SUPPLEMENTARY MATERIAL FOR

"AN UNBIASED PREDICTOR FOR SKEWED RESPONSE VARIABLE WITH MEASUREMENT ERROR IN COVARIATE"

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This supplementary material is structured as follows. Regularity conditions are given in Section S1. Further simulation and application results are given in Sections S2 and S3, respectively. Section S4 contains multivariate covariate setup. We provide proofs of theorems and some technical derivations in Sections S5 and S6.

S1 Regularity conditions

We give three regularity conditions as follows:

Condition 1. $\{(y_i, W_i, \psi_i)\}$ is a sequence of independent and identically distributed random vectors, and there exist positive constants ψ_L and ψ_U

such that $0 < \psi_L \le \inf_{1\le i\le m}\psi_i \le \sup_{1\le i\le m}\psi_i \le \psi_U < \infty$ for $i = 1, ..., m$.

Condition 2. $\boldsymbol{\omega} = (\beta_0, \beta_1, \sigma_v^2)' \in \Theta$ where Θ is a compact set such that $\Theta \subset (\mathbb{R}, \mathbb{R}, \mathbb{R}_+)$ and $\hat{\boldsymbol{\omega}} \xrightarrow{P} \boldsymbol{\omega}$.

Condition 3. (i) $\tilde{U}(\boldsymbol{\omega})$ exists almost surely in probability and $E\{\tilde{U}(\boldsymbol{\omega})\} = 0$. (ii) $\tilde{U}'(\boldsymbol{\omega})$ is a continuous function where $E\{\tilde{U}'(\boldsymbol{\omega})\}$ is uniformly bounded away from zero. (iii) $E\{|\tilde{U}(\boldsymbol{\omega})|^{4+\delta}\}$, $E\{|\tilde{U}'(\boldsymbol{\omega})|^{4+\delta}\}$, and $E\{\sup_{c\in(-\epsilon,\epsilon)}|\tilde{U}''(\boldsymbol{\omega})|^{4+\delta}\}$ are uniformly bounded under some $\epsilon > 0$ and $\delta > 0$.

S2 Further simulation results

In this Section, we provide the empirical MSE of predictors as well as $\hat{R}_{1i}$ and mse_J for multiple values of small areas related to the simulation Section of the paper. The results are listed in Table S2.1.

S3 Further application results

In this Section, we provide three figures related to the application Section of the paper. Figures S3.1 and S3.2 depict the distributions of the Census of Governments based on 4000 and 8000 sample sizes. Figure S3.3 shows the scatter plots for the SAIPE data set. Figure S3.4 displays the box-plots of two predictors from the SAIPE data set.

Table S2.1: Empirical MSE as well as $\hat{R}_{1i}$ and mse_J of predictors averaged by the values of C_i for all possible values of k. We assume $m \in \{20, 50, 100\}$, and the numerical values are in the logarithmic scale.

k	C_i	EMSE(y_i)	EMSE($\tilde{\theta}_{i,\text{No-ME}}$)	EMSE($\tilde{\theta}_{i,A}$)	EMSE($\tilde{\theta}_{i,B}$)	$\hat{R}_{1i}(\tilde{\theta}_{i,B})$	$mse_J(\tilde{\theta}_{i,B})$
				$m = 20$			
25	0	44.678	33.385	33.385	33.385	54.275	57.497
	2	48.742	37.031	50.733	38.925	90.202	80.191
50	0	48.514	38.684	38.684	38.684	63.755	63.852
	2	42.568	36.193	44.793	34.408	73.667	72.889
80	0	47.321	40.189	40.189	40.189	62.135	65.010
	2	43.966	37.164	46.019	35.167	79.058	78.690
100	2	45.600	38.923	47.445	37.089	81.210	78.617
				$m = 50$			
25	0	48.779	37.851	37.851	37.851	66.433	68.337
	2	47.535	37.699	49.199	39.195	90.517	89.474
50	0	47.239	37.227	37.227	37.227	63.285	65.029
	2	50.841	41.380	52.488	42.141	96.313	94.237
80	0	49.722	40.308	40.308	40.308	71.310	74.125
	2	48.903	41.514	50.419	40.469	90.451	88.179
100	2	48.613	42.374	49.999	41.082	89.500	88.080
				$m = 100$			
25	0	40.311	28.807	28.807	28.807	49.891	52.212
	2	49.897	38.359	51.599	40.349	92.106	89.288
50	0	46.091	35.481	35.481	35.481	62.508	64.791
	2	39.885	29.954	41.941	31.010	71.395	68.307
80	0	55.042	46.172	46.172	46.172	77.347	77.868
	2	39.988	33.310	41.700	32.434	72.332	71.142
100	2	42.977	36.106	44.535	35.132	78.235	77.868

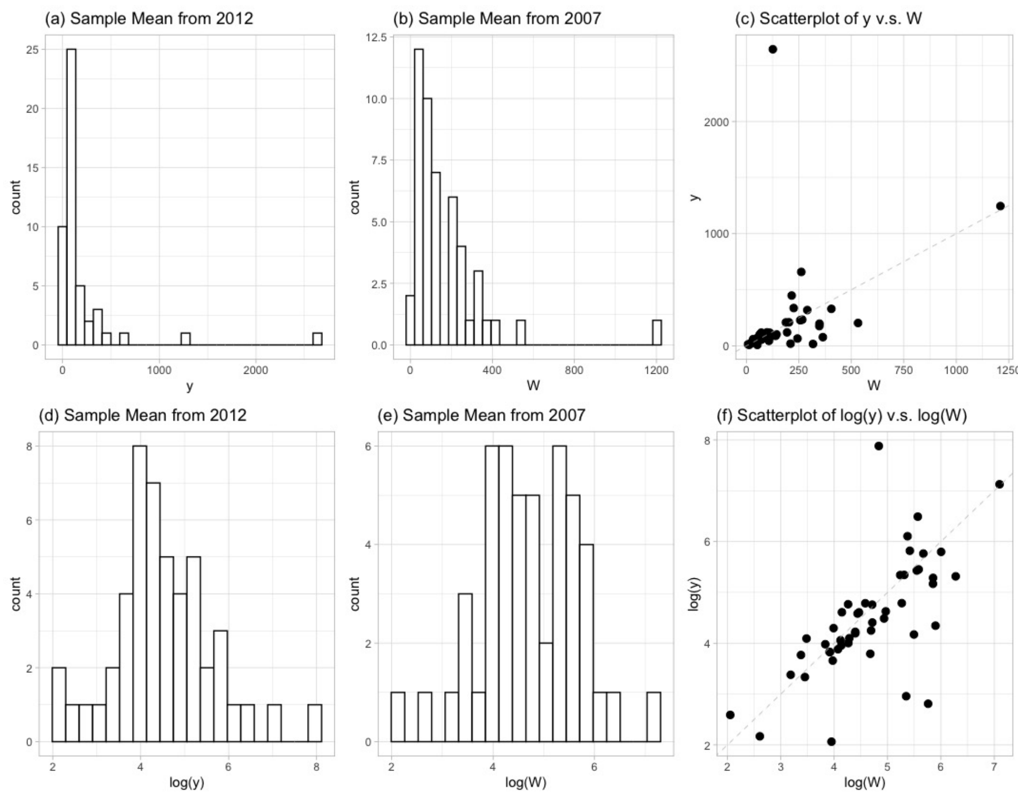

Figure S3.1: Histograms for the Census of Governments based on 4000 sample size. In both plots (a) and (b), the distributions of covariate and response are highly skewed to the right side. After transformations and in plots (d) and (e), we observe a stabilized distribution. Plots (c) and (f) display the regression relationship between the response variable and covariate before and after transformation.

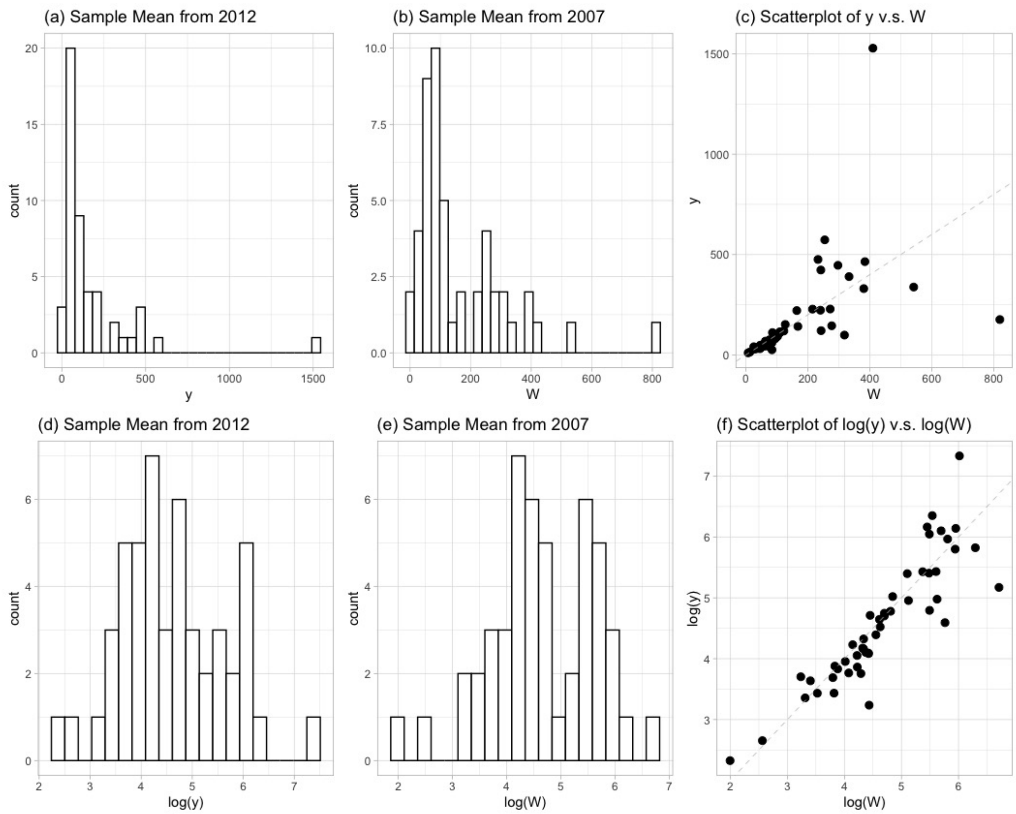

Figure S3.2: Histograms for the Census of Governments based on 8000 sample size. In both plots (a) and (b), the distributions of covariate and response are highly skewed to the right side. After transformations and in plots (d) and (e), we observe a stabilized distribution. Plots (c) and (f) display the regression relationship between the response variable and covariate before and after transformation.

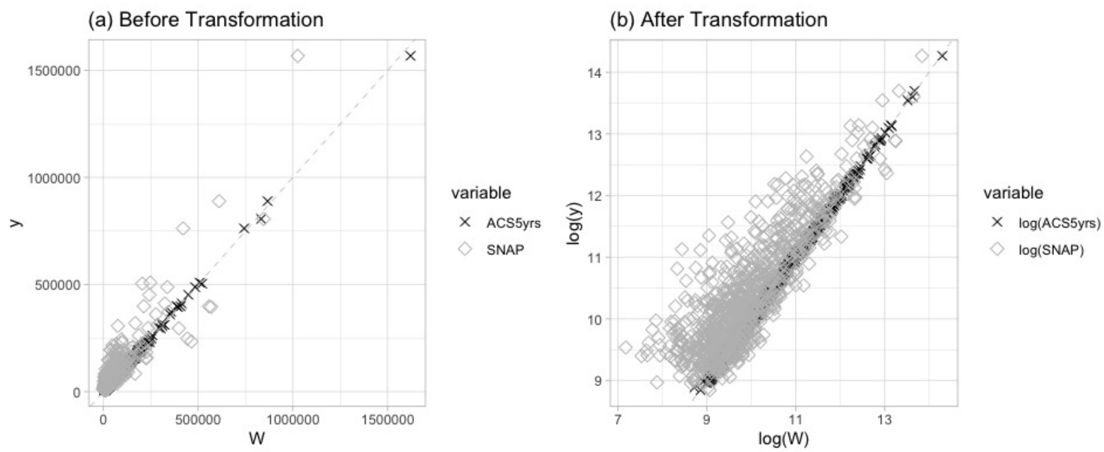

Figure S3.3: Scatter plots of response variable versus covariates (a) before and (b) after transformations for the SAIPE data set.

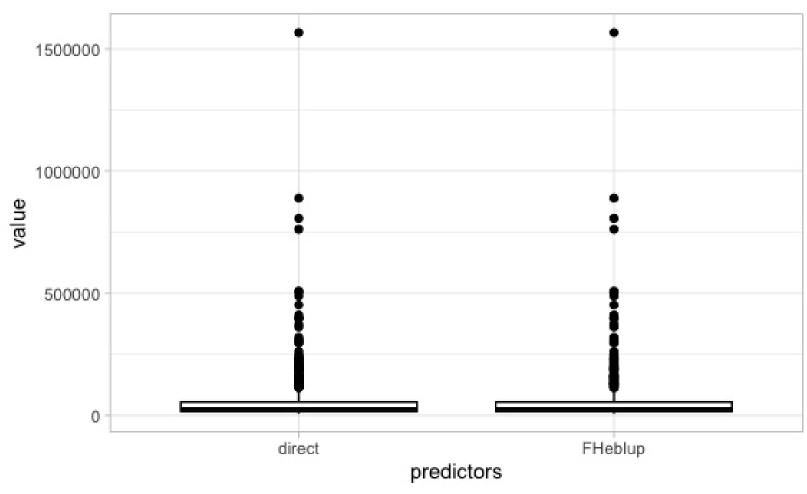

Figure S3.4: Box-plots of direct and FHeblup predictors for the SAIPE data set.

S4 Multivariate extension

In this Section, we give some details of formulation and forms of predictors A and B for multivariate covariate set-up. Let's assume the following hierarchical set-up

$$\begin{aligned} z_i|\phi_i &\sim N(\phi_i, \psi_i) \\ \phi_i &\sim N(\beta_0 + \boldsymbol{\beta}'\boldsymbol{x}_i, \sigma_v^2) \\ \boldsymbol{W_i} &\sim MVN(\boldsymbol{x}_i, \boldsymbol{C}_i), \end{aligned}$$

where $\boldsymbol{\beta}' = (\beta_1, \beta_2, ..., \beta_p)$, $\boldsymbol{x}_i = (x_{i1}, x_{i2}, ..., x_{ip})'$, $\boldsymbol{W}_i = (W_{i1}, W_{i2}, ..., W_{ip})'$, and $\boldsymbol{C}_i = \text{diag}(C_{i1}, ..., C_{ip})$.

The parameter of interest is $\theta_i = \exp(\beta_0 + \boldsymbol{\beta}'\boldsymbol{x}_i + v_i)$. Following the same derivations given in the manuscript, predictor A can be defined as

$$\tilde{\theta}_{i,A} = \exp(\tilde{\gamma}_i z_i + (1 - \tilde{\gamma}_i)(\beta_0 + \boldsymbol{\beta}'\boldsymbol{W}_i) + \tilde{\gamma}_i \psi_i/2),$$

where $\tilde{\gamma}_i = (\boldsymbol{\beta}'\boldsymbol{C}_i\boldsymbol{\beta} + \sigma_v^2)/S_i(\boldsymbol{\beta}, \sigma_v^2)$ and $S_i(\boldsymbol{\beta}, \sigma_v^2) = (\boldsymbol{\beta}'\boldsymbol{C}_i\boldsymbol{\beta} + \sigma_v^2 + \psi_i)$. Predictor B can be defined as $\tilde{\theta}_{i,B} = \tilde{\theta}_{i,A} \exp(-\frac{1}{2}d_i)$, where $d_i = 2\psi_i\boldsymbol{\beta}'\boldsymbol{C}_i\boldsymbol{\beta}/S_i(\boldsymbol{\beta}, \sigma_v^2)$. The vector of unknown parameters can be estimated along the same lines of the manuscript.

S5 Proofs of theorems

In this Section, we provide proofs of theorems.

Proof of Theorem 1:

$$\begin{aligned}
E[\tilde{\theta}_{i,A}] &= E[\exp(\tilde{\gamma}_i z_i)]E[\exp\{(1-\tilde{\gamma}_i)(\beta_0+\beta_1 W_i)\}]\exp(\tilde{\gamma}_i\psi_i/2) \\
&= \exp[\tilde{\gamma}_i(\beta_0+\beta_1 x_i+\frac{1}{2}\tilde{\gamma}_i^2(\sigma_v^2+\psi_i))] \\
&\quad \times \exp[(1-\tilde{\gamma}_i)(\beta_0+\beta_1 x_i+\frac{1}{2}(1-\tilde{\gamma}_i)^2(\beta_1^2 C_i))]\exp(\tilde{\gamma}_i\psi_i/2) \\
&= \exp[\beta_0+\beta_1 x_i+\frac{1}{2}\{\tilde{\gamma}_i^2(\sigma_v^2+\psi_i)+(1-\tilde{\gamma}_i)^2(\beta_1^2 C_i)+\tilde{\gamma}_i\psi_i\}]. \quad \text{(S5.1)}
\end{aligned}$$

Next, we simplify

$$\begin{aligned}
&\tilde{\gamma}_i^2(\sigma_v^2+\psi_i)+(1-\tilde{\gamma}_i)^2\beta_1^2 C_i+\tilde{\gamma}_i\psi_i \\
&= \tilde{\gamma}_i^2(\sigma_v^2+\psi_i+\beta_1^2 C_i)-2\tilde{\gamma}_i\beta_1^2 C_i+\beta_1^2 C_i+\tilde{\gamma}_i\psi_i \\
&= S_i^{-1}(\beta_1,\sigma_v^2)(\beta_1^2 C_i+\sigma_v^2)^2-2\tilde{\gamma}_i\beta_1^2 C_i+\beta_1^2 C_i+\tilde{\gamma}_i\psi_i \\
&= \tilde{\gamma}_i(\beta_1^2 C_i+\sigma_v^2)-2\tilde{\gamma}_i\beta_1^2 C_i+\beta_1^2 C_i+\tilde{\gamma}_i\psi_i \\
&= \tilde{\gamma}_i(\sigma_v^2+\psi_i)+(1-\tilde{\gamma}_i)\beta_1^2 C_i. \qquad \text{(S5.2)}
\end{aligned}$$

The result follows from (S5.1) and (S5.2).

Proof of Theorem 2: We use the equations from the expressions (4.2) of the manuscript to find the matrix $I_{\boldsymbol{\omega}}$ as follows

$$I_{\boldsymbol{\omega}} = \begin{bmatrix} var(\tilde{U}_1(\boldsymbol{\omega})) & cov(\tilde{U}_1(\boldsymbol{\omega}), \tilde{U}_2(\boldsymbol{\omega})) & cov(\tilde{U}_1(\boldsymbol{\omega}), \tilde{U}_3(\boldsymbol{\omega})) \\ cov(\tilde{U}_1(\boldsymbol{\omega}), \tilde{U}_2(\boldsymbol{\omega})) & var(\tilde{U}_2(\boldsymbol{\omega})) & cov(\tilde{U}_2(\boldsymbol{\omega}), \tilde{U}_3(\boldsymbol{\omega})) \\ cov(\tilde{U}_1(\boldsymbol{\omega}), \tilde{U}_3(\boldsymbol{\omega})) & cov(\tilde{U}_2(\boldsymbol{\omega}), \tilde{U}_3(\boldsymbol{\omega})) & var(\tilde{U}_3(\boldsymbol{\omega})) \end{bmatrix}.$$

The elements of the matrix are as follows

$$\begin{aligned} var(\tilde{U}_1(\boldsymbol{\omega})) &= \sum_{i=1}^{m} S_i^{-1}(\beta_1, \sigma_v^2), \\ cov(\tilde{U}_1(\boldsymbol{\omega}), \tilde{U}_2(\boldsymbol{\omega})) &= \sum_{i=1}^{m} S_i^{-2}(\beta_1, \sigma_v^2) cov[W_i \tau_i(\beta_0, \beta_1), \tau_i(\beta_0, \beta_1)] \\ &= \sum_{i=1}^{m} S_i^{-2}(\beta_1, \sigma_v^2) E[(W_i - x_i + x_i)\tau_i^2(\beta_0, \beta_1)] = \sum_{i=1}^{m} S_i^{-1}(\beta_1, \sigma_v^2) x_i, \end{aligned}$$

$$\begin{aligned} cov(\tilde{U}_1(\boldsymbol{\omega}), \tilde{U}_3(\boldsymbol{\omega})) &= 0, \\ var(\tilde{U}_2(\boldsymbol{\omega})) &= \sum_{i=1}^{m} S_i^{-2}(\beta_1, \sigma_v^2) var[W_i \tau_i(\beta_0, \beta_1)] + \beta_1^2 \sum_{i=1}^{m} S_i^{-4}(\beta_1, \sigma_v^2) C_i^2 \, var[\tau_i^2(\beta_0, \beta_1)] \\ &\quad + 2\beta_1 \sum_{i=1}^{m} S_i^{-3}(\beta_1, \sigma_v^2) C_i cov[W_i \tau_i(\beta_0, \beta_1), \tau_i^2(\beta_0, \beta_1)]. \end{aligned}$$

Note that we have

(i) $var[W_i \tau_i(\beta_0, \beta_1)] = (x_i^2 + C_i) S_i(\beta_1, \sigma_v^2) + \beta_1^2 C_i^2$,

(ii) $var[\tau_i^2(\beta_0, \beta_1)] = 2S_i^2(\beta_1, \sigma_v^2)$, and

(iii) $cov[W_i \tau_i(\beta_0, \beta_1), \tau_i^2(\beta_0, \beta_1)] = -2\beta_1 C_i S_i(\beta_1, \sigma_v^2)$.

Therefore,

$$var(\tilde{U}_2(\boldsymbol{\omega})) = \sum_{i=1}^{m} S_i^{-2}(\beta_1, \sigma_v^2)(\sigma_v^2 + \psi_i) C_i + \sum_{i=1}^{m} S_i^{-1}(\beta_1, \sigma_v^2) x_i^2 = \sum_{i=1}^{m} S_i^{-1}(\beta_1, \sigma_v^2)(x_i^2 + \tilde{\sigma}_{ci}^2),$$

$$
\begin{aligned}
cov(\tilde{U}_2(\boldsymbol{\omega}), \tilde{U}_3(\boldsymbol{\omega})) &= \frac{1}{2} cov[\sum_{i=1}^{m} S_i^{-1}(\beta_1, \sigma_v^2) W_i \tau_i(\beta_0, \beta_1) + \beta_1 \sum_{i=1}^{m} S_i^{-2}(\beta_1, \sigma_v^2) C_i \tau_i^2(\beta_0, \beta_1), \\
&\sum_{i=1}^{m} S_i^{-1}(\beta_1, \sigma_v^2) \tau_i^2(\beta_0, \beta_1)] \\
&= \frac{1}{2} \sum_{i=1}^{m} S_i^{-3}(\beta_1, \sigma_v^2) cov[W_i \tau_i(\beta_0, \beta_1), \tau_i^2(\beta_0, \beta_1)] \\
&\quad + \frac{1}{2} \beta_1 \sum_{i=1}^{m} S_i^{-4}(\beta_1, \sigma_v^2) C_i var[\tau_i^2(\beta_0, \beta_1)] = -\beta_1 \sum_{i=1}^{m} S_i^{-2}(\beta_1, \sigma_v^2) C_i \\
&\quad + \beta_1 \sum_{i=1}^{m} S_i^{-2}(\beta_1, \sigma_v^2) C_i = 0, \\
var(\tilde{U}_3(\boldsymbol{\omega})) &= \frac{1}{4} \sum_{i=1}^{m} S_i^{-4}(\beta_1, \sigma_v^2) var[\tau_i^2(\beta_0, \beta_1)] = \frac{1}{2} \sum_{i=1}^{m} S_i^{-2}(\beta_1, \sigma_v^2).
\end{aligned}
$$

As a final result, we get

$$
I_{\boldsymbol{\omega}} = \begin{bmatrix} \sum_{i=1}^{m} S_i^{-1}(\beta_1, \sigma_v^2) & \sum_{i=1}^{m} S_i^{-1}(\beta_1, \sigma_v^2) x_i & 0 \\ \sum_{i=1}^{m} S_i^{-1}(\beta_1, \sigma_v^2) x_i & \sum_{i=1}^{m} S_i^{-1}(\beta_1, \sigma_v^2)(x_i^2 + \tilde{\sigma}_{ci}^2) & 0 \\ 0 & 0 & \frac{1}{2} \sum_{i=1}^{m} S_i^{-2}(\beta_1, \sigma_v^2) \end{bmatrix}.
$$

S6 Details of derivations for $\hat{R}_{1i}$

Recall that $R_{1i} := M_{1i}(\boldsymbol{\omega}) M_{2i}(\boldsymbol{\omega})$. In order to estimate R_{1i}, one can define

$$
\begin{aligned}
&E[M_{1i}(\hat{\boldsymbol{\omega}}) M_{2i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega}) M_{2i}(\boldsymbol{\omega})]^2 := E[\{M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})\}\{M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega})\} \\
&\quad + M_{2i}(\boldsymbol{\omega})(M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})) + M_{1i}(\boldsymbol{\omega})(M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega}))]^2 \\
&= E[\{M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})\}^2 \{M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega})\}^2]
\end{aligned}
$$

$$+ M_{2i}^2(\boldsymbol{\omega})E[M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})]^2 + M_{1i}^2(\boldsymbol{\omega})E[M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega})]^2$$

$$+ 2E[(M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega}))^2(M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega}))]M_{2i}(\boldsymbol{\omega})$$

$$+ 2E[(M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega}))(M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega}))^2]M_{1i}(\boldsymbol{\omega})$$

$$+ 2M_{1i}(\boldsymbol{\omega})M_{2i}(\boldsymbol{\omega})E[(M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega}))(M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega}))]. \quad \text{(S6.1)}$$

Application of the Cauchy-Schwarz inequality yields

(i) $E[\{M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})\}^2\{M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega})\}^2]$

$$\leq E^{1/2}[M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})]^4 E^{1/2}[M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega})]^4 = O(1)O(m^{-1}) = O(m^{-1}),$$

(ii) $E[\{M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})\}^2\{M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega})\}]$

$$\leq E^{1/2}[M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})]^4 E^{1/2}[M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega})]^2 = O(1)O(m^{-1/2}) = O(m^{-1/2}),$$

(iii) $E[\{M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})\}\{M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega})\}^2]$

$$\leq E^{1/2}[M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})]^2 E^{1/2}[M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega})]^4 = O(1)O(m^{-1}) = O(m^{-1}),$$

(iv) $E[M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega})]^2 = O(m^{-1})$, and

(v) $E[\{M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})\}\{M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega})\}]$

$$\leq E^{1/2}[M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})]^2 E^{1/2}[M_{2i}(\hat{\boldsymbol{\omega}}) - M_{2i}(\boldsymbol{\omega})]^2 = O(1)O(m^{-1/2}) = O(m^{-1/2}).$$

Thus, we conclude that only the term $M_{2i}^2(\boldsymbol{\omega})E[M_{1i}(\hat{\boldsymbol{\omega}}) - M_{1i}(\boldsymbol{\omega})]^2$ from expression (S6.1) needs to be estimated. Therefore, the estimator of R_{1i} is the expression of $\hat{R}_{1i}$ given in the manuscript.