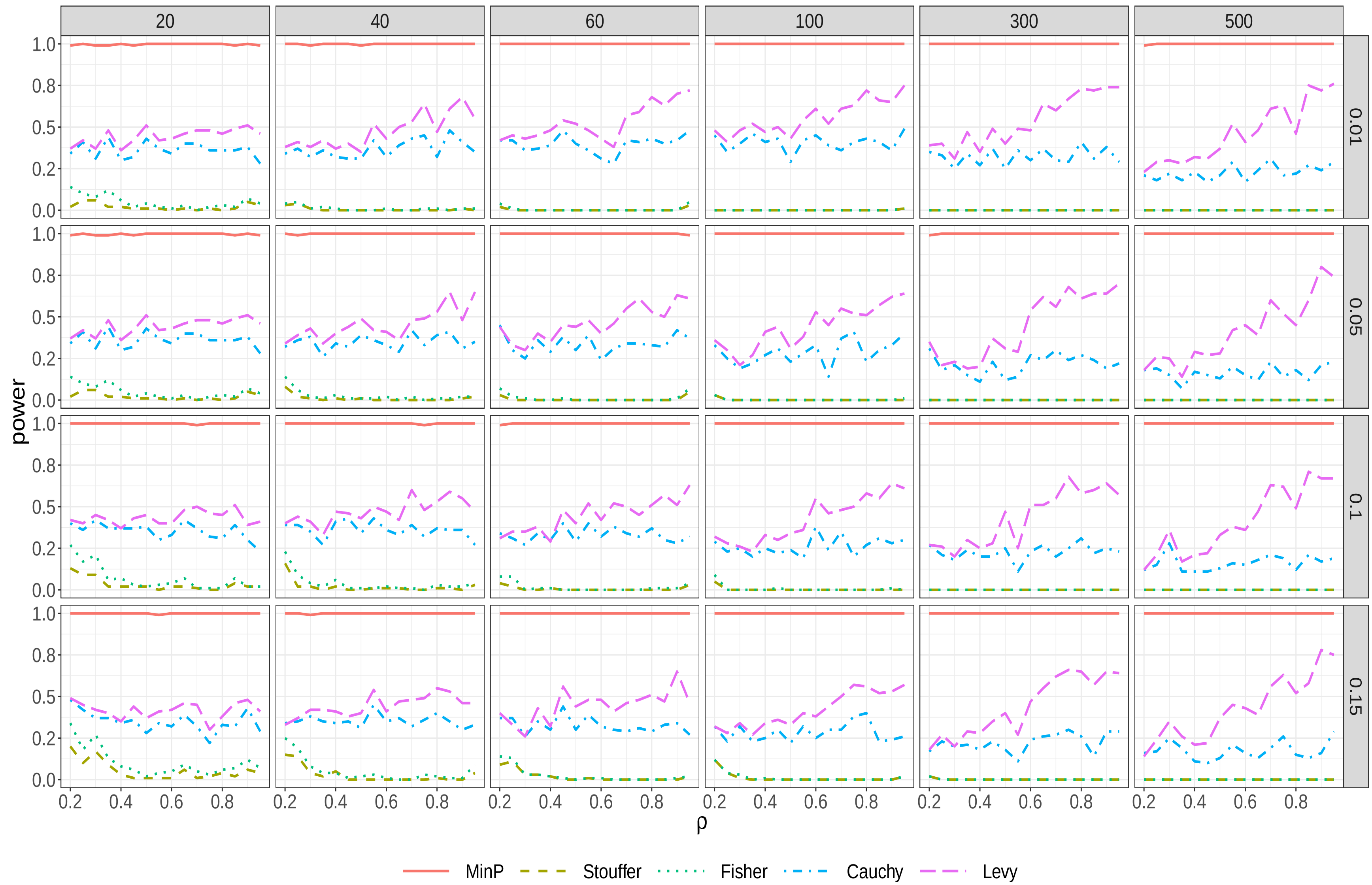

Figure S13: Size-adjusted power comparison for exponentially decaying case. The columns from left to right correspond to the dimension $d = 20, 40, 60, 100, 300, 500$. The rows from top to bottom correspond to the signal proportion $1\%, 5\%, 10\%, 15\%$.

zero-assumption. Following Efron's analysis, we want to use about 50% of test statistics for estimating the null density of test statistics. In our real data, we observe that the proportion of p-values larger than 0.4 is approximately 50% of the whole p-values. We provide a more detailed algorithm in Algorithm 1.

The simulated null p-values are generated as following Algorithm 1.

Algorithm 1 Generating Simulated null p-values

Require: p-values, $p_1, \ldots, p_d$, in each gene

Define $\mathcal{P} := \{p_i : p_i > 0.4,\ \text{for } i = 1, \ldots, d\}$ and $d^* = |\mathcal{P}|$.
Re-indexing $\mathcal{P} = \{p_1^*, \cdots, p_{d^*}^*\}$
for $b = 1$ to 500 **do**
    Sample $d^\dagger (= d^* \times 0.1)$ from $p_1^*, \cdots, p_{d^*}^*$ without replacement
    Generate $\mathcal{P}^\dagger := \{p_1^\dagger, \cdots, p_{d^\dagger}^\dagger\}$
    $p_i^{\dagger\dagger} \leftarrow (p_i^\dagger - \min(\mathcal{P}^\dagger))/(\max(\mathcal{P}^\dagger) - \min(\mathcal{P}^\dagger))$, for all $p_i^\dagger \in \mathcal{P}^\dagger$
    Calculate order statistics : $0 = p_{(1)}^{\dagger\dagger} \le p_{(2)}^{\dagger\dagger} \le \cdots \le p_{(d^\dagger-1)}^{\dagger\dagger} \le p_{(d^\dagger)}^{\dagger\dagger} = 1$
    Using $p_{(2)}^{\dagger\dagger}, \ldots, p_{(d^\dagger-1)}^{\dagger\dagger}$ to calculate combined p-values for Stouffer's, Cauchy, Lévy and minP combination methods, say $p_{global}^{\text{Cauchy}}(b), p_{global}^{\text{Stouffer}}(b), p_{global}^{\text{Levy}}(b)$ and $p_{global}^{\text{minP}}(b)$
end for
$U_1, \ldots, U_{500} \overset{i.i.d.}{\sim} U(0,1)$

Ensure: Quantile-Quantile plots

(i) between $p_{global}^{\text{Stouffer}}(1), \ldots p_{global}^{\text{Stouffer}}(500)$ and $U_1, \ldots, U_{500}$,

(ii) between $p_{global}^{\text{Cauchy}}(1), \ldots p_{global}^{\text{Cauchy}}(500)$ and $U_1, \ldots, U_{500}$,

(iii) between $p_{global}^{\text{Levy}}(1), \ldots p_{global}^{\text{Levy}}(500)$ and $U_1, \ldots, U_{500}$,

(iv) between $p_{global}^{\text{minP}}(1), \ldots p_{global}^{\text{minP}}(500)$ and $U_1, \ldots, U_{500}$

S5.6 Quantile-Quantile plots of Real Data

Figure S14 shows QQ plots represented by dotted lines for T_{Stouffer}, T_{Cauchy}, $T_{\text{Lévy}}$ and T_{minP} with uniform random variables for selected genes: "DOCK1", "GRID2", "GRIN2B", "LINC02306" and "MACROD2". In each figure, the solid line is the diagonal line, so if the dotted line is above the solid line, then the corresponding method fails in controlling a Type I error rate; otherwise, the method has smaller Type I error rate than the nominal level which means the decision is conservative. According to our simulation studies in the previous section, we observed that T_{Stouffer} is liberal in all cases. Figure S14 shows quite similar results to our simulation studies. For example, T_{Stouffer} has inflated Type I error rates for all five cases leading to failure in controlling given levels of Type I error rates. T_{Cauchy} obtains inflated Type I error for moderately correlated p-values and almost nominal level of Type I error for strongly correlated p-values. $T_{\text{Lévy}}$ and T_{minP} have almost nominal level for moderately correlated p-values. Furthermore, from the patterns of QQ-plots for T_{Stouffer}, T_{Cauchy}, $T_{\text{Lévy}}$ and T_{minP}, we may have some insight on the degree of correlations of p-values in real data. In "LINC02306" and "DOCK1" (1st and 4th column), Lévy and minP methods attain almost uniform distributed combined p-values while those of Stouffer's and Cauchy combination tend to be smaller than the uniform distribution. Hence, considering our simulation studies, we can conjecture that SNPs in "LINC02306" and "DOCK1" are correlated moderately leading to the result that $T_{\text{Lévy}}$ and T_{minP} produced more robust results to unknown correlations among SNPs compared to other methods. On the other hand, SNPs in "GRID2" can be considered to be highly correlated since T_{Cauchy} has almost diagonal QQ-plot which achieving almost nominal level of type I error rate whereas $T_{\text{Lévy}}$ and T_{minP} produces quite conservative results. Finally, all methods in "GRIN2B" (middle column) have combined p-values almost uniformly distributed indicating that the p-values in "GRIN2B" may be regarded as weakly dependent or almost independent.

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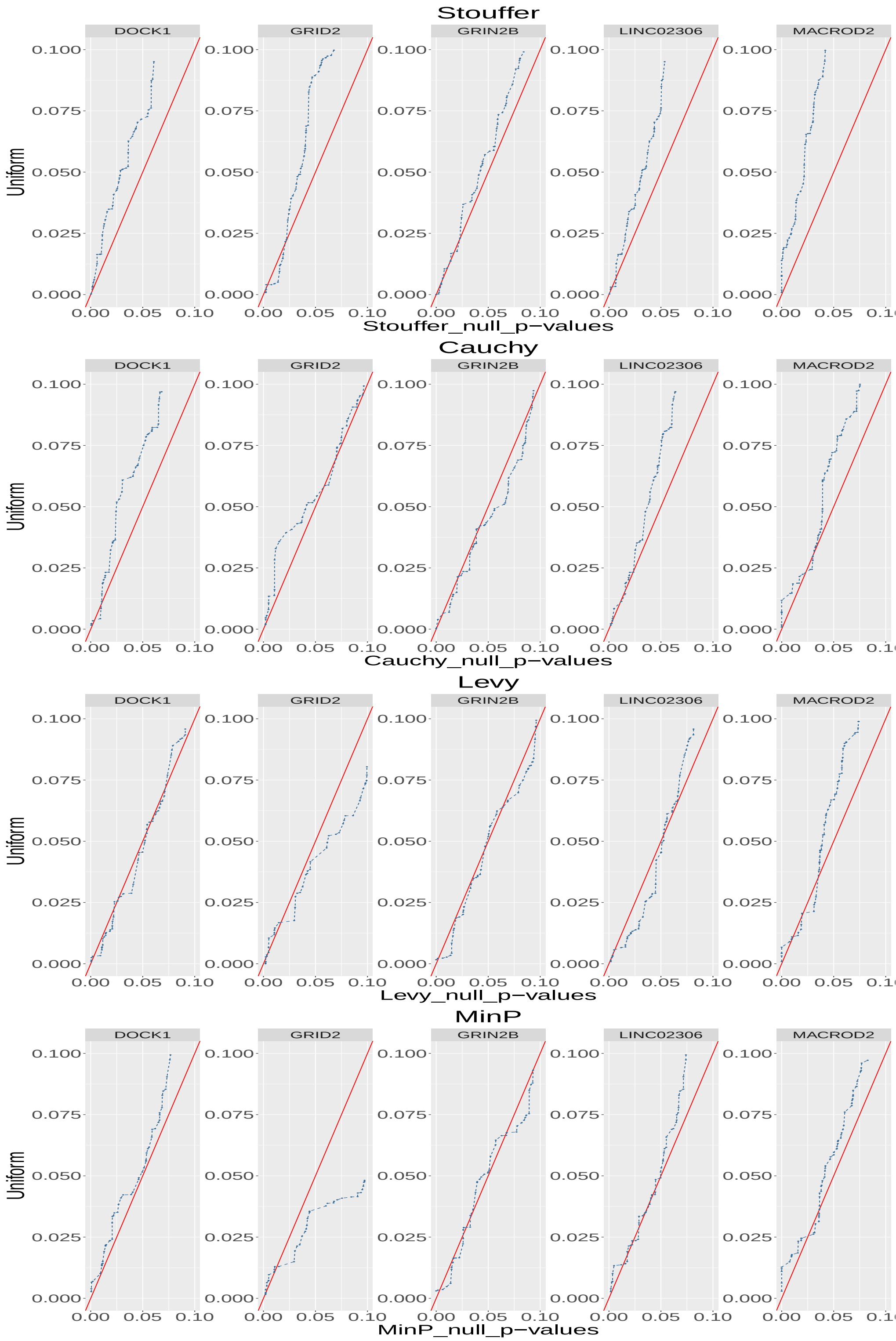

Figure S14: Quantile-Quantile plots of simulated null p-values of T_{Stouffer}, T_{Cauchy}, $T_{\text{Lévy}}$ and T_{minP} from the top to bottom for "DOCK1", "GRID2", "GRIN2B", "LINC02306" and "MACROD2" genes from the left to right panels. X-axis in each panel is quantiles of p-values of each method and Y-axis indicates quantiles of uniform random variables.