

A NEW CLASS OF ORTHOGONAL DESIGNS WITH GOOD LOW DIMENSIONAL SPACE-FILLING PROPERTIES

Chunyan Wang¹, Dennis K. J. Lin² and Min-Qian Liu³

¹*Renmin University of China*, ²*Purdue University* and ³*Nankai University*

Supplementary Material

In this supplemental file, we provide the proofs of Lemma 1 and Theorems 1 and 2, as well as five tables, where Tables S.1, S.2 and S.3 lists the OA(32, 9, 4, 2), OA(4, 3, 2, 2) and OD(64, 16¹⁶) in Example 1, respectively. Table S.4 lists the OD(64, 8¹⁸) in Example 3, and Table S.5 lists the OD(64, 16¹²8⁶) in Example 4.

S1. Proof of Lemma 1

(i) We show that if (d_1, d_2) is an OA($n, 2, s, 2$), then the arrays $(\alpha_0 \oplus d_1, \alpha_0 \oplus d_2, \alpha_s \oplus d_2)$ and $(\alpha_s \oplus d_1, \alpha_0 \oplus d_2, \alpha_s \oplus d_2)$ are OA($sn, 3, s, 3$)'s.

a) Consider the first array $(\alpha_0 \oplus d_1, \alpha_0 \oplus d_2, \alpha_s \oplus d_2)$, it can be written as

$$\begin{bmatrix} d_1 & d_2 & d_2 \\ d_1 & d_2 & d_2 + 1 \\ \vdots & \vdots & \vdots \\ d_1 & d_2 & d_2 + (s - 1) \end{bmatrix}.$$

As (d_1, d_2) is an orthogonal array of strength two, each level combination (α, β) with $\alpha, \beta \in \{0, 1, \dots, s - 1\}$ occurs in the array with the same frequency. And for any level combination (α, β, β) in (d_1, d_2, d_2) , there is a corresponding level combination

Corresponding author: Min-Qian Liu, School of Statistics and Data Science, LPMC & KLMDASR, Nankai University, Tianjin 300071, China. E-mail: mqliu@nankai.edu.cn.

$(\alpha, \beta, \beta + l)$ in $(d_1, d_2, d_2 + l)$ for $l = 1, \dots, s - 1$, where $\beta, \beta + 1, \dots, \beta + (s - 1)$ are distinct with each other and they are a permutation on $\{0, 1, \dots, s - 1\}$. Thus $(\alpha_0 \oplus d_1, \alpha_0 \oplus d_2, \alpha_s \oplus d_2)$ is an orthogonal array of strength three.

b) Now we consider the array $(\alpha_s \oplus d_1, \alpha_0 \oplus d_2, \alpha_s \oplus d_2)$, which can be written as

$$\begin{bmatrix} d_1 & d_2 & d_2 \\ d_1 + 1 & d_2 & d_2 + 1 \\ \vdots & \vdots & \vdots \\ d_1 + (s - 1) & d_2 & d_2 + (s - 1) \end{bmatrix}.$$

We know that (d_1, d_2) is an orthogonal array of strength two, then $(d_1 + l, d_2)$ must be an orthogonal array of strength two for $l = 1, \dots, s - 1$. Thus for each of these arrays, each level combination (α, β) with $\alpha, \beta \in \{0, 1, \dots, s - 1\}$ occurs with the same frequency. And for any level combination (α, β, β) in (d_1, d_2, d_2) , there is a corresponding level combination $(\alpha, \beta, \beta + l)$ in $(d_1 + l, d_2, d_2 + l)$ for $l = 1, \dots, s - 1$, where $\beta, \beta + 1, \dots, \beta + (s - 1)$ are distinct with each other and they are a permutation on $\{0, 1, \dots, s - 1\}$. Thus $(\alpha_s \oplus d_1, \alpha_0 \oplus d_2, \alpha_s \oplus d_2)$ is an orthogonal array of strength three.

Hence, each of the two arrays $(\alpha_0 \oplus d_1, \alpha_0 \oplus d_2, \alpha_s \oplus d_2)$ and $(\alpha_s \oplus d_1, \alpha_0 \oplus d_2, \alpha_s \oplus d_2)$ is an $\text{OA}(sn, 3, s, 3)$.

(ii) We show that if (d_1, d_2, d_3) is an $\text{OA}(n, 3, s, 2)$, then the array $(\alpha_0 \oplus d_1, \alpha_0 \oplus d_2, \alpha_s \oplus d_3)$ is an $\text{OA}(sn, 3, s, 3)$.

Note that $(\alpha_0 \oplus d_1, \alpha_0 \oplus d_2, \alpha_s \oplus d_3)$ can be written as

$$\begin{bmatrix} d_1 & d_2 & d_3 \\ d_1 & d_2 & d_3 + 1 \\ \vdots & \vdots & \vdots \\ d_1 & d_2 & d_3 + (s-1) \end{bmatrix}.$$

As (d_1, d_2) is an orthogonal array of strength two, each level combination (α, β) with $\alpha, \beta \in \{0, 1, \dots, s-1\}$ occurs in the array with the same frequency. And for any level combination (α, β, γ) in (d_1, d_2, d_3) , there is a corresponding level combination $(\alpha, \beta, \gamma + l)$ in $(d_1, d_2, d_3 + l)$ for $l = 1, \dots, s-1$, where $\gamma, \gamma + 1, \dots, \gamma + (s-1)$ are distinct with each other and they are a permutation on $\{0, 1, \dots, s-1\}$. Thus $(\alpha_0 \oplus d_1, \alpha_0 \oplus d_2, \alpha_s \oplus d_3)$ is an orthogonal array of strength three.

(iii) We show that if (d_1, d_2, d_3) is an $\text{OA}(n, 3, s, 3)$, then the arrays $(\alpha_0 \oplus d_1, \alpha_0 \oplus d_2, \alpha_0 \oplus d_3, \alpha_s \oplus d_3)$, $(\alpha_0 \oplus d_1, \alpha_s \oplus d_2, \alpha_0 \oplus d_3, \alpha_s \oplus d_3)$ and $(\alpha_s \oplus d_1, \alpha_s \oplus d_2, \alpha_0 \oplus d_3, \alpha_s \oplus d_3)$ are $\text{OA}(sn, 4, s, 4)$'s.

a) Consider the first array $(\alpha_0 \oplus d_1, \alpha_0 \oplus d_2, \alpha_0 \oplus d_3, \alpha_s \oplus d_3)$, it can be written as

$$\begin{bmatrix} d_1 & d_2 & d_3 & d_3 \\ d_1 & d_2 & d_3 & d_3 + 1 \\ \vdots & \vdots & \vdots & \vdots \\ d_1 & d_2 & d_3 & d_3 + (s-1) \end{bmatrix}.$$

As (d_1, d_2, d_3) is an orthogonal array of strength three, each level combination (α, β, γ) with $\alpha, \beta, \gamma \in \{0, 1, \dots, s-1\}$ occurs in the array with the same frequency. And for any level combination $(\alpha, \beta, \gamma, \gamma)$ in (d_1, d_2, d_3, d_3) , there is a corresponding level combination $(\alpha, \beta, \gamma, \gamma + l)$ in $(d_1, d_2, d_3, d_3 + l)$ for $l = 1, \dots, s-1$, where $\gamma, \gamma + 1, \dots, \gamma +$

$(s-1)$ are distinct with each other and they are a permutation on $\{0, 1, \dots, s-1\}$.

Thus $(\alpha_0 \oplus d_1, \alpha_0 \oplus d_2, \alpha_0 \oplus d_3, \alpha_s \oplus d_3)$ is an orthogonal array of strength four.

b) Now we consider the array $(\alpha_0 \oplus d_1, \alpha_s \oplus d_2, \alpha_0 \oplus d_3, \alpha_s \oplus d_3)$, which can be written as

$$\begin{bmatrix} d_1 & d_2 & d_3 & d_3 \\ d_1 & d_2 + 1 & d_3 & d_3 + 1 \\ \vdots & \vdots & \vdots & \vdots \\ d_1 & d_2 + (s-1) & d_3 & d_3 + (s-1) \end{bmatrix}.$$

We know that (d_1, d_2, d_3) is an orthogonal array of strength three, then $(d_1, d_2 + l, d_3)$ must be an orthogonal array of strength three for $l = 1, \dots, s-1$. Thus for each of these arrays, each level combination (α, β, γ) with $\alpha, \beta, \gamma \in \{0, 1, \dots, s-1\}$ occurs with the same frequency. And for any level combination $(\alpha, \beta, \gamma, \gamma)$ in (d_1, d_2, d_3, d_3) , there is a corresponding level combination $(\alpha, \beta, \gamma, \gamma + l)$ in $(d_1, d_2 + l, d_3, d_3 + l)$ for $l = 1, \dots, s-1$, where $\gamma, \gamma + 1, \dots, \gamma + (s-1)$ are distinct with each other and they are a permutation on $\{0, 1, \dots, s-1\}$. Thus $(\alpha_0 \oplus d_1, \alpha_s \oplus d_2, \alpha_0 \oplus d_3, \alpha_s \oplus d_3)$ is an orthogonal array of strength four.

c) Consider the array $(\alpha_s \oplus d_1, \alpha_s \oplus d_2, \alpha_0 \oplus d_3, \alpha_s \oplus d_3)$, which can be written as

$$\begin{bmatrix} d_1 & d_2 & d_3 & d_3 \\ d_1 + 1 & d_2 + 1 & d_3 & d_3 + 1 \\ \vdots & \vdots & \vdots & \vdots \\ d_1 + (s-1) & d_2 + (s-1) & d_3 & d_3 + (s-1) \end{bmatrix}.$$

We know that (d_1, d_2, d_3) is an orthogonal array of strength three, then $(d_1 + l, d_2 + l, d_3)$ must be an orthogonal array of strength three for $l = 1, \dots, s-1$. Thus for each

of these arrays, each level combination (α, β, γ) with $\alpha, \beta, \gamma \in \{0, 1, \dots, s-1\}$ occurs with the same frequency. And for any level combination $(\alpha, \beta, \gamma, \gamma)$ in (d_1, d_2, d_3, d_3) , there is a corresponding level combination $(\alpha, \beta, \gamma, \gamma + l)$ in $(d_1 + l, d_2 + l, d_3, d_3 + l)$ for $l = 1, \dots, s-1$, where $\gamma, \gamma + 1, \dots, \gamma + (s-1)$ are distinct with each other and they are a permutation on $\{0, 1, \dots, s-1\}$. Thus $(\alpha_s \oplus d_1, \alpha_s \oplus d_2, \alpha_0 \oplus d_3, \alpha_s \oplus d_3)$ is an orthogonal array of strength four.

Thus, each of the three arrays $(\alpha_0 \oplus d_1, \alpha_0 \oplus d_2, \alpha_0 \oplus d_3, \alpha_s \oplus d_3)$, $(\alpha_0 \oplus d_1, \alpha_s \oplus d_2, \alpha_0 \oplus d_3, \alpha_s \oplus d_3)$ and $(\alpha_s \oplus d_1, \alpha_s \oplus d_2, \alpha_0 \oplus d_3, \alpha_s \oplus d_3)$ is an $\text{OA}(sn, 4, s, 4)$.

S2. Proof of Theorem 1

We first consider the level of design \tilde{X} . For any column x of \tilde{X} , $x \in X_{(l)}$ for some l , it can be written as

$$x = \begin{cases} s^3(\alpha_0 \oplus c)^* + s^2(\alpha_s \oplus c')^* + s(\alpha_0 \oplus c')^* \pm (\alpha_0 \oplus c'')^*, & \text{if } x \text{ is an odd-numbered column of } X_{(l)}, \\ s^2(\alpha_s \oplus c)^* - s^3(\alpha_0 \oplus c')^* + s(\alpha_0 \oplus c)^* \pm (\alpha_0 \oplus c'')^*, & \text{if } x \text{ is an even-numbered column of } X_{(l)}, \end{cases}$$

where c and c' are the two columns in some C_{ij} with $C_{ij} \subset C_{(l)}$, c'' is a column in $C_{(l)}$ but not in C_{ij} , the coefficient of term $(\alpha_0 \oplus c'')^*$ is -1 when c'' is among the first two columns of $C_{(l)}$ and 1 when c'' is among the last two columns of $C_{(l)}$. Note that two columns of (c, c', c'') are in the same group and the remaining one is in another group of C . Then (c, c', c'') must be an $\text{OA}(n, 3, s, 3)$ according to Lemma 2. Furthermore, we can get that the following two arrays

$$(\alpha_0 \oplus c, \alpha_s \oplus c', \alpha_0 \oplus c', \alpha_0 \oplus c'') \text{ and } (\alpha_s \oplus c, \alpha_0 \oplus c', \alpha_0 \oplus c, \alpha_0 \oplus c'')$$

must be $\text{OA}(sn, 4, s, 4)$'s according to Lemma 1(iii). Therefore each column of \tilde{X} is populated by s^4 levels with the same frequency.

Then we prove the column-orthogonality of \tilde{X} . For any two columns x_1 and x_2 of \tilde{X} , $x_1 \in X_{(l_1)}$ and $x_2 \in X_{(l_2)}$ for some l_1 and l_2 , they can be written as

$$x_t = \begin{cases} s^3(\alpha_0 \oplus c_t)^* + s^2(\alpha_s \oplus c'_t)^* + s(\alpha_0 \oplus c'_t)^* \pm (\alpha_0 \oplus c''_t)^*, & \text{if } x \text{ is an odd-numbered column of } X_{(l_t)}, \\ s^2(\alpha_s \oplus c_t)^* - s^3(\alpha_0 \oplus c'_t)^* + s(\alpha_0 \oplus c_t)^* \pm (\alpha_0 \oplus c''_t)^*, & \text{if } x \text{ is an even-numbered column of } X_{(l_t)}, \end{cases}$$

for $t = 1$ and 2 , where c_1 and c'_1 are the two columns in some $C_{i_1j_1}$ with $C_{i_1j_1} \subset C_{(l_1)}$, c''_1 is a column in $C_{(l_1)}$ but not in $C_{i_1j_1}$, meanwhile c_2 and c'_2 are the two columns in some $C_{i_2j_2}$ with $C_{i_2j_2} \subset C_{(l_2)}$, c''_2 is a column in $C_{(l_2)}$ but not in $C_{i_2j_2}$. And we have $C_{i_1j_1} \subset C_{i_1}$, $C_{i_2j_2} \subset C_{i_2}$. There are three possible cases for the two columns (x_1, x_2) as shown below.

(i) First we consider the case that the leading columns of x_1 and x_2 are in the same group of \tilde{C} , and they correspond to the same C_{ij} . That is $l_1 = l_2$, $i_1 = i_2$, $j_1 = j_2$. In this case

$$c_1 = c_2 \text{ and } c'_1 = c'_2,$$

and one of x_1 and x_2 is of the odd-numbered column of $X_{(l_1)}$ and the other one is of the even-numbered column of $X_{(l_1)}$. Without loss of generality, suppose x_1 is of the odd-numbered column and x_2 is of the even-numbered column. Then we have

$$\begin{aligned} x_1^T x_2 &= (s^3(\alpha_0 \oplus c_1)^* + s^2(\alpha_s \oplus c'_1)^* + s(\alpha_0 \oplus c'_1)^* + (\alpha_0 \oplus c''_1)^*)^T \\ &\quad (s^2(\alpha_s \oplus c_1)^* - s^3(\alpha_0 \oplus c'_1)^* + s(\alpha_0 \oplus c_1)^* + (\alpha_0 \oplus c''_2)^*) \\ &= s^4[(\alpha_0 \oplus c_1)^*]^T (\alpha_0 \oplus c_1)^* - s^4[(\alpha_0 \oplus c'_1)^*]^T (\alpha_0 \oplus c'_1)^* \\ &= 0. \end{aligned}$$

(ii) Next we consider the case that the leading columns of x_1 and x_2 are in the same group of \tilde{C} , but they correspond to different C_{ij} . That is $l_1 = l_2$, and $C_{i_1j_1} \neq C_{i_2j_2}$.

There are three cases a) x_1 and x_2 are of odd-numbered columns of $X_{(l_1)}$; b) x_1 and x_2 are of even-numbered columns of $X_{(l_1)}$; and c) x_1 is of odd-numbered column of $X_{(l_1)}$ and x_2 is of even-numbered column of $X_{(l_1)}$, or x_1 is of even-numbered column of $X_{(l_1)}$ and x_2 is of odd-numbered column of $X_{(l_1)}$. For case a), we have $c_2 = c_1''$ and $c_2'' = c_1$, and (c_1, c_1', c_2, c_2') is an orthogonal array of strength two. Then

$$\begin{aligned}
 x_1^T x_2 &= (s^3(\alpha_0 \oplus c_1)^* + s^2(\alpha_s \oplus c_1')^* + s(\alpha_0 \oplus c_1')^* + (\alpha_0 \oplus c_1'')^*)^T \\
 &\quad (s^3(\alpha_0 \oplus c_1'')^* + s^2(\alpha_s \oplus c_2')^* + s(\alpha_0 \oplus c_2')^* - (\alpha_0 \oplus c_1)^*) \\
 &= -s^3[(\alpha_0 \oplus c_1)^*]^T (\alpha_0 \oplus c_1)^* + s^3[(\alpha_0 \oplus c_1'')^*]^T (\alpha_0 \oplus c_1'')^* \\
 &= 0.
 \end{aligned}$$

For case b), $c_2' = c_1''$, $c_2'' = c_1'$ and (c_1, c_1', c_2, c_2') is an orthogonal array of strength two.

Then

$$\begin{aligned}
 x_1^T x_2 &= (s^2(\alpha_s \oplus c_1)^* - s^3(\alpha_0 \oplus c_1')^* + s(\alpha_0 \oplus c_1)^* + (\alpha_0 \oplus c_1'')^*)^T \\
 &\quad (s^2(\alpha_s \oplus c_2)^* - s^3(\alpha_0 \oplus c_1'')^* + s(\alpha_0 \oplus c_2)^* - (\alpha_0 \oplus c_1')^*) \\
 &= s^3[(\alpha_0 \oplus c_1')^*]^T (\alpha_0 \oplus c_1')^* - s^3[(\alpha_0 \oplus c_1'')^*]^T (\alpha_0 \oplus c_1'')^* \\
 &= 0.
 \end{aligned}$$

For case c), let us assume x_1 is of odd-numbered column of $X_{(l_1)}$ and x_2 is of even-numbered column of $X_{(l_1)}$. Then we have $c_2 = c_1''$ and $c_2'' = c_1'$ and (c_1, c_1', c_2, c_2') is an

orthogonal array of strength two. Thus

$$\begin{aligned}
x_1^T x_2 &= (s^3(\alpha_0 \oplus c_1)^* + s^2(\alpha_s \oplus c'_1)^* + s(\alpha_0 \oplus c'_1)^* + (\alpha_0 \oplus c''_1)^*)^T \\
&\quad (s^2(\alpha_s \oplus c''_1)^* - s^3(\alpha_0 \oplus c'_2)^* + s(\alpha_0 \oplus c''_1)^* - (\alpha_0 \oplus c'_1)^*) \\
&= -s[(\alpha_0 \oplus c'_1)^*]^T (\alpha_0 \oplus c'_1)^* + s[(\alpha_0 \oplus c''_1)^*]^T (\alpha_0 \oplus c''_1)^* \\
&= 0.
\end{aligned}$$

(iii) Consider the case that the leading columns of x_1 and x_2 are from different group of \tilde{C} , where $c_1, c'_1 \in C_{(l_1)}$ and $c_2, c'_2 \in C_{(l_2)}$ with $l_1 \neq l_2$. It is easy to see that $(c_1, c'_1, c''_1, c_2, c'_2, c''_2)$ is an orthogonal array of strength two, indicating that $x_1^T x_2 = 0$.

Therefore, design \tilde{X} is an $\text{OD}(sn, (s^4)^{4q})$.

S3. Proof of Theorem 2

(i) We first consider the space-filling property of any two distinct columns of X . By inspecting the relation between the columns of \tilde{X} and \tilde{C} , we can see that for any column x of \tilde{X} , $x \in X_{(l)}$ (for some l) can be written as

$$x = \begin{cases} s^3(\alpha_0 \oplus c)^* + s^2(\alpha_s \oplus c')^* + s(\alpha_0 \oplus c')^* \pm (\alpha_0 \oplus c'')^*, & \text{if } x \text{ is an odd-numbered column of } X_{(l)}, \\ s^2(\alpha_s \oplus c)^* - s^3(\alpha_0 \oplus c')^* + s(\alpha_0 \oplus c)^* \pm (\alpha_0 \oplus c'')^*, & \text{if } x \text{ is an even-numbered column of } X_{(l)}, \end{cases}$$

where c and c' are the two columns in some C_{ij} with $C_{ij} \subset C_{(l)}$, c'' is a column in $C_{(l)}$ but not in C_{ij} . The coefficient of term $(\alpha_0 \oplus c'')^*$ is -1 when c'' is among the first two columns of $C_{(l)}$ and 1 when c'' is among the last two columns of $C_{(l)}$. We call c the leading column of x if x is the odd-numbered column of $X_{(l)}$, and c' the leading column of x if x is the even-numbered column of $X_{(l)}$ to facilitate later study.

Recall that X is obtained from \tilde{X} by reorganizing the $4q$ columns according to the order of their leading columns in the original groups C_1, \dots, C_g , and the order of the

columns of X follows that of C while the order of the columns of \tilde{X} follows that of \tilde{C} .

Any two columns x_1 and x_2 of X , $x_1 \in X_{(l_1)}$ and $x_2 \in X_{(l_2)}$ for some l_1 and l_2 can be written as

$$x_t = \begin{cases} s^3(\alpha_0 \oplus c_t)^* + s^2(\alpha_s \oplus c'_t)^* + s(\alpha_0 \oplus c'_t)^* \pm (\alpha_0 \oplus c''_t)^*, & \text{if } x_t \text{ is an odd-numbered column of } X_{(l_t)}, \\ s^2(\alpha_s \oplus c_t)^* - s^3(\alpha_0 \oplus c'_t)^* + s(\alpha_0 \oplus c_t)^* \pm (\alpha_0 \oplus c''_t)^*, & \text{if } x_t \text{ is an even-numbered column of } X_{(l_t)}, \end{cases} \quad (\text{S.1})$$

for $t = 1, 2$, where c_t and c'_t are the two columns in some $C_{i_t j_t}$ with $C_{i_t j_t} \subset C_{(l_t)}$ and c''_t is a column in $C_{(l_t)}$ but not in this $C_{i_t j_t}$.

Collapse the s^4 levels in $\Omega(s^4)$ of x_1 into the s levels in $\Omega(s)$ via the mapping

$$f(z) = \left\lfloor \frac{z + (s^4 - 1)/2}{s^3} \right\rfloor - (s - 1)/2 \text{ for } z \in \Omega(s^4), \quad (\text{S.2})$$

and collapse the s^4 levels of x_2 into the s^2 levels in $\Omega(s^2)$ via the mapping

$$h(z) = \left\lfloor \frac{z + (s^4 - 1)/2}{s^2} \right\rfloor - (s^2 - 1)/2 \text{ for } z \in \Omega(s^4), \quad (\text{S.3})$$

then we have

$$f(x_1) = \begin{cases} (\alpha_0 \oplus c_1)^*, & \text{if } x_1 \text{ is an odd-numbered column of } X_{(l_1)}, \\ -(\alpha_0 \oplus c'_1)^*, & \text{if } x_1 \text{ is an even-numbered column of } X_{(l_1)}, \end{cases} \quad (\text{S.4})$$

and

$$h(x_2) = \begin{cases} s(\alpha_0 \oplus c_2)^* + (\alpha_s \oplus c'_2)^*, & \text{if } x_2 \text{ is an odd-numbered column of } X_{(l_2)}, \\ (\alpha_s \oplus c_2)^* - s(\alpha_0 \oplus c'_2)^*, & \text{if } x_2 \text{ is an even-numbered column of } X_{(l_2)}. \end{cases}$$

There are three possible cases for two columns x_1 and x_2 .

a) x_1 and x_2 are from the same group of X , and (c_1, c'_1, c_2, c'_2) has two distinct columns, i.e. $i_1 = i_2$ and $j_1 = j_2$, while $c_1 = c_2$ and $c'_1 = c'_2 \in C_{i_1}$ are the two columns of $C_{i_1 j_1}$. Thus $(f(x_1), h(x_2))$ takes $((\alpha_0 \oplus c_1)^*, (\alpha_s \oplus c_1)^* - s(\alpha_0 \oplus c'_1)^*)$ or

$(-(\alpha_0 \oplus c'_1)^*, s(\alpha_0 \oplus c_1)^* + (\alpha_s \oplus c'_1)^*)$. It is easy to see that (c_1, c'_1) is an orthogonal array of strength two. Then, by Lemma 1(i), each of the arrays $(\alpha_0 \oplus c_1, \alpha_s \oplus c_1, \alpha_0 \oplus c'_1)$ and $(\alpha_0 \oplus c'_1, \alpha_0 \oplus c_1, \alpha_s \oplus c'_1)$ is an $\text{OA}(sn, 3, s, 3)$. Thus, $(f(x_1), h(x_2))$ must be an $\text{OA}(sn, 2, s \times s^2, 2)$.

b) x_1 and x_2 are from the same group of X , and the four columns of (c_1, c'_1, c_2, c'_2) are distinct, i.e. $i_1 = i_2$ and $j_1 \neq j_2$, while $c_1, c'_1 \in C_{i_1 j_1}$ and $c_2, c'_2 \in C_{i_1 j_2}$ are the four columns of $C_{i_1 j_1}$ and $C_{i_1 j_2}$. Then $(f(x_1), h(x_2))$ takes $((\alpha_0 \oplus c_1)^*, s(\alpha_0 \oplus c_2)^* + (\alpha_s \oplus c'_2)^*)$, $((\alpha_0 \oplus c_1)^*, (\alpha_s \oplus c_2)^* - s(\alpha_0 \oplus c'_2)^*)$, $(-(\alpha_0 \oplus c'_1)^*, s(\alpha_0 \oplus c_2)^* + (\alpha_s \oplus c'_2)^*)$ or $(-(\alpha_0 \oplus c'_1)^*, (\alpha_s \oplus c_2)^* - s(\alpha_0 \oplus c'_2)^*)$. It is clear that (c_1, c'_1, c_2, c'_2) forms an orthogonal array of strength two. Then, according to Lemma 1(ii), each of the arrays $(\alpha_0 \oplus c_1, \alpha_0 \oplus c_2, \alpha_s \oplus c'_2)$, $(\alpha_0 \oplus c_1, \alpha_s \oplus c_2, \alpha_0 \oplus c'_2)$, $(\alpha_0 \oplus c'_1, \alpha_0 \oplus c_2, \alpha_s \oplus c'_2)$ and $(\alpha_0 \oplus c'_1, \alpha_s \oplus c_2, \alpha_0 \oplus c'_2)$ is an $\text{OA}(sn, 3, s, 3)$, indicating that $(f(x_1), h(x_2))$ must form an $\text{OA}(sn, 2, s \times s^2, 2)$.

c) x_1 and x_2 are from different groups of X , i.e. $i_1 \neq i_2$. Then $(c_1, c'_1) \in C_{i_1}$ while $(c_2, c'_2) \in C_{i_2}$. According to Lemma 2, (c_1, c'_1, c_2, c'_2) is an $\text{OA}(n, 4, s, 4)$, indicating that each of arrays (c_1, c_2, c'_2) and (c'_1, c_2, c'_2) is an $\text{OA}(n, 3, s, 3)$. Thus $(f(x_1), h(x_2))$ must form an $\text{OA}(sn, 2, s \times s^2, 2)$.

Thus, for any two columns x_1 and x_2 of X , $(f(x_1), h(x_2))$ must form an $\text{OA}(sn, 2, s \times s^2, 2)$. Let $\hat{x}_1 = x_2$ and $\hat{x}_2 = x_1$, then $(f(\hat{x}_1), h(\hat{x}_2)) = (f(x_2), h(x_1))$ must form an $\text{OA}(sn, 2, s \times s^2, 2)$. Thus any two distinct columns of X achieve stratifications on $s \times s^2$ and $s^2 \times s$ grids.

(ii) We now consider the space-filling property of two columns from different groups. Let x_1 and x_2 be two columns from different groups of X , as given in (S.1), where

$x_1 \in X_{i_1}$ and $x_2 \in X_{i_2}$ for some $i_1 \neq i_2$. Collapse the s^4 levels in $\Omega(s^4)$ of x_1 and x_2 into the s^2 levels in $\Omega(s^2)$ using mapping (S.3). Then we have

$$h(x_t) = \begin{cases} s(\alpha_0 \oplus c_t)^* + (\alpha_s \oplus c'_t)^*, & \text{if } x_t \text{ is an odd-numbered column of } X_{(l_t)}, \\ (\alpha_s \oplus c_t)^* - s(\alpha_0 \oplus c'_t)^*, & \text{if } x_t \text{ is an even-numbered column of } X_{(l_t)}, \end{cases}$$

for $t = 1, 2$. As we have discussed, (c_1, c'_1, c_2, c'_2) form an $\text{OA}(n, 4, s, 4)$, which means that $(h(x_1), h(x_2))$ must be an $\text{OA}(sn, 2, s^2, 2)$.

Collapse the s^4 levels in $\Omega(s^4)$ of x_1 into the s levels in $\Omega(s)$ using mapping (S.2), and the s^4 levels of x_2 into the s^3 levels in $\Omega(s^3)$ using the mapping

$$r(z) = \left\lfloor \frac{z + (s^4 - 1)/2}{s} \right\rfloor - (s^3 - 1)/2 \text{ for } z \in \Omega(s^4).$$

Then we have

$$r(x_2) = \begin{cases} s^2(\alpha_0 \oplus c_2)^* + s(\alpha_s \oplus c'_2)^* + (\alpha_0 \oplus c'_2)^*, & \text{if } x_2 \text{ is an odd-numbered column of } X_{(l_2)}, \\ s(\alpha_s \oplus c_2)^* - s^2(\alpha_0 \oplus c'_2)^* + (\alpha_0 \oplus c_2)^*, & \text{if } x_2 \text{ is an even-numbered column of } X_{(l_2)}, \end{cases}$$

and $f(x_1)$ has the form shown in (S.4). As previously discussed, each of the arrays (c_1, c_2, c'_2) (c'_1, c_2, c'_2) is an $\text{OA}(n, 3, s, 3)$. Then each of the arrays $(\alpha_0 \oplus c_1, \alpha_0 \oplus c_2, \alpha_s \oplus c'_2, \alpha_0 \oplus c'_2)$, $(\alpha_0 \oplus c_1, \alpha_s \oplus c_2, \alpha_0 \oplus c'_2, \alpha_0 \oplus c_2)$, $(\alpha_0 \oplus c'_1, \alpha_0 \oplus c_2, \alpha_s \oplus c'_2, \alpha_0 \oplus c'_2)$ and $(\alpha_0 \oplus c'_1, \alpha_s \oplus c_2, \alpha_0 \oplus c'_2, \alpha_0 \oplus c_2)$ is an $\text{OA}(sn, 4, s, 4)$, via Lemma 1(iii), which indicates that $(f(x_1), r(x_2))$ must be an $\text{OA}(sn, 2, s \times s^3, 2)$.

Thus any two columns from different groups of X achieve stratifications on $s \times s^3$, $s^2 \times s^2$ and $s^3 \times s$ grids.

(iii) Next we investigate the space-filling property in three dimensions for design X . Consider three columns x_1, x_2, x_3 of X , where x_t has the form as in (S.1) with $c_t, c'_t \in C_{itjt}$ for $t = 1, 2, 3$. As shown in (S.4), x_t becomes $(\alpha_0 \oplus c_t)^*$ or $-(\alpha_0 \oplus c'_t)^*$ after

collapsing the s^4 levels in $\Omega(s^4)$ into the s levels in $\Omega(s)$ for $t = 1, 2, 3$. From Lemma 2, if the three columns c_1, c_2 and c_3 are from two different groups of C , then they must form an orthogonal array of strength three. Thus any three distinct columns from two different groups of X achieve a stratification on an $s \times s \times s$ grid.

S4. Five Tables

Table S.1: The $\text{OA}(32, 9, 4, 2)$ $A = (a_1, \dots, a_9)$ in Example 1.

Run	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1	0	0	3	3	3	3	1	2	0
2	0	1	2	0	2	1	3	1	3
3	0	2	1	2	0	0	2	3	2
4	0	3	0	1	1	2	0	0	1
5	1	0	2	2	1	3	3	3	1
6	1	1	3	1	0	1	1	0	2
7	1	2	0	3	2	0	0	2	3
8	1	3	1	0	3	2	2	1	0
9	2	0	1	3	0	2	3	0	3
10	2	1	0	0	1	0	1	3	0
11	2	2	3	2	3	1	0	1	1
12	2	3	2	1	2	3	2	2	2
13	3	0	0	2	2	2	1	1	2
14	3	1	1	1	3	0	3	2	1
15	3	2	2	3	1	1	2	0	0
16	3	3	3	0	0	3	0	3	3
17	0	0	3	1	1	0	2	1	3
18	0	1	2	2	0	2	0	2	0
19	0	2	1	0	2	3	1	0	1
20	0	3	0	3	3	1	3	3	2
21	1	0	2	0	3	0	0	0	2
22	1	1	3	3	2	2	2	3	1
23	1	2	0	1	0	3	3	1	0
24	1	3	1	2	1	1	1	2	3
25	2	0	1	1	2	1	0	3	0
26	2	1	0	2	3	3	2	0	3
27	2	2	3	0	1	2	3	2	2
28	2	3	2	3	0	0	1	1	1
29	3	0	0	0	0	1	2	2	1
30	3	1	1	3	1	3	0	1	2
31	3	2	2	1	3	2	1	3	3
32	3	3	3	2	2	0	3	0	0

Table S.2: The $\text{OA}(4, 3, 2, 2)$ $B = (b_1, b_2, b_3)$ in Example 1.

b_1	b_2	b_3
0	0	0
0	1	1
1	0	1
1	1	0

Table S.3: The $\text{OD}(64, 16^{16})$ $X = (X_1, \dots, X_8)$ in Example 1.

Run	X_1		X_2		X_3		X_4		X_5		X_6		X_7		X_8	
1	-7.5	0.5	-6.5	1.5	7.5	-0.5	6.5	-1.5	7.5	-0.5	6.5	-1.5	-0.5	-7.5	1.5	6.5
2	-7.5	1.5	-0.5	-6.5	0.5	6.5	-7.5	1.5	0.5	7.5	-1.5	-6.5	6.5	-0.5	-1.5	-7.5
3	-6.5	0.5	1.5	7.5	-0.5	-7.5	1.5	6.5	-7.5	0.5	-6.5	1.5	1.5	7.5	6.5	-0.5
4	-6.5	1.5	7.5	-0.5	-7.5	1.5	-0.5	-6.5	-0.5	-7.5	1.5	6.5	-7.5	0.5	-6.5	1.5
5	-1.5	-7.5	-6.5	0.5	1.5	6.5	0.5	7.5	-0.5	-6.5	7.5	-1.5	7.5	-0.5	6.5	-1.5
6	-1.5	-6.5	-0.5	-7.5	6.5	-0.5	-1.5	-7.5	-7.5	1.5	-0.5	-6.5	-1.5	-7.5	-6.5	0.5
7	-0.5	-7.5	1.5	6.5	-6.5	1.5	7.5	-0.5	0.5	6.5	-7.5	1.5	-6.5	0.5	1.5	7.5
8	-0.5	-6.5	7.5	-1.5	-1.5	-7.5	-6.5	0.5	7.5	-1.5	0.5	6.5	0.5	7.5	-1.5	-6.5
9	0.5	6.5	-7.5	1.5	-0.5	-6.5	7.5	-1.5	-6.5	0.5	1.5	7.5	6.5	-1.5	-7.5	0.5
10	0.5	7.5	-1.5	-6.5	-7.5	0.5	-6.5	1.5	-1.5	-7.5	-6.5	0.5	-0.5	-6.5	7.5	-1.5
11	1.5	6.5	0.5	7.5	7.5	-1.5	0.5	6.5	6.5	-0.5	-1.5	-7.5	-7.5	1.5	-0.5	-6.5
12	1.5	7.5	6.5	-0.5	0.5	7.5	-1.5	-6.5	1.5	7.5	6.5	-0.5	1.5	6.5	0.5	7.5
13	6.5	-1.5	-7.5	0.5	-6.5	0.5	1.5	7.5	1.5	6.5	0.5	7.5	-1.5	-6.5	-0.5	-7.5
14	6.5	-0.5	-1.5	-7.5	-1.5	-6.5	-0.5	-7.5	6.5	-1.5	-7.5	0.5	7.5	-1.5	0.5	6.5
15	7.5	-1.5	0.5	6.5	1.5	7.5	6.5	-0.5	-1.5	-6.5	-0.5	-7.5	0.5	6.5	-7.5	1.5
16	7.5	-0.5	6.5	-1.5	6.5	-1.5	-7.5	0.5	-6.5	1.5	7.5	-0.5	-6.5	1.5	7.5	-0.5
17	-7.5	0.5	-6.5	1.5	6.5	-0.5	-1.5	-7.5	-1.5	-7.5	-6.5	0.5	0.5	7.5	-1.5	-6.5
18	-7.5	1.5	-0.5	-6.5	1.5	6.5	0.5	7.5	-6.5	0.5	1.5	7.5	-6.5	0.5	1.5	7.5
19	-6.5	0.5	1.5	7.5	-1.5	-7.5	-6.5	0.5	1.5	7.5	6.5	-0.5	-1.5	-7.5	-6.5	0.5
20	-6.5	1.5	7.5	-0.5	-6.5	1.5	7.5	-0.5	6.5	-0.5	-1.5	-7.5	7.5	-0.5	6.5	-1.5
21	-1.5	-7.5	-6.5	0.5	0.5	6.5	-7.5	1.5	6.5	-1.5	-7.5	0.5	-7.5	0.5	-6.5	1.5
22	-1.5	-6.5	-0.5	-7.5	7.5	-0.5	6.5	-1.5	1.5	6.5	0.5	7.5	1.5	7.5	6.5	-0.5
23	-0.5	-7.5	1.5	6.5	-7.5	1.5	-0.5	-6.5	-6.5	1.5	7.5	-0.5	6.5	-0.5	-1.5	-7.5
24	-0.5	-6.5	7.5	-1.5	-0.5	-7.5	1.5	6.5	-1.5	-6.5	-0.5	-7.5	-0.5	-7.5	1.5	6.5
25	0.5	6.5	-7.5	1.5	-1.5	-6.5	-0.5	-7.5	0.5	7.5	-1.5	-6.5	-6.5	1.5	7.5	-0.5
26	0.5	7.5	-1.5	-6.5	-6.5	0.5	1.5	7.5	7.5	-0.5	6.5	-1.5	0.5	6.5	-7.5	1.5
27	1.5	6.5	0.5	7.5	6.5	-1.5	-7.5	0.5	-0.5	-7.5	1.5	6.5	7.5	-1.5	0.5	6.5
28	1.5	7.5	6.5	-0.5	1.5	7.5	6.5	-0.5	-7.5	0.5	-6.5	1.5	-1.5	-6.5	-0.5	-7.5
29	6.5	-1.5	-7.5	0.5	-7.5	0.5	-6.5	1.5	-7.5	1.5	-0.5	-6.5	1.5	6.5	0.5	7.5
30	6.5	-0.5	-1.5	-7.5	-0.5	-6.5	7.5	-1.5	-0.5	-6.5	7.5	-1.5	-7.5	1.5	-0.5	-6.5
31	7.5	-1.5	0.5	6.5	0.5	7.5	-1.5	-6.5	7.5	-1.5	0.5	6.5	-0.5	-6.5	7.5	-1.5
32	7.5	-0.5	6.5	-1.5	7.5	-1.5	0.5	6.5	0.5	6.5	-7.5	1.5	6.5	-1.5	-7.5	0.5
33	-3.5	4.5	-2.5	5.5	3.5	-4.5	2.5	-5.5	3.5	-4.5	2.5	-5.5	-4.5	-3.5	5.5	2.5
34	-3.5	5.5	-4.5	-2.5	4.5	2.5	-3.5	5.5	4.5	3.5	-5.5	-2.5	2.5	-4.5	-5.5	-3.5
35	-2.5	4.5	5.5	3.5	-4.5	-3.5	5.5	2.5	-3.5	4.5	-2.5	5.5	5.5	3.5	2.5	-4.5
36	-2.5	5.5	3.5	-4.5	-3.5	5.5	-4.5	-2.5	-4.5	-3.5	5.5	2.5	-3.5	4.5	-2.5	5.5
37	-5.5	-3.5	-2.5	4.5	5.5	2.5	4.5	3.5	-4.5	-2.5	3.5	-5.5	3.5	-4.5	2.5	-5.5
38	-5.5	-2.5	-4.5	-3.5	2.5	-4.5	-5.5	-3.5	-3.5	5.5	-4.5	-2.5	-5.5	-3.5	-2.5	4.5
39	-4.5	-3.5	5.5	2.5	-2.5	5.5	3.5	-4.5	4.5	2.5	-3.5	5.5	-2.5	4.5	5.5	3.5
40	-4.5	-2.5	3.5	-5.5	-5.5	-3.5	-2.5	4.5	3.5	-5.5	4.5	2.5	4.5	3.5	-5.5	-2.5
41	4.5	2.5	-3.5	5.5	-4.5	-2.5	3.5	-5.5	-2.5	4.5	5.5	3.5	2.5	-5.5	-3.5	4.5
42	4.5	3.5	-5.5	-2.5	-3.5	4.5	-2.5	5.5	-5.5	-3.5	-2.5	4.5	-4.5	-2.5	3.5	-5.5
43	5.5	2.5	4.5	3.5	3.5	-5.5	4.5	2.5	2.5	-4.5	-5.5	-3.5	-3.5	5.5	-4.5	-2.5
44	5.5	3.5	2.5	-4.5	4.5	3.5	-5.5	-2.5	5.5	3.5	2.5	-4.5	5.5	2.5	4.5	3.5
45	2.5	-5.5	-3.5	4.5	-2.5	4.5	5.5	3.5	5.5	2.5	4.5	3.5	-5.5	-2.5	-4.5	-3.5
46	2.5	-4.5	-5.5	-3.5	-5.5	-2.5	-4.5	-3.5	2.5	-5.5	-3.5	4.5	3.5	-5.5	4.5	2.5
47	3.5	-5.5	4.5	2.5	5.5	3.5	2.5	-4.5	-5.5	-2.5	-4.5	-3.5	4.5	2.5	-3.5	5.5
48	3.5	-4.5	2.5	-5.5	2.5	-5.5	-3.5	4.5	-2.5	5.5	3.5	-4.5	-2.5	5.5	3.5	-4.5
49	-3.5	4.5	-2.5	5.5	2.5	-4.5	-5.5	-3.5	-5.5	-3.5	-2.5	4.5	4.5	3.5	-5.5	-2.5
50	-3.5	5.5	-4.5	-2.5	5.5	2.5	4.5	3.5	-2.5	4.5	5.5	3.5	-2.5	4.5	5.5	3.5
51	-2.5	4.5	5.5	3.5	-5.5	-3.5	-2.5	4.5	5.5	3.5	2.5	-4.5	-5.5	-3.5	-2.5	4.5
52	-2.5	5.5	3.5	-4.5	-2.5	5.5	3.5	-4.5	2.5	-4.5	-5.5	-3.5	3.5	-4.5	2.5	-5.5
53	-5.5	-3.5	-2.5	4.5	4.5	2.5	-3.5	5.5	2.5	-5.5	-3.5	4.5	-3.5	4.5	-2.5	5.5
54	-5.5	-2.5	-4.5	-3.5	3.5	-4.5	2.5	-5.5	5.5	2.5	4.5	3.5	5.5	3.5	2.5	-4.5
55	-4.5	-3.5	5.5	2.5	-3.5	5.5	-4.5	-2.5	-2.5	5.5	3.5	-4.5	2.5	-4.5	-5.5	-3.5
56	-4.5	-2.5	3.5	-5.5	-4.5	-3.5	5.5	2.5	-5.5	-2.5	-4.5	-3.5	-4.5	-3.5	5.5	2.5
57	4.5	2.5	-3.5	5.5	-5.5	-2.5	-4.5	-3.5	4.5	3.5	-5.5	-2.5	-2.5	5.5	3.5	-4.5
58	4.5	3.5	-5.5	-2.5	-2.5	4.5	5.5	3.5	3.5	-4.5	2.5	-5.5	4.5	2.5	-3.5	5.5
59	5.5	2.5	4.5	3.5	2.5	-5.5	-3.5	4.5	-4.5	-3.5	5.5	2.5	3.5	-5.5	4.5	2.5
60	5.5	3.5	2.5	-4.5	5.5	3.5	2.5	-4.5	-3.5	4.5	-2.5	5.5	-5.5	-2.5	-4.5	-3.5
61	2.5	-5.5	-3.5	4.5	-3.5	4.5	-2.5	5.5	-3.5	5.5	-4.5	-2.5	5.5	2.5	4.5	3.5
62	2.5	-4.5	-5.5	-3.5	-4.5	-2.5	3.5	-5.5	-4.5	-2.5	3.5	-5.5	-3.5	5.5	-4.5	-2.5
63	3.5	-5.5	4.5	2.5	4.5	3.5	-5.5	-2.5	3.5	-5.5	4.5	2.5	-4.5	-2.5	3.5	-5.5
64	3.5	-4.5	2.5	-5.5	3.5	-5.5	4.5	2.5	4.5	2.5	-3.5	5.5	2.5	-5.5	-3.5	4.5

Table S.4: The $OD(64, 8^{18})$ $Y = (Y_1, \dots, X_9)$ in Example 3.

Run	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9
1	-3.5	0.5	-3.5	0.5	3.5	-0.5	3.5	-0.5	3.5
2	-3.5	0.5	-0.5	-3.5	0.5	3.5	-3.5	0.5	3.5
3	-3.5	0.5	0.5	3.5	-0.5	-3.5	0.5	3.5	3.5
4	-3.5	0.5	3.5	-0.5	-3.5	0.5	-0.5	-3.5	0.5
5	-0.5	-3.5	-3.5	0.5	0.5	3.5	-0.5	-3.5	3.5
6	-0.5	-3.5	-0.5	-3.5	3.5	-0.5	-3.5	-0.5	-3.5
7	-0.5	-3.5	0.5	3.5	-3.5	0.5	3.5	-0.5	3.5
8	-0.5	-3.5	3.5	-0.5	-0.5	-3.5	0.5	3.5	-0.5
9	0.5	3.5	-3.5	0.5	-0.5	-3.5	3.5	-0.5	3.5
10	0.5	3.5	-0.5	-3.5	-3.5	0.5	-0.5	-3.5	3.5
11	0.5	3.5	0.5	3.5	3.5	-0.5	-0.5	-3.5	-0.5
12	0.5	3.5	3.5	-0.5	0.5	3.5	0.5	3.5	0.5
13	3.5	-0.5	-3.5	0.5	-3.5	0.5	3.5	-0.5	-3.5
14	3.5	-0.5	-0.5	-3.5	-0.5	-3.5	0.5	3.5	-0.5
15	3.5	-0.5	0.5	3.5	0.5	-0.5	-3.5	0.5	3.5
16	3.5	-0.5	3.5	-0.5	-3.5	0.5	-3.5	0.5	3.5
17	-3.5	0.5	-3.5	0.5	3.5	-0.5	-3.5	0.5	3.5
18	-3.5	0.5	-0.5	-3.5	0.5	3.5	-3.5	0.5	3.5
19	-3.5	0.5	0.5	3.5	-0.5	-3.5	0.5	3.5	-0.5
20	-3.5	0.5	3.5	-0.5	-3.5	0.5	-0.5	-3.5	3.5
21	-0.5	-3.5	-3.5	0.5	0.5	3.5	-0.5	-3.5	0.5
22	-0.5	-3.5	-0.5	-3.5	3.5	-0.5	0.5	3.5	-0.5
23	-0.5	-3.5	0.5	3.5	-3.5	0.5	3.5	-0.5	-3.5
24	-0.5	-3.5	3.5	-0.5	-0.5	-3.5	-0.5	-3.5	3.5
25	0.5	3.5	-3.5	0.5	-0.5	-3.5	-3.5	0.5	3.5
26	0.5	3.5	-0.5	-3.5	-3.5	0.5	0.5	3.5	-0.5
27	0.5	3.5	0.5	3.5	3.5	-0.5	0.5	3.5	0.5
28	0.5	3.5	3.5	-0.5	0.5	3.5	-0.5	-3.5	-0.5
29	3.5	-0.5	-3.5	0.5	-3.5	0.5	0.5	3.5	-0.5
30	3.5	-0.5	-0.5	-3.5	-0.5	-3.5	3.5	-0.5	0.5
31	3.5	-0.5	0.5	3.5	-0.5	-3.5	0.5	-3.5	3.5
32	3.5	-0.5	3.5	-0.5	0.5	3.5	-3.5	0.5	-3.5
33	-1.5	2.5	-1.5	2.5	1.5	-2.5	1.5	-2.5	1.5
34	-1.5	2.5	-2.5	-1.5	2.5	1.5	-2.5	-1.5	1.5
35	-1.5	2.5	2.5	1.5	-2.5	-1.5	2.5	1.5	-2.5
36	-1.5	2.5	1.5	-2.5	-1.5	2.5	1.5	-2.5	-1.5
37	-2.5	-1.5	-1.5	2.5	1.5	-2.5	-1.5	2.5	-2.5
38	-2.5	-1.5	-2.5	-1.5	1.5	-2.5	-2.5	-1.5	1.5
39	-2.5	-1.5	2.5	1.5	-1.5	2.5	-1.5	2.5	1.5
40	-2.5	-1.5	1.5	-2.5	-2.5	-1.5	2.5	1.5	-2.5
41	2.5	1.5	-1.5	2.5	-2.5	-1.5	1.5	-2.5	1.5
42	2.5	1.5	-2.5	-1.5	-1.5	2.5	-2.5	-1.5	1.5
43	2.5	1.5	2.5	1.5	1.5	-2.5	-1.5	-2.5	-1.5
44	2.5	1.5	1.5	-2.5	2.5	1.5	1.5	2.5	1.5
45	1.5	-2.5	-1.5	2.5	-1.5	2.5	-2.5	-1.5	2.5
46	1.5	-2.5	-2.5	-1.5	-2.5	-1.5	1.5	-2.5	-1.5
47	1.5	-2.5	2.5	1.5	2.5	-1.5	2.5	1.5	-2.5
48	1.5	-2.5	1.5	-2.5	1.5	-2.5	-1.5	2.5	1.5
49	-1.5	2.5	-1.5	2.5	1.5	-2.5	-2.5	-1.5	1.5
50	-1.5	2.5	-2.5	-1.5	2.5	1.5	2.5	1.5	-2.5
51	-1.5	2.5	2.5	1.5	-2.5	-1.5	1.5	-2.5	-2.5
52	-1.5	2.5	1.5	-2.5	-1.5	2.5	-2.5	-1.5	1.5
53	-2.5	-1.5	-1.5	2.5	1.5	-1.5	2.5	-1.5	2.5
54	-2.5	-1.5	-2.5	-1.5	1.5	-2.5	2.5	1.5	-2.5
55	-2.5	-1.5	2.5	1.5	-1.5	-2.5	1.5	-2.5	-1.5
56	-2.5	-1.5	1.5	-2.5	-2.5	-1.5	-2.5	-1.5	2.5
57	2.5	1.5	-1.5	2.5	-2.5	-1.5	-1.5	2.5	-1.5
58	2.5	1.5	-2.5	-1.5	-1.5	2.5	2.5	1.5	-2.5
59	2.5	1.5	2.5	1.5	1.5	-2.5	1.5	-2.5	2.5
60	2.5	1.5	1.5	-2.5	2.5	1.5	-2.5	-1.5	-2.5
61	1.5	-2.5	-1.5	2.5	-1.5	2.5	-2.5	-1.5	2.5
62	1.5	-2.5	-2.5	-1.5	-2.5	-1.5	1.5	-2.5	2.5
63	1.5	-2.5	2.5	1.5	2.5	1.5	-2.5	-1.5	1.5
64	1.5	-2.5	1.5	-2.5	1.5	-2.5	1.5	2.5	-1.5

Table S.5: The OD(64, 16¹²8⁶) $W = (W_1, \dots, W_9)$ in Example 4.

Run	W_1		W_2		W_3		W_4		W_5		W_6		W_7		W_8		W_9	
1	-7.5	0.5	-6.5	1.5	7.5	-0.5	6.5	-1.5	7.5	-0.5	6.5	-1.5	-0.5	-3.5	0.5	3.5	-3.5	0.5
2	-7.5	1.5	-0.5	-6.5	0.5	6.5	-7.5	1.5	0.5	7.5	-1.5	-6.5	3.5	-0.5	-0.5	-3.5	3.5	-0.5
3	-6.5	0.5	1.5	7.5	-0.5	-7.5	1.5	6.5	-7.5	0.5	-6.5	1.5	0.5	3.5	3.5	-0.5	0.5	3.5
4	-6.5	1.5	7.5	-0.5	-7.5	1.5	-0.5	-6.5	-0.5	-7.5	1.5	6.5	-3.5	0.5	-3.5	0.5	-0.5	-3.5
5	-1.5	-7.5	-6.5	0.5	1.5	6.5	0.5	7.5	-0.5	-6.5	7.5	-1.5	3.5	-0.5	3.5	-0.5	-0.5	-3.5
6	-1.5	-6.5	-0.5	-7.5	6.5	-0.5	-1.5	-7.5	-7.5	1.5	-0.5	-6.5	-0.5	-3.5	-3.5	0.5	0.5	3.5
7	-0.5	-7.5	1.5	6.5	-6.5	1.5	7.5	-0.5	0.5	6.5	-7.5	1.5	-3.5	0.5	0.5	3.5	3.5	-0.5
8	-0.5	-6.5	7.5	-1.5	-1.5	-7.5	-6.5	0.5	7.5	-1.5	0.5	6.5	0.5	3.5	-0.5	-3.5	-3.5	0.5
9	0.5	6.5	-7.5	1.5	-0.5	-6.5	7.5	-1.5	-6.5	0.5	1.5	7.5	3.5	-0.5	-3.5	0.5	3.5	-0.5
10	0.5	7.5	-1.5	-6.5	-7.5	0.5	-6.5	1.5	-1.5	-7.5	-6.5	0.5	-0.5	-3.5	3.5	-0.5	-3.5	0.5
11	1.5	6.5	0.5	7.5	7.5	-1.5	0.5	6.5	6.5	-0.5	-1.5	-7.5	-3.5	0.5	-0.5	-3.5	-0.5	-3.5
12	1.5	7.5	6.5	-0.5	0.5	7.5	-1.5	-6.5	1.5	7.5	6.5	-0.5	0.5	3.5	0.5	3.5	0.5	3.5
13	6.5	-1.5	-7.5	0.5	-6.5	0.5	1.5	7.5	1.5	6.5	0.5	7.5	-0.5	-3.5	-0.5	-3.5	0.5	3.5
14	6.5	-0.5	-1.5	-7.5	-1.5	-6.5	-0.5	-7.5	6.5	-1.5	-7.5	0.5	3.5	-0.5	0.5	3.5	-0.5	-3.5
15	7.5	-1.5	0.5	6.5	1.5	7.5	6.5	-0.5	-1.5	-6.5	-0.5	-7.5	0.5	3.5	-3.5	0.5	-3.5	0.5
16	7.5	-0.5	6.5	-1.5	6.5	-1.5	-7.5	0.5	-6.5	1.5	7.5	-0.5	-3.5	0.5	3.5	-0.5	3.5	-0.5
17	-7.5	0.5	-6.5	1.5	6.5	-0.5	-1.5	-7.5	-1.5	-7.5	-6.5	0.5	0.5	3.5	-0.5	-3.5	3.5	-0.5
18	-7.5	1.5	-0.5	-6.5	1.5	6.5	0.5	7.5	-6.5	0.5	1.5	7.5	-3.5	0.5	0.5	3.5	-3.5	0.5
19	-6.5	0.5	1.5	7.5	-1.5	-7.5	-6.5	0.5	1.5	7.5	6.5	-0.5	-0.5	-3.5	-3.5	0.5	-0.5	-3.5
20	-6.5	1.5	7.5	-0.5	-6.5	1.5	7.5	-0.5	6.5	-0.5	-1.5	-7.5	3.5	-0.5	3.5	-0.5	0.5	3.5
21	-1.5	-7.5	-6.5	0.5	0.5	6.5	-7.5	1.5	6.5	-1.5	-7.5	0.5	-3.5	0.5	-3.5	0.5	0.5	3.5
22	-1.5	-6.5	-0.5	-7.5	7.5	-0.5	6.5	-1.5	1.5	6.5	0.5	7.5	0.5	3.5	3.5	-0.5	-0.5	-3.5
23	-0.5	-7.5	1.5	6.5	-7.5	1.5	-0.5	-6.5	-6.5	1.5	7.5	-0.5	3.5	-0.5	-0.5	-3.5	-3.5	0.5
24	-0.5	-6.5	7.5	-1.5	-0.5	-7.5	1.5	6.5	-1.5	-6.5	-0.5	-7.5	-0.5	-3.5	0.5	3.5	3.5	-0.5
25	0.5	6.5	-7.5	1.5	-1.5	-6.5	-0.5	-7.5	0.5	7.5	-1.5	-6.5	-3.5	0.5	3.5	-0.5	-3.5	0.5
26	0.5	7.5	-1.5	-6.5	-6.5	0.5	1.5	7.5	7.5	-0.5	6.5	-1.5	0.5	3.5	-3.5	0.5	3.5	-0.5
27	1.5	6.5	0.5	7.5	6.5	-1.5	-7.5	0.5	-0.5	-7.5	1.5	6.5	3.5	-0.5	0.5	3.5	0.5	3.5
28	1.5	7.5	6.5	-0.5	1.5	7.5	6.5	-0.5	-7.5	0.5	-6.5	1.5	-0.5	-3.5	-0.5	-3.5	-0.5	-3.5
29	6.5	-1.5	-7.5	0.5	-7.5	0.5	-6.5	1.5	-7.5	1.5	-0.5	-6.5	0.5	3.5	0.5	3.5	-0.5	-3.5
30	6.5	-0.5	-1.5	-7.5	-0.5	-6.5	7.5	-1.5	-0.5	-6.5	7.5	-1.5	-3.5	0.5	-0.5	-3.5	0.5	3.5
31	7.5	-1.5	0.5	6.5	0.5	7.5	-1.5	-6.5	7.5	-1.5	0.5	6.5	-0.5	-3.5	3.5	-0.5	3.5	-0.5
32	7.5	-0.5	6.5	-1.5	7.5	-1.5	0.5	6.5	0.5	6.5	-7.5	1.5	3.5	-0.5	-3.5	0.5	-3.5	0.5
33	-3.5	4.5	-2.5	5.5	3.5	-4.5	2.5	-5.5	3.5	-4.5	2.5	-5.5	-2.5	-1.5	2.5	1.5	-1.5	2.5
34	-3.5	5.5	-4.5	-2.5	4.5	2.5	-3.5	5.5	4.5	3.5	-5.5	-2.5	1.5	-2.5	-2.5	-1.5	1.5	-2.5
35	-2.5	4.5	5.5	3.5	-4.5	-3.5	5.5	2.5	-3.5	4.5	-2.5	5.5	2.5	1.5	1.5	-2.5	2.5	1.5
36	-2.5	5.5	3.5	-4.5	-3.5	5.5	-4.5	-2.5	-4.5	-3.5	5.5	2.5	-1.5	2.5	-1.5	2.5	-2.5	-1.5
37	-5.5	-3.5	-2.5	4.5	5.5	2.5	4.5	3.5	-4.5	-2.5	3.5	-5.5	1.5	-2.5	1.5	-2.5	-2.5	-1.5
38	-5.5	-2.5	-4.5	-3.5	2.5	-4.5	-5.5	-3.5	-3.5	5.5	-4.5	-2.5	-2.5	-1.5	-1.5	2.5	2.5	1.5
39	-4.5	-3.5	5.5	2.5	-2.5	5.5	3.5	-4.5	4.5	2.5	-3.5	5.5	-1.5	2.5	2.5	1.5	1.5	-2.5
40	-4.5	-2.5	3.5	-5.5	-5.5	-3.5	-2.5	4.5	3.5	-5.5	4.5	2.5	2.5	1.5	-2.5	-1.5	-1.5	2.5
41	4.5	2.5	-3.5	5.5	-4.5	-2.5	3.5	-5.5	-2.5	4.5	5.5	3.5	1.5	-2.5	-1.5	2.5	1.5	-2.5
42	4.5	3.5	-5.5	-2.5	-3.5	4.5	-2.5	5.5	-5.5	-3.5	-2.5	4.5	-2.5	-1.5	1.5	-2.5	-1.5	2.5
43	5.5	2.5	4.5	3.5	3.5	-5.5	4.5	2.5	2.5	-4.5	-5.5	-3.5	-1.5	2.5	-2.5	-1.5	-2.5	-1.5
44	5.5	3.5	2.5	-4.5	4.5	3.5	-5.5	-2.5	5.5	3.5	2.5	-4.5	2.5	1.5	2.5	1.5	2.5	1.5
45	2.5	-5.5	-3.5	4.5	-2.5	4.5	5.5	3.5	5.5	2.5	4.5	3.5	-2.5	-1.5	-2.5	-1.5	2.5	1.5
46	2.5	-4.5	-5.5	-3.5	-5.5	-2.5	-4.5	-3.5	2.5	-5.5	-3.5	4.5	1.5	-2.5	2.5	1.5	-2.5	-1.5
47	3.5	-5.5	4.5	2.5	5.5	3.5	2.5	-4.5	-5.5	-2.5	-4.5	-3.5	2.5	1.5	-1.5	2.5	-1.5	2.5
48	3.5	-4.5	2.5	-5.5	2.5	-5.5	-3.5	4.5	-2.5	5.5	3.5	-4.5	-1.5	2.5	1.5	-2.5	1.5	-2.5
49	-3.5	4.5	-2.5	5.5	2.5	-4.5	-5.5	-3.5	-5.5	-3.5	-2.5	4.5	2.5	1.5	-2.5	-1.5	1.5	-2.5
50	-3.5	5.5	-4.5	-2.5	5.5	2.5	4.5	3.5	-2.5	4.5	5.5	3.5	-1.5	2.5	2.5	1.5	-1.5	2.5
51	-2.5	4.5	5.5	3.5	-5.5	-3.5	-2.5	4.5	5.5	3.5	2.5	-4.5	-2.5	-1.5	-1.5	2.5	-2.5	-1.5
52	-2.5	5.5	3.5	-4.5	-2.5	5.5	3.5	-4.5	2.5	-4.5	-5.5	-3.5	1.5	-2.5	1.5	-2.5	2.5	1.5
53	-5.5	-3.5	-2.5	4.5	4.5	2.5	-3.5	5.5	2.5	-5.5	-3.5	4.5	-1.5	2.5	-1.5	2.5	2.5	1.5
54	-5.5	-2.5	-4.5	-3.5	3.5	-4.5	2.5	-5.5	5.5	2.5	4.5	3.5	2.5	1.5	1.5	-2.5	-2.5	-1.5
55	-4.5	-3.5	5.5	2.5	-3.5	5.5	-4.5	-2.5	-2.5	5.5	3.5	-4.5	1.5	-2.5	-2.5	-1.5	-1.5	2.5
56	-4.5	-2.5	3.5	-5.5	-4.5	-3.5	5.5	2.5	-5.5	-2.5	-4.5	-3.5	-2.5	-1.5	2.5	1.5	1.5	-2.5
57	4.5	2.5	-3.5	5.5	-5.5	-2.5	-4.5	-3.5	4.5	3.5	-5.5	-2.5	-1.5	2.5	1.5	-2.5	-1.5	2.5
58	4.5	3.5	-5.5	-2.5	-2.5	4.5	5.5	3.5	3.5	-4.5	2.5	-5.5	2.5	1.5	-1.5	2.5	1.5	-2.5
59	5.5	2.5	4.5	3.5	2.5	-5.5	-3.5	4.5	-4.5	-3.5	5.5	2.5	1.5	-2.5	2.5	1.5	2.5	1.5
60	5.5	3.5	2.5	-4.5	5.5	3.5	2.5	-4.5	-3.5	4.5	-2.5	5.5	-2.5	-1.5	-2.5	-1.5	-2.5	-1.5
61	2.5	-5.5	-3.5	4.5	-3.5	4.5	-2.5	5.5	-3.5	5.5	-4.5	-2.5	2.5	1.5	2.5	1.5	-2.5	-1.5
62	2.5	-4.5	-5.5	-3.5	-4.5	-2.5	3.5	-5.5	-4.5	-2.5	3.5	-5.5	-1.5	2.5	-2.5	-1.5	2.5	1.5
63	3.5	-5.5	4.5	2.5	4.5	3.5	-5.5	-2.5	3.5	-5.5	4.5	2.5	-2.5	-1.5	1.5	-2.5	1.5	-2.5
64	3.5	-4.5	2.5	-5.5	3.5	-5.5	4.5	2.5	4.5	2.5	-3.5	5.5	1.5	-2.5	-1.5	2.5	-1.5	2.5