VOLATILITY ANALYSIS WITH HIGH-FREQUENCY AND LOW-FREQUENCY HISTORICAL DATA, AND OPTIONS-IMPLIED INFORMATION

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Supplementary Material

The online Supplementary Material contains the proofs of the proposition and theorems, and autocorrelation function (ACF) plots for the residuals of each model during the first, second, and whole period are presented in Figure 1 and 3, repectively.

S1 Proof of Proposition 1

Proof of Proposition 1. We define

$$R(k) = \int_{i-1}^{i} \frac{(i-t)^k}{k!}\sigma_t^2 dt = \frac{\sigma_{i-1}^2}{(k+1)!} + \int_{i-1}^{i} \frac{(i-t)^{k+1}}{(k+1)!} d\sigma_t^2$$

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for any $k, i \in \mathbb{N}$. From Definition 1 and Itô lemma, it holds that

$$d\sigma_t^2 = [\omega + (\gamma - 1)\sigma_{i-1}^2 + \alpha O_{i-1} + \beta\sigma_t^2]dt + 2\beta\left(\int_{i-1}^t \sigma_s dB_s\right)\sigma_t dB_t,$$

for $i - 1 < t < i$,

$$\begin{aligned} R(k) =& \frac{\omega}{(k+2)!} + \frac{(k+\gamma+1)\sigma_{i-1}^2}{(k+2)!} + \frac{\alpha O_{i-1}}{(k+2)!} + \beta R(k+1) \\ & + 2\beta\int_{i-1}^i \frac{(i-t)^{k+1}}{(k+1)!}\left(\int_{i-1}^t \sigma_s dB_s\right)\sigma_t dB_t \\ & := R_1(k) + R_2(k) + R_3(k) + \beta R(k+1), \end{aligned} \tag{S1.1}$$

where

$$R_1(k) = \frac{\omega + \alpha O_{i-1}}{(k+2)!}, \qquad \text{and} \qquad R_2(k) = \frac{(k+\gamma+1)\sigma_{i-1}^2}{(k+2)!},$$
$$R_3(k) = 2\beta\int_{i-1}^i \frac{(i-t)^{k+1}}{(k+1)!}\left(\int_{i-1}^t \sigma_s dB_s\right)\sigma_t dB_t.$$

Since $R(k) \leq \int_{i-1}^i \sigma_t^2 dt$ and $0 < \beta < 1$, it holds that $\beta^k R(k) \to 0$ almost surely as $k \to \infty$ and, by iterating the formula at (S1.1), we have

$$R(0) = \int_{i-1}^i \sigma_t^2 dt = \sum_{k=0}^{\infty} \beta^k R_1(k) + \sum_{k=0}^{\infty} \beta^k R_2(k) + \sum_{k=0}^{\infty} \beta^k R_3(k) \tag{S1.2}$$

with probability one. By some algebra, it can be verified that

$$\begin{aligned} \sum_{k=0}^{\infty} \beta^k R_1(k) &= \beta^{-2}(e^\beta - 1 - \beta)(\omega + \alpha O_{i-1}), \\ \sum_{k=0}^{\infty} \beta^k R_2(k) &= [\beta^{-2}(\gamma - 1)(e^\beta - 1 - \beta) + \beta^{-1}(e^\beta - 1)]\sigma_{i-1}^2, \\ \sum_{k=0}^{\infty} \beta^k R_3(k) &= 2\int_{i-1}^i (e^{(i-t)\beta} - 1)\int_{i-1}^t \sigma_s dB_s \sigma_t dB_t \\ &:= D_i. \end{aligned} \tag{S1.3}$$

Moreover, the instantaneous volatility σ_t^2 at integer time point i has the form of $\sigma_i^2 = \omega + \gamma\sigma_{i-1}^2 + \alpha O_{i-1} + \beta Z_i^2$, where

$$Z_i = \int_{i-1}^{i} \sigma_s dB_s.$$

This, together with (S1.2) and (S1.3), implies the equation at (2.4).

(b) It is an immediate consequence of $E[D_i|\mathcal{F}_{i-1}] = 0$.

(c) By Itô's lemma we conclude

$$\begin{aligned} E\left[Z_i^2\right] &= 2E\left[\int_{i-1}^{i}\int_{i-1}^{t} \sigma_s dB_s \sigma_t dB_t\right] + E\left[\int_{i-1}^{i} \sigma_t^2 dt\right] \\ &= E\left[g_i(\theta)\right], \end{aligned}$$

where the last equality is due to the fact that $\int_{i-1}^{i}\int_{i-1}^{t} \sigma_{\mathrm{s}} dB_{\mathrm{s}} \sigma_{\mathrm{t}} dB_t$ is a martingale difference. Then, we have

$$E\left[g_i(\theta)\right] = \omega^g + (\gamma + \beta^g)\, E\left[g_{i-1}(\theta)\right] + (\eta^g + \xi^g)E[O_i] = \frac{\omega^g + (\eta^g + \xi^g)E[O_i]}{1 - \gamma - \beta^g}.$$

Similarly, we can show that

$$\begin{aligned} E\left[\sigma_i^2\right] &= \omega + \gamma E\left[\sigma_{i-1}^2\right] + \alpha E[O_i] + \beta \frac{\omega^g + (\eta^g + \xi^g)E[O_i]}{1 - \gamma - \beta^g} \\ &= \frac{\omega(1 - \beta^g - \gamma) + \beta\omega^g + \left[\alpha(1 - \beta^g - \gamma) + \beta(\eta^g + \xi^g)\right] E[O_i]}{(1 - \beta^g - \gamma)(1 - \gamma)}. \end{aligned}$$

□

S2 Proof of Theorem 1 - 2

Proof of Theorem 1. Let

$$\begin{aligned}
\widehat{L}_{n,m}(\theta) &= -\frac{1}{2n}\sum_{i=1}^{n}\log(g_i(\theta)) - \frac{1}{2n}\sum_{i=1}^{n}\frac{RV_i}{g_i(\theta)} \equiv -\frac{1}{2n}\sum_{i=1}^{n}\widehat{\ell}_i(\theta) \quad \textit{and} \\
\widehat{\psi}_{n,m}(\theta) &= \frac{\partial \widehat{L}_{n,m}(\theta)}{\partial\theta}, \\
\widehat{L}_n(\theta) &= -\frac{1}{2n}\sum_{i=1}^{n}\log(g_i(\theta)) - \frac{1}{2n}\sum_{i=1}^{n}\frac{\int_{i-1}^{i}\sigma_t^2 dt}{g_i(\theta)} \quad \textit{and} \quad \widehat{\psi}_n(\theta) = \frac{\partial \widehat{L}_n(\theta)}{\partial\theta}, \\
L_n(\theta) &= -\frac{1}{2n}\sum_{i=1}^{n}\log(g_i(\theta)) - \frac{1}{2n}\sum_{i=1}^{n}\frac{g_i(\theta_0)}{g_i(\theta)} \quad \textit{and} \quad \psi_n(\theta) = \frac{\partial L_n(\theta)}{\partial\theta},
\end{aligned}$$

Let (ω_l^g, ω_u^g), (β_l^g, β_u^g), (η_l^g, η_u^g), and (ξ_l^g, ξ_u^g) be the lower bound and the upper bound of ω^g, β^g, η^g and ξ^g. To ease notations, we denote derivatives of any function g at x_0 by

$$\frac{\partial g(x_0)}{\partial x} = \frac{\partial g(x)}{\partial x}\bigg|_{x=x_0}.$$

(a) First, we prove that there is a unique maximizer of $L_n(\theta)$. By the definition of $L_n(\theta)$, we have

$$\max_{\theta\in\Theta} L_n(\theta) \le -\frac{1}{2n}\sum_{i=1}^{n}\min_{\theta_i\in\Theta}\left(\log(g_i(\theta_i)) + \frac{g_i(\theta_0)}{g_i(\theta_i)}\right).$$

If θ_{0i} satisfies $g_i(\theta_{0i}) = g_i(\theta_0)$, θ_{0i} is the minimizer of $\log(g_i(\theta_i)) + \frac{g_i(\theta_0)}{g_i(\theta_i)}$. Thus, if $\theta^* \in \Theta$ satisfies $g_i(\theta^*) = g_i(\theta_0)$ for all $i = 1, 2, \ldots, n$, θ^* is the maximizer of $L_n(\theta)$. Note that θ_0 is one of the candidates θ^*. Below we show that θ^* must be equal θ_0 a.s. Since

$$g_i(\theta) = \omega^g + \gamma g_{i-1}(\theta) + \beta^g Z_{i-1}^2 + \eta^g O_{i-1} + \xi^g O_{i-2},$$

θ^* and θ_0 satisfy the following equation,

$$\begin{pmatrix} 1 & g_1(\theta_0) & Z_1^2 & O_1 & O_0 \\ 1 & g_2(\theta_0) & Z_2^2 & O_2 & O_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & g_{n-1}(\theta_0) & Z_{n-1}^2 & O_{n-1} & O_{n-2} \end{pmatrix} \begin{pmatrix} \omega^{*g} - \omega_0^g \\ \gamma^* - \gamma_0 \\ \beta^{*g} - \beta_0^g \\ \eta^{*g} - \eta_0^g \\ \xi^{*g} - \xi_0^g \end{pmatrix} \equiv \mathbf{M} \begin{pmatrix} \omega^{*g} - \omega_0^g \\ \gamma^* - \gamma_0 \\ \beta^{*g} - \beta_0^g \\ \eta^{*g} - \eta_0^g \\ \xi^{*g} - \xi_0^g \end{pmatrix} = \mathbf{0} \quad \text{a.s.},$$

where $\omega^{*g} = \beta^{*-1}(e^{\beta^*} - 1)\omega^*$, $\beta^{*g} = \beta^{*-1}(\gamma^* - 1)(e^{\beta^*} - 1 - \beta^*) + e^{\beta^*} - 1$, $\eta^{*g} = \beta^{*-2}(e^{\beta^*} - 1 - \beta^*)\alpha^*$, and $\xi^{*g} = [\beta^{*-1}(e^{\beta^*} - 1) - \beta^{*-2}(e^{\beta^*} - 1 - \beta^*)]\alpha^*$. Since Z_i's and O_i's are nondegenerate, $\mathbf{M}$ is of full rank, which implies that $\mathbf{M}^\top\mathbf{M}$ is invertible and

$$\begin{pmatrix} \omega^{*g} - \omega_0^g \\ \gamma^* - \gamma_0 \\ \beta^{*g} - \beta_0^g \\ \eta^{*g} - \eta_0^g \\ \xi^{*g} - \xi_0^g \end{pmatrix} = \mathbf{0} \quad \text{a.s.}$$

For given γ, β^g is strictly increasing function with respect to β and for given β, both η^g and ξ^{*g} are strictly increasing function with respect to α. Then, we have $\theta^* = \theta_0$, i.e., there is a unique maximizer of $L_n(\theta)$. Then, since

$L_n(\theta)$ is a continuous function, for any $\varepsilon > 0$, there is a constant c, such that

$$L_n(\theta_0) - \max_{\theta\in\Theta:\|\theta-\theta_0\|_{max}\geq\varepsilon} L_n(\theta) > c \quad \text{a.s.}$$

With the help of Theorem 1 in Xiu (2010) and Lemma 2, we can derive the conclusion.

(b) Applying Taylor expansion and Rolle mean value theorem, we have

$$\widehat{\psi}_{n,m}(\widehat{\theta}) - \widehat{\psi}_{n,m}(\theta_0) = -\widehat{\psi}_{n,m}(\theta_0) = \nabla\widehat{\psi}_{n,m}(\theta^*)(\widehat{\theta} - \theta_0)$$

where θ^* is between θ_0 and $\widehat{\theta}$. According to Lemma 3 (b), $-\nabla\psi_n(\theta_0)$ is a positive matrix. If we can show $-\nabla\widehat{\psi}_{n,m}(\theta^*) \xrightarrow{p} -\nabla\psi_n(\theta_0)$, the convergence rate of $\widehat{\theta} - \theta_0$ is the same as that of $\widehat{\psi}_{n,m}(\theta_0)$.

We first show that

$$\widehat{\psi}_{n,m}(\theta_0) = O_p(m^{-1/4}) + O_p(n^{-1/2}).$$

For any $j \in \{1, 2, 3, 4\}$, by Lemma 1 and Hölder's inequality, we have

$$\begin{aligned}\left\|\widehat{\psi}_{n,m}(\theta_0) - \widehat{\psi}_n(\theta_0)\right\|_{L_1} =& \left\|\frac{1}{2n}\sum_{i=1}^{n}\frac{\partial g_i(\theta_0)}{\partial\theta_j}g_i(\theta_0)^{-2}\left(RV_i - \int_{i-1}^{i}\sigma_s^2dt\right)\right\|_{L_1}\\ \leq& C\frac{1}{n}\sum_{i-1}^{n}\left\|\frac{\partial g_i(\theta_0)}{\partial\theta_j}g_i(\theta_0)^{-1}\right\|_{L_q}\left\|RV_i - \int_{i=1}^{i}\sigma_s^2dt\right\|_{L_p}\\ \leq& Cm^{-1/4} \end{aligned} \tag{S2.4}$$

where $1 < p \leq 1 + \delta$ and $1/p + 1/q = 1$ and the last inequality is due to

Assumption 1 (g). Then, we have

$$\widehat{\psi}_{n,m}(\theta_0) = \frac{1}{2n}\sum_{i=1}^{n}\frac{\partial g_i(\theta_0)}{\partial\theta}g_i(\theta_0)^{-1}\frac{D_i}{g_i(\theta_0)} + O_p(m^{-1/4})$$

Applying Itô's lemma and Itô isometry, we have for any $j \in \{1, 2, 3, 4\}$,

$$\begin{aligned}
&E\left[\left(\frac{1}{2n}\sum_{i=1}^{n}\frac{\partial g_i(\theta_0)}{\partial\theta_j}g_i(\theta_0)^{-1}\frac{D_i}{g_i(\theta_0)}\right)^2\right] \\
=&\frac{1}{4n^2}\sum_{i=1}^{n}E\left[\left(\frac{\partial g_i(\theta_0)}{\partial\theta_j}\right)^2 g_i(\theta_0)^{-2}\frac{D_i^2}{g_i^2(\theta_0)}\right] \\
=&\frac{1}{4n^2}\sum_{i=1}^{n}E\left[\left(\frac{\partial g_i(\theta_0)}{\partial\theta_j}\right)^2 g_i(\theta_0)^{-2}\frac{E[D_i^2|\mathcal{F}_{i-1}]}{g_i^2(\theta_0)}\right] \\
\leq&C\frac{1}{n^2}\sum_{i=1}^{n}E\left[\left(\frac{\partial g_i(\theta_0)}{\partial\theta_j}\right)^2 g_i(\theta_0)^{-2}\frac{E[Z_i^4|\mathcal{F}_{i-1}]}{g_i^2(\theta_0)}\right].
\end{aligned} \tag{S2.5}$$

According to Assumption 1 (c) and Lemma 1 (b), we know that (S2.5) is of order n^{-1}. Thus, we further have

$$\widehat{\psi}_{n,m}(\theta_0) = O_p(m^{-1/4}) + O_p(n^{-1/2}).$$

Then, we show that

$$\left\|\nabla\widehat{\psi}_{n,m}(\theta^*) - \nabla\psi_n(\theta_0)\right\|_{\max} = o_p(1).$$

By the triangular inequality, we have

$$\begin{aligned}
\left\|\nabla\widehat{\psi}_{n,m}(\theta^*) - \nabla\psi_n(\theta_0)\right\|_{\max} \leq& \left\|\nabla\widehat{\psi}_{n,m}(\theta^*) - \nabla\widehat{\psi}_{n,m}(\theta_0)\right\|_{\max} \\
&+ \left\|\nabla\widehat{\psi}_{n,m}(\theta_0) - \nabla\psi_n(\theta_0)\right\|_{\max}.
\end{aligned} \tag{S2.6}$$

For the first term on the right side of (S2.6), noticing Theorem 1 (a) and Lemma 3 (a), we have

$$\left\|\nabla\widehat{\psi}_{n,m}(\theta^*)-\nabla\widehat{\psi}_{n,m}(\theta_0)\right\|_{\max}\leq\frac{C}{n}\sum_{i=1}^{n}\max_{j,k,v\in(1,2,3,4)^3}\sup_{\theta\in B(\theta_0)}\left|\frac{\partial^3\widehat{\ell}_i(\theta)}{\partial\theta_j\partial\theta_k\partial\theta_v}\right|\|\theta^*-\theta_0\|_{\max}$$
$$=o_p(1).$$

For the second term on the right side of (S2.6), similar to the proof of (S2.4), by Hölder's inequality and Lemma 1, we have

$$\left\|\nabla\widehat{\psi}_{n,m}(\theta_0)-\nabla\widehat{\psi}_{n}(\theta_0)\right\|_{\max}=O_p(m^{-1/4})$$

Therefore, we derive that

$$\nabla\widehat{\psi}_{n,m}(\theta_0)=\nabla\widehat{\psi}_{n}(\theta_0)+O_p(m^{1/4})=\nabla\psi_n(\theta_0)+\zeta_n+O_p(m^{-1/4})\quad\text{a.s.},$$

where $\zeta_n=\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{\partial^2 g_i(\theta_0)}{\partial\theta\partial\theta^\top}g_i(\theta_0)^{-1}\frac{-D_i}{g_i(\theta_0)}+\frac{\partial g_i(\theta_0)}{\partial\theta}\left(\frac{\partial g_i(\theta_0)}{\partial\theta}\right)^\top g_i(\theta_0)^{-2}\frac{2D_i}{g_i(\theta_0)}\right)$ is a martingale. Similar to the proof of (S2.5), we can show $\|\zeta_n\|_{\max}=O_p(n^{-1/2})$. Then,

$$\left\|\nabla\widehat{\psi}_{n,m}(\theta^*)-\nabla\psi_{n}(\theta_0)\right\|_{\max}=o_p(1).$$

Finally, with the help of the above two results, we have $\left\|\widehat{\theta}-\theta_0\right\|_{\max}=O_p\left(m^{-1/4}+n^{-1/2}\right)$. □

Proof of Theorem 2. Similar to Theorem 1(b),

$$\widehat{\psi}_{n,m}(\widehat{\theta})-\widehat{\psi}_{n,m}(\theta_0)=-\widehat{\psi}_{n,m}(\theta_0)=\nabla\widehat{\psi}_{n,m}(\theta^*)(\widehat{\theta}-\theta_0),$$

where θ^* is between θ_0 and $\widehat{\theta}$, we have

$$\begin{aligned}\sqrt{n}(\widehat{\theta}-\theta_0) =& -\sqrt{n}\left(\nabla\widehat{\psi}_{n,m}(\theta^*)\right)^{-1}\widehat{\psi}_{n,m}(\theta_0)\\ =& -\sqrt{n}\left(\nabla\widehat{\psi}_{n,m}(\theta^*)\right)^{-1}\left(\widehat{\psi}_n(\theta_0)+O_p(m^{-1/4})\right)\\ =& -\sqrt{n}\left(\nabla\psi_n(\theta_0)+o_p(1)\right)^{-1}\widehat{\psi}_n(\theta_0)+o_p(1),\end{aligned}$$

where the second and third equality is due to the proof of Theorem 1(b).

As $\lambda^\top \frac{\partial g_i(\theta_0)}{\partial\theta} g_i(\theta_0)^{-1}\frac{D_i}{g_i(\theta_0)}$ is stationary and ergodic, we have

$$\sqrt{n}\widehat{\psi}_n(\theta_0) = \sqrt{n}\frac{1}{2n}\sum_{i=1}^{n}\frac{\partial g_i(\theta_0)}{\partial\theta} g_i(\theta_0)^{-1}\frac{D_i}{g_i(\theta_0)} \xrightarrow{d} N(0,A),$$

by using Cramér-Wold device and the martingale central limit theorem.

On the other hand, by the proof of Theorem 1(b),

$$-\nabla\psi_n(\theta_0) = \frac{1}{2n}\sum_{i=1}^{n}\left(\frac{1}{g_i^2(\theta_0)}\frac{\partial g_i(\theta_0)}{\partial\theta}\left(\frac{\partial g_i(\theta_0)}{\partial\theta}\right)^\top\right) \to B \quad \text{in probability.}$$

Therefore,

$$\sqrt{n}(\widehat{\theta}-\theta_0) \xrightarrow{d} N(0, B^{-1}AB^{-1}).$$

□

S3 Three useful lemmas

We provide three useful lemmas.

Lemma 1. *Under Assumption 1 (a), (b), for the GARCH-Itô-OI model, we have* *(a). there exists a neighborhood $B(\theta_0)$ of θ_0 such that for any $p \geq 1$,* $\sup_{i\in\mathbb{N}} \left\| \sup_{\theta\in B(\Theta_0)} \frac{g_i(\theta_0)}{g_i(\theta)} \right\|_{L_p} < \infty$ *and $B(\theta_0) \subset \Theta$.*

(b). for any $p \geq 1$, $\sup_{i\in\mathbb{N}} \left\| \sup_{\theta\in\Theta} \frac{\partial g_i(\theta)}{\partial\theta_j} \right\|_{L_p} \leq C$, $\sup_{i\in\mathbb{N}} \left\| \sup_{\theta\in\Theta} \frac{\partial^2 g_i(\theta)}{\partial\theta_j\partial\theta_k} \right\|_{L_p} \leq C$, *and* $\sup_{i\in\mathbb{N}} \left\| \sup_{\theta\in\Theta} \frac{\partial^3 g_i(\theta)}{\partial\theta_j\partial\theta_k\partial\theta_v} \right\|_{L_p} \leq C$ *for any $j, k, v \in \{1, 2, 3, 4\}$, where $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) = (\omega, \beta, \gamma, \alpha)$;*

Proof. (a) By the iteration of $g_i(\theta)$, we have

$$g_i(\theta) = \sum_{k=0}^{i-3} (\omega^g + \beta^g Z_{i-k-1}^2 + \eta^g O_{i-k-1} + \xi^g O_{i-k-2})\gamma^k + \gamma^{i-2} g_2(\theta),$$

where

$$\begin{aligned} g_2(\theta) =& \beta^{-2}(e^\beta - 1 - \beta)(\omega + \alpha O_1) + [\beta^{-2}(\gamma - 1)(e^\beta - 1 - \beta) + \beta^{-1}(e^\beta - 1)]\sigma_1^2 \\ &< \infty. \end{aligned} \tag{S3.7}$$

Choose $s \in [0, 1]$ such that $\sup_{i\in\mathbb{N}} E(Z_i^{2ps}) < \infty$. Then, for $0 < \delta < \frac{1-\gamma_u^s}{\gamma_u^s}$, there exists a neighborhood $B(\theta_0) \in \Theta$ such that $\gamma_0 \leq (1 + \delta)\gamma$ for any $\theta \in B(\theta_0)$ and $\rho = (1+\delta)\gamma_u^s < 1$. Then, similar to the proof of Lemma 2(d) of Kim and Wang (2016), we have

$$\begin{aligned}
\sup_{\theta\in B(\theta_0)} \frac{g_i(\theta_0)}{g_i(\theta)} \leq & C + \frac{\beta_0^g}{\beta_l^g} \sum_{k=0}^{i-3} \sup_{\theta\in B(\theta_0)} \left(\frac{\gamma_0}{\gamma}\right)^k \frac{\gamma^k \beta^g Z_{i-k-1}^2}{\gamma^k \beta^g Z_{i-k-1}^2 + \omega^g} \\
\leq & C + C \sum_{k=0}^{i-3} \sup_{\theta\in B(\theta_0)} \left(\frac{\gamma_0}{\gamma}\right)^k \gamma^{ks} \left(\frac{\beta^g Z_{i-k-1}^2}{\omega^g}\right)^s \\
\leq & C + C \sum_{k=0}^{i-3} \rho^k Z_{i-k-1}^{2s},
\end{aligned}$$

where the second inequality holds by noticing that $x/(x+1) \leq x^s$ for any $x \geq 0$ and any $s \in [0,1]$. Furthermore, since $\sup_{i\in\mathbb{N}} E(Z_{i-k-1}^{2ps}) < \infty$ and $|\rho| < 1$, we have

$$\sup_{i\in\mathbb{N}} \left\| \sup_{\theta\in B(\Theta_0)} \frac{g_i(\theta_0)}{g_i(\theta)} \right\|_{L_p} < \infty.$$

(b) We first prove that the first order derivatives of $g_i(\theta)$ are finite.

$$\begin{aligned}
\frac{\partial g_i(\theta)}{\partial \alpha} = & \sum_{k=0}^{i-3} \frac{\gamma^k(\eta^g O_{i-k-1} + \xi^g O_{i-k-2})}{\alpha} + \gamma^{i-2}\frac{\partial g_2(\theta)}{\partial \alpha} \\
\leq & \sum_{k=0}^{i-3} \frac{\gamma^k(\eta^g O_{i-k-1} + \xi^g O_{i-k-2})}{\alpha} + C.
\end{aligned}$$

By noticing that $x/(x+1) \leq x^s$ for any $x \geq 0$ and any $s \in [0,1]$, we can show

$$\begin{aligned}
g_i(\theta)^{-1}\frac{\partial g_i(\theta)}{\partial \alpha} \leq & \sum_{k=0}^{i-3} \frac{\gamma^k(\eta^g O_{i-k-1} + \xi^g O_{i-k-2})}{\alpha\omega^g + \gamma^k(\eta^g O_{i-k-1} + \xi^g O_{i-k-2})} + C \\
\leq & C \sum_{k=0}^{i-3} \left(\frac{\gamma^k(\eta^g O_{i-k-1} + \xi^g O_{i-k-2})}{\alpha\omega^g}\right)^s + C \\
\leq & C \sum_{k=0}^{i-3} \gamma_u^{ks}(\eta_u^g O_{i-k-1} + \xi_u^g O_{i-k-2})^s + C.
\end{aligned}$$

Under Assumption 1 (b), we have

$$\sup_{i\in\mathbb{N}}\left\|\sup_{\theta\in\Theta}\frac{\partial g_i(\theta)}{\partial\alpha}\right\|_{L_p}\le C.$$

Applying the same argument, we can also prove that

$$\sup_{i\in\mathbb{N}}\left\|\sup_{\theta\in\Theta}\frac{\partial g_i(\theta)}{\partial\omega}\right\|_{L_p}\le C,\quad \sup_{i\in\mathbb{N}}\left\|\sup_{\theta\in\Theta}\frac{\partial g_i(\theta)}{\partial\beta}\right\|_{L_p}\le C.$$

Additionally, we have

$$\begin{aligned}\frac{\partial g_i(\theta)}{\partial\gamma}=&\sum_{k=0}^{i-3}k\gamma^{k-1}(\omega^g+\beta^g Z_{i-k-1}^2+\eta^g O_{i-k-1}+\xi^g O_{i-k-2})\\&+\sum_{k=0}^{i-3}\gamma^k\beta^{-1}(e^\beta-1-\beta)Z_{i-k-1}^2\\&+(i-2)\gamma^{i-3}g_2(\theta)+\gamma^{i-2}\frac{\partial g_2(\theta)}{\partial\gamma}\\\le&\sum_{k=0}^{i-3}k\gamma^{k-1}(\omega^g+\beta^g Z_{i-k-1}^2+\eta^g O_{i-k-1}+\xi^g O_{i-k-2})+C.\end{aligned}$$

The last inequality holds, as Z_i^2 is stationary and $|\gamma|<1$. Choose $s\in[0,1]$ such that $\sup_{i\in\mathbb{N}}E(\omega_u^g+\beta_u^g Z_{i-k-1}^2+\eta_u^g O_{i-k-1}+\xi_u^g O_{i-k-2})^s<\infty$. By noticing that $x/(x+1)\le x^s$ for any $x\ge 0$ and any $s\in[0,1]$, we can show

$$\begin{aligned}g_i(\theta)^{-1}\frac{\partial g_i(\theta)}{\partial\gamma}\le&\sum_{k=0}^{i-3}k\frac{\gamma^{k-1}(\omega^g+\beta^g Z_{i-k-1}^2+\eta^g O_{i-k-1}+\xi^g O_{i-k-2})}{\gamma^{i-2}g_2+\gamma^k(\omega^g+\beta^g Z_{i-k-1}^2+\eta^g O_{i-k-1}+\xi^g O_{i-k-2})}+C\\\le&\gamma^{-1}\sum_{k=0}^{i-3}k\frac{\gamma^{k-i+2}(\omega^g+\beta^g Z_{i-k-1}^2+\eta^g O_{i-k-1}+\xi^g O_{i-k-2})}{g_2+\gamma^{k-i+2}(\omega^g+\beta^g Z_{i-k-1}^2+\eta^g O_{i-k-1}+\xi^g O_{i-k-2})}+C\\\le&C\sum_{k=0}^{i-3}k\left[\frac{\gamma^{k-i+2}}{g_2}(\omega^g+\beta^g Z_{i-k-1}^2+\eta^g O_{i-k-1}+\xi^g O_{i-k-2})\right]^s+C\\\le&C\sum_{k=0}^{i-3}(\omega_u^g+\beta_u^g Z_{i-k-1}^2+\eta_u^g O_{i-k-1}+\xi_u^g O_{i-k-2})^s+C.\end{aligned}$$

We further have

$$\sup_{i\in\mathbb{N}}\left\|\sup_{\theta\in\Theta}\frac{\partial g_i(\theta)}{\partial\gamma}\right\|_{L_p}\le C.$$

Finally, we can similarly show the boundedness for the second order, and third order derivatives. □

Lemma 2. *Under Assumption 1 (a), (b), (d), (f) and (g), we have*

$$\sup_{\theta\in\Theta}\left|\widehat{L}_{n,m}(\theta)-L_n(\theta)\right|=O_p(m^{-1/4})+o_p(1).$$

Proof. The integrated volatilities in the GARCH-Itô-OI model and the GARCH-Itô model of Kim and Wang (2016) have the same martingale difference term. Furthermore, the exogenous option-implied information acts like a time-varying "constant" term in GARCH-Itô-OI model. Therefore, similar to the proof of Lemma 3 of Kim and Wang (2016), we can obtain the result. □

Lemma 3. *Under Assumption 1 (a), (b) and (h), we have*

(a). there exists a neighborhood $B(\theta_0)$ of θ such that $\sup_{i\in\mathbb{N}}\left\|\sup_{\theta\in B(\theta_0)}\frac{\partial^3\widehat{\ell}_i(\theta)}{\partial\theta_j\partial\theta_k\partial\theta_v}\right\|_{L_1}$ $<\infty$ *for any $j,k,v\in\{1,2,3,4\}$, where $\theta=(\theta_1,\theta_2,\theta_3,\theta_4)=(\omega,\beta,\gamma,\alpha)$.*

(b). $-\nabla\psi_n(\theta_0)$ is a positive definite matrix for $n\ge 4$.

Proof. (a) For any $j, k, v \in \{1, 2, 3, 4\}$, we have

$$\begin{aligned}
\frac{\partial^3 \widehat{\ell}_i(\theta)}{\partial\theta_j \partial\theta_k \partial\theta_v} = & \left\{1 - \frac{RV_i}{g_i(\theta)}\right\} \left\{\frac{1}{g_i(\theta)} \frac{\partial^3 g_i(\theta)}{\partial\theta_j \partial\theta_k \partial\theta_v}\right\} \\
& + \left\{2\frac{RV_i}{g_i(\theta)} - 1\right\} \left\{\frac{1}{g_i(\theta)} \frac{\partial g_i(\theta)}{\partial\theta_j}\right\} \left\{\frac{1}{g_i(\theta)} \frac{\partial^2 g_i(\theta)}{\partial\theta_k \partial\theta_v}\right\} \\
& + \left\{2\frac{RV_i}{g_i(\theta)} - 1\right\} \left\{\frac{1}{g_i(\theta)} \frac{\partial g_i(\theta)}{\partial\theta_k}\right\} \left\{\frac{1}{g_i(\theta)} \frac{\partial^2 g_i(\theta)}{\partial\theta_j \partial\theta_v}\right\} \\
& + \left\{2\frac{RV_i}{g_i(\theta)} - 1\right\} \left\{\frac{1}{g_i(\theta)} \frac{\partial g_i(\theta)}{\partial\theta_v}\right\} \left\{\frac{1}{g_i(\theta)} \frac{\partial^2 g_i(\theta)}{\partial\theta_j \partial\theta_k}\right\} \\
& + \left\{2 - 6\frac{RV_i}{g_i(\theta)}\right\} \left\{\frac{1}{g_i(\theta)} \frac{\partial g_i(\theta)}{\partial\theta_j}\right\} \left\{\frac{1}{g_i(\theta)} \frac{\partial g_i(\theta)}{\partial\theta_k}\right\} \left\{\frac{1}{g_i(\theta)} \frac{\partial g_i(\theta)}{\partial\theta_v}\right\}
\end{aligned}$$

By Assumption 1 (h),

$$E[RV_i|\mathcal{F}_{i-1}] \leq CE\left[\int_{i-1}^{i} \sigma_t^2 dt|\mathcal{F}_{i-1}\right] + C \quad a.s..$$

Then, by Lemma 1, the tower property and Hölder's inequality, we have

$$\begin{aligned}
& E\left[\sup_{\theta\in B(\theta_0)} \left|\frac{RV_i}{g_i(\theta)} \left\{\frac{1}{g_i(\theta)} \frac{\partial^3 g_i(\theta)}{\partial\theta_j \partial\theta_k \partial\theta_v}\right\}\right|\right] \\
\leq & CE\left[\sup_{\theta\in B(\theta_0)} \frac{g_i(\theta_0)}{g_i(\theta)} \left|\frac{1}{g_i(\theta)} \frac{\partial^3 g_i(\theta)}{\partial\theta_j \partial\theta_k \partial\theta_v}\right|\right] + C \\
\leq & C\left\|\sup_{\theta\in B(\theta_0)} \frac{g_i(\theta_0)}{g_i(\theta)}\right\|_{L_p} \left\|\sup_{\theta\in B(\theta_0)} \left|\frac{1}{g_i(\theta)} \frac{\partial^3 g_i(\theta)}{\partial\theta_j \partial\theta_k \partial\theta_v}\right|\right\|_{L_q} + C \leq C < \infty
\end{aligned}$$

where $1/p + 1/q = 1, p > 1$ and $q > 1$. Similarly, we can prove that other terms are also bounded.

(b) It is easy to show that

$$-\nabla\psi_n(\theta_0) = \frac{1}{2n} \sum_{i=1}^{n} \frac{\partial g_i(\theta_0)}{\partial\theta} \frac{\partial g_i(\theta_0)^\top}{\partial\theta} g_i(\theta_0)^{-2} = \frac{1}{2n} \sum_{i=1}^{n} g_{\theta,i} g_{\theta,i}^\top$$

where $g_{\theta,i} = \frac{\partial g_i(\theta_0)}{\partial \theta} g_i(\theta_0)^{-1}$. Suppose that $-\nabla\psi_n(\theta_0)$ is not a positive definite matrix. Then, there exists $\lambda \neq \mathbf{0}$ such that $\frac{1}{2n}\sum_{i=1}^n \lambda^\top g_{\theta,i} g_{\theta,i}^\top \lambda = 0$, which further implies

$$g_{\theta,i}^\top \lambda = 0 \quad a.s. \quad \text{for all} \quad i = 1, \ldots, n.$$

Since $g_i(\theta_0)$ stays away from zero, we have

$$\begin{pmatrix} \frac{\partial g_1(\theta_0)}{\partial \omega} & \frac{\partial g_1(\theta_0)}{\partial \beta} & \frac{\partial g_1(\theta_0)}{\partial \gamma} & \frac{\partial g_1(\theta_0)}{\partial \alpha} \\ \frac{\partial g_2(\theta_0)}{\partial \omega} & \frac{\partial g_2(\theta_0)}{\partial \beta} & \frac{\partial g_2(\theta_0)}{\partial \gamma} & \frac{\partial g_2(\theta_0)}{\partial \alpha} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_n(\theta_0)}{\partial \omega} & \frac{\partial g_n(\theta_0)}{\partial \beta} & \frac{\partial g_n(\theta_0)}{\partial \gamma} & \frac{\partial g_n(\theta_0)}{\partial \alpha} \end{pmatrix} \lambda = \mathbf{0} \quad a.s.,$$

where

$$\begin{aligned} \frac{\partial g_{i+1}(\theta_0)}{\partial \omega} &= \frac{\partial \omega_0^g}{\partial \omega} + \gamma \frac{\partial g_i(\theta_0)}{\partial \omega}, \\ \frac{\partial g_{i+1}(\theta_0)}{\partial \beta} &= \frac{\partial \omega_0^g}{\partial \beta} + \gamma \frac{\partial g_i(\theta_0)}{\partial \beta} + \frac{\partial \beta_0^g}{\partial \beta} Z_i^2 + \frac{\partial \eta_0^g}{\partial \beta} O_i + \frac{\partial \xi_0^g}{\partial \beta} O_{i-1}, \\ \frac{\partial g_{i+1}(\theta_0)}{\partial \gamma} &= g_i(\theta_0) + \gamma \frac{\partial g_i(\theta_0)}{\partial \gamma} + \frac{\partial \beta_0^g}{\partial \gamma} Z_i^2, \\ \frac{\partial g_{i+1}(\theta_0)}{\partial \alpha} &= \gamma \frac{\partial g_i(\theta_0)}{\partial \alpha} + \frac{\partial \eta_0^g}{\partial \alpha} O_i + \frac{\partial \xi_0^g}{\partial \alpha} O_{i-1}, \end{aligned}$$

and $\frac{\partial \omega_0^g}{\partial \omega} = \beta_0^{-1}(e^{\beta_0} - 1)$, $\frac{\partial \omega_0^g}{\partial \beta} = \beta_0^{-2}(1 - e^{\beta_0})\omega_0 + \beta_0^{-1} e^{\beta_0} \omega_0$, $\frac{\partial \beta_0^g}{\partial \beta} = (\gamma_0 - 1)(\beta_0^{-1} e^{\beta_0} - \beta_0^{-2} e^{\beta_0} + \beta_0^{-2}) + e^{\beta_0}$, $\frac{\partial \eta_0^g}{\partial \beta} = (\beta_0^{-2} e^{\beta_0} - 2\beta_0^{-3} e^{\beta_0} + 2\beta_0^{-3} + \beta_0^{-2})\alpha$, $\frac{\partial \xi_0^g}{\partial \beta} = (\beta_0^{-1} e^{\beta_0} - 2\beta_0^{-2} e^{\beta_0} + 2\beta_0^{-3} e^{\beta_0} - 2\beta_0^{-3})\alpha$, $\frac{\partial \beta_0^g}{\partial \gamma} = \beta_0^{-1}(e^{\beta_0} - 1 - \beta_0)$, $\frac{\partial \eta_0^g}{\partial \alpha} = \beta_0^{-2}(e^{\beta_0} - 1 - \beta_0)$, $\frac{\partial \xi_0^g}{\partial \alpha} = \beta_0^{-1}(e^{\beta_0} - 1) - \beta_0^{-2}(e^{\beta_0} - 1 - \beta_0)$. Since Z_i's and O_i's are nondegenerate, the matrix on the left hand side is of full rank a.s., which implies $\lambda = \mathbf{0}$ a.s. Thus, it is a contradiction to the initial assumption. $\square$

S4 Autocorrelation Function Plots during the Three Time Periods

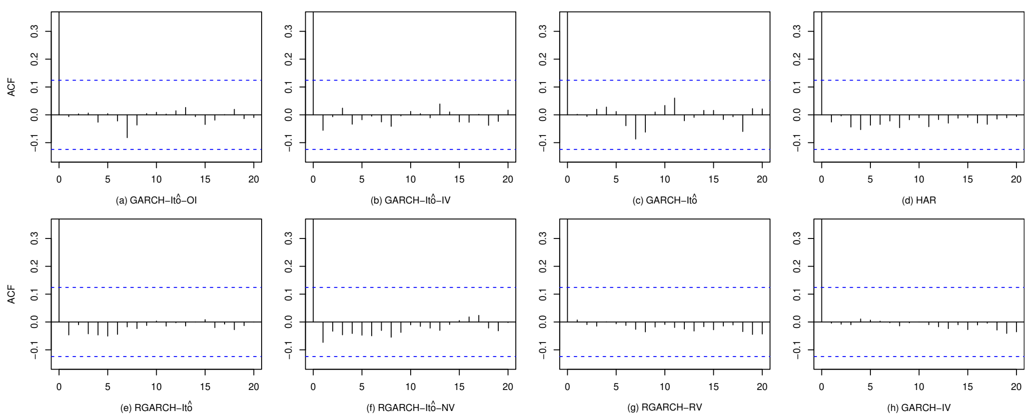

Figure 1: ACF plots for the residuals of the compared models during the first period.

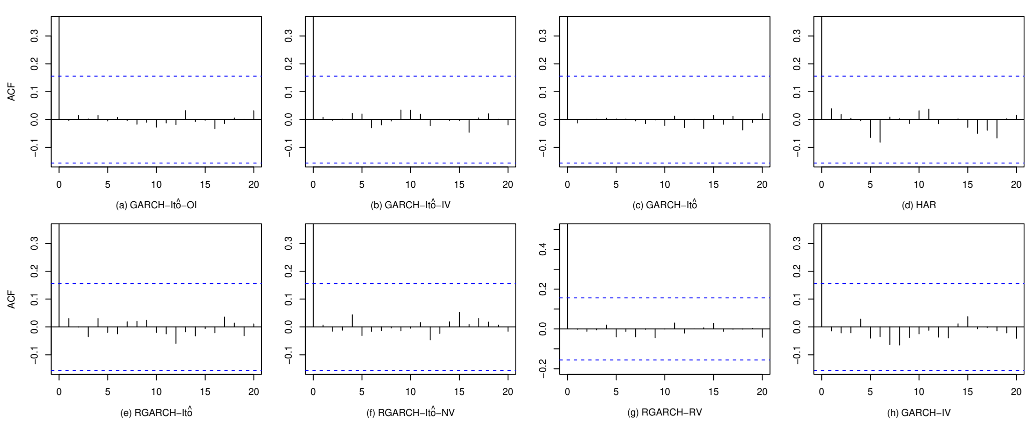

Figure 2: ACF plots for the residuals of the compared models during the second period.

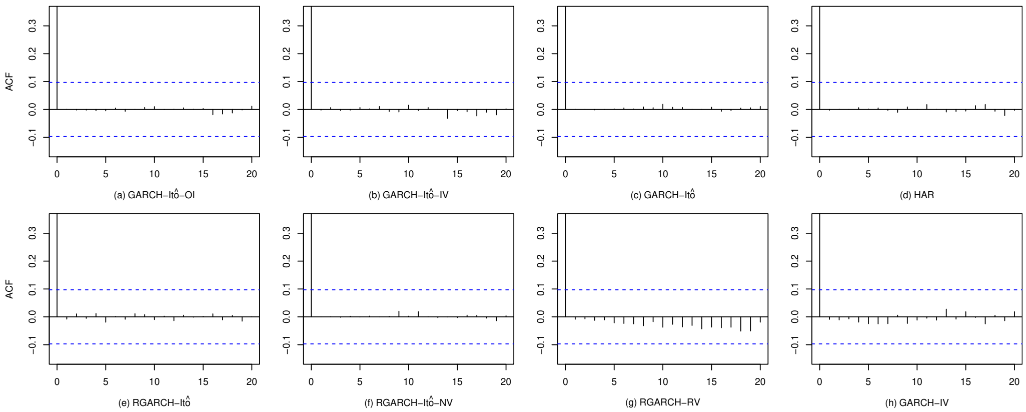

Figure 3: ACF plots for the residuals of the compared models during the whole period.

Bibliography

Kim D., Wang Y. (2016). Unified discrete-time and continuous-time models and statistical inferences for merged low-frequency and high-frequency financial data. *Journal of Econometrics* **194**, 220-230.

Xiu, D. (2010). Quasi-maximum likelihood estimation of volatility with high frequency data. *Journal of Econometrics* **159**, 235-250.