Nonparametric Comparisons of Multiple Distributions under Uniform Stochastic Ordering

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Supplementary Material

This supplementary article consists of two appendices. In Appendix S1, we provide the theoretical results, including proofs of the lemmas and theorems in the equality test in Section 2 (with a list of critical values), GOF test in Section 3, and distinguishing distributions methods in Section 4 in the Manuscript. All supplementary explanations in the Manuscript are provided in the Remarks. In Appendix S2, we provide a list of critical values $t_{kp,\alpha}$ and $u_{kp,\alpha}$ for the equality test in Section 2. We further include more simulation results, including comparisons with empirical-likelihood-based tests and p-value adjusted methods (Bonferroni, Benjamini and Yekutieli's methods, and Cauchy combination test) of the proposed methods with selected numbers of samples $k = 3, 4$ and 5, and 10 with equal sample sizes $n = 60, 100, 200$ and $p = 1, 2, \infty$.

S1 Proof of Lemmas and Theorems

We provide theoretical justifications for the proposed equality tests, GOF tests, and distinguishing distribution methods in the following Sections S1.1 to S1.3. We denote $n = \min_{1\le i\le k} n_i$ and convergences in probability, in distribution, and in law by $\xrightarrow{p}$, $\xrightarrow{d}$, and $\xrightarrow{w}$, respectively. Throughout this work, we assume that the inverse function F_i^{-1} exists and equals the quantile function.

For $1 \le i < k$, recall that $\mathbb{F}_i$ is the empirical distribution of the ith sample and $\mathbb{F}_{i+1}^{-1}$ as the empirical quantile of the $(i+1)$th sample. We first demonstrate that the empirical version of $\hat{R}_i$ only depends on the corresponding R_i.

Remark S1.1. *Assume that all the distributions F_j are continuous and invertible, then the sampling distribution of $\hat{R}_i$ only depends on the sample sizes and R_i, but not directly from the distributions F_i or F_{i+1}.*

For $1 \le j \le k$, since $\{F_j(X_{j1}), \ldots, F_j(X_{jn_j})\}$ are independent random samples from the uniform distribution with support $(0, 1)$, we can rewrite $X_{ij} = F_i^{-1}(U_{ij})$ by assuming that the inverse of F_i exists such that F_i^{-1} is identical to the quantile function of F_i over $(0, 1)$. So, we define $\mathbb{U}_{jn_j}$ as the uniform empirical distributions and $\mathbb{U}_{jn_j}^{-1}$ as the corresponding empirical quantile functions from a random sample

$\{F_j(X_{j1}), \ldots, F_j(X_{jn_j})\}$. Therefore,

$$\begin{aligned}
\mathbb{F}_i(t) &= n_i^{-1} \sum_{k=1}^{n_i} I(X_{ik} \le t) = n_i^{-1} \sum_{k=1}^{n_i} I(F_i^{-1}(U_{ik}) \le t) \\
&= n_i^{-1} \sum_{j=1}^{n_i} I\{U_{ik} \le F_i(t)\} = \mathbb{U}_{j,n_j}\{F_i(t)\}, \\
\mathbb{F}_{i+1}^{-1}(u) &= \inf\{t : \mathbb{F}_{i+1}(t) \ge u\} = \inf\left\{t : n_i^{-1} \sum_{k=1}^{n_{i+1}} I\{U_{(i+1)k} \le F_{i+1}(t)\} \ge u\right\} \\
&= \inf\left\{F_{i+1}^{-1}(w) : n_{i+1}^{-1} \sum_{j=1}^{n_{i+1}} I\{U_{(i+1)k} \le w\} \ge u\right\} \\
&= F_{i+1}^{-1}\left[\inf\left\{w : n_{i+1}^{-1} \sum_{j=1}^{n_{i+1}} I\{U_{(i+1)k} \le w\} \ge u\right\}\right] = F_{i+1}^{-1}\{\mathbb{U}_{i+1,n_{i+1}}^{-1}(u)\},
\end{aligned}$$

where the last equality holds because F_{i+1}^{-1} is assumed to be continuous. Hence, for $1 \le i < k$,

$$\hat{R}_i(u) = \mathbb{F}_i\{\mathbb{F}_{i+1}^{-1}(u)\} = \mathbb{U}_{j,n_j}\{F_i[F_{i+1}^{-1}\{\mathbb{U}_{i+1,n_{i+1}}^{-1}(u)\}]\} = \mathbb{U}_{j,n_j}[R_i\{\mathbb{U}_{i+1,n_{i+1}}^{-1}(u)\}],$$

where $0 \le u \le 1$. In other words, if there exist distributions G_i and G_{i+1} satisfy that $R_i = G_i\{G_{i+1}^{-1}\}$, even $G_i \ne F_i$ and $G_{i+1} \ne F_{i+1}$, the approach above still follows such that the distribution of $\hat{R}_i(u)$ are identical.

The following lemma provides the asymptotic joint behavior of the empirical estimators $\hat{R}_i(u) = \mathbb{F}_i\{\mathbb{F}_{i+1}^{-1}(u)\}$ for $0 \le u \le 1$.

Lemma S1.1. *Assume that, for all $1 \le i < k$, R_i have continuous first derivatives R_i' over $[0,1]$. There exist independent standard Brownian bridges $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ such that*

$$\sup_{0 \le u \le 1} \left| C_i\{\hat{R}_i(u) - R_i(u)\} - \left[\lambda_i^{1/2}\mathcal{B}_i\{R_i(u)\} - (1-\lambda_i)^{1/2} R_i'(u)\mathcal{B}_{i+1}(u)\right]\right|$$

converges to 0 almost surely as $n \to \infty$, where $C_i = \sqrt{n_i n_{i+1}/(n_i + n_{i+1})}$ and the sample fractions $n_{i+1}/(n_i + n_{i+1})$ converge to $0 < \lambda_i < 1$ as $n \to \infty$.

Proof of Lemma S1.1. Here, we follow the same notations produced in the proof of Remark S1.1. To study the asymptotic behavior of $\hat{R}_i$, we subtract and adding $R_i\{\mathbb{U}_{i+1,n_{i+1}}^{-1}(u)\}$ in $C_i\{\hat{R}_i(u) - R_i(u)\}$ and obtain

$$\begin{aligned}
&C_i\{\hat{R}_i(u) - R_i(u)\} \\
&= C_i\left(\mathbb{U}_{j,n_j}[R_i\{\mathbb{U}_{i+1,n_{i+1}}^{-1}(u)\}] - R_i\{\mathbb{U}_{i+1,n_{i+1}}^{-1}(u)\}\right) \qquad \text{(S1.1)} \\
&\quad + C_i[R_i\{\mathbb{U}_{i+1,n_{i+1}}^{-1}(u)\} - R_i(u)]. \qquad \text{(S1.2)}
\end{aligned}$$

According to Theorem 3.1.1 and Theorem 3.1.3 in Shorack and Wellner (1986), as $n \to \infty$, we obtain

$$\sup_{0 \le u \le 1} |\mathbb{U}_{j,n_j}(u) - u| = \sup_{0 \le u \le 1} |\mathbb{U}_{j,n_j}^{-1}(u) - u| \to 0, \quad \text{a.s.}, \tag{S1.3}$$

$$\sup_{0 \le u \le 1} |\sqrt{n_i}\{\mathbb{U}_{j,n_j}(u) - u\} - \mathcal{B}_i(u)| \to 0, \quad \text{a.s.}, \tag{S1.4}$$

$$\sup_{0 \le u \le 1} |\sqrt{n_i}\{\mathbb{U}_{j,n_j}^{-1}(u) - u\} + \mathcal{B}_i(u)| \to 0, \quad \text{a.s.}, \tag{S1.5}$$

where $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ are the independent standard Brownian bridges since the k samples are mutually independent. Therefore, combining (S1.3) and (S1.4), (S1.1) converges to $\lambda_i^{1/2}\mathcal{B}_i\{R_i(u)\}$ in sup-norm over $[0,1]$ a.s.. On the other hand, applying the mean value theorem and (S1.5), (S1.2) converges to $-(1-\lambda_i)^{1/2}R_i'(u) \cdot \mathcal{B}_{i+1}(u)$ in sup-norm over $[0,1]$ a.s.. Hence,

$$\sup_{0 \le u \le 1} \left| C_i\{\hat{R}_i(u) - R_i(u)\} - \left[\lambda_i^{1/2}\mathcal{B}_i\{R_i(u)\} - (1-\lambda_i)^{1/2}R_i'(u)\mathcal{B}_{i+1}(u)\right]\right|$$

converges to 0 almost surely as $n \to \infty$.

□

Lemma S1.1 gives the foundation of all the asymptotic results in this work, the asymptotic joint behavior of $\hat{R}_1, \hat{R}_2 \ldots, \hat{R}_{k-1}$ through $\mathcal{B}_1, \ldots, \mathcal{B}_k$. Hereafter, we denote $\mathcal{T}_i(u) = \lambda_i^{1/2}\mathcal{B}_i\{R_i(u)\} - (1-\lambda_i)^{1/2}R_i'(u)\mathcal{B}_{i+1}(u)$ for $0 \le u \le 1$. Comparing with Theorem 2.2 in Hsieh and Turnbull (1996), the negative sign before $(1-\lambda_i)^{1/2}R_i'(u)\mathcal{B}_{i+1}(u)$ in $\mathcal{T}_i$ is required when $k > 2$ because the quantile function approximation in (S1.5) and the empirical distribution approximation in (S1.4) share the same Brownian bridge $\mathcal{B}_i$ but with opposite sign.

S1.1 Proofs and Lemmas in Section 2

Define $\mathcal{T}_{i0}$ and $\mathcal{J}_{i0}$ by

$$\mathcal{T}_{i0}(u) = \lambda_i^{1/2}\mathcal{B}_i(u) - (1-\lambda_i)^{1/2}\mathcal{B}_{i+1}(u), 0 \le u \le 1,$$
$$\mathcal{J}_{i0}(u) = \{\sup_{0 \le v \le u} \mathcal{T}_{i0}(v)/(1-v)\}(1-u), 0 \le u < 1, \quad \text{and} \quad \mathcal{J}_{i0}(1) = 0.$$

When $F_i = F_{i+1}$ such that $R_i = R_0$, Lemma S1.1 gives

$$\sup_{0 \le u \le 1} \left| C_i\{\hat{R}_i(u) - R_i(u)\} - \mathcal{T}_{i0}(u)\right| \to 0 \quad \text{a.s..}$$

as $n \to \infty$. The following Lemma S1.2 provides the limiting distributions of $T_{kp} = \sum_{1 \le i < k} \Delta_{ip}$ and $U_{kp} = \max_{1 \le i < k} \Delta_{ip}$ under H_0, where $\Delta_{ip} = C_i\|\mathcal{M}\hat{R}_i - R_0\|_p$.

Lemma S1.2. *Under H_0, for every $p \in [1, \infty]$, T_{kp} and U_{kp} converge in distribution to $\sum_{1 \le i < k} \|\mathcal{J}_{i0}\|_p$ and $\max_{1 \le i < k} \|\mathcal{J}_{i0}\|_p$, respectively, as $n \to \infty$.*

Proof of Lemma S1.2. Under $H_0 : F_1 = F_2 = \cdots = F_k$, we have $R_i = R_0$ and $\mathcal{M}R_i = R_0$ for all $1 \le i < k$ where $R_0(u) = u$ over $u \in [0,1]$ is the equal distribution line. According to Lemma 5 in Tang et al. (2017), the functional operator $\mathcal{M}$ is Hadamard directional differentiable, so the functional delta method and the continuous mapping theorem can be applied. Therefore, apply Lemma 4 in Tang et al. (2017), as $n \to \infty$, the difference $\Delta_{ip} = C_i\|\mathcal{M}\hat{R}_i - R_0\|_p = C_i\|\mathcal{M}\hat{R}_i - \mathcal{M}R_i\|_p$ converges to $\|\mathcal{J}_{i0}\|_p$ in distribution for $1 \le p \le \infty$. Applying continuous mapping theorem again, T_{kp} and U_{kp} converge to $\sum_{1\le i<k} \|\mathcal{J}_{i0}\|_p$ and $\max_{1\le i<k} \|\mathcal{J}_{i0}\|_p$ in distribution, respectively, as $n \to \infty$. □

Proof of Theorem 1. Under H_0, from Lemma S1.1, it is clear that both $\text{pr}(U_{kp} > u_{kp,\alpha}) = \alpha$ and $\text{pr}(T_{kp} > t_{kp,\alpha}) = \alpha$ by definition. Under H_1, to show the consistency for proposed equal tests, we first show that the quantile values $t_{kp,\alpha}$ and $u_{kp,\alpha}$ are bounded asymptotically. It suffices to show that both $\sum_{1\le i<k} \|\mathcal{J}_{i0}\|_p$ and $\max_{1\le i<k} \|\mathcal{J}_{i0}\|_p$ are bounded in probability. By definition, it is clear that $\mathcal{J}_{i0}(u) \ge 0$ for all $0 \le u \le 1$, then we have

$$\begin{aligned}
0 \le \mathcal{J}_{i0}(u) = & \sup_{0\le v\le u} \{\mathcal{T}_{i0}(v)/(1-v)\}(1-u) \\
& \le \sup_{0\le v\le u} \{\mathcal{T}_{i0}(v)/(1-u)\}(1-u) = \sup_{0\le v\le u} \mathcal{T}_{i0}(u) \le \|\mathcal{T}_{i0}\|_\infty.
\end{aligned}$$

Hence, $\|\mathcal{J}_{i0}\|_p \le \|\mathcal{J}_{i0}\|_\infty \le \|\mathcal{T}_{i0}\|_\infty \le \lambda_i^{1/2}\|\mathcal{B}_i\|_\infty + (1-\lambda_i)^{1/2}\|\mathcal{B}_{i+1}\|_\infty \le \|\mathcal{B}_i\|_\infty + \|\mathcal{B}_{i+1}\|_\infty$ because $0 < \lambda_i < 1$ for $1 \le i < k$. Note that the Brownian bridge is bounded with probability one. The boundedness holds the same for $\|\mathcal{J}_{i0}\|_p$, $\sum_{1\le i<k} \|\mathcal{J}_{i0}\|_p$ and $\max_{1\le i<k} \|\mathcal{J}_{i0}\|_p$. Therefore, both quantile values $t_{kp,\alpha}$ and $u_{kp,\alpha}$ are bounded.

Under H_1, there exists at least a pair of consecutive distributions, say F_i and F_{i+1}, such that $F_i \prec F_{i+1}$ and $\|R_i - R_0\|_p > 0$. To show the powers $\text{pr}(U_{kp} > u_{kp,\alpha})$ and $\text{pr}(T_{kp} > t_{kp,\alpha})$ approach 1 as $n \to \infty$, it suffices to consider $\text{pr}(\Delta_{ip} > u_{kp,\alpha})$ and $\text{pr}(\Delta_{ip} > t_{kp,\alpha})$ because both T_{kp} and U_{kp} are larger than Δ_{ip} almost surely. In general, we will show that, given $t > 0$, $\text{pr}(\Delta_{ip} > t)$ converges to 1 as $n \to \infty$.

Since $\mathcal{M}R_i = R_i$ under H_1, apply the Minkowski inequality and obtain

$$\begin{aligned}
\Delta_{ip} &= C_i\|\mathcal{M}\hat{R}_i - \mathcal{M}R_i + R_i - R_0\|_p \ge \ C_i\|R_i - R_0\|_p - C_i\|\mathcal{M}\hat{R}_i - \mathcal{M}R_i\|_p \\
&\ge C_i\|R_i - R_0\|_p - C_i\|\mathcal{M}\hat{R}_i - \mathcal{M}R_i\|_\infty \\
&\ge C_i\|R_i - R_0\|_p - C_i\|\hat{R}_i - R_i\|_\infty,
\end{aligned}$$

where the last inequality holds because of the continuity of $\mathcal{M}$ according to Lemma 3 in Tang et al. (2017). Hence,

$$\text{pr}(\Delta_{ip} > t) \ge \text{pr}\left(C_i\|R_i - R_0\|_p > t + C_i\|\hat{R}_i - R_i\|_\infty\right).$$

Since $\|R_i - R_0\|_p > 0$, then $C_i\|R_i - R_0\|_p \to \infty$ as $C_i \to \infty$. Therefore, to show $\text{pr}(\Delta_{ip} > t) \to 1$, it suffices to show that $C_i\|\hat{R}_i - R_i\|_\infty = O_P(1)$, that is, the boundedness of $\|\mathcal{T}_i\|_\infty$ equivalently by Lemma S1.1. Recall that $\mathcal{T}_i(u) = \lambda_i^{1/2}\mathcal{B}_i\{R_i(u)\} + (1-\lambda_i)^{1/2}R_i'(u)\mathcal{B}_i(u)$, we have

$$\|\mathcal{T}_i\|_\infty \leq \lambda_i^{1/2}\|\mathcal{B}_i\|_\infty + (1-\lambda_i)^{1/2}\|R_i'\|_\infty\|\mathcal{B}_{i+1}\|_\infty.$$

Therefore, $\|\mathcal{T}_i\|_\infty$ is bounded in probability since $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ are bounded with probability one and R_i' is bounded as well.

□

Lastly, we provide the critical values $t_{kp,\alpha}$ and $u_{kp,\alpha}$ at significance level $\alpha = 0.05$ mentioned in Section 2 and applied in Section 5.3.

Table S1.1: Critical values $t_{kp,\alpha}$ and $u_{kp,\alpha}$ at significance level $\alpha = 0.05$.

		$t_{kp,\alpha}$			$u_{kp,\alpha}$		
		$p=1$	$p=2$	$p=\infty$	$p=1$	$p=2$	$p=\infty$
$k=3$	$n=60$	0.916	1.040	1.826	0.704	0.784	1.278
	$n=100$	0.915	1.036	1.838	0.716	0.793	1.343
	$n=200$	0.936	1.059	1.850	0.720	0.801	1.350
$k=4$	$n=60$	1.217	1.388	2.465	0.748	0.833	1.369
	$n=100$	1.236	1.407	2.475	0.758	0.843	1.414
	$n=200$	1.268	1.436	2.550	0.768	0.851	1.400
$k=5$	$n=60$	1.530	1.751	3.104	0.774	0.859	1.461
	$n=100$	1.549	1.763	3.111	0.785	0.870	1.414
	$n=200$	1.587	1.801	3.200	0.797	0.883	1.450

S1.2 Proofs and Lemmas for GOF tests in Section 3

Recall that $M_{ip} = C_i\|\mathcal{M}\hat{R}_i - \hat{R}_i\|_p$, according to Tang et al. (2017), the asymptotic distribution of the test statistics $S_{kp} = \sum_{1\leq i<k} M_{ip}$ and $W_{kp} = \max_{1\leq i<k} M_{ip}$ both depend on the shape of the ODCs R_i. Recall $r_i(u) = \{1-R_i(u)\}/(1-u)$ for $0 \leq u < 1$ and $r_i(1) = \lim_{u\to 1^-} r_i(u)$ for $1 \leq i < k$. Under $H_0^* : F_1 \preceq \cdots \preceq F_k$, all ODCs R_i are star-shaped for $1 \leq i < k$. For each star-shaped R_i, define the non-strictly-star-shaped region $\mathcal{S}_{i0} = \{u \in [0,1] : r_i(u) = r_i(u-) \text{ or } r_i(u) = r_i(u+)\}$. The strictly-star-shaped region is defined by $\mathcal{S}_{i1} = [0,1] \cap \mathcal{S}_{i0}^c$, that is, $r_i(u)$ decreases strictly in $u \in \mathcal{S}_{i1}$. If the non-strictly star-shaped region $\mathcal{S}_{i0}$ is nonempty, that is, there exists at least a nonempty closed interval, say $[a,b]$ with $0 \leq a < b \leq 1$, such that $r_i(u) = r_i(v)$ when $u, v \in [a,b]$, then R_i is called non-strictly star-shaped. If $\mathcal{S}_{i0}$ is empty, then R_i is called strictly star-shaped. One can further write $\mathcal{S}_{i0}$ in terms of a countable union of disjoint closed intervals; i.e., $\mathcal{S}_{i0} = \cup_l [a_{il}, b_{il}]$ where r_i takes distinctive values among different interval $[a_{il}, b_{il}]$. For example, if $R_i = R_0$, then $\mathcal{S}_{i0} = [0,1]$ because $r_i(u)$ is constant over $[0,1]$. Hence R_i is non-strictly star-shaped. If $R_i(u) = u^{1/2}$, then $\mathcal{S}_{i0} = \emptyset$ because

$r_i(u) = (1 - u^{1/2})/(1 - u) = (1 + u^{1/2})^{-1}$ is strictly decreasing over $[0, 1]$ and therefore R_i is strictly star-shaped.

Following the same proof of Theorem 1 in Tang et al. (2017), we obtain the limiting distribution of M_{ip} stated in the following Lemma.

Lemma S1.3. *Under H_0^*, for $1 \leq p \leq \infty$,*

(a) if R_i is strictly star-shaped, then $M_{ip} \xrightarrow{p} 0$ as $n \to \infty$;

(b) if R_i is non-strictly star-shaped with nonempty non-strictly-star-shaped region $\mathcal{S}_{i0} = \cup_l [a_{il}, b_{il}]$, then $M_{ip} \xrightarrow{d} \|\mathcal{W}_{i\cdot}\|_p$ as $n \to \infty$, where $\mathcal{W}_{i\cdot} = \sum_l \mathcal{W}_{il}$ and

$$\mathcal{W}_{il}(u) = \left[\left\{ \sup_{a_{il} \leq v \leq u} \mathcal{T}_i(v)/(1 - v) \right\} (1 - u) - \mathcal{T}_i(u) \right] I(a_{il} \leq u \leq b_{il}),$$

where $I(\cdot)$ is the indicator function.

Lemma S1.3 not only suggests the asymptotic marginal distribution of M_{ip} under H_0^*, it gives asymptotic joint behavior of $M_{1p}, \ldots, M_{k-1,p}$ though $\mathcal{T}_1, \ldots, \mathcal{T}_{k-1}$, where $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$ are not necessarily independent because they share the same Brownian bridge $\mathcal{B}_{i+1}$. In addition, since $\mathcal{W}_{il}$ are constantly zero outside of the non-strictly star-shaped regions, the test statistics S_{kp} and W_{kp} only depend on the non-strictly-star-shaped region $\cup_i \cup_l [a_{il}, b_{il}]$.

Next, the asymptotic distribution of S_{kp} and W_{kp} under H_0^* can be obtained by applying the continuous mapping theorem. We state the results as a lemma below.

Lemma S1.4. *Under H_0^*, as $n \to \infty$,*

$$S_{kp} \xrightarrow{d} \sum_{i=1}^{k-1} \|\mathcal{W}_{i\cdot}\|_p \quad \text{and} \quad W_{kp} \xrightarrow{d} \max_{1 \leq i < k} \|\mathcal{W}_{i\cdot}\|_p,$$

where $\mathcal{W}_{i\cdot}$ are defined in Lemma S1.3 when R_i is non-strictly star-shaped and we define $\|\mathcal{W}_{i\cdot}\|_p = 0$ if R_i is strictly star-shaped.

Next, we provide the proof of Theorem 2 that the surrogate random variables $\tilde{S}_{kp}$ and $\tilde{W}_{kp}$ defined in Section 2 are stochastically larger than S_{kp} and W_{kp}, respectively.

Proof of Theorem 2. Under H_0^*, to show $\tilde{S}_{kp}$ and $\tilde{W}_{kp}$ are stochastically larger than S_{kp} and W_{kp}, respectively, it suffices to consider non-strictly star-shaped R_i because Δ_{ip} converges to 0 in probability when R_i is strictly star-shaped. Given non-strictly star-shaped R_i for $1 \leq i < k$ with nonempty non-strictly star-shaped region $\mathcal{S}_{i0}$, one can check that $R_i'(u) = r_i(u)$ when $u \in \mathcal{S}_{i0}$. We replace R_i and R_i' by $\mathcal{M}R_i$ and r_i in $\mathcal{T}_i$, respectively, and define

$$\mathcal{L}_i(u) = \lambda_i^{1/2} \mathcal{B}_i\{\mathcal{M}R_i(u)\} - (1 - \lambda_i)^{1/2} r_i(u) \mathcal{B}_{i+1}(u),$$

for $0 \le u < 1$ and $\mathcal{L}_i(1) = 0$. Hence, $\mathcal{L}_i(u)$ agrees with $\mathcal{T}_i(u)$ over $\mathcal{S}_{i0}$, that is,

$$\mathcal{L}_i(u) I(u \in \mathcal{S}_{i0}) = \mathcal{T}_i(u) I(u \in \mathcal{S}_{i0}).$$

Next, recall

$$\mathcal{V}_i(u) = \left\{ \sup_{0 \le v \le u} \mathcal{L}_i(v)/(1-v) \right\} (1-u) - \mathcal{L}_i(u)$$

for $0 \le u < 1$ and $\mathcal{V}_i(1) = 0$. By definition, we have

$$\begin{aligned} & \mathcal{V}_i(u) I(a_{il} \le u \le b_{il}) \\ \ge & \left[\left\{ \sup_{a_{il} \le v \le u} \mathcal{L}_i(v)/(1-v) \right\} (1-u) - \mathcal{L}_i(u) \right] I(a_{il} \le u \le b_{il}) \\ = & \left[\left\{ \sup_{a_{il} \le v \le u} \mathcal{T}_i(v)/(1-v) \right\} (1-u) - \mathcal{T}_i(u) \right] I(a_{il} \le u \le b_{il}) \\ = & \mathcal{W}_{i\cdot}(u) I(a_{il} \le u \le b_{il}). \end{aligned} \tag{S1.6}$$

Hence, $\mathcal{V}_i(u) \ge \mathcal{W}_{i\cdot}(u)$ when $u \in \mathcal{S}_{i0}$. On the other hand, since $\mathcal{V}_i(u) \ge 0$ and $\mathcal{W}_{il}(u) = 0$ when $u \in \mathcal{S}_{i1}$. Therefore, $\mathcal{V}_i(u) \ge \mathcal{W}_{i\cdot}(u)$ for $0 \le u \le 1$ such that $\|\mathcal{V}_i\|_p \ge \|\mathcal{W}_{i\cdot}\|_p$, $\tilde{S}_{kp} = \sum_{1 \le i < k} \|\mathcal{V}_i\|_p \ge \sum_{1 \le i < k} \|\mathcal{W}_{i\cdot}\|_p$, and $\tilde{W}_{kp} = \max_{1 \le i < k} \|\mathcal{V}_i\|_p \ge \max_{1 \le i < k} \|\mathcal{W}_{i\cdot}\|_p$. Recall that $\tilde{s}_{kp,\alpha}$ and $\tilde{w}_{kp,\alpha}$ are α-th upper quantile of $\tilde{S}_{kp,\alpha}$ and $\tilde{W}_{kp,\alpha}$, hence,

$$\begin{aligned} \lim_{n \to \infty} \mathrm{pr}\left(S_{kp,\alpha} > \tilde{s}_{kp,\alpha}\right) &= \mathrm{pr}\Big(\sum_{1 \le i < k} \|\mathcal{W}_{i\cdot}\|_p > \tilde{s}_{kp,\alpha}\Big) \\ &\le \mathrm{pr}\Big(\sum_{1 \le i < k} \|\mathcal{V}_i\|_p > \tilde{s}_{kp,\alpha}\Big) = \mathrm{pr}(\tilde{S}_{kp,\alpha} > \tilde{s}_{kp,\alpha}) = \alpha, \\ \lim_{n \to \infty} \mathrm{pr}\left(W_{kp,\alpha} > \tilde{w}_{kp,\alpha}\right) &= \mathrm{pr}\Big(\max_{1 \le i < k} \|\mathcal{W}_{i\cdot}\|_p > \tilde{w}_{kp,\alpha}\Big) \\ &\le \mathrm{pr}\Big(\max_{1 \le i < k} \|\mathcal{V}_i\|_p > \tilde{w}_{kp,\alpha}\Big) = \mathrm{pr}(\tilde{W}_{kp,\alpha} > \tilde{w}_{kp,\alpha}) = \alpha. \end{aligned}$$

Under H_1^*, we wish to show that the critical values $\tilde{s}_{kp,\alpha}$ and $\tilde{w}_{kp,\alpha}$ are bounded. To show the boundedness of the critical values $\tilde{s}_{kp,\alpha}$ and $\tilde{w}_{kp,\alpha}$, it suffices to consider the boundedness of $\|\mathcal{V}_i\|_p$. We define a functional operator $\mathcal{M}_{[0,1]}^{(1,0)}$ as the least star-shaped majorant with kernel $(1,0)$, that is, given any bounded function h with support $[0,1]$,

$$\begin{aligned} \mathcal{M}_{[0,1]}^{(1,0)} h(u) &= 0 - \inf_{0 \le v \le u} \left\{ 0 - h(v)/(1-v) \right\} (1-u) \\ &= \sup_{0 \le v \le u} \left\{ h(v)/(1-v) \right\} (1-u) \end{aligned}$$

and $\mathcal{M}_{[0,1]}^{(1,0)} h(1) = 0$ (see Lemma 1 in Tang et al. (2017)). Therefore, we can write

$$\mathcal{V}_i(u) = \mathcal{M}_{[0,1]}^{(1,0)} \mathcal{L}_i(u) - \mathcal{L}_i(u).$$

Now, define $\mathcal{Z}(u) = 0$ for $u \in [0,1]$ and one can check that $\mathcal{M}_{[0,1]}^{(1,0)}\mathcal{Z}(u) = \mathcal{Z}(u)$ over $u \in [0,1]$. Therefore,

$$\begin{aligned}\|\mathcal{V}_i\|_p \leq \|\mathcal{V}_i\|_\infty &= \|\mathcal{M}_{[0,1]}^{(1,0)}\mathcal{L}_i - \mathcal{L}_i\|_\infty = \|\mathcal{M}_{[0,1]}^{(1,0)}\mathcal{L}_i - \mathcal{M}_{[0,1]}^{(1,0)}\mathcal{Z} + \mathcal{Z}_i - \mathcal{L}_i\|_\infty \\ &\leq \|\mathcal{M}_{[0,1]}^{(1,0)}\mathcal{L}_i - \mathcal{M}_{[0,1]}^{(1,0)}\mathcal{Z}\|_\infty + \|\mathcal{Z} - \mathcal{L}_i\|_\infty \leq 2\|\mathcal{L}_i - \mathcal{Z}\|_\infty = 2\|\mathcal{L}_i\|_\infty,\end{aligned}$$

because of the triangle inequality and the Lipschitz continuity of the operator $\mathcal{M}_{[0,1]}^{(1,0)}$ (Lemma 3 in Tang et al. (2017)). From the definition of $\mathcal{L}_i$, we can further bound $\|\mathcal{L}_i\|_\infty$ by

$$\begin{aligned}\|\mathcal{L}_i\|_\infty &\leq \lambda_i^{1/2}\|\mathcal{B}_i\{\mathcal{M}R_i\}\|_\infty + (1-\lambda_i)^{1/2}\|r_i\|_\infty\|\mathcal{B}_{i+1}\|_\infty \\ &= \lambda_i^{1/2}\|\mathcal{B}_i\|_\infty + (1-\lambda_i)^{1/2}\|r_i\|_\infty\|\mathcal{B}_{i+1}\|_\infty.\end{aligned}$$

One can check that $1 \geq \mathcal{M}R_i(u) \geq R_0(u) = u$ for all $u \in [0,1]$, therefore,

$$0 \leq \sup_{u\in[0,1)} \frac{1-1}{1-u} \leq \sup_{u\in[0,1)} \frac{1-\mathcal{M}R_i(u)}{1-u} \leq \sup_{u\in[0,1)} \frac{1-u}{1-u} = 1$$

for $u \in [0,1)$. Therefore, $r_i(1) = r_i(1-)$ is between 0 and 1. Hence, $0 \leq r_i(u) \leq 1$ for $0 \leq u \leq 1$ such that $\|r_i\|_\infty \leq 1$. Since the standard Brownian bridges are bounded with probability one, then $\|\mathcal{L}_i\|_\infty = O_P(1)$ such that $\|\mathcal{V}_i\|_\infty = O_P(1)$, $\tilde{S}_{kp} = \sum_{1\leq i<k}\|\mathcal{V}_i\|_p = O_P(1)$, and $\tilde{W}_{kp} = \max_{1\leq i<k}\|\mathcal{V}_i\|_p = O_P(1)$. Therefore, $\tilde{s}_{kp,\alpha}$ and $\tilde{w}_{kp,\alpha}$ are bounded under H_1^*.

To show the consistency of the proposed GOF tests, we follow a similar idea in the proof of Theorem 1. Under H_1^*, there exists at least a pair of consecutive distributions, say F_i and F_{i+1}, such that F_i and F_{i+1} are not USO with $\|\mathcal{M}R_i - R_i\|_p > 0$. Apply the Minkowski inequality and obtain

$$\begin{aligned}M_{ip} &= C_i\|\mathcal{M}\hat{R}_i - \mathcal{M}R_i + \mathcal{M}R_i - R_i + R_i - \hat{R}_i\|_p \\ &\geq -C_i\|\mathcal{M}\hat{R}_i - \mathcal{M}R_i\|_p + C_i\|\mathcal{M}R_i - R_i\|_p - C_i\|R_i - \hat{R}_i\|_p \\ &\geq -2C_i\|\hat{R}_i - R_i\|_p + C_i\|\mathcal{M}R_i - R_i\|_p, \end{aligned} \tag{S1.7}$$

where the last inequality holds because of the Lipschitz continuity of $\mathcal{M}$ according to Lemma 3 in Tang et al. (2017). Hence, we have the lower bound of the probability of event $M_{ip} > t$ below:

$$\text{pr}(M_{ip} > t) \geq \text{pr}\left(C_i\|\mathcal{M}R_i - R_i\|_p > t + 2C_i\|\hat{R}_i - R_i\|_p\right). \tag{S1.8}$$

Since $\|\mathcal{M}R_i - R_i\|_p > 0$, then $C_i\|\mathcal{M}R_i - R_i\|_p \to \infty$ as $n \to \infty$. To further show that $\text{pr}(M_{ip} > t) \to 1$ as $n \to \infty$, it suffices to show that $C_i\|\hat{R}_i - R_i\|_\infty = O_P(1)$, or equivalently, the boundedness of $\|\mathcal{T}_i\|_\infty$. Recall that $\mathcal{T}_i(u) = \lambda_i^{1/2}\mathcal{B}_i\{R_i(u)\} + (1-\lambda_i)^{1/2}R_i'(u)\mathcal{B}_i(u)$, we have

$$\|\mathcal{T}_i\|_\infty \leq \lambda_i^{1/2} \cdot \|\mathcal{B}_i\|_\infty + (1-\lambda_i)^{1/2} \cdot \|R_i'\|_\infty \cdot \|\mathcal{B}_{i+1}\|_\infty.$$

Therefore, $\|\mathcal{T}_i\|_\infty$ is bounded in probability since $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ are bounded with probability one and R_i' is bounded as well. Subsequently, since $\tilde{s}_{kp,\alpha}$ and $\tilde{w}_{kp,\alpha}$ are fixed and bounded, we have $\mathrm{pr}(S_{kp} > \tilde{s}_{kp,\alpha}) \geq \mathrm{pr}(M_{ip} > \tilde{s}_{kp,\alpha}) \to 1$ and $\mathrm{pr}(W_{kp} > \tilde{w}_{kp,\alpha}) \geq \mathrm{pr}(M_{ip} > \tilde{w}_{kp,\alpha}) \to 1$ as $n \to \infty$.

□

In the following, we use the same relationship between $\mathcal{V}_i$ and $\mathcal{W}_i$ in S1.6 to show that $\tilde{M}_{ip}$ defined in Section 3.2 is asymptotically larger than M_{ip} stochastically.

Remark S1.2. *Under H_0^*, $\lim_{n\to\infty} \mathrm{pr}(M_{ip} > t) \leq \mathrm{pr}(\tilde{M}_{ip} > t)$ holds at any t.*

Proof of Remark S1.2. Here we follow the same notations and assumptions in Lemma S1.3. By continuous mapping theorem, for finite $p \geq 1$, we have

$$\begin{aligned}
M_{ip} &\xrightarrow{d} \left[\sum_k \int_{a_k}^{b_k} \left\{\sup_{a_k \leq v \leq u} \left(\frac{\mathcal{T}_i(v)}{1-v}\right)(1-u) - \mathcal{T}_i(u)\right\}^p du\right]^{1/p} \\
&= \left[\sum_k \int_{a_k}^{b_k} \left\{\sup_{a_k \leq v \leq u} \left(\frac{\mathcal{L}_i(v)}{1-v}\right)(1-u) - \mathcal{L}_i(u)\right\}^p du\right]^{1/p} \\
&\leq \left[\int_0^1 \left\{\sup_{0 \leq v \leq u} \left(\frac{\mathcal{L}_i(v)}{1-v}\right)(1-u) - \mathcal{L}_i(u)\right\}^p du\right]^{1/p} = \tilde{M}_{ip}, \\
M_{i\infty} &\xrightarrow{d} \max_k \left\{\sup_{a_k \leq u \leq b_k} \sup_{a_k \leq v \leq u} \left(\frac{\mathcal{T}_i(v)}{1-v}\right)(1-u) - \mathcal{T}_i(u)\right\} \\
&\leq \sup_{0 \leq u \leq 1} \sup_{0 \leq v \leq u} \left(\frac{\mathcal{T}_i(v)}{1-v}\right)(1-u) - \mathcal{T}_i(u) = \tilde{M}_{i\infty},
\end{aligned}$$

where $0/0$ is defined by 0 in the supremum, and both the inequalities above are because of the definition of the supremum. Therefore, we can conclude that

$$\lim_{n\to\infty} \mathrm{pr}(M_{ip} > t) \leq \mathrm{pr}(\tilde{M}_{ip} > t) \text{ holds at any } t.$$

Similarly, we can define

$$\tilde{M}_{i\infty} = \sup_{0 \leq u \leq 1} \left[\sup_{0 \leq v \leq u} \left(\frac{\mathcal{L}_i(v)}{1-v}\right)(1-u) - \mathcal{L}_i(u)\right],$$

such that $\lim_{n\to\infty} \mathrm{pr}(M_{i\infty} > t) \leq \mathrm{pr}(\tilde{M}_{i\infty} > t)$ for all $t \geq 0$. □

Lemma S1.5. *Under H_0^*, for every $1 \leq p \leq \infty$, as $n \to \infty$,*

$$\hat{S}_{kp}^* \xrightarrow{d} \tilde{S}_{kp} \quad \textit{and} \quad \hat{W}_{kp}^* \xrightarrow{d} \tilde{W}_{kp}.$$

Proof of Lemma S1.5. Define $\mathcal{V}_i^*(u) = \sup_{0\le v\le u}\{\mathcal{L}_i^*(v)/(1-v)\}(1-u) - \mathcal{L}_i^*(u)$ where $\mathcal{L}_i^*(u) = \lambda_i^{1/2}\mathcal{B}_i^*\{\mathcal{M}R_i(u)\} - (1-\lambda_i)^{1/2} r_i(u)\mathcal{B}_{i+1}^*(u)$ for $0 \le u \le 1$ and corresponding $S_{kp}^* = \sum_{1\le i<k}\|\mathcal{V}_i^*\|$, and $W_{kp}^* = \max_{1\le i<k}\|\mathcal{V}_i^*\|$. By construction, $\tilde{S}_{kp}$ and S_{kp}^* share the same distribution. Similarly, $\tilde{W}_{kp}$ and W_{kp}^* share the same distribution. Therefore, it suffices to show that $\hat{S}_{kp}^* \xrightarrow{d} S_{kp}^*$ and $\hat{W}_{kp}^* \xrightarrow{d} W_{kp}^*$ as $n\to\infty$.

Since S_{kp}^* and W_{kp}^* contain $\mathcal{L}_i^*$ and $\hat{S}_{kp}^*$ and $\hat{W}_{kp}^*$ contains $\widehat{\mathcal{L}}_i^*$, we firstly show that the difference between $\widehat{\mathcal{L}}_i^*$ and $\mathcal{L}_i^*$ are negligible, that is, $\|\widehat{\mathcal{L}}_i^* - \mathcal{L}_i^*\|_\infty = o_P(1)$. Note that

$$
\begin{aligned}
\|\widehat{\mathcal{L}}_i^* - \mathcal{L}_i^*\|_\infty = \sup_{0\le u\le 1}\Big| & C_i[n_i^{-1/2}\mathcal{B}_i^*\{\mathcal{M}\hat{R}_i(u)\} - n_{i+1}^{-1/2}\hat{r}_i(u)\mathcal{B}_{i+1}^*(u)] \\
& - [\lambda_i^{1/2}\mathcal{B}_i^*\{R_i(u)\} - (1-\lambda_i)^{1/2} r_i(u)\mathcal{B}_{i+1}^*(u)]\Big| \\
\le \sup_{0\le u\le 1} & \left|C_i n_i^{-1/2}\mathcal{B}_i^*\{\mathcal{M}\hat{R}_i(u)\} - \lambda_i^{1/2}\mathcal{B}_i^*\{R_i(u)\}\right| \\
& + \sup_{0\le u\le 1}\left|-C_i n_{i+1}^{-1/2}\hat{r}_i(u)\mathcal{B}_{i+1}^*(u) + (1-\lambda_i)^{1/2} r_i(u)\mathcal{B}_{i+1}^*(u)\right| \\
=: & \, I_{1n} + I_{2n}.
\end{aligned}
$$

For the first term I_{1n}, since $C_i n_i^{-1/2}$ converges to a constant $\lambda_i^{1/2}$ as $n\to\infty$ by assumption, it suffices to show that $\sup_{0\le u\le 1}|\mathcal{B}_i^*\{\mathcal{M}\hat{R}_i(u)\} - \mathcal{B}_i^*\{R_i(u)\}| = o_P(1)$, under H_0^*. According to the Lipschitz continuity of $\mathcal{M}$, $\|\mathcal{M}\hat{R}_i - R_i\|_\infty = \|\mathcal{M}\hat{R}_i - \mathcal{M}R_i\|_\infty \le \|\hat{R}_i - R_i\|_\infty$ since $R_i = \mathcal{M}R_i$. On the other hand, from Theorem 2.1 in Hsieh and Turnbull (1996), $\|\hat{R}_i - R_i\|_\infty$ converges to zero almost surely, then $\sup_{0\le u\le 1}|\mathcal{B}_i^*\{\mathcal{M}\hat{R}_i(u)\} - \mathcal{B}_i^*\{R_i(u)\}|$ converges to zero almost surely because of $\mathcal{B}_i^*$ is uniformly continuous almost surely, hence, $I_{1n} = o_P(1)$.

For the second term I_{2n}, since $C_i n_{i+1}^{-1/2}$ converges to $(1-\lambda_i)^{1/2}$ as $n\to\infty$, it suffices to show that $\sup_{0\le u\le 1}|\{\hat{r}_i(u) - r_i(u)\}\mathcal{B}_i^*(u)| = o_P(1)$. Given $0<\delta<1$, because $0 \le \hat{r}_i \le 1$ and $0 \le r_i \le 1$, then $\sup_{1-\delta\le u\le 1}|\{\hat{r}_i(u) - r_i(u)\}\mathcal{B}_i^*(u)| \le 2\sup_{1-\delta\le u\le 1}|\mathcal{B}_i^*(u)|$. Define $\mathcal{W}_i^*(u) = \mathcal{B}_i^*(u) + \zeta u$ for $u\in[0,1]$ where ζ follows the standard normal distribution and is independent of $\mathcal{B}_i^*$. One can show that $\mathcal{W}_i^*$ is the standard Wiener process over $[0,1]$. Because of the symmetry of $\mathcal{B}_i^*$, we have $\sup_{1-\delta\le u\le 1}|\mathcal{B}_i^*(u)| \overset{d}{=} \sup_{0\le u\le\delta}|\mathcal{B}_i^*(u)|$. Therefore, given $\eta>0$,

$$
\begin{aligned}
\mathrm{pr}\left(\sup_{1-\delta\le u\le 1}|\mathcal{B}_i^*(u)| > \eta\right) &= \mathrm{pr}\left(\sup_{0\le u\le\delta}|\mathcal{B}_i^*(u)| > \eta\right) \\
&\le \mathrm{pr}\left(\sup_{0\le u\le\delta}|\mathcal{W}_i^*(u)| + \delta|\zeta| > \eta\right) \\
&= \mathrm{pr}\left(\sqrt{\delta}\left(\sup_{0\le u\le 1}|\mathcal{W}_i^*(u)|\right) + \delta|\zeta| > \eta\right),
\end{aligned}
$$

where the last equality holds because $\mathcal{W}_i^*$ is a standard Wiener process and independent from ζ. Note that both $\sup_{0\le u\le 1}|\mathcal{W}_i^*(u)|$ and $|\zeta|$ are bounded in probability, then given $\epsilon>0$ and $\eta>0$, we can choose small enough $\delta=\delta(\eta,\epsilon)$ such that

$$\mathrm{pr}\left(\sup_{1-\delta\le u\le 1}|\mathcal{B}_i^*(u)|>\eta\right)<\epsilon.$$

Then we conclude that $\sup_{1-\delta\le u\le 1}|\{\hat{r}_i(u)-r_i(u)\}\mathcal{B}_i^*(u)|=o_P(1)$.

On the other hand, with the same choice of δ above, note that

$$\begin{aligned}|\{\hat{r}_i(u)-r_i(u)\}\mathcal{B}_i^*(u)| &\le \left|\frac{1-\mathcal{M}\hat{R}_i(u)}{1-u}-\frac{1-R_i(u)}{1-u}\right||\mathcal{B}_i^*(u)|\\ &= \frac{|\mathcal{M}\hat{R}_i(u)-R_i(u)|}{1-u}|\mathcal{B}_i^*(u)|.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{0\le u\le 1-\delta}|\{\hat{r}_i(u)-r_i(u)\}\mathcal{B}_i^*(u)| &\le \frac{1}{\delta}\|\mathcal{M}\hat{R}_i-R_i\|_\infty\|\mathcal{B}_i^*\|_\infty\\ &\le \frac{1}{\delta}\|\hat{R}_i-R_i\|_\infty\|\mathcal{B}_i^*\|_\infty,\end{aligned}$$

where the last inequality is because of $\mathcal{M}R_i=R_i$ under H_0 and the Lipschitz continuity of $\mathcal{M}$. Since $\|\hat{R}_i-R_i\|_\infty$ converges to zero almost surely from Theorem 2.1 Hsieh and Turnbull (1996) and $\mathcal{B}_i$ is bounded almost surely, we conclude that $\sup_{0\le u\le 1-\delta}|\{\hat{r}_i(u)-r_i(u)\}\mathcal{B}_i(u)|=o_P(1)$ and $I_{2n}=o_P(1)$. Subsequently, we have $\|\widehat{\mathcal{L}}_i^*-\mathcal{L}_i^*\|_\infty=o_P(1)$.

Next, we will show that $\|\widehat{\mathcal{V}}_i^*-\mathcal{V}_i^*\|_p=o_P(1)$. From the definitions of $\mathcal{V}_i$ and $\widehat{\mathcal{V}}_i^*$, we can write $\widehat{\mathcal{V}}_i^*(u)$ and $\mathcal{V}_i^*(u)$ in terms of the functional operator $\mathcal{M}_{[0,1]}^{(1,0)}$ defined in the proof of Theorem 3:

$$\widehat{\mathcal{V}}_i^*(u)-\mathcal{V}_i^*(u)=\mathcal{M}_{[0,1]}^{(1,0)}\widehat{\mathcal{L}}_i^*(u)-\mathcal{M}_{[0,1]}^{(1,0)}\mathcal{L}_i^*(u)-\{\widehat{\mathcal{L}}_i^*(u)-\mathcal{L}_i^*(u)\}$$

and then we have

$$\begin{aligned}\|\widehat{\mathcal{V}}_i^*-\mathcal{V}_i^*\|_p\le\|\widehat{\mathcal{V}}_i^*-\mathcal{V}_i^*\|_\infty &\le \left\|\mathcal{M}_{[0,1]}^{(1,0)}\widehat{\mathcal{L}}_i^*-\mathcal{M}_{[0,1]}^{(1,0)}\mathcal{L}_i^*\right\|_\infty+\|\widehat{\mathcal{L}}_i^*-\mathcal{L}_i\|_\infty\\ &\le 2\|\widehat{\mathcal{L}}_i^*-\mathcal{L}_i^*\|_\infty,\end{aligned}$$

where the last inequality holds because of the Lipschitz continuity of $\mathcal{M}_{[0,1]}^{(1,0)}$.

Lastly, because of the triangle inequality, we have

$$\left|\|\widehat{\mathcal{V}}_i^*\|_p-\|\mathcal{V}_i^*\|_p\right|\le\|\widehat{\mathcal{V}}_i^*-\mathcal{V}_i^*\|_p\le\|\widehat{\mathcal{V}}_i^*-\mathcal{V}_i^*\|_\infty=o_P(1),$$

which implies that $\|\widehat{\mathcal{V}}_i^*\|_p$ and $\|\mathcal{V}_i^*\|_p$ are asymptotic identical in distribution. Recall that $\hat{S}_{kp}^*$ is the sum and $\hat{W}_{kp}^*$ is the maximum of $\|\widehat{\mathcal{V}}_i^*\|_p$, respectively. Therefore, applying the continuous mapping theorem, $\hat{S}_{kp}^*\xrightarrow{d}\sum_{i=1}^{k-1}\|\mathcal{V}_i\|_p=S_{kp}^*$ and $\hat{W}_{kp}^*\xrightarrow{d}\max_{1\le i<k}\|\mathcal{V}_i^*\|_p=W_{kp}^*$.

□

From Theorem 2 and Lemma S1.5, the upper α-th quantile values of $\hat{S}^*_{kp}$ and $\hat{W}^*_{kp}$ are reasonable choices of critical values of the GOF tests. We conclude this section by providing the proof of Theorem 3 that the upper α-th quantiles $\hat{s}^*_{k,p,\alpha}$ and $\hat{w}^*_{k,p,\alpha}$ of $\hat{S}^*_{kp}$ and $\hat{W}^*_{kp}$, respectively, control the type I error under α asymptotically and provide consistency of the proposed tests.

Proof of Theorem 3. Under H^*_0, Lemma S1.5, shows that $\hat{S}^*_{kp}$ and $\hat{W}^*_{kp}$ converges in distribution to $\tilde{S}_{kp}$ and $\tilde{W}_{kp}$, respectively. For consistency of the proposed GOF tests, we wish to show that the critical values $\hat{s}^*_{kp,\alpha}$ and $\hat{w}^*_{kp,\alpha}$ are finite with probability one and the test statistics diverge to positive infinity. To show the boundedness of the critical values $\hat{s}^*_{kp,\alpha}$ and $\hat{w}^*_{kp,\alpha}$, it suffices to show the boundedness of $\|\widehat{\mathcal{V}}^*_i\|_p$. Recall that the process $\widehat{\mathcal{V}}^*_i$ is defined by $\widehat{\mathcal{V}}^*_i(u) = \mathcal{M}^{(1,0)}_{[0,1]}\widehat{\mathcal{L}}^*_i(u) - \widehat{\mathcal{L}}^*_i(u)$ for $0 \le u < 1$ and $\widehat{\mathcal{V}}^*_i(1) = 0$. The process $\widehat{\mathcal{L}}^*_i$ is defined by $\widehat{\mathcal{L}}^*_i(u) = C_i[n_i^{-1/2}\mathcal{B}_i\{\mathcal{M}\hat{R}_i(u)\} - n_{i+1}^{-1/2}\hat{r}_i(u)\mathcal{B}_{i+1}(u)]$ for $0 \le u \le 1$. It is clear that $\|\mathcal{M}^{(1,0)}_{[0,1]}\widehat{\mathcal{L}}^*_i\|_\infty$ is bounded by $\|\widehat{\mathcal{L}}^*_i(u)\|_\infty$, then we have

$$\|\widehat{\mathcal{V}}^*_i\|_p \le \|\widehat{\mathcal{V}}^*_i\|_\infty \le \|\mathcal{M}^{(1,0)}_{[0,1]}\widehat{\mathcal{L}}^*_i\|_\infty + \|\widehat{\mathcal{L}}^*_i\|_\infty \le 2\|\widehat{\mathcal{L}}^*_i\|_\infty.$$

Because $0 \le \hat{r}_i \le 1$, $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ are bounded with probability one, then $\|\widehat{\mathcal{V}}^*_i\|_\infty$ is bounded with probability one since $C_i n_i^{-1/2}$ and $C_i n_{i+1}^{-1/2}$ are bounded, too. Therefore, the asymptotic distribution of $\hat{S}^*_{kp}$ and $\hat{W}^*_{kp}$ are bounded such that the corresponding upper α-th quantile; i.e., the critical values $\hat{s}^*_{kp}$ and $\hat{w}^*_{kp}$ are bounded, too.

Now, we follow the same proof in Theorem 2 to show the consistency of the GOF test. Under H^*_1, there exists at least one i such that $\|\mathcal{M}R_i - R_i\|_p > 0$. Set $t = \hat{s}^*_{kp,\alpha}$, according to (S1.7) and (S1.8), we have $\mathrm{pr}(S_{kp} > \hat{s}^*_{kp,\alpha}) \ge \mathrm{pr}(M_{ip} > \hat{s}^*_{kp,\alpha}) \to 1$. Similarly, set $t = \hat{w}^*_{kp,\alpha}$, we have $\mathrm{pr}(W_{kp} > \hat{w}^*_{kp,\alpha}) \ge \mathrm{pr}(M_{ip} > \hat{w}^*_{kp,\alpha}) \to 1$ as $n \to \infty$. □

S1.3 Proofs and Lemmas for Jump Detection in Section 4

Proof of Theorem 4. Under H^*_0, recall that $J = \{1 \le i < k : F_i \prec F_{i+1}\}$. By definition of $u_{kp,\alpha}$, the probability that J^0_p incorrectly detects jump points with probability

$$\begin{aligned}\mathrm{pr}(J^0_p \ne \emptyset) &= \mathrm{pr}(\Delta_{ip} > u_{kp,\alpha} \text{ for some } 1 \le i < k)\\ &= 1 - \mathrm{pr}(\Delta_{ip} \le u_{kp,\alpha} \text{ for all } 1 \le i < k) = 1 - \alpha\end{aligned}$$

for all finite sample sizes.

If $J \ne \emptyset$, that is H_1 true, then

$$\begin{aligned}\mathrm{pr}(J^0_p \supseteq J) &= \mathrm{pr}(\Delta_{ip} > u_{kp,\alpha}, \text{ for all } i \in J)\\ &= 1 - \mathrm{pr}(\Delta_{ip} \le u_{kp,\alpha} \text{ for some } j \in J)\\ &\ge 1 - \sum_{j\in J}\mathrm{pr}(\Delta_{ip} \le u_{kp,\alpha} \text{ for } j \in J),\end{aligned} \tag{S1.9}$$

where $\mathrm{pr}(\Delta_{ip} \le u_{kp,\alpha}) \to 0$ as $n \to \infty$ since Δ_{ip} is a consistent test statistic against $F_i = F_{i+1}$. Therefore, $\mathrm{pr}(J_p^0 \supseteq J) \to 1$ as $n \to \infty$. Further, we denote $J^c = \{1, \ldots, (k-1)\}/J$. If $J^c = \emptyset$, then $J = \{1, \ldots, (k-1)\}$ and

$$\mathrm{pr}(J_p^0 = J) = \mathrm{pr}(\Delta_{ip} > u_{kp,\alpha} \text{ for all } i \in J) \ge \sum_{i=1}^{k-1} \mathrm{pr}(\Delta_{ip} > u_{kp,\alpha}) - (k-2).$$

Note that $\mathrm{pr}(\Delta_{ip} > u_{kp,\alpha}) \to 1$ as $n \to \infty$, $\mathrm{pr}(J_p^0 = J) \to 1$ as well. If $J^c \ne \emptyset$, then

$$\begin{aligned}\mathrm{pr}(J_p^0 \supset J) &= \mathrm{pr}(\Delta_{ip} > u_{kp,\alpha} \text{ for all } i \in J \text{ and } \Delta_{lp} > u_{kp,\alpha} \text{ for some } l \in J^c) \\ &\le \mathrm{pr}(\Delta_{lp} > u_{kp,\alpha} \text{ for some } l \in J^c) \\ &= \mathrm{pr}(\max_{l \in J^c} \Delta_{lp} > u_{kp,\alpha}).\end{aligned}$$

Now, we generate k random samples independently with sample sizes n_i from $\mathcal{U}(0,1)$ and obtain Δ_{lp}^* for $1 \le l < k$. Therefore, $\mathrm{pr}(\max_{l \in J^c} \Delta_{lp} > u_{kp,\alpha}) = \mathrm{pr}(\max_{l \in J^c} \Delta_{lp}^* > u_{kp,\alpha})$ because $\max_{l \in J^c} \Delta_{lp}$ is clearly distribution free when all $R_l = R_0$ for $l \in J^c$. Then we have

$$\mathrm{pr}(J_p^0 \supset J) \le \mathrm{pr}(\max_{l \in J^c} \Delta_{lp}^* > u_{kp,\alpha}) \le \mathrm{pr}(\max_{1 \le l < k} \Delta_{lp}^* > u_{kp,\alpha}) = \alpha. \qquad \text{(S1.10)}$$

since $\max_{1 \le l < k} \Delta_{lp}^*$ under $F_1 = F_2 = \cdots = F_k$ is distribution-free, too. From (S1.9) and (S1.10),

$$\lim_{n \to \infty} \mathrm{pr}(J_p^0 = J) = \lim_{n \to \infty} \mathrm{pr}(J_p^0 \supseteq J) - \lim_{n \to \infty} \mathrm{pr}(J_p^0 \supset J) \ge 1 - \alpha.$$

□

Proof of Theorem 5. Recall that $J = \{1 \le i < k : F_i \prec F_{i+1}\}$ and $\mathcal{E} = \{1 \le i < k : F_i = F_{i+1}\}$. It suffices to consider $J \ne \emptyset$ and $\mathcal{E} \ne \emptyset$. When $i \in J$, the probability of the event $\Delta_{ip} > \delta_i$ can be bounded below by:

$$\begin{aligned}\mathrm{pr}(\Delta_{ip} > \delta_i) &= \mathrm{pr}(C_i \|\mathcal{M}\hat{R}_i - R_0\|_p > \delta_i) \\ &= \mathrm{pr}(C_i \|\mathcal{M}\hat{R}_i - R_i + R_i - R_0\|_p > \delta_i) \\ &\ge \mathrm{pr}(\big|C_i \|\mathcal{M}\hat{R}_i - R_i\|_p - C_i \|R_i - R_0\|_p\big| > \delta_i) \\ &\ge \mathrm{pr}(C_i \|\mathcal{M}\hat{R}_i - R_i\|_p + \delta_i < C_i \|R_i - R_0\|_p) \\ &= \mathrm{pr}(\|\mathcal{M}\hat{R}_i - R_i\|_p + \delta_i / C_i < \|R_i - R_0\|_p).\end{aligned}$$

From Lemma 3 in Tang et al. (2017), since $i \in J$ such that $\mathcal{M}R_i = R_i$ and $\|\mathcal{M}\hat{R}_i - R_i\|_\infty = \|\mathcal{M}\hat{R}_i - \mathcal{M}R_i\|_\infty \le \|\hat{R}_i - R_i\|_\infty$, then

$$\|\mathcal{M}\hat{R}_i - R_i\|_p \le \|\mathcal{M}\hat{R}_i - R_i\|_\infty \le \|\hat{R}_i - R_i\|_\infty = o_P(1). \qquad \text{(S1.11)}$$

Note that $\delta_i/C_i \to 0$, then $\|\mathcal{M}\hat{R}_i - R_i\|_p + \delta_i/C_i = o_P(1)$, too. Because $\|R_i - R_0\|_p > 0$, the probability $\text{pr}(\|\mathcal{M}\hat{R}_i - R_i\|_p + \delta_i/C_i < C_i\|R_i - R_0\|_p)$ converges to 1 as $n \to \infty$. Since the number of elements in J is finite, $\text{pr}(\cap_{i\in J}\{\Delta_{ip} > \delta_i\}) \to 1$, too. On the other hand, when $i \in \mathcal{E}$, $\Delta_{ip} = C_i\|\mathcal{M}\hat{R}_i - R_0\|_p = O_P(1)$, therefore, $\text{pr}(\cap_{i\in\mathcal{E}}\{\Delta_{ip} \le \delta_i\}) \to 1$ since $\delta_i \to \infty$ and number of elements in $\mathcal{E}$ is finite. Then

$$\begin{aligned} &\text{pr}(J_p^0 = J) \\ =\ &\text{pr}(\cap_{i\in J}\{\Delta_{ip} > \delta_i\}, \cap_{i\in\mathcal{E}}\{\Delta_{ip} \le \delta_i\}) \\ \ge\ &[\text{pr}(\cap_{i\in J}\{\Delta_{ip} > \delta_i\}) - 1] + \text{pr}(\cap_{i\in\mathcal{E}}\{\Delta_{ip} \le \delta_i\}) \to 1, \text{ as } n \to \infty. \end{aligned}$$

□

Proof of Theorem 6. Here we define

$$\begin{aligned} Q_{ip}(\eta) = &\|\mathcal{M}\hat{R}_i - R_0\|_p I\{i \notin J_p(\eta)\} \\ &+ \left(\|\mathcal{M}\hat{R}_i - \hat{R}_i\|_p + \frac{\log C_i}{C_i} d_{ip}\right) I\{i \in J_p(\eta)\} \end{aligned}$$

such that $Q_p(\eta) = \sum_{i=1}^{k-1} Q_{ip}(\eta)$. Recall Section 4, since $Q_p(\eta)$ is a step function of η, it suffices to consider

$$\eta_p^* = \arg\min_{\eta\in\{\eta_0^\dagger, \eta_1^\dagger, \ldots, \eta_{k-1}^\dagger\}} Q_p(\eta)$$

where $\eta_i^\dagger = \Delta_{ip}$ for $1 \le i < k$ and $\eta_0^\dagger = 0$.

When all distributions are identical ($J = \emptyset$), according to previous discussion in Theorems 1 and 3, we have $\|\mathcal{M}\hat{R}_i - R_i\|_p = O_P(C_i^{-1})$ and $\|\mathcal{M}\hat{R}_i - \hat{R}_i\|_p = O_P(C_i^{-1})$, for all $1 \le i < k$, Therefore, $\|\mathcal{M}\hat{R}_i - R_i\|_p = o_P(\log C_i/C_i)$ and $\|\mathcal{M}\hat{R}_i - \hat{R}_i\|_p = o_P(\log C_i/C_i)$ such that the penalty term $d_{ip}\log C_i/C_i$, where $d_{ip} > c$ for some $c > 0$, dominates Q_{ip}. Therefore, $I\{i \in J_p^*(\eta_p^*)\} = 0$ is preferred and then $\eta_p^* \ge \eta_i^\dagger$ is suggested. Hence, the largest $\eta_i^\dagger$ minimizes $Q(\eta)$ with $\eta_p^* = \max_i(\eta_i^\dagger)$ such that $J_p^* = \emptyset$ with probability approaches 1 as $n \to \infty$.

When J is not empty, assume that $i \in J$. According to previous discussion in Theorem 1 and 3, we have $\|\mathcal{M}\hat{R}_i - R_i\|_p = O_P(C_i)$ but not $o_P(C_i)$ because $\|\mathcal{M}R_i - R_i\|_p > 0$. On the other hand, in $Q_{ip}(\eta)$, $\|\mathcal{M}\hat{R}_i - \hat{R}_i\|_p = O_P(C_i^{-1}) = o_P(C_i)$ and $d_{ip}\log C_i/C_i = o_P(C_i)$ because $d_{ip} = o_P(\log C_i)$. Therefore, $\|\mathcal{M}\hat{R}_i - R_i\|_p$ dominates Q_{ip} so that $I\{i \in J_p^*(\eta_p^*)\} = 1$ is preferred such that $\eta_p^* < \eta_i^\dagger$ is suggested. Therefore, $\eta_p^* < \min_{i\in J}\{\eta_i^\dagger\}$ with large probability approaching 1. On the other hand, if $i \notin J$. Then we have we have $\|\mathcal{M}\hat{R}_i - R_i\|_p = o_P(\log C_i/C_i)$, $\|\mathcal{M}\hat{R}_i - \hat{R}_i\|_p = o_P(\log C_i/C_i)$. Therefore, $d_{ip}\log C_i/C_i$ dominates Q_{ip} such that $I\{i \in J_p^*(\eta_p^*)\} = 0$ is preferred such that $\eta_p^* \ge \eta_i^\dagger$ is suggested. Therefore, $\eta_p^* \ge \max_{i\notin J, 1\le i<k}\{\eta_i^\dagger, 0\}$ with large probability

approaching 1 as $n \to \infty$. Here we define $\max_{i \notin J, 1 \leq i < k} \{\eta_i^\dagger, 0\} = 0$ if $J = \{1, \ldots, (k-1)\}$. In conclusion, the optimized η_p^* satisfies

$$\max_{i \notin J, 1 \leq i < k} \{\eta_i^\dagger, 0\} \leq \eta_p^* < \min_{i \in J} \{\eta_i^\dagger\}$$

with probability approaching 1 as $n \to \infty$, where η_p^* exists because $\eta_j^\dagger = O_p(1)$ for $j \notin J$ but $\eta_j^\dagger$ diverges to ∞ for $j \in J$. Therefore, $\max_{i \notin J, 1 \leq i < k} \{\eta_i^*, 0\} < \min_{i \in J} \{\eta_i^*\}$ with probability approaching 1 as $n \to \infty$. □

S2 Supplementary numerical results

In this section, we provide more simulation comparisons for $k = 3, 4, 5$ and 10 with sample sizes $n = 60, 100, 200$ and $p = 1, 2, \infty$ for the proposed equal tests, GOF tests, and distinguish distribution methods in Sections 2, 3, and 4, respectively.

p-value Adjusted Methods

Here, we include how to calculate p-value for the equality and GOF tests. For the equality test, the two-sample equality test examines the hypotheses $H_{0i} : F_i = F_{i+1}$ versus $H_{1i} : F_i \preceq F_{i+1}$ but not $F_i = F_{i+1}$. Recall that the scaled L^p difference between the star-shaped estimator and the equal distribution line is given by $\Delta_{ip} = \|\mathcal{M}\hat{R}_i - R_0\|_p$, which is also the test statistic for H_{0i} versus H_{1i}. According to the data, denote the observed test statistic by δ_{ip}, then the p-value is given by $p_{\Delta_{ip}} = \mathrm{pr}(\Delta_{ip} > \delta_{ip})$ when $F_i = F_{i+1}$. Since we reject the null hypothesis when any consecutive pairs of samples reject the null hypothesis, Holm–Bonferroni method, Hochberg's correction, Benjamini-Hochberg adjustment, and Bonferroni's methods are identical. We also compare Bonferroni's methods with Benjamini and Yekutieli's (BY) (Benjamini and Yekutieli, 2001) adjustment. From the p-value $p_{\Delta_{ip}}$, we also consider the equally weighted Cauchy combination (Liu and Xie, 2020) test statistic by

$$\sum_{i=1}^{k-1} \frac{\tan\{(0.5 - p_{\Delta_{ip}})\pi\}}{k-1}.$$

and reject H_0 when the test statistic is larger than the upper αth quantile of the standard Cauchy distribution.

For the GOF test, we follow the same idea in Section 3.1 to obtain the p-value based on the least favorable configuration when testing H_{0i}^* versus H_{1i}^*. For each test for H_{0i}^*, one can obtain p-values according to $F_i = F_{i+1}$ by $p_{M_{ip}} = \mathrm{pr}(\|\mathcal{D}\|_p > d_{ip})$, where m_{ip} is a realization of M_{ip}. Similar to the quality test, we consider Bonferroni's and BY's p-value adjustment methods. Then, one can consider the equally weighted Cauchy combination test statistic by

$$\sum_{i=1}^{k-1} \frac{\tan\{(0.5 - p_{M_{ip}})\pi\}}{k-1}.$$

Again, we reject H_0^* when the test statistic is larger than the upper αth quantile of the standard Cauchy distribution.

Data generation and ODCs

For the assessments of the proposed equality tests, GOF tests, and jump detection method, we follow the same idea in Sections 5.2 to generate data. We also choose ODCs $(R_1, R_2, \ldots, R_{k-1}) = (G_{q_1}, G_{q_2}, \ldots, G_{q_{k-1}})$ from the family of ODC G_q with $-1 \leq q \leq 1$. For power curve comparisons for GOF tests, we extend the three-sample cases $\{(K_\delta, R_0)\}_{\delta=0}^9$ and $\{(K_\delta, K_\delta)\}_{\delta=0}^0$ to $k = 4, 5$ cases by adding equal distributions, that is, $\{(K_\delta, R_0, R_0)\}_{\delta=0}^9$ and $\{(K_\delta, K_\delta, R_0)\}_{\delta=0}^0$ for $k = 4$; $\{(K_\delta, R_0, R_0, R_0)\}_{\delta=0}^9$ and $\{(K_\delta, K_\delta, R_0, R_0)\}_{\delta=0}^0$ for $k = 5$.

S2.1 Equality Tests

Here, we provide extra numerical comparisons for our proposed methods, T_{kp} and U_{ip}, see Tables S2.1, S2.4, S2.7, and S2.10, for $k = 3, 4, 5$, and 10, respectively. In general, a larger number of samples k leads to lower power. Similar to the discussion in Section 5.3, both T_{kp} and U_{kp} have reasonable sizes. But T_{kp} has better power than U_{kp} of gathering departure from H_0. Interestingly, when it comes to the robustness, U_{kp} has better power than T_{kp} if the last ODC is non-star-shaped and violates both H_0 and H_1. See Tables S2.3, S2.6, and S2.9 for $k = 3, 4$, and 5, respectively. .

We also compare with the empirical likelihood approach proposed by El Barmi and McKeague (2016), denoted by ME-B. See Tables S2.1, S2.4, and S2.7. In general, ME-B outperforms when the departure from the null hypothesis is significant. However, when the departure is mild and harder to detect, our tests perform better than ME-B. One can also find similar powers of T_{kp} when the accumulated departure $\sum D_0(R_i, p)$, defined in the Manuscript, are close. For U_{kp} with the same $\max D_0(R_i, p)$, the power is lower when the number of zero individual departure $D_0(R_i, p)$ is larger.

In addition, we compare the p-value adjusted tests, including the Cauchy combination test, BY, and Bonferroni adjustments. See Tables S2.2, S2.5, S2.8, and S2.11. Bonferroni methods outperformed the BY p-value adjustment methods, and under most scenarios, especially when there is more than one ODC violating H_0, the Cauchy combination tests were better than the Bonferroni method. Lastly, neither of these two methods surpasses our proposed tests.

S2.2 GOF Tests

For GOF tests, we provide more power comparisons for S_{kp} and W_{kp} with $k = 3, 4, 5$, and 10 S2.12, S2.14, S2.16, and S2.18. Similar to Section 5.4, the sizes of both tests are well-controlled. A larger number of samples k leads to lower power, and S_{kp} has better power of gathering departure from H_0^*.

We also compare the p-value adjusted tests, including the Cauchy combination test, BY, and Bonferroni adjustments for the GOF tests. See Tables S2.13, S2.15, S2.17, and S2.19 for $k = 3, 4, 5$, and 10, respectively. Similar to the equality test, Bonferroni's methods outperformed the BY methods. When more than one ODC violates H_1, the Cauchy combination tests have better power than the Bonferroni methods. Again, neither of these two methods surpasses our proposed tests.

A similar discussion of accumulated departure can be applied here. Similar powers of S_{kp} can be found when the accumulated departure $\sum D^*(R_i, p)$, defined in the Manuscript, are close. For W_{kp} with the same $\max D^*(R_i, p)$, the power is lower when the number of zero individual departure $D^*(R_i, p)$ is larger.

S2.3 Jumps detection

Lastly, Tables S2.20, S2.22, and S2.24 provide detailed assessment for the jump detection method J_p^0 while Tables S2.21, S2.23, and S2.25 report the assessment for J_p^*. Similar to the equality test, the larger number of samples k leads to lower correctness. Larger sample sizes help to have better performance as expected. Most of the findings are similar to the discussion in Section 5.5 in the Manuscript,

Table S2.1: Size and power comparisons for equality tests with $k = 3$ test statistics T_{kp}, U_{kp}, and ME-B.

n	(q_1, q_2)	T_{31}	U_{31}	T_{32}	U_{32}	$T_{3\infty}$	$U_{3\infty}$	ME-B
60	(0.0,0.0)	0.051	0.048	0.043	0.049	0.044	0.051	0.050
	(0.2,0.0)	0.184	0.152	0.196	0.170	0.209	0.197	0.183
	(0.6,0.0)	0.678	0.573	0.746	0.673	0.781	0.750	0.811
	(1.0,0.0)	0.948	0.887	0.973	0.936	0.990	0.973	0.995
	(0.2,0.2)	0.428	0.230	0.462	0.269	0.466	0.312	0.458
	(0.4,0.2)	0.700	0.412	0.734	0.482	0.747	0.567	0.777
	(0.6,0.4)	0.954	0.758	0.974	0.856	0.974	0.907	0.990
100	(0.0,0.0)	0.067	0.051	0.067	0.059	0.065	0.047	0.058
	(0.2,0.0)	0.273	0.173	0.302	0.212	0.324	0.213	0.260
	(0.6,0.0)	0.901	0.781	0.944	0.863	0.957	0.911	0.963
	(1.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(0.2,0.2)	0.599	0.337	0.648	0.403	0.657	0.399	0.672
	(0.4,0.2)	0.893	0.600	0.929	0.711	0.934	0.741	0.948
	(0.6,0.4)	0.998	0.940	1.000	0.975	1.000	0.990	1.000
200	(0.0,0.0)	0.048	0.053	0.046	0.056	0.050	0.054	0.042
	(0.2,0.0)	0.432	0.340	0.476	0.404	0.506	0.443	0.474
	(0.6,0.0)	0.998	0.984	1.000	0.995	1.000	0.999	1.000
	(1.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(0.2,0.2)	0.889	0.574	0.921	0.678	0.927	0.720	0.948
	(0.4,0.2)	0.995	0.924	0.997	0.966	0.997	0.989	1.000
	(0.6,0.4)	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.2: Size and power comparisons with $k = 3$ and $p = 1, 2, \infty$ for equality tests with adjusted p-values, including Cauchy, BY, and Bonferroni.

n	(q_1, q_2)	C_{31}	Y_{31}	B_{31}	C_{32}	Y_{32}	B_{32}	$C_{3\infty}$	$Y_{3\infty}$	$B_{3\infty}$
60	$(0.0, 0.0)$	0.042	0.031	0.047	0.042	0.031	0.047	0.048	0.047	0.051
	$(0.2, 0.0)$	0.138	0.108	0.150	0.162	0.126	0.167	0.184	0.169	0.197
	$(0.6, 0.0)$	0.575	0.500	0.571	0.682	0.593	0.667	0.759	0.718	0.750
	$(1.0, 0.0)$	0.894	0.850	0.883	0.943	0.917	0.935	0.975	0.964	0.973
	$(0.2, 0.2)$	0.234	0.167	0.226	0.278	0.202	0.263	0.332	0.269	0.312
	$(0.4, 0.2)$	0.449	0.341	0.405	0.522	0.400	0.472	0.607	0.526	0.567
	$(0.6, 0.4)$	0.811	0.688	0.757	0.879	0.793	0.846	0.926	0.886	0.907
100	$(0.0, 0.0)$	0.048	0.037	0.054	0.052	0.032	0.060	0.049	0.033	0.055
	$(0.2, 0.0)$	0.178	0.139	0.184	0.209	0.156	0.216	0.245	0.186	0.241
	$(0.6, 0.0)$	0.807	0.740	0.788	0.882	0.833	0.868	0.927	0.900	0.919
	$(1.0, 0.0)$	0.990	0.974	0.983	0.999	0.996	0.997	1.000	0.999	0.999
	$(0.2, 0.2)$	0.380	0.274	0.359	0.438	0.318	0.409	0.471	0.368	0.442
	$(0.4, 0.2)$	0.676	0.535	0.623	0.766	0.632	0.719	0.817	0.703	0.785
	$(0.6, 0.4)$	0.970	0.927	0.945	0.992	0.973	0.976	0.997	0.989	0.992
200	$(0.0, 0.0)$	0.039	0.035	0.053	0.041	0.038	0.055	0.047	0.037	0.054
	$(0.2, 0.0)$	0.326	0.250	0.333	0.385	0.310	0.402	0.439	0.364	0.443
	$(0.6, 0.0)$	0.989	0.974	0.983	0.996	0.990	0.995	0.999	0.996	0.999
	$(1.0, 0.0)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$(0.2, 0.2)$	0.633	0.461	0.570	0.715	0.553	0.676	0.766	0.643	0.720
	$(0.4, 0.2)$	0.952	0.877	0.924	0.985	0.950	0.966	0.995	0.976	0.989
	$(0.6, 0.4)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.3: Robustness comparisons for equality tests with $k = 3$ test statistics T_{kp}, U_{kp}, and ME-B.

n		T_{31}	U_{31}	T_{32}	U_{32}	$T_{3\infty}$	$U_{3\infty}$	ME-B
60	(0.0,-0.2)	0.018	0.030	0.016	0.028	0.018	0.031	0.014
	(0.2,-0.2)	0.109	0.134	0.116	0.149	0.121	0.177	0.062
	(0.4,-0.2)	0.299	0.323	0.348	0.371	0.374	0.450	0.247
	(0.6,-0.2)	0.541	0.559	0.623	0.657	0.669	0.735	0.599
100	(0.0,-0.2)	0.020	0.023	0.020	0.026	0.022	0.020	0.012
	(0.2,-0.2)	0.153	0.145	0.163	0.179	0.184	0.186	0.082
	(0.4,-0.2)	0.464	0.442	0.527	0.549	0.573	0.602	0.398
	(0.6,-0.2)	0.796	0.769	0.862	0.848	0.892	0.898	0.831
200	(0.0,-0.2)	0.016	0.031	0.016	0.032	0.017	0.035	0.006
	(0.2,-0.2)	0.263	0.318	0.314	0.381	0.321	0.425	0.121
	(0.4,-0.2)	0.795	0.807	0.870	0.883	0.907	0.937	0.760
	(0.6,-0.2)	0.986	0.980	0.995	0.991	1.000	0.998	0.995

Table S2.4: Size and power comparisons for equality tests with $k = 4$ test statistics T_{kp}, U_{kp}, and ME-B.

n	(q_1, q_2, q_3)	T_{41}	U_{41}	T_{42}	U_{42}	$T_{4\infty}$	$U_{4\infty}$	ME-B
60	(0.0,0.0,0.0)	0.048	0.054	0.047	0.049	0.051	0.049	0.064
	(0.2,0.0,0.0)	0.174	0.116	0.181	0.127	0.171	0.130	0.185
	(0.6,0.0,0.0)	0.634	0.466	0.697	0.570	0.700	0.668	0.795
	(1.0,0.0,0.0)	0.934	0.846	0.961	0.916	0.973	0.959	0.995
	(0.4,0.2,0.0)	0.656	0.317	0.695	0.383	0.680	0.444	0.821
	(0.6,0.4,0.0)	0.945	0.648	0.963	0.746	0.964	0.830	0.994
	(0.6,0.4,0.2)	0.979	0.684	0.988	0.776	0.988	0.856	1.000
100	(0.0,0.0,0.0)	0.050	0.053	0.053	0.050	0.046	0.039	0.042
	(0.2,0.0,0.0)	0.233	0.157	0.261	0.167	0.240	0.163	0.247
	(0.6,0.0,0.0)	0.846	0.730	0.898	0.825	0.914	0.880	0.963
	(1.0,0.0,0.0)	0.995	0.981	0.999	0.996	0.999	1.000	1.000
	(0.4,0.2,0.0)	0.858	0.525	0.883	0.625	0.872	0.672	0.960
	(0.6,0.4,0.0)	0.997	0.894	0.998	0.952	0.999	0.972	1.000
	(0.6,0.4,0.2)	1.000	0.909	1.000	0.963	1.000	0.979	1.000
200	(0.0,0.0,0.0)	0.057	0.052	0.055	0.057	0.059	0.057	0.058
	(0.2,0.0,0.0)	0.372	0.262	0.417	0.317	0.401	0.355	0.453
	(0.6,0.0,0.0)	0.995	0.966	0.997	0.985	0.998	0.995	1.000
	(1.0,0.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(0.4,0.2,0.0)	0.998	0.859	0.998	0.931	0.998	0.969	1.000
	(0.6,0.4,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(0.6,0.4,0.2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.5: Size and power comparisons with $k = 4$ and $p = 1, 2, \infty$ for equality tests with adjusted p-values, including Cauchy combination (C_{4p}), BY (Y_{4p}), and Bonferroni corrected methods (B_{4p}).

n	(q_1, q_2, q_3)	C_{41}	Y_{41}	B_{41}	C_{42}	Y_{42}	B_{42}	$C_{4\infty}$	$Y_{4\infty}$	$B_{4\infty}$
60	(0.0,0.0,0.0)	0.045	0.036	0.056	0.045	0.028	0.050	0.049	0.031	0.069
	(0.2,0.0,0.0)	0.117	0.088	0.120	0.122	0.086	0.129	0.143	0.099	0.163
	(0.6,0.0,0.0)	0.476	0.396	0.469	0.582	0.459	0.578	0.695	0.600	0.712
	(1.0,0.0,0.0)	0.854	0.784	0.848	0.925	0.875	0.917	0.965	0.946	0.965
	(0.4,0.2,0.0)	0.336	0.250	0.320	0.411	0.279	0.385	0.504	0.362	0.496
	(0.6,0.4,0.0)	0.704	0.556	0.652	0.810	0.636	0.753	0.892	0.770	0.871
	(0.6,0.4,0.2)	0.762	0.587	0.686	0.865	0.671	0.783	0.921	0.802	0.892
100	(0.0,0.0,0.0)	0.051	0.026	0.054	0.046	0.029	0.050	0.043	0.026	0.048
	(0.2,0.0,0.0)	0.149	0.104	0.158	0.164	0.125	0.169	0.179	0.127	0.184
	(0.6,0.0,0.0)	0.741	0.638	0.731	0.834	0.766	0.828	0.897	0.855	0.892
	(1.0,0.0,0.0)	0.985	0.960	0.981	0.995	0.991	0.996	1.000	1.000	1.000
	(0.4,0.2,0.0)	0.578	0.376	0.530	0.674	0.499	0.627	0.750	0.602	0.717
	(0.6,0.4,0.0)	0.930	0.826	0.894	0.973	0.919	0.954	0.990	0.963	0.982
	(0.6,0.4,0.2)	0.960	0.850	0.909	0.986	0.932	0.965	0.995	0.973	0.987
200	(0.0,0.0,0.0)	0.046	0.031	0.051	0.041	0.029	0.051	0.050	0.033	0.054
	(0.2,0.0,0.0)	0.251	0.179	0.261	0.294	0.223	0.300	0.343	0.259	0.345
	(0.6,0.0,0.0)	0.971	0.942	0.966	0.986	0.973	0.984	0.997	0.992	0.995
	(1.0,0.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(0.4,0.2,0.0)	0.893	0.734	0.857	0.960	0.871	0.924	0.983	0.932	0.965
	(0.6,0.4,0.0)	1.000	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(0.6,0.4,0.2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.6: Robustness comparisons for equality tests with $k = 4$ test statistics T_{kp}, U_{kp}, and ME-B.

n	(q_1, q_2, q_3)	T_{41}	U_{41}	T_{42}	U_{42}	$T_{4\infty}$	$U_{4\infty}$	ME-B
60	(0.0,0.0,-0.2)	0.021	0.039	0.024	0.037	0.023	0.037	0.023
	(0.2,0.0,-0.2)	0.088	0.104	0.095	0.118	0.098	0.120	0.098
	(0.4,0.0,-0.2)	0.259	0.244	0.292	0.296	0.283	0.346	0.320
	(0.2,0.2,-0.2)	0.247	0.168	0.271	0.198	0.257	0.215	0.304
	(0.6,0.4,-0.2)	0.912	0.641	0.935	0.743	0.936	0.828	0.975
100	(0.0,0.0,-0.2)	0.019	0.037	0.016	0.036	0.017	0.031	0.011
	(0.2,0.0,-0.2)	0.131	0.141	0.147	0.153	0.138	0.156	0.121
	(0.4,0.0,-0.2)	0.420	0.413	0.472	0.502	0.482	0.570	0.503
	(0.2,0.2,-0.2)	0.409	0.260	0.450	0.293	0.428	0.300	0.453
	(0.6,0.4,-0.2)	0.992	0.888	0.995	0.949	0.997	0.972	1.000
200	(0.0,0.0,-0.2)	0.018	0.032	0.017	0.037	0.020	0.038	0.011
	(0.2,0.0,-0.2)	0.199	0.247	0.243	0.304	0.232	0.344	0.177
	(0.4,0.0,-0.2)	0.720	0.740	0.806	0.838	0.825	0.892	0.817
	(0.2,0.2,-0.2)	0.706	0.447	0.765	0.534	0.767	0.608	0.769
	(0.6,0.4,-0.2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.7: Size and power comparisons for equality tests with $k = 5$ test statistics T_{kp}, U_{kp}, and ME-B.

n	(q_1, q_2, q_3, q_4)	T_{51}	U_{51}	T_{52}	U_{52}	$T_{5\infty}$	$U_{5\infty}$	ME-B
	(0.0,0.0,0.0,0.0)	0.048	0.054	0.051	0.051	0.051	0.048	0.058
	(0.2,0.0,0.0,0.0)	0.158	0.103	0.164	0.112	0.161	0.125	0.179
	(0.6,0.0,0.0,0.0)	0.585	0.466	0.641	0.565	0.640	0.665	0.784
	(1.0,0.0,0.0,0.0)	0.892	0.818	0.930	0.896	0.947	0.948	0.997
60	(0.4,0.2,0.0,0.0)	0.611	0.313	0.632	0.374	0.616	0.432	0.818
	(0.6,0.4,0.0,0.0)	0.934	0.609	0.959	0.721	0.951	0.816	0.998
	(0.8,0.6,0.4,0.0)	1.000	0.904	1.000	0.966	1.000	0.986	1.000
	(0.8,0.6,0.4,0.2)	1.000	0.916	1.000	0.972	1.000	0.991	1.000
	(0.0,0.0,0.0,0.0)	0.044	0.051	0.045	0.052	0.036	0.043	0.047
	(0.2,0.0,0.0,0.0)	0.204	0.145	0.227	0.171	0.208	0.180	0.252
	(0.6,0.0,0.0,0.0)	0.803	0.674	0.864	0.802	0.875	0.867	0.959
	(1.0,0.0,0.0,0.0)	0.990	0.961	1.000	0.996	0.999	1.000	1.000
100	(0.4,0.2,0.0,0.0)	0.814	0.458	0.857	0.561	0.838	0.627	0.969
	(0.6,0.4,0.0,0.0)	0.997	0.848	1.000	0.940	1.000	0.970	1.000
	(0.8,0.6,0.4,0.0)	1.000	0.996	1.000	1.000	1.000	1.000	1.000
	(0.8,0.6,0.4,0.2)	1.000	0.996	1.000	1.000	1.000	1.000	1.000
	(0.0,0.0,0.0,0.0)	0.049	0.059	0.049	0.055	0.045	0.053	0.057
	(0.2,0.0,0.0,0.0)	0.332	0.246	0.365	0.282	0.373	0.316	0.407
	(0.6,0.0,0.0,0.0)	0.986	0.959	0.997	0.981	0.996	0.992	1.000
	(1.0,0.0,0.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
200	(0.4,0.2,0.0,0.0)	0.992	0.812	0.997	0.896	0.997	0.938	1.000
	(0.6,0.4,0.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(0.8,0.6,0.4,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(0.8,0.6,0.4,0.2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.8: Size and power comparisons with $k = 5$ and $p = 1, 2, \infty$ for equality tests with adjusted p-values, Cauchy combination (C_{5p}), BY (Y_{5p}), and Bonferroni corrected methods (B_{5p}).

n	(q_1, q_2, q_3, q_4)	C_{51}	Y_{51}	B_{51}	C_{52}	Y_{52}	B_{52}	$C_{5\infty}$	$Y_{5\infty}$	$B_{5\infty}$
60	(0.0,0.0,0.0,0.0)	0.042	0.016	0.047	0.039	0.019	0.047	0.044	0.023	0.048
	(0.2,0.0,0.0,0.0)	0.088	0.037	0.089	0.097	0.047	0.101	0.117	0.068	0.125
	(0.6,0.0,0.0,0.0)	0.454	0.300	0.444	0.547	0.403	0.544	0.662	0.553	0.665
	(1.0,0.0,0.0,0.0)	0.815	0.677	0.797	0.896	0.814	0.886	0.952	0.916	0.948
	(0.4,0.2,0.0,0.0)	0.296	0.164	0.284	0.360	0.210	0.345	0.443	0.300	0.432
	(0.6,0.4,0.0,0.0)	0.641	0.410	0.586	0.750	0.539	0.693	0.845	0.703	0.816
	(0.8,0.6,0.4,0.0)	0.952	0.762	0.884	0.992	0.895	0.951	0.999	0.961	0.986
	(0.8,0.6,0.4,0.2)	0.971	0.780	0.896	0.996	0.909	0.961	1.000	0.973	0.991
100	(0.0,0.0,0.0,0.0)	0.047	0.032	0.051	0.045	0.029	0.055	0.040	0.029	0.049
	(0.2,0.0,0.0,0.0)	0.139	0.099	0.146	0.171	0.115	0.176	0.184	0.133	0.191
	(0.6,0.0,0.0,0.0)	0.678	0.582	0.680	0.805	0.729	0.804	0.878	0.826	0.873
	(1.0,0.0,0.0,0.0)	0.968	0.941	0.963	0.996	0.983	0.996	1.000	0.999	1.000
	(0.4,0.2,0.0,0.0)	0.481	0.343	0.460	0.586	0.437	0.569	0.685	0.533	0.644
	(0.6,0.4,0.0,0.0)	0.897	0.755	0.854	0.966	0.885	0.940	0.987	0.952	0.972
	(0.8,0.6,0.4,0.0)	1.000	0.987	0.997	1.000	1.000	1.000	1.000	1.000	1.000
	(0.8,0.6,0.4,0.2)	1.000	0.987	0.997	1.000	1.000	1.000	1.000	1.000	1.000
200	(0.0,0.0,0.0,0.0)	0.047	0.026	0.058	0.042	0.024	0.052	0.043	0.031	0.051
	(0.2,0.0,0.0,0.0)	0.222	0.157	0.240	0.258	0.185	0.270	0.308	0.224	0.309
	(0.6,0.0,0.0,0.0)	0.960	0.922	0.957	0.985	0.981	0.970	0.994	0.992	0.985
	(1.0,0.0,0.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(0.4,0.2,0.0,0.0)	0.845	0.694	0.810	0.920	0.806	0.886	0.955	0.888	0.935
	(0.6,0.4,0.0,0.0)	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(0.8,0.6,0.4,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(0.8,0.6,0.4,0.2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.9: Robustness comparisons for equality tests with $k = 4$ test statistics T_{kp}, U_{kp}, and ME-B.

n	(q_1, q_2, q_3, q_4)	T_{51}	U_{51}	T_{52}	U_{52}	$T_{5\infty}$	$U_{5\infty}$	ME-B
60	(0.0,0.0,0.0,-0.2)	0.019	0.047	0.016	0.045	0.024	0.041	0.030
	(0.2,0.0,0.0,-0.2)	0.080	0.096	0.086	0.106	0.086	0.118	0.086
	(0.2,0.2,0.0,-0.2)	0.221	0.153	0.227	0.179	0.223	0.192	0.343
	(0.4,0.2,0.0,-0.2)	0.472	0.306	0.516	0.368	0.501	0.426	0.694
	(0.6,0.4,0.2,-0.2)	0.970	0.631	0.982	0.742	0.976	0.835	0.999
100	(0.0,0.0,0.0,-0.2)	0.016	0.041	0.016	0.045	0.013	0.036	0.015
	(0.2,0.0,0.0,-0.2)	0.107	0.135	0.113	0.164	0.110	0.174	0.118
	(0.2,0.2,0.0,-0.2)	0.351	0.229	0.387	0.273	0.376	0.294	0.527
	(0.4,0.2,0.0,-0.2)	0.722	0.455	0.774	0.559	0.748	0.623	0.897
	(0.6,0.4,0.2,-0.2)	1.000	0.873	1.000	0.953	1.000	0.977	1.000
200	(0.0,0.0,0.0,-0.2)	0.017	0.045	0.016	0.039	0.022	0.038	0.017
	(0.2,0.0,0.0,-0.2)	0.198	0.237	0.221	0.271	0.235	0.303	0.186
	(0.2,0.2,0.0,-0.2)	0.664	0.400	0.718	0.472	0.703	0.534	0.852
	(0.4,0.2,0.0,-0.2)	0.975	0.809	0.988	0.895	0.981	0.937	0.999
	(0.6,0.4,0.2,-0.2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.10: Size and power comparisons for equality tests with $k = 10$ test statistics T_{kp}, U_{kp}.

n	(q_1, q_2, q_3, q_4)	$T_{10,1}$	$U_{10,1}$	$T_{10,2}$	$U_{10,2}$	$T_{10,\infty}$	$U_{10,\infty}$
60	(0.0,0.0,0.0,0.0)	0.049	0.049	0.048	0.047	0.050	0.040
	(0.2,0.0,0.0,0.0)	0.129	0.074	0.115	0.079	0.118	0.079
	(0.6,0.0,0.0,0.0)	0.440	0.345	0.476	0.455	0.454	0.511
	(1.0,0.0,0.0,0.0)	0.774	0.724	0.822	0.846	0.820	0.918
	(0.4,0.2,0.0,0.0)	0.443	0.207	0.469	0.256	0.418	0.277
	(0.6,0.4,0.0,0.0)	0.831	0.456	0.863	0.566	0.818	0.634
	(0.8,0.6,0.4,0.0)	1.000	0.792	1.000	0.886	0.999	0.932
	(0.8,0.6,0.4,0.2)	1.000	0.805	1.000	0.893	1.000	0.942
100	(0.0,0.0,0.0,0.0)	0.041	0.061	0.044	0.058	0.039	0.055
	(0.2,0.0,0.0,0.0)	0.156	0.118	0.159	0.130	0.160	0.132
	(0.6,0.0,0.0,0.0)	0.633	0.576	0.698	0.705	0.690	0.815
	(1.0,0.0,0.0,0.0)	0.942	0.937	0.972	0.980	0.967	0.998
	(0.4,0.2,0.0,0.0)	0.651	0.339	0.695	0.409	0.652	0.501
	(0.6,0.4,0.0,0.0)	0.972	0.736	0.985	0.855	0.974	0.932
	(0.8,0.6,0.4,0.0)	1.000	0.981	1.000	0.999	1.000	1.000
	(0.8,0.6,0.4,0.2)	1.000	1.000	1.000	1.000	1.000	1.000
200	(0.0,0.0,0.0,0.0)	0.057	0.056	0.055	0.055	0.055	0.059
	(0.2,0.0,0.0,0.0)	0.275	0.180	0.276	0.205	0.248	0.249
	(0.6,0.0,0.0,0.0)	0.927	0.931	0.954	0.968	0.946	0.985
	(1.0,0.0,0.0,0.0)	1.000	0.999	1.000	1.000	1.000	1.000
	(0.4,0.2,0.0,0.0)	0.944	0.698	0.962	0.812	0.932	0.891
	(0.6,0.4,0.0,0.0)	1.000	0.997	1.000	0.999	1.000	1.000
	(0.8,0.6,0.4,0.0)	1.000	1.000	1.000	1.000	1.000	1.000
	(0.8,0.6,0.4,0.2)	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.11: Size and power comparisons with $k = 10$ and $p = 1, 2, \infty$ for equality tests with adjusted p-values, Cauchy combination ($C_{10,p}$), BY ($Y_{10,p}$), and Bonferroni corrected methods ($B_{10,p}$). For simplicity, we set $R_i = R_0$ for $i = 6, \ldots, 9$ for all cases.

n	(q_1, q_2, q_3, q_4)	$C_{10,1}$	$Y_{10,1}$	$B_{10,1}$	$C_{10,2}$	$Y_{10,2}$	$B_{10,2}$	$C_{10,\infty}$	$Y_{10,\infty}$	$B_{10,\infty}$
60	(0.0,0.0,0.0,0.0)	0.032	0.012	0.036	0.036	0.012	0.041	0.044	0.022	0.049
	(0.2,0.0,0.0,0.0)	0.054	0.026	0.055	0.067	0.029	0.071	0.094	0.040	0.094
	(0.6,0.0,0.0,0.0)	0.305	0.208	0.297	0.406	0.284	0.411	0.552	0.421	0.556
	(1.0,0.0,0.0,0.0)	0.690	0.578	0.689	0.830	0.724	0.825	0.933	0.873	0.927
	(0.4,0.2,0.0,0.0)	0.182	0.103	0.179	0.237	0.136	0.234	0.339	0.188	0.330
	(0.6,0.4,0.0,0.0)	0.434	0.279	0.397	0.552	0.365	0.521	0.709	0.523	0.694
	(0.8,0.6,0.4,0.0)	0.822	0.568	0.733	0.919	0.729	0.867	0.972	0.887	0.947
	(0.8,0.6,0.4,0.2)	0.840	0.577	0.746	0.937	0.739	0.874	0.980	0.895	0.956
100	(0.0,0.0,0.0,0.0)	0.056	0.020	0.064	0.054	0.021	0.058	0.050	0.023	0.055
	(0.2,0.0,0.0,0.0)	0.115	0.052	0.124	0.123	0.058	0.130	0.129	0.078	0.132
	(0.6,0.0,0.0,0.0)	0.584	0.387	0.584	0.709	0.532	0.707	0.822	0.712	0.815
	(1.0,0.0,0.0,0.0)	0.946	0.860	0.942	0.981	0.952	0.981	0.999	0.989	0.998
	(0.4,0.2,0.0,0.0)	0.347	0.175	0.346	0.434	0.222	0.414	0.537	0.345	0.501
	(0.6,0.4,0.0,0.0)	0.774	0.531	0.742	0.886	0.685	0.858	0.953	0.856	0.932
	(0.8,0.6,0.4,0.0)	0.991	0.920	0.982	1.000	0.983	0.999	1.000	1.000	1.000
	(0.8,0.6,0.4,0.2)	1.000	0.924	1.000	1.000	0.984	1.000	1.000	1.000	1.000
200	(0.0,0.0,0.0,0.0)	0.048	0.026	0.054	0.050	0.026	0.061	0.051	0.024	0.063
	(0.2,0.0,0.0,0.0)	0.172	0.098	0.170	0.209	0.120	0.219	0.254	0.132	0.264
	(0.6,0.0,0.0,0.0)	0.927	0.858	0.925	0.970	0.939	0.968	0.989	0.973	0.987
	(1.0,0.0,0.0,0.0)	1.000	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000
	(0.4,0.2,0.0,0.0)	0.730	0.520	0.676	0.848	0.668	0.824	0.925	0.769	0.901
	(0.6,0.4,0.0,0.0)	0.998	0.995	0.995	1.000	0.999	0.999	1.000	1.000	1.000
	(0.8,0.6,0.4,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(0.8,0.6,0.4,0.2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.12: Size and power comparisons for GOF tests with test statistics S_{kp} and W_{kp} when $k = 3$.

n	(q_1, q_2)	S_{31}	W_{31}	S_{32}	W_{32}	$S_{3\infty}$	$W_{3\infty}$
60	(0.0,0.0)	0.066	0.067	0.062	0.062	0.027	0.053
	(0.2,0.0)	0.033	0.045	0.028	0.037	0.012	0.032
	(0.4,0.0)	0.021	0.048	0.022	0.040	0.009	0.030
	(0.4,0.2)	0.004	0.012	0.005	0.011	0.001	0.008
	(-0.2,0.0)	0.220	0.205	0.205	0.216	0.130	0.178
	(-0.6,0.0)	0.791	0.711	0.819	0.765	0.768	0.798
	(-1.0,0.0)	0.984	0.969	0.993	0.984	0.984	0.989
	(-0.2,0.2)	0.149	0.199	0.141	0.207	0.083	0.168
	(-0.4,0.2)	0.425	0.450	0.430	0.505	0.343	0.484
	(-0.2,-0.2)	0.449	0.309	0.466	0.326	0.357	0.284
	(-0.4,-0.2)	0.754	0.533	0.772	0.574	0.696	0.568
	(-0.6,-0.4)	0.980	0.871	0.986	0.909	0.979	0.920
100	(0.0,0.0)	0.048	0.060	0.041	0.057	0.023	0.048
	(0.2,0.0)	0.023	0.037	0.018	0.037	0.014	0.034
	(0.4,0.0)	0.018	0.035	0.016	0.036	0.010	0.034
	(0.4,0.2)	0.001	0.004	0.002	0.006	0.005	0.006
	(-0.2,0.0)	0.285	0.260	0.292	0.269	0.216	0.256
	(-0.6,0.0)	0.953	0.915	0.974	0.950	0.968	0.970
	(-1.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000
	(-0.2,0.2)	0.197	0.254	0.202	0.258	0.151	0.250
	(-0.4,0.2)	0.619	0.645	0.662	0.720	0.609	0.751
	(-0.2,-0.2)	0.653	0.419	0.669	0.456	0.607	0.426
	(-0.4,-0.2)	0.927	0.744	0.950	0.799	0.924	0.815
	(-0.6,-0.4)	0.998	0.982	0.999	0.993	1.000	0.995
200	(0.0,0.0)	0.063	0.059	0.052	0.057	0.049	0.048
	(0.2,0.0)	0.018	0.036	0.022	0.037	0.021	0.038
	(0.4,0.0)	0.015	0.040	0.018	0.039	0.016	0.042
	(0.4,0.2)	0.000	0.001	0.001	0.002	0.004	0.009
	(-0.2,0.0)	0.500	0.402	0.530	0.464	0.492	0.508
	(-0.6,0.0)	1.000	0.999	1.000	1.000	1.000	1.000
	(-1.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000
	(-0.2,0.2)	0.357	0.405	0.395	0.464	0.379	0.511
	(-0.4,0.2)	0.905	0.906	0.933	0.945	0.937	0.976
	(-0.2,-0.2)	0.905	0.649	0.939	0.740	0.928	0.786
	(-0.4,-0.2)	0.999	0.959	0.999	0.982	0.999	0.995
	(-0.6,-0.4)	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.13: Size comparisons with $k = 3$ and $p = 1, 2, \infty$ for GOF tests with adjusted p-values, including Cauchy combination (C^*_{3p}), BY (Y^*_{3p}), and Bonferroni corrected methods (B^*_{3p}).

n	(q_1, q_2)	C^*_{31}	Y^*_{31}	B^*_{31}	C^*_{32}	Y^*_{32}	B^*_{32}	$C^*_{3\infty}$	$Y^*_{3\infty}$	$B^*_{3\infty}$
	(0.0,0.0)	0.040	0.037	0.047	0.044	0.035	0.049	0.046	0.032	0.053
	(0.2,0.0)	0.024	0.022	0.028	0.021	0.020	0.026	0.026	0.021	0.030
	(0.4,0.0)	0.020	0.021	0.026	0.019	0.020	0.024	0.023	0.021	0.029
	(0.4,0.2)	0.004	0.004	0.006	0.004	0.005	0.006	0.004	0.003	0.007
	(-0.2,0.0)	0.162	0.113	0.165	0.169	0.117	0.172	0.176	0.108	0.177
60	(-0.6,0.0)	0.685	0.605	0.669	0.755	0.684	0.749	0.822	0.745	0.815
	(-1.0,0.0)	0.964	0.929	0.956	0.983	0.962	0.981	0.992	0.976	0.991
	(-0.2,0.2)	0.122	0.113	0.145	0.134	0.117	0.154	0.136	0.108	0.155
	(-0.4,0.2)	0.350	0.317	0.371	0.408	0.364	0.449	0.469	0.419	0.497
	(-0.2,-0.2)	0.280	0.217	0.256	0.301	0.232	0.291	0.340	0.229	0.310
	(-0.4,-0.2)	0.528	0.416	0.473	0.603	0.472	0.563	0.653	0.526	0.613
	(-0.6,-0.4)	0.888	0.787	0.835	0.930	0.863	0.910	0.957	0.894	0.933
	(0.0,0.0)	0.042	0.036	0.048	0.041	0.032	0.048	0.039	0.033	0.043
	(0.2,0.0)	0.022	0.021	0.026	0.023	0.019	0.029	0.023	0.020	0.028
	(0.4,0.0)	0.019	0.020	0.025	0.018	0.019	0.028	0.018	0.026	0.026
	(0.4,0.2)	0.001	0.002	0.003	0.002	0.001	0.003	0.004	0.003	0.004
	(-0.2,0.0)	0.201	0.117	0.215	0.232	0.194	0.249	0.250	0.205	0.259
100	(-0.6,0.0)	0.895	0.847	0.890	0.944	0.914	0.936	0.976	0.952	0.973
	(-1.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(-0.2,0.2)	0.158	0.159	0.193	0.187	0.175	0.225	0.213	0.191	0.238
	(-0.4,0.2)	0.551	0.524	0.574	0.642	0.598	0.672	0.708	0.659	0.735
	(-0.2,-0.2)	0.424	0.322	0.379	0.474	0.349	0.433	0.496	0.362	0.453
	(-0.4,-0.2)	0.762	0.658	0.701	0.835	0.723	0.787	0.875	0.752	0.831
	(-0.6,-0.4)	0.991	0.958	0.974	0.998	0.984	0.990	0.998	0.992	0.995
	(0.0,0.0)	0.036	0.027	0.046	0.037	0.030	0.044	0.039	0.026	0.043
	(0.2,0.0)	0.012	0.015	0.023	0.014	0.017	0.023	0.019	0.015	0.027
	(0.4,0.0)	0.011	0.014	0.022	0.012	0.016	0.022	0.015	0.013	0.025
	(0.4,0.2)	0.000	0.000	0.000	0.001	0.000	0.001	0.004	0.002	0.006
	(-0.2,0.0)	0.369	0.302	0.373	0.441	0.360	0.432	0.505	0.423	0.503
200	(-0.6,0.0)	0.999	0.991	0.997	1.000	0.999	1.000	1.000	1.000	1.000
	(-1.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(-0.2,0.2)	0.305	0.288	0.352	0.372	0.344	0.412	0.458	0.411	0.486
	(-0.4,0.2)	0.869	0.830	0.884	0.926	0.910	0.937	0.965	0.954	0.972
	(-0.2,-0.2)	0.711	0.550	0.633	0.774	0.644	0.715	0.85	0.717	0.788
	(-0.4,-0.2)	0.976	0.930	0.953	0.989	0.970	0.980	0.998	0.989	0.996
	(-0.6,-0.4)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.14: Size and power comparisons for GOF tests with test statistics S_{kp} and W_{kp} for $k = 4$.

n	(q_1, q_2, q_3)	S_{41}	W_{41}	S_{42}	W_{42}	$S_{4\infty}$	$W_{4\infty}$
60	(0.0,0.0,0.0)	0.054	0.075	0.043	0.067	0.024	0.046
	(0.2,0.0,0.0)	0.018	0.051	0.014	0.045	0.010	0.031
	(0.4,0.0,0.0)	0.011	0.052	0.011	0.044	0.010	0.032
	(0.4,0.2,0.0)	0.003	0.025	0.004	0.023	0.002	0.019
	(0.2,0.2,0.2)	0.001	0.009	0.002	0.008	0.001	0.009
	(-0.2,0.0,0.0)	0.190	0.179	0.180	0.182	0.093	0.151
	(-0.6,0.0,0.0)	0.713	0.652	0.730	0.708	0.592	0.743
	(-1.0,0.0,0.0)	0.959	0.950	0.968	0.977	0.939	0.985
	(-0.2,0.0,0.2)	0.126	0.171	0.112	0.178	0.051	0.151
	(-0.4,0.0,0.2)	0.363	0.418	0.359	0.461	0.229	0.461
	(-0.2,-0.2,0.0)	0.404	0.280	0.394	0.283	0.258	0.246
	(-0.4,-0.2,0.0)	0.710	0.494	0.711	0.530	0.557	0.523
	(-0.6,-0.4,-0.2)	0.986	0.822	0.989	0.871	0.981	0.885
100	(0.0,0.0,0.0)	0.045	0.065	0.041	0.062	0.020	0.042
	(0.2,0.0,0.0)	0.022	0.051	0.023	0.046	0.013	0.039
	(0.4,0.0,0.0)	0.019	0.054	0.016	0.048	0.006	0.043
	(0.4,0.2,0.0)	0.008	0.028	0.010	0.027	0.000	0.025
	(0.2,0.2,0.2)	0.000	0.002	0.000	0.006	0.001	0.004
	(-0.2,0.0,0.0)	0.254	0.216	0.251	0.232	0.165	0.223
	(-0.6,0.0,0.0)	0.911	0.875	0.936	0.927	0.898	0.955
	(-1.0,0.0,0.0)	1.000	0.998	1.000	1.000	0.999	1.000
	(-0.2,0.0,0.2)	0.165	0.209	0.154	0.226	0.109	0.223
	(-0.4,0.0,0.2)	0.563	0.613	0.599	0.675	0.455	0.696
	(-0.2,-0.2,0.0)	0.611	0.362	0.619	0.392	0.488	0.390
	(-0.4,-0.2,0.0)	0.913	0.695	0.927	0.747	0.867	0.768
	(-0.6,-0.4,-0.2)	1.000	0.982	1.000	0.994	1.000	0.997
200	(0.0,0.0,0.0)	0.059	0.062	0.051	0.060	0.035	0.044
	(0.2,0.0,0.0)	0.018	0.051	0.017	0.045	0.014	0.039
	(0.4,0.0,0.0)	0.011	0.056	0.014	0.051	0.011	0.041
	(0.4,0.2,0.0)	0.003	0.023	0.004	0.027	0.004	0.031
	(0.2,0.2,0.2)	0.000	0.003	0.000	0.003	0.001	0.016
	(-0.2,0.0,0.0)	0.448	0.336	0.460	0.390	0.384	0.434
	(-0.6,0.0,0.0)	1.000	0.993	1.000	0.998	1.000	1.000
	(-1.0,0.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000
	(-0.2,0.0,0.2)	0.328	0.332	0.347	0.394	0.291	0.437
	(-0.4,0.0,0.2)	0.883	0.862	0.912	0.925	0.889	0.96
	(-0.2,-0.2,0.0)	0.923	0.609	0.924	0.68	0.885	0.716
	(-0.4,-0.2,0.0)	1.000	0.937	1.000	0.969	0.999	0.989
	(-0.6,-0.4,-0.2)	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.15: Size comparisons with $k = 4$ and $p = 1, 2, \infty$ for GOF tests with adjusted p-values, including Cauchy combination (C^*_{4p}), BY (Y^*_{4p}), and Bonferroni corrected methods (B^*_{4p}).

n	(q_1, q_2, q_3)	C^*_{41}	Y^*_{41}	B^*_{41}	C^*_{42}	Y^*_{42}	B^*_{42}	$C^*_{4\infty}$	$Y^*_{4\infty}$	$B^*_{4\infty}$
60	(0.0,0.0,0.0)	0.039	0.026	0.054	0.041	0.025	0.052	0.043	0.024	0.051
	(0.2,0.0,0.0)	0.020	0.015	0.037	0.021	0.013	0.031	0.024	0.017	0.032
	(0.4,0.0,0.0)	0.014	0.014	0.034	0.016	0.012	0.029	0.019	0.017	0.031
	(0.4,0.2,0.0)	0.006	0.006	0.018	0.005	0.005	0.015	0.007	0.009	0.018
	(0.2,0.2,0.2)	0.002	0.002	0.005	0.001	0.002	0.003	0.004	0.004	0.006
	(-0.2,0.0,0.0)	0.124	0.092	0.140	0.132	0.097	0.141	0.162	0.111	0.167
	(-0.6,0.0,0.0)	0.609	0.527	0.613	0.677	0.600	0.675	0.755	0.666	0.755
	(-1.0,0.0,0.0)	0.927	0.889	0.926	0.970	0.940	0.965	0.987	0.974	0.988
	(-0.2,0.0,0.2)	0.104	0.086	0.127	0.115	0.093	0.130	0.143	0.106	0.157
	(-0.4,0.0,0.2)	0.319	0.259	0.345	0.370	0.304	0.392	0.426	0.308	0.456
	(-0.2,-0.2,0.0)	0.220	0.154	0.226	0.240	0.172	0.237	0.285	0.189	0.279
	(-0.4,-0.2,0.0)	0.452	0.318	0.429	0.500	0.376	0.484	0.584	0.424	0.553
	(-0.6,-0.4,-0.2)	0.873	0.703	0.788	0.920	0.787	0.851	0.954	0.847	0.912
100	(0.0,0.0,0.0)	0.036	0.024	0.049	0.037	0.027	0.045	0.036	0.021	0.045
	(0.2,0.0,0.0)	0.020	0.019	0.037	0.024	0.022	0.034	0.027	0.018	0.036
	(0.4,0.0,0.0)	0.018	0.019	0.037	0.022	0.022	0.034	0.023	0.018	0.035
	(0.4,0.2,0.0)	0.010	0.010	0.019	0.012	0.013	0.018	0.011	0.011	0.022
	(0.2,0.2,0.2)	0.000	0.001	0.001	0.000	0.001	0.001	0.000	0.000	0.004
	(-0.2,0.0,0.0)	0.170	0.120	0.189	0.190	0.138	0.201	0.216	0.135	0.235
	(-0.6,0.0,0.0)	0.860	0.755	0.855	0.916	0.866	0.913	0.956	0.923	0.955
	(-1.0,0.0,0.0)	0.997	0.996	0.996	1.000	1.000	1.000	1.000	1.000	1.000
	(-0.2,0.0,0.2)	0.133	0.112	0.174	0.158	0.127	0.187	0.190	0.125	0.220
	(-0.4,0.0,0.2)	0.487	0.380	0.543	0.579	0.490	0.626	0.666	0.556	0.683
	(-0.2,-0.2,0.0)	0.342	0.209	0.329	0.374	0.249	0.362	0.450	0.270	0.423
	(-0.4,-0.2,0.0)	0.692	0.472	0.650	0.775	0.585	0.729	0.834	0.652	0.791
	(-0.6,-0.4,-0.2)	0.994	0.943	0.976	0.999	0.983	0.994	0.999	0.993	0.998
200	(0.0,0.0,0.0)	0.035	0.022	0.043	0.032	0.025	0.044	0.031	0.025	0.037
	(0.2,0.0,0.0)	0.020	0.013	0.028	0.020	0.016	0.030	0.021	0.017	0.029
	(0.4,0.0,0.0)	0.017	0.013	0.028	0.018	0.016	0.030	0.019	0.017	0.028
	(0.4,0.2,0.0)	0.007	0.006	0.016	0.008	0.009	0.016	0.010	0.009	0.018
	(0.2,0.2,0.2)	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.001	0.006
	(-0.2,0.0,0.0)	0.273	0.195	0.285	0.338	0.256	0.348	0.418	0.314	0.423
	(-0.6,0.0,0.0)	0.993	0.974	0.990	0.999	0.993	0.998	1.000	1.000	1.000
	(-1.0,0.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(-0.2,0.0,0.2)	0.240	0.188	0.271	0.311	0.246	0.334	0.373	0.309	0.413
	(-0.4,0.0,0.2)	0.784	0.740	0.817	0.890	0.824	0.901	0.946	0.919	0.951
	(-0.2,-0.2,0.0)	0.609	0.387	0.538	0.698	0.512	0.638	0.781	0.598	0.718
	(-0.4,-0.2,0.0)	0.956	0.854	0.913	0.983	0.927	0.958	0.993	0.965	0.987
	(-0.6,-0.4,-0.2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.16: Size and power comparisons for GOF tests with test statistics S_{kp} and W_{kp} for $k = 5$.

n	(q_1, q_2, q_3, q_4)	S_{51}	W_{51}	S_{52}	W_{52}	$S_{4\infty}$	$W_{4\infty}$
60	(0.0,0.0,0.0,0.0)	0.046	0.073	0.038	0.062	0.020	0.049
	(0.2,0.0,0.0,0.0)	0.020	0.056	0.019	0.050	0.009	0.043
	(0.4,0.0,0.0,0.0)	0.020	0.058	0.014	0.047	0.005	0.042
	(0.4,0.2,0.0,0.0)	0.008	0.044	0.010	0.041	0.003	0.033
	(0.2,0.2,0.2,0.2)	0.002	0.012	0.002	0.012	0.000	0.009
	(-0.2,0.0,0.0,0.0)	0.176	0.156	0.151	0.146	0.082	0.117
	(-0.6,0.0,0.0,0.0)	0.655	0.636	0.652	0.689	0.477	0.702
	(-1.0,0.0,0.0,0.0)	0.946	0.941	0.959	0.967	0.895	0.978
	(-0.2,0.0,0.0,0.2)	0.110	0.147	0.096	0.141	0.044	0.113
	(-0.4,0.0,0.0,0.2)	0.312	0.361	0.301	0.399	0.160	0.388
	(-0.2,-0.2,0.0,0.0)	0.371	0.231	0.359	0.233	0.176	0.194
	(-0.4,-0.2,0.0,0.0)	0.645	0.427	0.634	0.463	0.436	0.446
	(-0.6,-0.4,-0.2,0.0)	0.991	0.800	0.994	0.847	0.973	0.851
	(-0.8,-0.6,-0.4,-0.2)	1.000	0.979	1.000	0.993	1.000	0.988
100	(0.0,0.0,0.0,0.0)	0.042	0.072	0.037	0.064	0.019	0.044
	(0.2,0.0,0.0,0.0)	0.016	0.059	0.013	0.058	0.004	0.042
	(0.4,0.0,0.0,0.0)	0.015	0.061	0.012	0.060	0.003	0.044
	(0.4,0.2,0.0,0.0)	0.005	0.043	0.004	0.047	0.001	0.037
	(0.2,0.2,0.2,0.2)	0.000	0.003	0.000	0.004	0.000	0.008
	(-0.2,0.0,0.0,0.0)	0.224	0.183	0.210	0.201	0.133	0.187
	(-0.6,0.0,0.0,0.0)	0.870	0.841	0.897	0.906	0.812	0.937
	(-1.0,0.0,0.0,0.0)	0.996	0.995	1.000	0.999	0.992	0.999
	(-0.2,0.0,0.0,0.2)	0.130	0.178	0.132	0.196	0.085	0.183
	(-0.4,0.0,0.0,0.2)	0.481	0.534	0.495	0.596	0.359	0.625
	(-0.2,-0.2,0.0,0.0)	0.553	0.317	0.534	0.339	0.374	0.323
	(-0.4,-0.2,0.0,0.0)	0.868	0.632	0.885	0.691	0.768	0.710
	(-0.6,-0.4,-0.2,0.0)	1.000	0.956	1.000	0.983	1.000	0.989
	(-0.8,-0.6,-0.4,-0.2)	1.000	1.000	1.000	1.000	1.000	1.000
200	(0.0,0.0,0.0,0.0)	0.049	0.065	0.045	0.060	0.035	0.060
	(0.2,0.0,0.0,0.0)	0.023	0.051	0.016	0.048	0.021	0.049
	(0.4,0.0,0.0,0.0)	0.022	0.054	0.016	0.049	0.016	0.051
	(0.4,0.2,0.0,0.0)	0.005	0.040	0.005	0.037	0.008	0.037
	(0.2,0.2,0.2,0.2)	0.007	0.038	0.005	0.036	0.010	0.035
	(-0.2,0.0,0.0,0.0)	0.408	0.322	0.394	0.355	0.294	0.391
	(-0.6,0.0,0.0,0.0)	0.995	0.989	0.998	0.996	0.995	0.999
	(-1.0,0.0,0.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000
	(-0.2,0.0,0.0,0.2)	0.281	0.325	0.286	0.356	0.203	0.400
	(-0.4,0.0,0.0,0.2)	0.828	0.832	0.879	0.902	0.821	0.945
	(-0.2,-0.2,0.0,0.0)	0.890	0.571	0.904	0.640	0.820	0.679
	(-0.4,-0.2,0.0,0.0)	0.994	0.920	0.998	0.962	0.996	0.985
	(-0.6,-0.4,-0.2,0.0)	1.000	1.000	1.000	1.000	1.000	1.000
	(-0.8,-0.6,-0.4,-0.2)	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.17: Size comparisons with $k = 5$ and $p = 1, 2, \infty$ for GOF tests with adjusted p-values, including Cauchy combination (C^*_{5p}), BY (Y^*_{5p}), and Bonferroni corrected methods (B^*_{5p}).

n	(q_1, q_2, q_3, q_4)	C^*_{51}	Y^*_{51}	B^*_{51}	C^*_{52}	Y^*_{52}	B^*_{52}	$C^*_{5\infty}$	$Y^*_{5\infty}$	$B^*_{5\infty}$
60	(0.0,0.0,0.0,0.0)	0.040	0.028	0.043	0.040	0.024	0.045	0.048	0.026	0.052
	(0.2,0.0,0.0,0.0)	0.030	0.021	0.033	0.028	0.018	0.035	0.037	0.020	0.042
	(0.4,0.0,0.0,0.0)	0.026	0.020	0.031	0.024	0.017	0.032	0.032	0.020	0.039
	(0.4,0.2,0.0,0.0)	0.018	0.012	0.022	0.015	0.011	0.023	0.023	0.015	0.031
	(0.2,0.2,0.2,0.2)	0.002	0.004	0.009	0.004	0.002	0.008	0.005	0.003	0.009
	(-0.2,0.0,0.0,0.0)	0.102	0.064	0.104	0.114	0.065	0.116	0.131	0.072	0.129
	(-0.6,0.0,0.0,0.0)	0.567	0.454	0.567	0.647	0.530	0.645	0.727	0.623	0.725
	(-1.0,0.0,0.0,0.0)	0.922	0.874	0.919	0.957	0.926	0.958	0.985	0.963	0.982
	(-0.2,0.0,0.0,0.2)	0.086	0.057	0.093	0.096	0.059	0.105	0.112	0.063	0.117
	(-0.4,0.0,0.0,0.2)	0.276	0.205	0.293	0.325	0.233	0.339	0.388	0.295	0.392
	(-0.2,-0.2,0.0,0.0)	0.174	0.102	0.176	0.190	0.105	0.197	0.228	0.117	0.219
	(-0.4,-0.2,0.0,0.0)	0.373	0.248	0.363	0.430	0.283	0.416	0.494	0.341	0.475
	(-0.6,-0.4,-0.2,0.0)	0.824	0.614	0.752	0.884	0.706	0.832	0.928	0.795	0.887
	(-0.8,-0.6,-0.4,-0.2)	0.996	0.914	0.968	0.999	0.963	0.989	1.000	0.988	0.993
100	(0.0,0.0,0.0,0.0)	0.039	0.025	0.046	0.034	0.025	0.045	0.036	0.018	0.044
	(0.2,0.0,0.0,0.0)	0.024	0.021	0.037	0.023	0.019	0.038	0.022	0.012	0.036
	(0.4,0.0,0.0,0.0)	0.023	0.021	0.037	0.022	0.019	0.038	0.020	0.012	0.035
	(0.4,0.2,0.0,0.0)	0.010	0.013	0.022	0.012	0.012	0.024	0.015	0.008	0.027
	(0.2,0.2,0.2,0.2)	0.001	0.001	0.002	0.001	0.001	0.002	0.001	0.000	0.004
	(-0.2,0.0,0.0,0.0)	0.137	0.084	0.149	0.159	0.094	0.168	0.191	0.107	0.199
	(-0.6,0.0,0.0,0.0)	0.791	0.698	0.787	0.886	0.799	0.886	0.945	0.884	0.944
	(-1.0,0.0,0.0,0.0)	0.993	0.987	0.992	0.999	0.996	0.999	0.999	0.999	0.999
	(-0.2,0.0,0.0,0.2)	0.121	0.080	0.141	0.142	0.090	0.159	0.171	0.105	0.189
	(-0.4,0.0,0.0,0.2)	0.433	0.323	0.454	0.511	0.320	0.540	0.607	0.499	0.624
	(-0.2,-0.2,0.0,0.0)	0.263	0.155	0.255	0.318	0.180	0.301	0.359	0.209	0.341
	(-0.4,-0.2,0.0,0.0)	0.596	0.390	0.553	0.687	0.505	0.651	0.741	0.590	0.718
	(-0.6,-0.4,-0.2,0.0)	0.975	0.880	0.937	0.991	0.952	0.982	0.996	0.978	0.991
	(-0.8,-0.6,-0.4,-0.2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
200	(0.0,0.0,0.0,0.0)	0.043	0.033	0.046	0.048	0.029	0.050	0.054	0.030	0.058
	(0.2,0.0,0.0,0.0)	0.027	0.024	0.035	0.031	0.021	0.037	0.038	0.021	0.046
	(0.4,0.0,0.0,0.0)	0.025	0.024	0.035	0.030	0.021	0.037	0.033	0.021	0.046
	(0.4,0.2,0.0,0.0)	0.016	0.014	0.022	0.016	0.012	0.024	0.018	0.011	0.031
	(0.2,0.2,0.2,0.2)	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001
	(-0.2,0.0,0.0,0.0)	0.248	0.163	0.253	0.307	0.221	0.318	0.388	0.277	0.394
	(-0.6,0.0,0.0,0.0)	0.989	0.974	0.985	0.996	0.991	0.996	0.998	0.999	0.999
	(-1.0,0.0,0.0,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000
	(-0.2,0.0,0.0,0.2)	0.217	0.156	0.245	0.282	0.214	0.309	0.358	0.274	0.385
	(-0.4,0.0,0.0,0.2)	0.768	0.683	0.788	0.868	0.805	0.880	0.938	0.910	0.944
	(-0.2,-0.2,0.0,0.0)	0.529	0.340	0.479	0.644	0.439	0.590	0.745	0.558	0.694
	(-0.4,-0.2,0.0,0.0)	0.930	0.794	0.895	0.970	0.898	0.955	0.993	0.968	0.985
	(-0.6,-0.4,-0.2,0.0)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(-0.8,-0.6,-0.4,-0.2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.18: Size and power comparisons for GOF tests with test statistics S_{kp} and W_{kp} for $k = 10$.

n	(q_1, q_2, q_3, q_4)	$S_{10,1}$	$W_{10,1}$	$S_{10,2}$	$W_{10,2}$	$S_{10,\infty}$	$W_{10,\infty}$
60	(0.0,0.0,0.0,0.0)	0.042	0.070	0.038	0.061	0.007	0.045
	(0.2,0.0,0.0,0.0)	0.022	0.068	0.017	0.053	0.006	0.04
	(0.4,0.0,0.0,0.0)	0.017	0.069	0.012	0.056	0.006	0.041
	(0.4,0.2,0.0,0.0)	0.009	0.062	0.008	0.051	0.005	0.038
	(0.2,0.2,0.2,0.2)	0.000	0.011	0.000	0.007	0.000	0.007
	(-0.2,0.0,0.0,0.0)	0.124	0.118	0.096	0.110	0.024	0.097
	(-0.6,0.0,0.0,0.0)	0.448	0.523	0.421	0.589	0.172	0.620
	(-1.0,0.0,0.0,0.0)	0.795	0.897	0.791	0.943	0.479	0.957
	(-0.2,0.0,0.0,0.2)	0.084	0.115	0.072	0.111	0.014	0.097
	(-0.4,0.0,0.0,0.2)	0.200	0.285	0.179	0.307	0.058	0.300
	(-0.2,-0.2,0.0,0.0)	0.236	0.172	0.201	0.179	0.062	0.145
	(-0.4,-0.2,0.0,0.0)	0.439	0.330	0.394	0.365	0.161	0.334
	(-0.6,-0.4,-0.2,0.0)	0.926	0.681	0.922	0.760	0.670	0.781
	(-0.8,-0.6,-0.4,-0.2)	1.000	0.935	1.000	0.975	0.990	0.974
100	(0.0,0.0,0.0,0.0)	0.062	0.088	0.046	0.077	0.010	0.051
	(0.2,0.0,0.0,0.0)	0.029	0.081	0.026	0.073	0.006	0.049
	(0.4,0.0,0.0,0.0)	0.028	0.081	0.024	0.076	0.007	0.049
	(0.4,0.2,0.0,0.0)	0.014	0.078	0.010	0.070	0.002	0.051
	(0.2,0.2,0.2,0.2)	0.000	0.007	0.000	0.007	0.000	0.005
	(-0.2,0.0,0.0,0.0)	0.171	0.153	0.150	0.159	0.056	0.143
	(-0.6,0.0,0.0,0.0)	0.679	0.743	0.664	0.839	0.432	0.880
	(-1.0,0.0,0.0,0.0)	0.967	0.986	0.967	0.998	0.857	0.999
	(-0.2,0.0,0.0,0.2)	0.129	0.154	0.111	0.159	0.038	0.142
	(-0.4,0.0,0.0,0.2)	0.324	0.416	0.303	0.484	0.142	0.525
	(-0.2,-0.2,0.0,0.0)	0.365	0.238	0.319	0.252	0.147	0.244
	(-0.4,-0.2,0.0,0.0)	0.669	0.488	0.647	0.558	0.371	0.601
	(-0.6,-0.4,-0.2,0.0)	0.998	0.904	0.999	0.958	0.970	0.974
	(-0.8,-0.6,-0.4,-0.2)	1.000	0.999	1.000	1.000	1.000	1.000
200	(0.0,0.0,0.0,0.0)	0.067	0.076	0.061	0.076	0.034	0.064
	(0.2,0.0,0.0,0.0)	0.037	0.068	0.043	0.071	0.029	0.057
	(0.4,0.0,0.0,0.0)	0.037	0.070	0.036	0.073	0.024	0.058
	(0.4,0.2,0.0,0.0)	0.026	0.064	0.025	0.069	0.013	0.053
	(0.2,0.2,0.2,0.2)	0.000	0.001	0.000	0.001	0.000	0.005
	(-0.2,0.0,0.0,0.0)	0.321	0.248	0.291	0.288	0.183	0.322
	(-0.6,0.0,0.0,0.0)	0.961	0.981	0.966	0.992	0.905	0.999
	(-1.0,0.0,0.0,0.0)	1.000	1.000	1.000	1.000	0.999	1.000
	(-0.2,0.0,0.0,0.2)	0.233	0.244	0.214	0.282	0.141	0.321
	(-0.4,0.0,0.0,0.2)	0.678	0.754	0.680	0.845	0.488	0.904
	(-0.2,-0.2,0.0,0.0)	0.734	0.420	0.713	0.490	0.489	0.551
	(-0.4,-0.2,0.0,0.0)	0.966	0.845	0.964	0.919	0.888	0.955
	(-0.6,-0.4,-0.2,0.0)	1.000	1.000	1.000	1.000	1.000	1.000
	(-0.8,-0.6,-0.4,-0.2)	1.000	1.000	1.000	1.000	1.000	1.000

Table S2.19: Size comparisons with $k = 10$ and $p = 1, 2, \infty$ for GOF tests with adjusted p-values, including Cauchy combination ($C^*_{10,p}$), BY ($Y^*_{10,p}$), and Bonferroni corrected methods ($B^*_{10,p}$). For simplicity, we set $R_i = R_0$ for $i = 6, \ldots, 9$ for all cases.

n	(q_1, q_2, q_3, q_4)	$C^*_{10,1}$	$Y^*_{10,1}$	$B^*_{10,1}$	$C^*_{10,2}$	$Y^*_{10,2}$	$B^*_{10,2}$	$C^*_{10,\infty}$	$Y^*_{10,\infty}$	$B^*_{10,\infty}$
60	(0.0,0.0,0.0,0.0)	0.040	0.019	0.048	0.038	0.019	0.047	0.040	0.016	0.045
	(0.2,0.0,0.0,0.0)	0.033	0.015	0.040	0.030	0.015	0.040	0.032	0.014	0.039
	(0.4,0.0,0.0,0.0)	0.030	0.015	0.039	0.029	0.015	0.039	0.032	0.014	0.039
	(0.4,0.2,0.0,0.0)	0.023	0.012	0.035	0.024	0.012	0.036	0.030	0.013	0.035
	(0.2,0.2,0.2,0.2)	0.002	0.000	0.005	0.003	0.001	0.004	0.003	0.001	0.004
	(-0.2,0.0,0.0,0.0)	0.078	0.039	0.087	0.086	0.037	0.094	0.084	0.033	0.090
	(-0.6,0.0,0.0,0.0)	0.454	0.303	0.469	0.539	0.383	0.543	0.622	0.440	0.623
	(-1.0,0.0,0.0,0.0)	0.872	0.737	0.870	0.926	0.844	0.928	0.958	0.907	0.959
	(-0.2,0.0,0.0,0.2)	0.071	0.038	0.083	0.077	0.036	0.090	0.079	0.033	0.091
	(-0.4,0.0,0.0,0.2)	0.207	0.125	0.221	0.254	0.142	0.262	0.290	0.157	0.293
	(-0.2,-0.2,0.0,0.0)	0.129	0.061	0.132	0.144	0.059	0.146	0.145	0.053	0.144
	(-0.4,-0.2,0.0,0.0)	0.267	0.148	0.270	0.320	0.166	0.317	0.360	0.178	0.300
	(-0.6,-0.4,-0.2,0.0)	0.663	0.427	0.641	0.762	0.516	0.718	0.830	0.570	0.791
	(-0.8,-0.6,-0.4,-0.2)	0.960	0.775	0.904	0.988	0.871	0.963	0.991	0.933	0.982
100	(0.0,0.0,0.0,0.0)	0.050	0.016	0.063	0.052	0.019	0.063	0.039	0.018	0.039
	(0.2,0.0,0.0,0.0)	0.043	0.015	0.057	0.045	0.016	0.057	0.034	0.016	0.037
	(0.4,0.0,0.0,0.0)	0.038	0.015	0.056	0.041	0.016	0.056	0.032	0.016	0.036
	(0.4,0.2,0.0,0.0)	0.031	0.014	0.049	0.034	0.015	0.048	0.030	0.015	0.035
	(0.2,0.2,0.2,0.2)	0.000	0.000	0.004	0.000	0.000	0.002	0.001	0.000	0.002
	(-0.2,0.0,0.0,0.0)	0.109	0.039	0.119	0.131	0.049	0.134	0.132	0.057	0.118
	(-0.6,0.0,0.0,0.0)	0.680	0.476	0.689	0.785	0.619	0.798	0.866	0.752	0.863
	(-1.0,0.0,0.0,0.0)	0.985	0.939	0.984	0.997	0.983	0.998	0.998	0.995	0.999
	(-0.2,0.0,0.0,0.2)	0.099	0.037	0.116	0.117	0.047	0.131	0.120	0.055	0.115
	(-0.4,0.0,0.0,0.2)	0.336	0.170	0.347	0.411	0.246	0.433	0.476	0.315	0.483
	(-0.2,-0.2,0.0,0.0)	0.197	0.078	0.194	0.229	0.097	0.223	0.247	0.108	0.214
	(-0.4,-0.2,0.0,0.0)	0.421	0.209	0.419	0.524	0.294	0.511	0.597	0.365	0.559
	(-0.6,-0.4,-0.2,0.0)	0.900	0.664	0.864	0.961	0.799	0.943	0.984	0.896	0.969
	(-0.8,-0.6,-0.4,-0.2)	1.000	0.967	0.996	1.000	0.995	1.000	0.999	0.999	1.000
200	(0.0,0.0,0.0,0.0)	0.050	0.028	0.063	0.052	0.025	0.063	0.039	0.019	0.039
	(0.2,0.0,0.0,0.0)	0.043	0.025	0.057	0.045	0.021	0.057	0.034	0.016	0.037
	(0.4,0.0,0.0,0.0)	0.038	0.025	0.056	0.041	0.021	0.056	0.032	0.016	0.036
	(0.4,0.2,0.0,0.0)	0.031	0.023	0.049	0.034	0.018	0.048	0.030	0.014	0.035
	(0.2,0.2,0.2,0.2)	0.000	0.001	0.004	0.000	0.001	0.002	0.001	0.000	0.002
	(-0.2,0.0,0.0,0.0)	0.205	0.118	0.211	0.244	0.147	0.241	0.322	0.186	0.320
	(-0.6,0.0,0.0,0.0)	0.976	0.944	0.979	0.990	0.981	0.989	0.999	0.994	0.999
	(-1.0,0.0,0.0,0.0)	0.999	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000
	(-0.2,0.0,0.0,0.2)	0.187	0.113	0.201	0.224	0.143	0.233	0.301	0.183	0.313
	(-0.4,0.0,0.0,0.2)	0.687	0.573	0.703	0.815	0.715	0.815	0.903	0.823	0.902
	(-0.2,-0.2,0.0,0.0)	0.388	0.230	0.374	0.478	0.281	0.437	0.586	0.358	0.561
	(-0.4,-0.2,0.0,0.0)	0.832	0.662	0.802	0.927	0.803	0.901	0.964	0.896	0.955
	(-0.6,-0.4,-0.2,0.0)	1.000	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(-0.8,-0.6,-0.4,-0.2)	1.000	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000

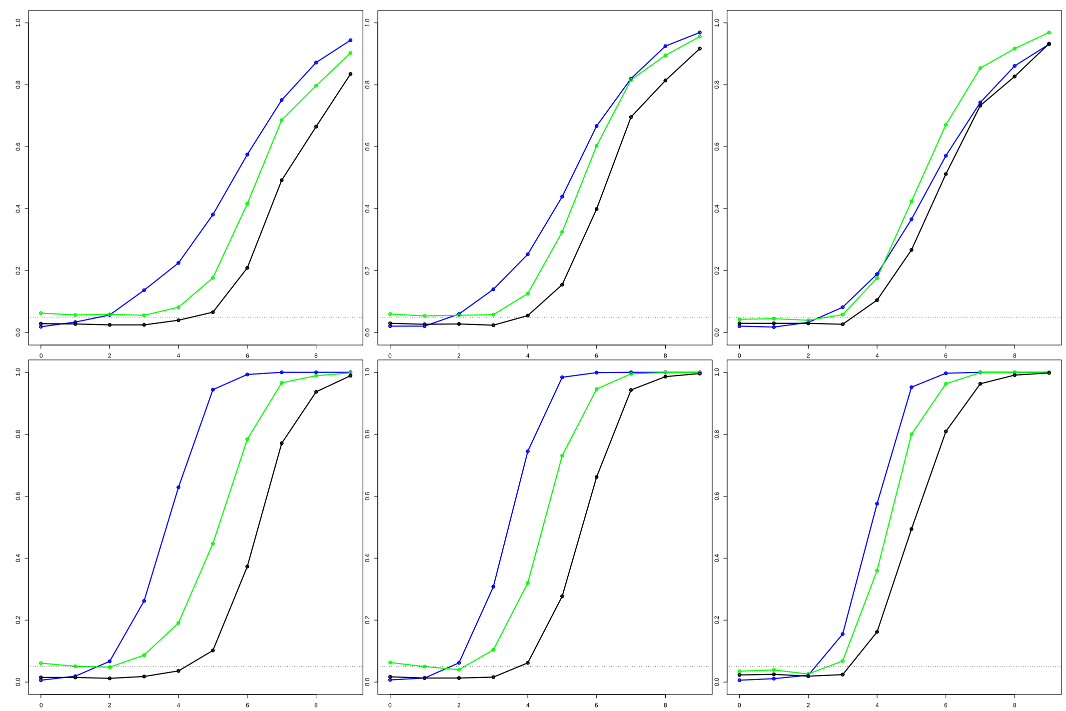

Figure S2.1: Power curves comparison for GOF tests with $k = 4$ and $n = 200$. The probability of rejecting H_0^* for δ from 0 to 9 with test statistics approaches, S_{kp} (in green using circle), W_{kp} (in blue using triangle), and the Bonferroni-corrected test (dashed line) for $\{(K_\delta, R_0, R_0)\}_{\delta=0}^{9}$ (first row) and $\{(K_\delta, K_\delta, R_0)\}_{\delta=0}^{9}$ (second row) with $p = 1$ (first column), $p = 2$ (second column), and $p = \infty$ (third column).

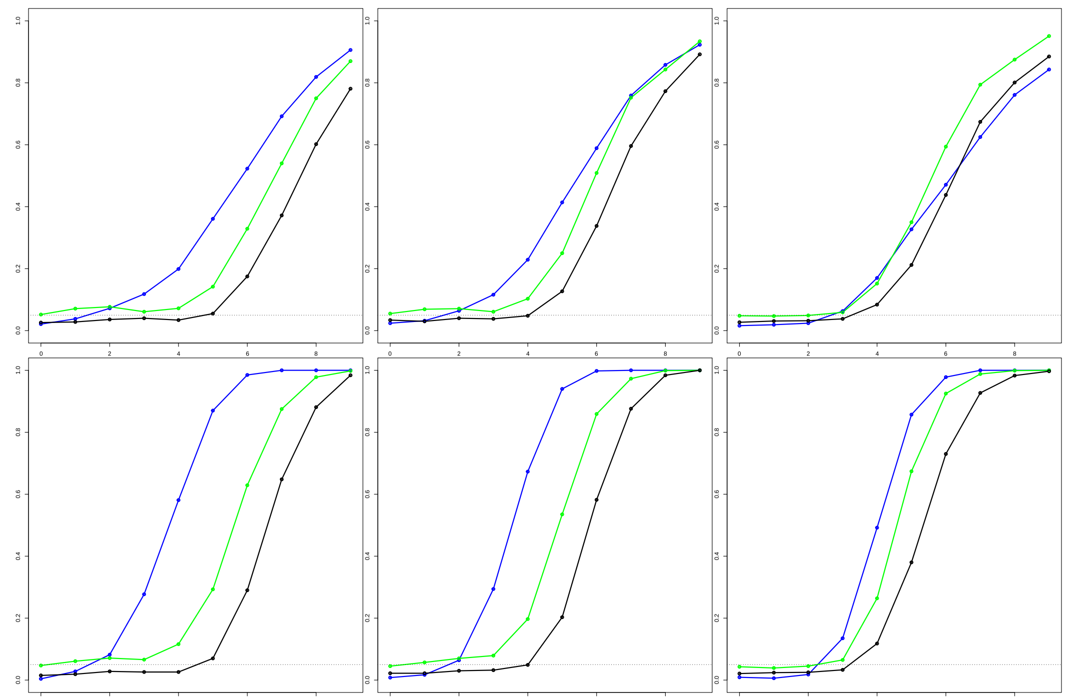

Figure S2.2: Power curves comparison for GOF tests with $k = 5$ and $n = 200$. The probability of rejecting H_0^* for δ from 0 to 9 with test statistics approaches, S_{kp} (in green using circle), W_{kp} (in blue using triangle), and the Bonferroni-corrected test (dashed line) for $\{(K_\delta, R_0, R_0, R_0)\}_{\delta=0}^{9}$ (first row) and $\{(K_\delta, K_\delta, R_0, R_0)\}_{\delta=0}^{9}$ (second row) with $p = 1$ (first column), $p = 2$ (second column), and $p = \infty$ (third column).

Table S2.20: Performance of J_p^0 evaluated with the correct rate (C), true positive average (TA), and false positive average (FA).

			$p=1$			$p=2$			$p=\infty$	
n	(q_1,q_2)	C	TA	FA	C	TA	FA	C	TA	FA
60	(0.0,0.0)	0.952	0.000	0.048	0.951	0.000	0.049	0.949	0.000	0.051
	(0.4,0.0)	0.320	0.320	0.021	0.371	0.371	0.021	0.447	0.449	0.022
	(0.8,0.0)	0.747	0.752	0.021	0.831	0.840	0.021	0.893	0.904	0.022
	(1.0,0.0)	0.866	0.874	0.021	0.915	0.929	0.021	0.951	0.969	0.022
	(0.6,0.4)	0.100	0.858	0.000	0.168	1.024	0.000	0.248	1.155	0.000
	(0.8,0.6)	0.343	1.275	0.000	0.488	1.464	0.000	0.624	1.616	0.000
	(1.0,0.8)	0.618	1.609	0.000	0.757	1.755	0.000	0.867	1.867	0.000
	(1.0,1.0)	0.738	1.731	0.000	0.870	1.869	0.000	0.946	1.946	0.000
100	(0.0,0.0)	0.949	0.000	0.051	0.941	0.000	0.059	0.953	0.000	0.047
	(0.4,0.0)	0.432	0.439	0.031	0.533	0.546	0.036	0.593	0.599	0.030
	(0.8,0.0)	0.894	0.912	0.031	0.937	0.965	0.036	0.964	0.990	0.030
	(1.0,0.0)	0.951	0.977	0.031	0.961	0.996	0.036	0.969	0.998	0.030
	(0.6,0.4)	0.340	1.280	0.000	0.471	1.446	0.000	0.579	1.569	0.000
	(0.8,0.6)	0.719	1.715	0.000	0.841	1.841	0.000	0.907	1.907	0.000
	(1.0,0.8)	0.912	1.912	0.000	0.962	1.962	0.000	0.983	1.983	0.000
	(1.0,1.0)	0.952	1.952	0.000	0.990	1.990	0.000	0.997	1.997	0.000
200	(0.0,0.0)	0.947	0.000	0.053	0.944	0.000	0.056	0.946	0.000	0.054
	(0.4,0.0)	0.801	0.806	0.023	0.871	0.883	0.025	0.920	0.936	0.022
	(0.8,0.0)	0.975	0.996	0.023	0.975	1.000	0.025	0.978	1.000	0.022
	(1.0,0.0)	0.977	1.000	0.023	0.975	1.000	0.025	0.978	1.000	0.022
	(0.6,0.4)	0.754	1.754	0.000	0.845	1.845	0.000	0.900	1.900	0.000
	(0.8,0.6)	0.965	1.965	0.000	0.986	1.986	0.000	0.992	1.992	0.000
	(1.0,0.8)	0.996	1.996	0.000	1.000	2.000	0.000	1.000	2.000	0.000
	(1.0,1.0)	1.000	2.000	0.000	1.000	2.000	0.000	1.000	2.000	0.000

Table S2.21: Performance of J_p^* evaluated with the correct rate (C), true positive average (TA), and false positive average (FA).

		$p=1$			$p=2$			$p=\infty$		
n	(q_1, q_2)	C	TA	FA	C	TA	FA	C	TA	FA
60	(0.0,0.0)	0.961	0.000	0.039	0.968	0.000	0.032	0.952	0.000	0.048
	(0.4,0.0)	0.263	0.263	0.015	0.258	0.258	0.013	0.349	0.350	0.019
	(0.8,0.0)	0.670	0.673	0.015	0.713	0.716	0.013	0.806	0.816	0.020
	(1.0,0.0)	0.811	0.815	0.015	0.858	0.863	0.013	0.899	0.911	0.018
	(0.6,0.4)	0.057	0.723	0.000	0.067	0.760	0.000	0.147	0.950	0.000
	(0.8,0.6)	0.234	1.114	0.000	0.292	1.205	0.000	0.453	1.414	0.000
	(1.0,0.8)	0.488	1.462	0.000	0.589	1.577	0.000	0.714	1.707	0.000
	(1.0,1.0)	0.618	1.607	0.000	0.715	1.707	0.000	0.823	1.820	0.000
100	(0.0,0.0)	0.996	0.000	0.004	0.997	0.000	0.003	0.995	0.000	0.005
	(0.4,0.0)	0.086	0.086	0.002	0.101	0.101	0.002	0.211	0.211	0.003
	(0.8,0.0)	0.527	0.527	0.002	0.646	0.646	0.002	0.823	0.823	0.003
	(1.0,0.0)	0.753	0.753	0.002	0.849	0.849	0.002	0.937	0.937	0.000
	(0.6,0.4)	0.011	0.411	0.000	0.020	0.494	0.000	0.093	0.811	0.000
	(0.8,0.6)	0.115	0.879	0.000	0.209	1.067	0.000	0.481	1.444	0.000
	(1.0,0.8)	0.402	1.348	0.000	0.577	1.558	0.000	0.789	1.788	0.000
	(1.0,1.0)	0.571	1.550	0.000	0.733	1.727	0.000	0.895	1.895	0.000
200	(0.0,0.0)	1.000	0.000	0.000	1.000	0.000	0.000	1.000	0.000	0.000
	(0.4,0.0)	0.030	0.030	0.000	0.041	0.041	0.000	0.167	0.167	0.000
	(0.8,0.0)	0.539	0.539	0.000	0.713	0.713	0.000	0.929	0.929	0.000
	(1.0,0.0)	0.819	0.819	0.000	0.933	0.933	0.000	0.989	0.989	0.000
	(0.6,0.4)	0.001	0.224	0.000	0.002	0.343	0.000	0.061	0.801	0.000
	(0.8,0.6)	0.051	0.722	0.000	0.150	0.991	0.000	0.567	1.552	0.000
	(1.0,0.8)	0.382	1.341	0.000	0.630	1.624	0.000	0.902	1.902	0.000
	(1.0,1.0)	0.633	1.625	0.000	0.841	1.841	0.000	0.969	1.969	0.000

Table S2.22: Performance of J_p^0 evaluated with the correct rate (C), true positive average (TA), and false positive average (FA).

n	(q_1, q_2, q_3)	$p = 1$			$p = 2$			$p = \infty$		
		C	TA	FA	C	TA	FA	C	TA	FA
60	(0.0,0.0,0.0)	0.946	0.000	0.054	0.951	0.000	0.049	0.951	0.000	0.049
	(0.4,0.0,0.0)	0.221	0.227	0.034	0.271	0.277	0.034	0.320	0.325	0.035
	(0.8,0.0,0.0)	0.655	0.670	0.034	0.762	0.784	0.034	0.835	0.863	0.035
	(1.0,0.0,0.0)	0.812	0.833	0.034	0.882	0.911	0.034	0.924	0.956	0.035
	(0.6,0.4,0.0)	0.052	0.693	0.017	0.100	0.843	0.015	0.184	1.012	0.014
	(0.8,0.6,0.0)	0.257	1.129	0.017	0.402	1.348	0.015	0.533	1.508	0.014
	(1.0,0.8,0.0)	0.524	1.498	0.017	0.691	1.688	0.015	0.810	1.815	0.014
	(1.0,1.0,0.0)	0.669	1.658	0.017	0.827	1.832	0.015	0.914	1.925	0.014
	(1.0,0.8,0.6)	0.215	1.986	0.000	0.389	2.286	0.000	0.544	2.499	0.000
	(1.0,0.8,0.8)	0.331	2.193	0.000	0.538	2.493	0.000	0.709	2.692	0.000
	(1.0,1.0,1.0)	0.555	2.501	0.000	0.763	2.750	0.000	0.891	2.889	0.000
100	(0.0,0.0,0.0)	0.947	0.000	0.053	0.950	0.000	0.050	0.961	0.000	0.039
	(0.4,0.0,0.0)	0.388	0.397	0.034	0.479	0.488	0.031	0.548	0.561	0.024
	(0.8,0.0,0.0)	0.869	0.891	0.034	0.934	0.959	0.031	0.961	0.983	0.024
	(1.0,0.0,0.0)	0.947	0.977	0.034	0.965	0.996	0.031	0.976	1.000	0.024
	(0.6,0.4,0.0)	0.247	1.135	0.017	0.361	1.312	0.015	0.477	1.451	0.010
	(0.8,0.6,0.0)	0.637	1.633	0.017	0.788	1.792	0.015	0.855	1.860	0.010
	(1.0,0.8,0.0)	0.879	1.889	0.017	0.948	1.959	0.015	0.973	1.981	0.010
	(1.0,1.0,0.0)	0.936	1.949	0.017	0.976	1.990	0.015	0.989	1.998	0.010
	(1.0,0.8,0.6)	0.599	2.579	0.000	0.764	2.761	0.000	0.846	2.846	0.000
	(1.0,0.8,0.8)	0.789	2.784	0.000	0.914	2.914	0.000	0.960	2.960	0.000
	(1.0,1.0,1.0)	0.919	2.919	0.000	0.985	2.985	0.000	0.996	2.996	0.000
200	(0.0,0.0,0.0)	0.948	0.000	0.052	0.943	0.000	0.057	0.943	0.000	0.057
	(0.4,0.0,0.0)	0.710	0.730	0.033	0.802	0.826	0.037	0.855	0.885	0.038
	(0.8,0.0,0.0)	0.965	0.997	0.033	0.963	1.000	0.037	0.962	1.000	0.038
	(1.0,0.0,0.0)	0.967	1.000	0.033	0.963	1.000	0.037	0.962	1.000	0.038
	(0.6,0.4,0.0)	0.668	1.669	0.021	0.783	1.787	0.021	0.868	1.877	0.021
	(0.8,0.6,0.0)	0.942	1.952	0.021	0.963	1.982	0.021	0.973	1.993	0.021
	(1.0,0.8,0.0)	0.977	1.996	0.021	0.979	2.000	0.021	0.979	2.000	0.021
	(1.0,1.0,0.0)	0.979	2.000	0.021	0.979	2.000	0.021	0.979	2.000	0.021
	(1.0,0.8,0.6)	0.956	2.956	0.000	0.988	2.988	0.000	0.995	2.995	0.000
	(1.0,0.8,0.8)	0.992	2.992	0.000	0.999	2.999	0.000	1.000	3.000	0.000
	(1.0,1.0,1.0)	1.000	3.000	0.000	1.000	3.000	0.000	1.000	3.000	0.000

Table S2.23: Performance of J_p^* evaluated with the correct rate (C), true positive average (TA), and false positive average (FA).

			$p=1$			$p=2$			$p=\infty$	
n	(q_1,q_2,q_3)	C	TA	FA	C	TA	FA	C	TA	FA
60	(0.0,0.0,0.0)	0.938	0.000	0.062	0.950	0.000	0.050	0.930	0.000	0.070
	(0.4,0.0,0.0)	0.217	0.222	0.040	0.225	0.230	0.032	0.296	0.307	0.044
	(0.8,0.0,0.0)	0.637	0.655	0.040	0.687	0.704	0.032	0.776	0.813	0.043
	(1.0,0.0,0.0)	0.779	0.804	0.039	0.841	0.864	0.031	0.876	0.917	0.043
	(0.6,0.4,0.0)	0.050	0.681	0.018	0.056	0.712	0.014	0.151	0.933	0.024
	(0.8,0.6,0.0)	0.237	1.107	0.018	0.294	1.188	0.014	0.465	1.420	0.023
	(1.0,0.8,0.0)	0.490	1.457	0.017	0.587	1.572	0.014	0.711	1.713	0.023
	(1.0,1.0,0.0)	0.613	1.596	0.017	0.717	1.711	0.014	0.826	1.838	0.022
	(1.0,0.8,0.6)	0.184	1.922	0.000	0.273	2.093	0.000	0.431	2.343	0.000
	(1.0,0.8,0.8)	0.295	2.132	0.000	0.404	2.308	0.000	0.588	2.549	0.000
	(1.0,1.0,1.0)	0.493	2.424	0.000	0.630	2.592	0.000	0.777	2.767	0.000
100	(0.0,0.0,0.0)	0.995	0.000	0.005	0.996	0.000	0.004	0.993	0.000	0.007
	(0.4,0.0,0.0)	0.108	0.108	0.002	0.123	0.123	0.002	0.211	0.212	0.004
	(0.8,0.0,0.0)	0.556	0.556	0.002	0.660	0.660	0.002	0.825	0.826	0.004
	(1.0,0.0,0.0)	0.755	0.755	0.002	0.853	0.854	0.002	0.948	0.951	0.003
	(0.6,0.4,0.0)	0.014	0.413	0.000	0.019	0.484	0.000	0.092	0.807	0.001
	(0.8,0.6,0.0)	0.125	0.887	0.000	0.208	1.060	0.000	0.475	1.432	0.001
	(1.0,0.8,0.0)	0.394	1.335	0.000	0.559	1.538	0.000	0.808	1.805	0.001
	(1.0,1.0,0.0)	0.565	1.542	0.000	0.735	1.726	0.000	0.902	1.902	0.000
	(1.0,0.8,0.6)	0.079	1.633	0.000	0.159	1.898	0.000	0.417	2.347	0.000
	(1.0,0.8,0.8)	0.177	1.867	0.000	0.334	2.182	0.000	0.635	2.617	0.000
	(1.0,1.0,1.0)	0.419	2.315	0.000	0.627	2.586	0.000	0.851	2.848	0.000
200	(0.0,0.0,0.0)	0.999	0.000	0.001	0.999	0.000	0.001	0.999	0.000	0.001
	(0.4,0.0,0.0)	0.040	0.040	0.001	0.051	0.051	0.001	0.164	0.165	0.001
	(0.8,0.0,0.0)	0.525	0.526	0.001	0.686	0.687	0.001	0.921	0.922	0.001
	(1.0,0.0,0.0)	0.790	0.791	0.001	0.925	0.926	0.001	0.985	0.986	0.001
	(0.6,0.4,0.0)	0.000	0.213	0.001	0.002	0.329	0.001	0.063	0.781	0.001
	(0.8,0.6,0.0)	0.045	0.707	0.001	0.136	0.960	0.001	0.557	1.544	0.001
	(1.0,0.8,0.0)	0.357	1.308	0.001	0.614	1.607	0.001	0.892	1.892	0.001
	(1.0,1.0,0.0)	0.610	1.595	0.001	0.840	1.839	0.001	0.969	1.969	0.001
	(1.0,0.8,0.6)	0.035	1.530	0.000	0.137	1.925	0.000	0.579	2.564	0.000
	(1.0,0.8,0.8)	0.144	1.862	0.000	0.391	2.329	0.000	0.825	2.822	0.000
	(1.0,1.0,1.0)	0.477	2.419	0.000	0.771	2.768	0.000	0.957	2.957	0.000

Table S2.24: Performance of J_p^0 evaluated with the correct rate (C), true positive average (TA), and false positive average (FA).

		$p=1$			$p=2$			$p=\infty$		
n	(q_1,q_2,q_3,q_4)	C	TA	FA	C	TA	FA	C	TA	FA
60	(0.0,0.0,0.0,0.0)	0.946	0.000	0.054	0.949	0.000	0.051	0.952	0.000	0.048
	(0.4,0.0,0.0,0.0)	0.216	0.221	0.041	0.265	0.271	0.039	0.329	0.333	0.035
	(0.8,0.0,0.0,0.0)	0.624	0.640	0.041	0.735	0.757	0.039	0.808	0.834	0.035
	(1.0,0.0,0.0,0.0)	0.777	0.801	0.041	0.857	0.887	0.039	0.913	0.944	0.035
	(0.6,0.4,0.0,0.0)	0.040	0.636	0.024	0.074	0.786	0.023	0.135	0.947	0.022
	(0.8,0.6,0.0,0.0)	0.192	1.043	0.024	0.337	1.270	0.023	0.466	1.439	0.022
	(0.8,0.8,0.0,0.0)	0.343	1.261	0.024	0.550	1.522	0.023	0.681	1.688	0.022
	(1.0,1.0,0.0,0.0)	0.600	1.586	0.024	0.771	1.779	0.023	0.881	1.900	0.020
	(1.0,1.0,0.8,0.0)	0.355	2.234	0.007	0.588	2.555	0.007	0.763	2.760	0.008
	(1.0,1.0,0.8,0.8)	0.188	2.846	0.000	0.427	3.300	0.000	0.647	3.606	0.000
	(1.0,1.0,1.0,1.0)	0.352	3.183	0.000	0.623	3.578	0.000	0.814	3.807	0.000
100	(0.0,0.0,0.0,0.0)	0.949	0.000	0.051	0.948	0.000	0.052	0.957	0.000	0.043
	(0.4,0.0,0.0,0.0)	0.337	0.349	0.037	0.432	0.445	0.038	0.513	0.522	0.031
	(0.8,0.0,0.0,0.0)	0.846	0.873	0.037	0.917	0.949	0.038	0.952	0.981	0.031
	(1.0,0.0,0.0,0.0)	0.924	0.954	0.037	0.958	0.996	0.038	0.969	1.000	0.031
	(0.6,0.4,0.0,0.0)	0.185	1.033	0.025	0.303	1.247	0.023	0.405	1.380	0.020
	(0.8,0.6,0.0,0.0)	0.569	1.569	0.025	0.738	1.751	0.023	0.839	1.853	0.020
	(0.8,0.8,0.0,0.0)	0.742	1.755	0.025	0.884	1.902	0.023	0.943	1.961	0.020
	(1.0,1.0,0.0,0.0)	0.904	1.925	0.025	0.963	1.986	0.023	0.976	1.996	0.020
	(1.0,1.0,0.8,0.0)	0.784	2.783	0.014	0.920	2.926	0.012	0.962	2.969	0.010
	(1.0,1.0,0.8,0.8)	0.665	3.654	0.000	0.878	3.876	0.000	0.948	3.948	0.000
	(1.0,1.0,1.0,1.0)	0.842	3.841	0.000	0.968	3.968	0.000	0.989	3.989	0.000
200	(0.0,0.0,0.0,0.0)	0.941	0.000	0.059	0.945	0.000	0.055	0.947	0.000	0.053
	(0.4,0.0,0.0,0.0)	0.664	0.689	0.047	0.763	0.794	0.046	0.823	0.858	0.045
	(0.8,0.0,0.0,0.0)	0.949	0.992	0.047	0.954	1.000	0.046	0.955	1.000	0.045
	(1.0,0.0,0.0,0.0)	0.953	1.000	0.047	0.954	1.000	0.046	0.955	1.000	0.045
	(0.6,0.4,0.0,0.0)	0.630	1.640	0.030	0.761	1.777	0.032	0.832	1.854	0.033
	(0.8,0.6,0.0,0.0)	0.920	1.940	0.030	0.953	1.982	0.032	0.958	1.991	0.033
	(0.8,0.8,0.0,0.0)	0.956	1.986	0.030	0.965	1.997	0.032	0.967	2.000	0.033
	(1.0,1.0,0.0,0.0)	0.970	2.000	0.030	0.968	2.000	0.032	0.967	2.000	0.033
	(1.0,1.0,0.8,0.0)	0.983	2.996	0.014	0.982	2.999	0.017	0.982	3.000	0.018
	(1.0,1.0,0.8,0.8)	0.992	3.992	0.000	0.998	3.998	0.000	1.000	4.000	0.000
	(1.0,1.0,1.0,1.0)	1.000	4.000	0.000	1.000	4.000	0.000	1.000	4.000	0.000

Table S2.25: Performance of J_p^* evaluated with the correct rate (C), true positive average (TA), and false positive average (FA).

		$p=1$			$p=2$			$p=\infty$		
n	(q_1, q_2, q_3, q_4)	C	TA	FA	C	TA	FA	C	TA	FA
60	(0.0,0.0,0.0,0.0)	0.919	0.000	0.081	0.934	0.000	0.066	0.909	0.000	0.091
	(0.4,0.0,0.0,0.0)	0.242	0.252	0.062	0.245	0.256	0.050	0.326	0.345	0.072
	(0.8,0.0,0.0,0.0)	0.626	0.656	0.062	0.687	0.714	0.051	0.758	0.807	0.067
	(1.0,0.0,0.0,0.0)	0.758	0.799	0.063	0.816	0.851	0.049	0.843	0.901	0.064
	(0.6,0.4,0.0,0.0)	0.045	0.685	0.039	0.054	0.723	0.031	0.119	0.908	0.044
	(0.8,0.6,0.0,0.0)	0.206	1.084	0.038	0.272	1.194	0.033	0.435	1.403	0.043
	(0.8,0.8,0.0,0.0)	0.364	1.302	0.038	0.453	1.418	0.032	0.598	1.610	0.042
	(1.0,1.0,0.0,0.0)	0.599	1.599	0.038	0.700	1.710	0.030	0.795	1.826	0.036
	(1.0,1.0,0.8,0.0)	0.382	2.271	0.017	0.497	2.439	0.014	0.671	2.649	0.015
	(1.0,1.0,0.8,0.8)	0.217	2.919	0.000	0.324	3.131	0.000	0.540	3.448	0.000
	(1.0,1.0,1.0,1.0)	0.380	3.229	0.000	0.521	3.431	0.000	0.702	3.670	0.000
100	(0.0,0.0,0.0,0.0)	0.992	0.000	0.008	0.993	0.000	0.007	0.989	0.000	0.011
	(0.4,0.0,0.0,0.0)	0.118	0.118	0.007	0.135	0.135	0.006	0.233	0.235	0.010
	(0.8,0.0,0.0,0.0)	0.538	0.542	0.007	0.649	0.653	0.006	0.815	0.822	0.009
	(1.0,0.0,0.0,0.0)	0.757	0.761	0.007	0.860	0.865	0.006	0.937	0.945	0.008
	(0.6,0.4,0.0,0.0)	0.009	0.415	0.004	0.012	0.482	0.004	0.091	0.811	0.005
	(0.8,0.6,0.0,0.0)	0.126	0.865	0.004	0.207	1.050	0.004	0.456	1.419	0.006
	(0.8,0.8,0.0,0.0)	0.259	1.124	0.004	0.394	1.329	0.004	0.674	1.670	0.007
	(1.0,1.0,0.0,0.0)	0.562	1.543	0.004	0.741	1.739	0.004	0.899	1.904	0.006
	(1.0,1.0,0.8,0.0)	0.268	2.077	0.004	0.466	2.389	0.003	0.724	2.708	0.003
	(1.0,1.0,0.8,0.8)	0.118	2.621	0.000	0.276	3.048	0.000	0.587	3.537	0.000
	(1.0,1.0,1.0,1.0)	0.286	3.065	0.000	0.512	3.437	0.000	0.799	3.791	0.000
200	(0.0,0.0,0.0,0.0)	0.999	0.000	0.001	0.999	0.000	0.001	0.999	0.000	0.001
	(0.4,0.0,0.0,0.0)	0.022	0.022	0.001	0.030	0.030	0.001	0.164	0.164	0.001
	(0.8,0.0,0.0,0.0)	0.551	0.551	0.001	0.702	0.702	0.001	0.920	0.920	0.001
	(1.0,0.0,0.0,0.0)	0.810	0.810	0.001	0.929	0.929	0.001	0.989	0.989	0.001
	(0.6,0.4,0.0,0.0)	0.001	0.225	0.000	0.001	0.336	0.000	0.065	0.815	0.000
	(0.8,0.6,0.0,0.0)	0.049	0.736	0.000	0.136	0.975	0.000	0.573	1.564	0.000
	(0.8,0.8,0.0,0.0)	0.217	1.081	0.000	0.438	1.402	0.000	0.847	1.846	0.000
	(1.0,1.0,0.0,0.0)	0.648	1.637	0.000	0.849	1.849	0.000	0.972	1.972	0.000
	(1.0,1.0,0.8,0.0)	0.308	2.167	0.000	0.574	2.548	0.000	0.905	2.903	0.000
	(1.0,1.0,0.8,0.8)	0.113	2.670	0.000	0.326	3.197	0.000	0.812	3.805	0.000
	(1.0,1.0,1.0,1.0)	0.366	3.215	0.000	0.715	3.694	0.000	0.947	3.947	0.000

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