MOMENT DEVIATION SUBSPACES OF DIMENSION REDUCTION FOR HIGH-DIMENSIONAL DATA WITH CHANGE STRUCTURE

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Supplementary Material

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S1. Central κ -th moment deviation subspace and part of numerical analysis

S1.1 Central κ -th moment deviation subspace

In this section, we consider higher moments as we may have an interest in detecting change points in the contemporaneous mean or second- order moment change structures. Assume $X_i = (X_{i1}, \dots, X_{ip})^{\top}$, for $i = 1 \dots, n$, be independent *p*-dimensional random variable vectors. Define the new high-dimensional variables Z_i based on X_i as:

$$Z_{i} = (X_{i1}, ..., X_{ip}, X_{i1}^{2}, X_{i1}X_{i2}, ..., X_{i1}X_{ip}, X_{i2}^{2}, X_{i2}X_{i3}, ..., X_{i2}X_{ip},$$

$$\cdots X_{i1}^{\kappa}, X_{i1}^{\kappa-1}X_{i2}\cdots, X_{ip}^{\kappa})^{\top}, \qquad (S1.1)$$

where κ denotes some positive integer. Let p_Z denote the dimension of Z_i . Without loss of generality, assume that the sequence $\{Z_i\}_{i=1}^n$ of all means follows a piecewise constant structure with K+1 segments. In other words, there are K change points $1 \leq z_1 < z_2 < ... < z_K \leq n$ such that $E(Z_{z_{k-1}+j}) = \mu_Z^{(k)}$, and $\operatorname{Cov}(Z_{z_{k-1}+j}) = \Sigma_Z^{(k)}$, for $k = 1, \cdots, K+1$ and $1 \leq j \leq z_k - z_{k-1}$ where $z_0 = 0$ and $z_{K+1} = n$.

Definition S1.1. Span $\{\mu_Z^{(k)} - \mu_Z^{(l)}, \text{ for } k, l = 1, \cdots, K+1\}$ is called the

central κ -th moment deviation subspace of the sequence $\{X_i\}_{i=1}^n$ and is written as $S_{\{X_i\}_{i=1}^n}^{\kappa}$. $q_{\kappa} = \dim\{S_{\{X_i\}_{i=1}^n}^{\kappa}\}$ is called the structural dimension of $S_{\{X_i\}_{i=1}^n}^{\kappa}$.

The following theorem states a similar result as that in Theorem 2.1.

Theorem S1.1. For any basis matrix $B \in \mathcal{R}^{p_Z \times q_\kappa}$ of $S^{\kappa}_{\{X_i\}_{i=1}^n}$, both the sequences of $\{B^{\top}Z_i\}_{i=1}^n$ and $\{Z_i\}_{i=1}^n$ have the same locations of change points.

To get a consistent estimator of the basis matrix B about the subspace $S_{\{X_i\}_{i=1}^n}^{\kappa}$, we also consider the following Mahalanobis matrix of the sequence $\{Z_i\}_{i=1}^n$ as:

$$M_{Z,n} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{i \neq j} (Z_i - Z_j) (Z_i - Z_j)^{\top}.$$
 (S1.2)

Compute the expectation of $M_{Z,n}$ to get:

$$\begin{split} E(M_{Z,n}) &= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{i \neq j} E\left\{ (Z_i - Z_j)(Z_i - Z_j)^{\top} \right\} \\ &= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{i \neq j} \operatorname{Cov}(Z_i - Z_j) + \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{i \neq j} E(Z_i - Z_j) E(Z_i - Z_j)^{\top} \\ &= \sum_{k=1}^{K+1} \sum_{l \neq k, l \leq K+1} \frac{n_k n_l}{n(n-1)} (\Sigma_Z^{(k)} + \Sigma_Z^{(l)}) + \sum_{k=1}^{K+1} \frac{2n_k (n_k - 1)}{n(n-1)} \Sigma_Z^{(k)} \\ &+ \sum_{k=1}^{K+1} \sum_{l \neq k, l \leq K+1} \frac{n_k n_l}{n(n-1)} (\mu_Z^{(k)} - \mu_Z^{(l)}) (\mu_Z^{(k)} - \mu_Z^{(l)})^{\top}. \end{split}$$

As $n_k/n \to c_k > 0$, for $k = 1, 2, \dots, K+1$, and $\sum_{k=1}^{K+1} c_k = 1$, we have

$$E(M_{Z,n}) \rightarrow \sum_{k=1}^{K+1} \sum_{l \neq k, l \leq K+1} c_k c_l (\Sigma_Z^{(k)} + \Sigma_Z^{(l)}) + 2 \sum_{k=1}^{K+1} c_k^2 \Sigma_Z^{(k)} + \sum_{k=1}^{K+1} \sum_{l \neq k, l \leq K+1} c_k c_l (\mu_Z^{(k)} - \mu_Z^{(l)}) (\mu_Z^{(k)} - \mu_Z^{(l)})^\top = 2 \sum_{k=1}^{K+1} c_k \Sigma_Z^{(k)} + \sum_{k=1}^{K+1} \sum_{l \neq k, l \leq K+1} c_k c_l (\mu_Z^{(k)} - \mu_Z^{(l)}) (\mu_Z^{(k)} - \mu_Z^{(l)})^\top =: 2 \Sigma_{pooled}^Z + \Delta_Z = M_Z.$$

Theorem S1.2. Under the model (S1.1), we have $\operatorname{Span}(\Delta_Z) = \operatorname{S}_{\{X_i\}_{i=1}^n}^{\kappa}$. Furthermore, let $B = (v_1, \cdots, v_{q_{\kappa}})$ denote the eigenvectors of Δ_Z associated with the nonzero eigenvalues of Δ_Z , then B is the basis matrix of $\operatorname{S}_{\{X_i\}_{i=1}^n}^{\kappa}$, namely, $\operatorname{Span}(B) = \operatorname{S}_{\{X_i\}_{i=1}^n}^{\kappa}$. We need to consistently estimate the pooled covariance matrix Σ_{pooled}^Z to estimate the target matrix Δ_Z efficiently. Similarly, we adopt a "divideand-conquer" strategy. Divide the data into \tilde{K} segments with the subscript as: $S_m = \{(m-1)\beta_n + 1, \dots, m\beta_n\}$, for $m = 1, \dots, \tilde{K} - 1$ and $S_{\tilde{K}} =$ $\{(\tilde{K}-1)\beta_n + 1, \dots, n\}$. We calculate the covariance for each segment and then average them to get an estimator of the pooled covariance matrix Σ_{pooled}^Z as:

$$\Sigma_{pooled,n}^{Z} = \frac{1}{\tilde{K}} \sum_{m=1}^{K} \hat{\Sigma}_{Zm} \text{ with } \hat{\Sigma}_{Zm} = \frac{1}{\#\{\mathcal{S}_m\} - 1} \sum_{k \in \mathcal{S}_m} (Z_k - \bar{Z}_m) (Z_k - \bar{Z}_m) (\bar{S}_{1.3})$$

where $\bar{Z}_m = \frac{1}{\#\{S_m\}} \sum_{k \in S_i} Z_k$ and $\#\{S_m\}$ denotes the cardinality of the set S_m .

Together the formula (S1.2) with (S1.3), we can have an estimator of Δ_Z as:

$$\Delta_{Z,n} = M_{Z,n} - 2\Sigma_{pooled,n}^Z$$

The basis matrix $B \in \mathcal{R}^{p_Z \times q_\kappa}$ of $S^{\kappa}_{\{X_i\}_{i=1}^n}$ can be estimated as the eigenvectors $B_n = (\hat{v}_1, \cdots, \hat{v}_{q_\kappa})$ associated with the largest q_κ eigenvalues of $\Delta_{Z,n}$.

Theorem S1.3. Under the model (S1.1), assume that $X_i - E(X_i)$ are

independent random variables, and Assumptions S3.5, S3.6, S3.7 and S3.8 hold. Then,

$$||\Delta_n - \Delta||_F = O_p\left(\sqrt{\frac{p_Z}{n}}\right) + O_p\left(\frac{\sqrt{p_Z}\beta_n}{n}\right),$$

where $|| \cdot ||_F$ denotes the Frobenius norm of a matrix. Furthermore, we have

$$||B_n - B||_F = O_p\left(\sqrt{\frac{p_Z}{n}}\right) + O_p\left(\frac{\sqrt{p_Z}\beta_n}{n}\right).$$

Remark S1.1. Theorem S1.1 presents that based on the central 2-moment deviation subspace, it is a direct application to identify the locations of change points in the normal case, namely,

$$X_i \sim N(u_i, \Sigma_i), \ 1 \le i \le n.$$

There are K change points $1 \leq z_1 < z_2 < \dots < z_K \leq n$ such that $E(X_{z_{k-1}+j}) = \mu^{(k)}$, and $Cov(X_{z_{k-1}+j}) = \Sigma^{(k)}$, for $k = 1, \dots, K+1$ and $1 \leq j \leq z_k - z_{k-1}$.

Similarly, we can determine the dimension q_{κ} of the subspace $S^{\kappa}_{\{X_i\}_{i=1}^n}$ based on TRR in (2.5) of the main body. Then, based on the lowerdimensional sequence $\{B_n^{\top} Z_i\}_{i=1}^n$, we can estimate the locations $\{\hat{z}_1, \dots, \hat{z}_{\hat{K}}\}$ and the number \hat{K} by existing methods.

S1.2 Numerical Studies with changes in covariance matrix

Experiment 4: Changes in the covariance matrix. The data

are generated from the multivariate normal distributions:

$$G_0 = G_2 = G_4 = N(0_p, I_{p \times p}), \text{ and } G_1 = G_3 = N(0_p, \Sigma).$$

Consider the following four settings:

- Case 1: $\Sigma = (1 a)I_{p \times p}$ with a = 0.3, 0.5, the change points are located at 100*i* for i = 1, 2, 3, 4, respectively;
- Case 2: Σ = (σ_{ij}), where σ_{ij} = I(i = j) + aI(i ≠ j) with a = 0.3, 0.5, the settings of change points are the same as Case 1;
- Case 3: Σ = (σ_{ij}), where σ_{ij} = a^{|i-j|} with a = 0.3, 0.5, the settings of change points are the same as Case 1;
- Case 4: the setting of $\Sigma = (\sigma_{ij})$ is the same as Case 3, and change points located at 90, 250, 390, and 450.

Here different values of a indicate the magnitudes of changes. We design p = 5, 10 associated with $p_Z = 20, 65$, respectively. The results are reported in Table 1. From Table 1, E-Divisive_{dr} performs the best among the competitors. The dimension reduction-based methods perform significantly better than the three methods. The E-Divisive and KCP tend to underestimate the number of change points seriously, but ks-cp30 overestimates it in this experiment. Additionally, the dimension reduction-based methods are relatively robust against the different covariance matrices, while E-Divisive, KCP, and ks-cp30 are very sensitive. Thus, there are significant improvements over the three methods by reducing the dimension.

S2. Genetics data with mean changes

Analyze the array comparative genomic hybridization (aCGH) microarray data set, which was analyzed in Stransky et al. [2006] to detect mean changes in the data structure. The dataset includes 57 individuals with a bladder tumor. We use the processed data in the R package: *ecp* to choose 43 individuals out of the 57 individuals at 2215 different loci's on their genome, namely p = 43 and n = 2215. This empirical study aims to find unusual chromosomal characteristics.

Because the structural dimension is determined to be $\hat{q} = 1$ via the TRR criterion, Figure 1 plots the locations of change points using E-Divisive and SBS before and after dimension reduction. We found 34 and 45 change

points using E-Divisive_{dr} and SBS_{dr} while E-Divisive, SBS found 97 and 3 changes. It would say that E-Divisive has an overestimation issue, and SBS underestimates the number of change points. It would suggest that the results after dimension reduction are reasonable. Further, when the methods ks-cp3o, Multirank, KCP, Inspect, DCBS and GeomCP are used, we found that they respectively identify 1, 7, 55, 254, 6, and 27 change points. This shows that KCP, and Inspect would overestimate the number of changes. ks-cp3o, Multirank, DCBS would also underestimate the number of change points, GeomCP has a similar result with E-Divisive_{dr} for this data set.

S3. Regularity Conditions and Proofs of the theorems

S3.1 Regularity Conditions

To investigate the asymptotic properties, we list the following assumptions. Let $\epsilon_i = X_i - E(X_i)$, $\tilde{\epsilon}_i = Z_i - E(Z_i)$ and $\tilde{\Sigma}_i = Var(Z_i)$.

Assumption S3.1. $0 < \min_{1 \le i \le n} \lambda_{min}(\Sigma_i) \le \max_{1 \le i \le n} \lambda_{max}(\Sigma_i) < \infty$.

Assumption S3.2. $0 < \min_{1 \le i \le n} \lambda_{min}(Var(\epsilon_i \epsilon_i^{\top})) \le \max_{1 \le i \le n} \lambda_{max}(Var(\epsilon_i \epsilon_i^{\top})) < \infty.$

Assumption S3.3. $0 < \max_{1 \le i \le n} \lambda_{max} (E(\epsilon_i \epsilon_i^\top - \Sigma_i)^4) < \infty.$

Assumption S3.4. $0 \leq \max_{1 \leq k \leq K+1} |\alpha^\top \mu^{(k)}| < \infty$ for all $||\alpha|| = 1$.

Assumption S3.5. $0 < \min_{1 \le i \le n} \lambda_{min}(\tilde{\Sigma}_i) \le \max_{1 \le i \le n} \lambda_{max}(\tilde{\Sigma}_i) < \infty.$

Assumption S3.6. $0 < \min_{1 \le i \le n} \lambda_{min}(Var(\tilde{\epsilon}_i \tilde{\epsilon}_i^{\top})) \le \max_{1 \le i \le n} \lambda_{max}(Var(\tilde{\epsilon}_i \tilde{\epsilon}_i^{\top})) < \infty.$

Assumption S3.7. $0 < \max_{1 \le i \le n} \lambda_{max} (E(\tilde{\epsilon}_i \tilde{\epsilon}_i^\top - \tilde{\Sigma}_i)^4) < \infty.$

Assumption S3.8. $0 \leq \max_{1 \leq k \leq K+1} \left| \alpha^{\top} \mu_Z^{(i)} \right| < \infty$ for all $||\alpha|| = 1$.

Remark S3.1. These assumptions are satisfied in many situations we are interested in, such as $\Sigma_i = I_p$ and m-dependent cases. See relevant references in the literature such as Dette et al. [2022] and Chen et al. [2010].

S3.2 Appendix. Proofs of the theorems

In this section, we present the proofs of the theoretical results.

Lemma 1. Under the model (2.1), assume that $X_i - E(X_i)$ are independent random variables. Under Assumptions S3.1–S3.4, then

$$\max_{1 \le s \le p} \left| \hat{\lambda}_s(\Delta_n) - \lambda_s \right| = O_p\left(\sqrt{\frac{p}{n}}\right) + O_p\left(\frac{\sqrt{p}\beta_n}{n}\right).$$

The proof of this lemma can be similar as that in the proof of Theorem 2.3; we omit it here.

Proof of Theorem 2.1. For any basis matrix $B \in \mathcal{R}^{p \times q}$ of $S_{\{E(X_i)\}_{i=1}^n}$, we have $\operatorname{Span}(B) = S_{\{E(X_i)\}_{i=1}^n}$. Assume there are \overline{K} change points $1 \leq \overline{z}_1 < \overline{z}_1$ $\bar{z}_2 < ... < \bar{z}_{\bar{K}} \le n$ of the sequence $\{Y_i = B^\top X_i\}_{i=1}^n$ such that $E(Y_{\bar{z}_{k-1}+j}) = B^\top \mu_{\bar{z}_{k-1}+j}$, for $k = 1, \cdots, \bar{K}+1, 1 \le j \le \bar{z}_k - \bar{z}_{k-1}$ and $B^\top \mu_{\bar{z}_k} \ne B^\top \mu_{\bar{z}_{k+1}}$.

If the locations of change points in the sequence $\{B^{\top}X_i\}_{i=1}^n$ are not these in the sequence $\{X_i\}_{i=1}^n$, there exist k such that $\mu_{\bar{z}_k} = \mu_{\bar{z}_k+1}$. Then we have $B^{\top}\mu_{\bar{z}_k} = B^{\top}\mu_{\bar{z}_k+1}$. However, \bar{z}_k is a change point of the sequence $\{B^{\top}X_i\}_{i=1}^n$, which implies that $B^{\top}\mu_{\bar{z}_k} \neq B^{\top}\mu_{\bar{z}_k+1}$. This is a contradiction. Thus, the locations of change points in the sequence $\{B^{\top}X_i\}_{i=1}^n$ are those in the sequence $\{X_i\}_{i=1}^n$.

On the other hand, if the locations of change points in the sequence $\{X_i\}_{i=1}^n$ are not those in the sequence $\{B^{\top}X_i\}_{i=1}^n$, there exists a k such that $B^{\top}(\mu_{z_k} - \mu_{z_{k+1}}) = 0$ and $\mu_{z_k} \neq \mu_{z_{k+1}}$. Therefore, $\mu_{z_k} - \mu_{z_{k+1}}$ is vertical to the subspace Span(B), namely

$$\mu_{z_k} - \mu_{z_k+1} \perp \operatorname{Span}(B). \tag{S3.1}$$

By the definition of central mean deviation subspace $S_{\{E(X_i)\}_{i=1}^n}$, we have $\mu_{z_k+1} - \mu_{z_k} \in S_{\{E(X_i)\}_{i=1}^n}$. As $\text{Span}(B) = S_{\{E(X_i)\}_{i=1}^n}$, we conclude that

$$\mu_{z_k} - \mu_{z_k+1} \in \operatorname{Span}(B). \tag{S3.2}$$

Altogether the results in (S3.1) and (S3.2), we conclude that $\mu_{z_k} - \mu_{z_{k+1}} = 0$.

This produces the contradiction that z_k is the location of a change point in $\{X_i\}_{i=1}^n$, namely, $\mu_{z_k} \neq \mu_{z_k+1}$. Therefore, the locations of change points in the sequence $\{X_i\}_{i=1}^n$ are these in the sequence $\{B^{\top}X_i\}_{i=1}^n$. The proof is finished.

Proof of Theorem 2.2. Recall that

$$\Delta = \sum_{k=1}^{K+1} \sum_{l \neq k, l \leq K+1} c_k c_l (\mu^{(k)} - \mu^{(l)}) (\mu^{(k)} - \mu^{(l)})^\top = A A^\top,$$

where $A = (\sqrt{c_1 c_2} \{ \mu^{(1)} - \mu^{(2)} \}, \dots, \sqrt{c_{K+1} c_K} \{ \mu^{(K+1)} - \mu^{(K)} \})$, we have $\text{Span}(\Delta) \subseteq \text{Span}(A).$

As rank(A) = rank(AA^T) and Span(AA^T) = Span(A), we conclude that Span(Δ) = Span(A). By the definition of central mean deviation subspace $S_{\{E(X_i)\}_{i=1}^n}$, Span(A) = $S_{\{E(X_i)\}_{i=1}^n}$. Therefore, we can get Span(Δ) = $S_{\{E(X_i)\}_{i=1}^n}$.

Proof of Theorem 2.3. By the Taylor expansion, we have (see also, e.g., Sun [1988] that was used in Zhu and Fang [1996])

$$\hat{\lambda}_s(\Delta_n) - \lambda_s(\Delta) = \nu_s^\top (\Delta_n - \Delta)\nu_s + R_{1s}(\Delta^*)$$

and

$$\hat{\nu}_s(\Delta_n) - \nu_s(\Delta) = \sum_{t=1, t \neq s}^p \frac{\nu_t(\Delta)(\nu_s^\top(\Delta)(\Delta_n - \Delta)\nu_s(\Delta))}{\lambda_s(\Delta) - \lambda_t(\Delta)} + R_{2s}(\Delta^*),$$

where

$$R_{1s}(\Delta^{\star}) = \nu_s(\Delta^{\star})^{\top}(\Delta_n - \Delta)\nu_s(\Delta^{\star}) - \nu_s^{\top}(\Delta)(\Delta_n - \Delta)\nu_s(\Delta),$$

$$R_{2s}(\Delta^{\star}) = \sum_{t=1, t \neq s}^{p} \frac{\nu_t(\nu_s(\Delta^{\star})^{\top}(\Delta_n - \Delta)\nu_s(\Delta^{\star}))}{\lambda_s(\Delta) - \lambda_t(\Delta)} - \sum_{t=1, t \neq s}^{p} \frac{\nu_t(\Delta)(\nu_s^{\top}(\Delta - \Delta)\nu_s)}{\lambda_s(\Delta) - \lambda_t(\Delta)},$$

and $\Delta^{\star} - \Delta \rightarrow 0$ in probability.

Firstly, we know

$$\left\|\sum_{t=1,t\neq s}^{p} \frac{\nu_t(\Delta)}{\lambda_s(\Delta) - \lambda_t(\Delta)}\right\|_F^2 = O(p).$$

Now we focus on the term $\nu_s^{\top}(\Delta)(\Delta_n - \Delta)\nu_s(\Delta)$.

Here we give a general conclusion: we consider $\alpha^{\top}(\Delta_n - \Delta)\alpha$ for any $||\alpha|| = 1$. Recall that

$$\Delta_n - \Delta = M_n - M - 2(\Sigma_{pooled,n} - \Sigma_{pooled}) = M_n - (\Delta - 2\Sigma_{pooled}) - 2(\Sigma_{pooled,n} - \Sigma_{pooled}).$$

Part 1. We calculate the convergence rate of $M_n - \Delta - 2\Sigma_{pooled}$ in the

first step. Let $\epsilon_i = X_i - E(X_i)$ and $\delta_{ij} = E(X_i) - E(X_j)$. Then M_n can be decomposed as follows:

$$\begin{split} M_{n} &= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (X_{i} - X_{j}) (X_{i} - X_{j})^{\mathsf{T}} \\ &= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (\epsilon_{i} - \epsilon_{j} + \delta_{ij}) (\epsilon_{i} - \epsilon_{j} + \delta_{ij})^{\mathsf{T}} \\ &= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \epsilon_{i} \epsilon_{i}^{\mathsf{T}} + \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \epsilon_{j} \epsilon_{j}^{\mathsf{T}} \\ &+ \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (\epsilon_{i} - \epsilon_{j}) \delta_{ij}^{\mathsf{T}} + \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \delta_{ij} \delta_{ij}^{\mathsf{T}} \\ &= \frac{2}{n} \sum_{i=1}^{n} \epsilon_{i} \epsilon_{i}^{\mathsf{T}} + \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} E(X_{i}) \epsilon_{i}^{\mathsf{T}} + \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} E(X_{j}) \epsilon_{j}^{\mathsf{T}} \\ &- \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} E(X_{i}) \epsilon_{j}^{\mathsf{T}} - \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} E(X_{j}) \epsilon_{i}^{\mathsf{T}} + \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \delta_{ij} \delta_{ij}^{\mathsf{T}} \\ &= \frac{2}{n} \sum_{i=1}^{n} \epsilon_{i} \epsilon_{i}^{\mathsf{T}} + \frac{4}{n} \sum_{i=1}^{n} E(X_{i}) \epsilon_{i}^{\mathsf{T}} - \frac{4}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} E(X_{i}) \epsilon_{j}^{\mathsf{T}} + \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \delta_{ij} \delta_{ij}^{\mathsf{T}} \\ &= : 2M_{1n} + 4M_{2n} - 4M_{3n} + M_{4n}. \end{split}$$

We now deal with the fourth terms $\alpha^{\top} M_{in} \alpha$ one by one. For any fixed α with $||\alpha|| = 1$, we have

$$\alpha^{\top} M_{1n} \alpha = \frac{1}{n} \sum_{i=1}^{n} \alpha^{\top} \epsilon_i \epsilon_i^{\top} \alpha =: \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{ni},$$

where $Z_{ni} = \frac{1}{\sqrt{n}} \alpha^{\top} \epsilon_i \epsilon_i^{\top} \alpha$ is a double array sequence. Then we have

$$E(Z_{ni}) = \frac{1}{\sqrt{n}} \alpha^{\top} \Sigma_i \alpha, \quad Var\left(\sum_{i=1}^n Z_{ni}\right) = \frac{1}{n} \sum_{i=1}^n Var(\alpha^{\top} \epsilon_i \epsilon_i^{\top} \alpha) =: B^2$$

To verify the Lindeberg condition. For any $\eta > 0$, under Assumptions S3.2 and S3.3, we have

$$\begin{split} &\sum_{i=1}^{n} \frac{1}{B^{2}} \int_{|Z_{ni} - E(Z_{ni})| > \eta B} (Z_{ni} - E(Z_{ni}))^{2} dF_{ni} \\ &= \frac{1}{B^{2}} \sum_{i=1}^{n} E\left[(Z_{ni} - E(Z_{ni}))^{2} I(|Z_{ni} - E(Z_{ni})| > \eta B) \right] \\ &\leq \frac{n}{B^{2}} \max_{i} E\left[(Z_{ni} - E(Z_{ni}))^{4} \right]^{1/2} \left[P(|Z_{ni} - E(Z_{ni})| > \eta B) \right]^{1/2} \\ &= \frac{1}{B^{2}} \max_{i} E\left[(\alpha^{\top} \epsilon_{i} \epsilon_{i}^{\top} \alpha - \alpha^{\top} \Sigma_{i} \alpha)^{4} \right]^{1/2} \left[P(|Z_{ni} - E(Z_{ni})| > \eta B) \right]^{1/2} \\ &\leq \max_{i} \frac{E\left[(\alpha^{\top} \epsilon_{i} \epsilon_{i}^{\top} \alpha - \alpha^{\top} \Sigma_{i} \alpha)^{4} \right]^{1/2}}{B^{2}} \left[\frac{Var(Z_{ni})}{\eta^{2} B^{2}} \right]^{1/2} \\ &\leq \frac{\max_{i} \lambda_{max}^{1/2} (E(\epsilon_{i} \epsilon_{i}^{\top} - \Sigma_{i})^{4}) \max_{i} \lambda_{max}^{1/2} (Var(\epsilon_{i} \epsilon_{i}))}{\left[\min_{i} \lambda_{min} (Var(\epsilon_{i} \epsilon_{i}^{\top})) \right]^{3/2} \eta \sqrt{n}} \\ &= O\left(\left(\frac{1}{\sqrt{n}} \right) \to 0. \end{split}$$

Thus, $\alpha^{\top} M_{1n} \alpha - \frac{1}{n} \sum_{i=1}^{n} \alpha^{\top} \Sigma_i \alpha \xrightarrow{\mathcal{D}} \mathcal{N} \left(0, \frac{1}{n^2} \sum_{i=1}^{n} Var(\alpha^{\top} \epsilon_i \epsilon_i^{\top} \alpha) \right)$ which implies that $\alpha^{\top} M_{1n} \alpha - 2\alpha^{\top} \Sigma_{pooled} \alpha = O_p \left(\frac{1}{\sqrt{n}} \right)$ with $\frac{1}{n} \sum_{i=1}^{n} \alpha^{\top} \Sigma_i \alpha \xrightarrow{a.s.} \alpha^{\top} \Sigma_{pooled} \alpha$ and Assumption S3.1. Next, consider

$$\alpha^{\top} M_{2n} \alpha = \frac{1}{n} \sum_{i=1}^{n} \alpha^{\top} E(X_i) \epsilon_i^{\top} \alpha.$$

Note that if $\alpha^{\top} E(X_i) = 0$ for all i, then $\alpha^{\top} M_{2n} \alpha = 0$. Therefore, by the piecewise constant structure of the sequence $E(X_i)$'s, the number of $\alpha^{\top} E(X_i) \neq 0$ is not smaller than $\min\{n_i\} + 1$, where $\min\{n_i\}$ is the minimum number of data points between any two true changes. Let $\mathcal{N}_n =$ $\{i : \alpha^{\top} E(X_i) \neq 0\}$. The cardinality $\#\mathcal{N}_n$ of \mathcal{N}_n satisfies that $\min\{n_i\} + 1 \leq$ $\#\mathcal{N}_n \leq n$.

$$\alpha^{\top} M_{2n} \alpha = \frac{1}{n} \sum_{i=1}^{n} \alpha^{\top} E(X_i) \epsilon_i^{\top} \alpha = \frac{\# \mathcal{N}_n}{n} \frac{1}{\# \mathcal{N}_n} \sum_{i \in \mathcal{N}_n} \alpha^{\top} E(X_i) \epsilon_i^{\top} \alpha.$$

Thus, for every $\xi > 0$, when we take $M = \left[\frac{\max_{1 \le k \le K} |\alpha^\top \mu^{(k)}|^2 \max_{1 \le i \le n} \lambda_{max}(\Sigma_i)}{\xi}\right]^{1/2} + 1$, then under Assumptions S3.1 and S3.4,

$$\Pr\left(\sqrt{\#\mathcal{N}_n} \left| \frac{1}{\#\mathcal{N}_n} \sum_{i \in \mathcal{N}_n} \alpha^\top E(X_i) \epsilon_i^\top \alpha \right| > M\right) \leq \frac{\sum_{i \in \mathcal{N}_n} \left[\alpha^\top E(X_i) \right]^2 \alpha^\top E(\epsilon_i \epsilon_i^\top) \alpha}{\#\mathcal{N}_n M^2} \\ \leq \frac{\max_{1 \leq k \leq K} |\alpha^\top \mu^{(k)}|^2 \max_{1 \leq i \leq n} \lambda_{max}(\Sigma_i)}{M^2} < \xi$$

Thus, $\frac{1}{\#\mathcal{N}_n} \sum_{i \in \mathcal{N}_n} \alpha^\top E(X_i) \epsilon_i^\top \alpha = O_p\left(\frac{1}{\sqrt{\#\mathcal{N}_n}}\right)$. We could also obtain, since

 $\#\mathcal{N}_n \le n,$

$$\alpha^{\top} M_{2n} \alpha = \frac{\# \mathcal{N}_n}{n} \frac{1}{\# \mathcal{N}_n} \sum_{i \in \mathcal{N}_n} \alpha^{\top} E(X_i) \epsilon_i^{\top} \alpha = O_p\left(\frac{\sqrt{\# \mathcal{N}_n}}{n}\right) \le O_p\left(\frac{1}{\sqrt{n}}\right).$$

Consider the term $\alpha^{\top} M_{3n} \alpha$. Under Assumption S3.4, we have

$$\begin{aligned} \alpha^{\top} M_{3n} \alpha &= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \alpha^{\top} E(X_i) \epsilon_j^{\top} \alpha \\ &= \frac{1}{n(n-1)} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha^{\top} E(X_i) \epsilon_j^{\top} \alpha \right] - \frac{1}{n(n-1)} \sum_{i=1}^{n} \alpha^{\top} E(X_i) \epsilon_i^{\top} \alpha \\ &= \frac{n^2}{n(n-1)} \left[\frac{1}{n} \sum_{i=1}^{n} \alpha^{\top} E(X_i) \right] \left[\frac{1}{n} \sum_{j=1}^{n} \alpha^{\top} \epsilon_j \right] - \frac{1}{n-1} \alpha^{\top} M_{2n} \alpha \\ &\leq \frac{n}{n-1} \left| \frac{1}{n} \sum_{i=1}^{n} \alpha^{\top} E(X_i) \right| \left| \frac{1}{n} \sum_{j=1}^{n} \alpha^{\top} \epsilon_j \right| + O_p \left(\frac{1}{n\sqrt{n}} \right) \\ &\leq \frac{n}{n-1} \left| \max_{1 \leq k \leq K} \alpha^{\top} \mu^{(k)} \right| \left| \frac{1}{n} \sum_{j=1}^{n} \alpha^{\top} \epsilon_j \right| + O_p \left(\frac{1}{n\sqrt{n}} \right) \\ &= O_p \left(\frac{1}{\sqrt{n}} \right) + O_p \left(\frac{1}{n\sqrt{n}} \right) \end{aligned}$$

Next, we turn to consider the term $\alpha^{\top} M_{4n} \alpha$ by rewriting it as

$$\begin{aligned} \alpha^{\top} M_{4n} \alpha &= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \alpha^{\top} \delta_{ij} \delta_{ij}^{\top} \alpha \\ &= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \alpha^{\top} (E(X_i) - E(X_j)) (E(X_i) - E(X_j))^{\top} \alpha \\ &= \frac{1}{n(n-1)} \sum_{k=1}^{K+1} \sum_{l \neq k, l \leq K+1} \sum_{i=z_{k-1}+1} \sum_{j=z_{l-1}+1} \alpha^{\top} (\mu^{(k)} - \mu^{(l)}) (\mu^{(k)} - \mu^{(l)})^{\top} \alpha \\ &= \frac{1}{n(n-1)} \sum_{k=1}^{K+1} \sum_{l \neq k, l \leq K+1} n_k n_l \alpha^{\top} (\mu^{(k)} - \mu^{(l)}) (\mu^{(k)} - \mu^{(l)})^{\top} \alpha \\ &= \alpha^{\top} \Delta \alpha + o(1). \end{aligned}$$

To sum up, together with all the results of the four terms $\alpha^{\top} M_{in} \alpha$'s, we conclude that

$$\alpha^{\top}(M_n - \Delta - 2\Sigma_{pooled})\alpha \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{4}{n^2}\sum_{i=1}^n Var(\alpha^{\top}\epsilon_i\epsilon_i^{\top}\alpha)\right)$$

and then

$$\alpha^{\top}(M_n - \Delta - 2\Sigma_{pooled})\alpha = \alpha^{\top}(M_n - M)\alpha = O_p\left(\frac{1}{\sqrt{n}}\right)$$

for any fixed α .

Part 2. We decompose $\alpha^{\top} (\Sigma_{pooled,n} - \Sigma_{pooled}) \alpha$ as

$$\begin{aligned} \alpha^{\top} \left(\Sigma_{pooled,n} - \Sigma_{pooled} \right) \alpha \\ = \alpha^{\top} \left[\frac{1}{\tilde{K}} \sum_{m=1}^{\tilde{K}} (\hat{\Sigma}_m - \Sigma_{pooled}) \right] \alpha \\ = \alpha^{\top} \left(\frac{1}{\tilde{K}} \sum_{m=1}^{\tilde{K}} \frac{1}{2\beta_n(\beta_n - 1)} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} (X_k - X_l) (X_k - X_l)^{\top} - \Sigma_{pooled} \right) \alpha \\ = \alpha^{\top} \left(\frac{1}{\tilde{K}} \sum_{m=1}^{\tilde{K}} \frac{1}{2\beta_n(\beta_n - 1)} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} (\epsilon_k - \epsilon_l + \delta_{kl}) (\epsilon_k - \epsilon_l + \delta_{kl})^{\top} - \Sigma_{pooled} \right) \alpha \\ = \frac{1}{\tilde{K}} \sum_{m=1}^{\tilde{K}} \frac{1}{\beta_n} \sum_{k \in \mathcal{S}_m} \left(\alpha^{\top} \epsilon_m \epsilon_m^{\top} \alpha - \alpha^{\top} \Sigma_{pooled} \alpha \right) - \frac{1}{\tilde{K}} \sum_{m=1}^{\tilde{K}} \frac{1}{\beta_n(\beta_n - 1)} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} \alpha^{\top} \delta_{kl} (\epsilon_k - \epsilon_l)^{\top} \alpha \\ &+ \frac{1}{\tilde{K}} \sum_{m=1}^{\tilde{K}} \frac{1}{2\beta_n(\beta_n - 1)} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} \alpha^{\top} \delta_{kl} \delta_{kl}^{\top} \alpha \\ &+ \frac{1}{\tilde{K}} \sum_{m=1}^{\tilde{K}} \frac{1}{2\beta_n(\beta_n - 1)} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} \alpha^{\top} \delta_{kl} \delta_{kl}^{\top} \alpha \\ &= : \Sigma_{1n} - \Sigma_{2n} + \Sigma_{3n} + \Sigma_{4n}. \end{aligned}$$

It is easy to obtain that

$$\Sigma_{1n} = \frac{1}{\tilde{K}} \sum_{m=1}^{\tilde{K}} \frac{1}{\beta_n} \sum_{k \in \mathcal{S}_m} \left(\alpha^\top \epsilon_k \epsilon_k^\top \alpha - \alpha^\top \Sigma_{pooled} \alpha \right) = \frac{1}{n} \sum_{i=1}^n \left(\alpha^\top \epsilon_i \epsilon_i^\top \alpha - \alpha^\top \Sigma_{pooled} \alpha \right)$$
$$\stackrel{\mathcal{D}}{\to} \mathcal{N} \left(0, \frac{1}{n^2} \sum_{i=1}^n Var(\alpha^\top \epsilon_i \epsilon_i^\top \alpha) \right),$$

which implies that $\Sigma_{1n} = O_p\left(\frac{1}{\sqrt{n}}\right)$.

For Σ_{2n} , we can rewrite it as

$$\begin{split} \Sigma_{2n} = & \frac{1}{\tilde{K}} \sum_{m=1}^{\tilde{K}} \frac{1}{\beta_n(\beta_n - 1)} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} \alpha^\top \epsilon_k \epsilon_l^\top \alpha \\ = & \frac{1}{\sqrt{\tilde{K}\beta_n(\beta_n - 1)}} \sum_{m=1}^{\tilde{K}} \frac{1}{\sqrt{\tilde{K}}} \frac{1}{\sqrt{\tilde{K}}} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} \alpha^\top \epsilon_k \epsilon_l^\top \alpha \\ = & \frac{1}{\sqrt{\tilde{K}\beta_n(\beta_n - 1)}} \sum_{m=1}^{\tilde{K}} T_{\tilde{K}m}, \end{split}$$

where $T_{\tilde{K}m} = \frac{1}{\sqrt{\tilde{K}\beta_n(\beta_n-1)}} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} \alpha^\top \epsilon_k \epsilon_l^\top \alpha$. We can also derive that

$$\begin{split} E(T_{\tilde{K}m}) &= \frac{1}{\sqrt{\tilde{K}\beta_n(\beta_n - 1)}} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} E(\alpha^\top \epsilon_k) E(\epsilon_l^\top \alpha) = 0, \\ Var(T_{\tilde{K}m}) &= E(T_{\tilde{K}m}^2) = \left[\frac{1}{\sqrt{\tilde{K}\beta_n(\beta_n - 1)}} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} \alpha^\top \epsilon_k \epsilon_l^\top \alpha \right]^2 \\ &= \frac{1}{\tilde{K}\beta_n(\beta_n - 1)} \sum_{k_1 \in \mathcal{S}_m} \sum_{l_1 \neq k_1, l_1 \in \mathcal{S}_m} \sum_{k_2 \in \mathcal{S}_m} \sum_{l_2 \neq k_2, l_2 \in \mathcal{S}_m} E(\alpha^\top \epsilon_{k_1} \alpha^\top \epsilon_{l_1} \alpha^\top \epsilon_{k_2} \alpha^\top \epsilon_{l_2}) \\ &= \frac{2}{\tilde{K}\beta_n(\beta_n - 1)} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} E[(\alpha^\top \epsilon_k)^2] E[(\alpha^\top \epsilon_l)^2] \\ &= \frac{2}{\tilde{K}\beta_n(\beta_n - 1)} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} (\alpha^\top \Sigma_k \alpha) (\alpha^\top \Sigma_l \alpha) \end{split}$$

Thus, under Assumptions S3.1 and S3.2, we could calculate

$$\frac{2}{\tilde{K}}\min_{1\leq i\leq n}\lambda_{\min}^2(\Sigma_i)\leq Var(T_{\tilde{K}m})\leq \frac{2}{\tilde{K}}\max_{1\leq i\leq n}\lambda_{\max}^2(\Sigma_i)$$

and

$$2\min_{1\leq i\leq n}\lambda_{\min}^2(\Sigma_i)\leq Var\left(\sum_{i=1}^{\tilde{K}}T_{\tilde{K}i}\right)\leq 2\max_{1\leq i\leq n}\lambda_{\max}^2(\Sigma_i).$$

$$E\left[T_{\tilde{K}m}^{4}\right] = \frac{1}{[\tilde{K}\beta_{n}(\beta_{n}-1)]^{2}} E\left\{\sum_{k\in\mathcal{S}_{m}}\sum_{l\neq k,l\in\mathcal{S}_{m}}\alpha^{\top}\epsilon_{k}\epsilon_{l}^{\top}\alpha\right\}^{4}$$
$$= \frac{C_{1}}{[\tilde{K}\beta_{n}(\beta_{n}-1)]^{2}}\sum_{k\in\mathcal{S}_{m}}\sum_{l\neq k,l\in\mathcal{S}_{m}}E[(\alpha^{\top}\epsilon_{k})^{4}]E[(\epsilon_{l}^{\top}\alpha)^{4}]$$
$$+ \frac{C_{2}}{[\tilde{K}\beta_{n}(\beta_{n}-1)]^{2}}\sum_{k_{1}\in\mathcal{S}_{m}}\sum_{\substack{l_{1}\neq k_{1}\\l_{1}\in\mathcal{S}_{m}}}\sum_{\substack{k_{2}\neq k_{1}\neq l_{1}\\k_{2}\in\mathcal{S}_{m}}}\sum_{\substack{l_{2}\neq l_{1}\neq k_{1}\neq l_{1}\\l_{2}\in\mathcal{S}_{m}}}E[(\alpha^{\top}\epsilon_{k_{2}})^{2}]E[(\alpha^{\top}\epsilon_{l_{2}})^{2}]E[(\alpha^{\top}\epsilon_{l_{2}})^{2}]}$$
$$=O\left(\frac{1}{\tilde{K}^{2}}\right),$$

where C_1 and C_2 are positive integers that don't take a lot of effort to

calculate. For any $\eta > 0$, with Assumptions S3.1 and S3.2, we have

$$\begin{split} &\sum_{m=1}^{\tilde{K}} \frac{1}{Var\left(\sum_{m=1}^{\tilde{K}} T_{\tilde{K}m}\right)} \int_{|T_{\tilde{K}m}| > \sqrt{\eta Var\left(\sum_{m=1}^{\tilde{K}} T_{\tilde{K}m}\right)}} T_{\tilde{K}m}^{2} dF'_{\tilde{K}m} \\ &= \frac{1}{Var\left(\sum_{i=1}^{\tilde{K}} T_{\tilde{K}m}\right)} \sum_{m=1}^{\tilde{K}} E\left[T_{\tilde{K}m}^{2} I\left(|T_{\tilde{K}m}| > \sqrt{\eta Var\left(\sum_{m=1}^{\tilde{K}} T_{\tilde{K}m}\right)}\right)\right] \\ &\leq \frac{1}{Var\left(\sum_{m=1}^{\tilde{K}} T_{\tilde{K}m}\right)} \sum_{m=1}^{\tilde{K}} E\left[T_{\tilde{K}m}^{4}\right]^{1/2} P^{1/2}\left(|T_{\tilde{K}m}| > \sqrt{\eta Var\left(\sum_{m=1}^{\tilde{K}} T_{\tilde{K}m}\right)}\right) \\ &\leq \frac{\tilde{K} \max_{1 \le m \le \tilde{K}} E\left[T_{\tilde{K}m}^{4}\right]^{1/2} \max_{1 \le m \le \tilde{K}} Var(T_{\tilde{K}m})^{1/2}}{Var\left(\sum_{m=1}^{\tilde{K}} T_{\tilde{K}m}\right)^{3/2} \sqrt{\eta}} \\ &= O\left(\frac{1}{\sqrt{\eta \tilde{K}}}\right) \to 0. \end{split}$$

Then

$$\Sigma_{2n} = O_p\left(\frac{1}{\sqrt{\tilde{K}\beta_n(\beta_n - 1)}}\right) = O_p\left(\frac{1}{\sqrt{n\beta_n}}\right).$$

Recall the definition of S_m right above equation (2.3) in the main body of the paper. Write all those sets $\{S_m \text{ for } m = 1, \cdots, \tilde{K}\}$. Note that all S_m for $m = 1, \cdots, \tilde{K}$ are disjoint. Then we further split all sets into two disjoint subsets of sets where $S^c = \{S_m, \text{ where } S_m \text{ contains at least an index } m \text{ such that}$ $z_m \in S_m \text{ and } z_m + 1 \in S_m\}$, and the rest sets as S. The number of the set S^c is less than or equal to the number of change points K. Based on this definition, we can write Σ_{3n} as

$$\begin{split} \Sigma_{3n} &= \frac{1}{\tilde{K}} \sum_{m=1}^{\tilde{K}} \frac{1}{\beta_n (\beta_n - 1)} \sum_{k \in S_m} \sum_{l \neq k, l \in S_m} \alpha^\top \delta_{kl} (\epsilon_k - \epsilon_l)^\top \alpha \\ &= \frac{1}{\tilde{K}} \sum_{S_m \in S^c} \frac{1}{\beta_n (\beta_n - 1)} \sum_{k \in S_m} \sum_{l \neq k, l \in S_m} \alpha^\top \delta_{kl} (\epsilon_k - \epsilon_l)^\top \alpha \\ &+ \frac{1}{\tilde{K}} \sum_{S_m \in S^c} \frac{1}{\beta_n (\beta_n - 1)} \sum_{k \in S_m} \sum_{l \neq k, l \in S_m} \alpha^\top \delta_{kl} (\epsilon_k - \epsilon_l)^\top \alpha \\ &= \frac{1}{\tilde{K}} \sum_{S_m \in S^c} \frac{1}{\beta_n (\beta_n - 1)} \sum_{k \in S_m} \sum_{l \neq k, l \in S_m} \alpha^\top \delta_{kl} (\epsilon_k - \epsilon_l)^\top \alpha \\ &= \frac{1}{\tilde{K}} \sum_{S_m \in S^c} \frac{1}{\beta_n (\beta_n - 1)} \sum_{k \in S_m} \sum_{l \neq k, l \in S_m} \alpha^\top \delta_{kl} (\epsilon_k - \epsilon_l)^\top \alpha \\ &= \frac{2}{\tilde{K}} \sum_{S_m \in S^c} \frac{1}{\beta_n (\beta_n - 1)} \sum_{k \in S_m} \sum_{l \neq k, l \in S_m} \alpha^\top (E(X_k) - E(X_l)) (\epsilon_k - \epsilon_l)^\top \alpha \\ &= \frac{2}{\tilde{K}} \sum_{S_m \in S^c} \frac{1}{\beta_n} \sum_{k \in S_m} \alpha^\top E(X_k) \epsilon_k^\top \alpha \\ &+ \frac{2}{\tilde{K}} \sum_{S_m \in S^c} \frac{1}{\beta_n (\beta_n - 1)} \left[\sum_{k \in S_m} \sum_{l \in S_m} \alpha^\top E(X_k) \epsilon_l^\top \alpha - \sum_{k \in S_m} \alpha^\top E(X_k) \epsilon_k^\top \alpha \right] \\ &= \frac{2\# \{S^c\}}{\tilde{K}} O_p \left(\frac{1}{\sqrt{\# \{S^c\} \beta_n}} \right) + \frac{2\# \{S^c\}}{\tilde{K} (\beta_n - 1)} O_p \left(\frac{1}{\sqrt{\# \{S^c\} \beta_n}} \right) \\ &+ \frac{2}{\tilde{K}} \sum_{S_m \in S^c} \left[\frac{1}{\beta_n} \sum_{k \in S_m} \alpha^\top E(X_k) \right] \left[\frac{\beta_n}{\beta_n - 1} \frac{1}{\beta_n} \sum_{l \in S_m} \alpha^\top \epsilon_l \right] \\ &= \frac{2\# \{S^c\}}{\tilde{K}} O_p \left(\frac{1}{\sqrt{\# \{S^c\} \beta_n}} \right) + \frac{2\# \{S^c\}}{\tilde{K} (\beta_n - 1)} O_p \left(\frac{1}{\sqrt{\# \{S^c\} \beta_n} \right) \\ &+ O(1) \left[\frac{2}{\tilde{K}} \sum_{S_m \in S^c} \frac{\beta_n}{\beta_n - 1} \frac{1}{\beta_n} \sum_{l \in S_m} \alpha^\top \epsilon_l \right] \\ &= O_p \left(\frac{\sqrt{\# \{S^c\}}}{\tilde{K} \sqrt{\beta_n}} \right) = O_p \left(\frac{\sqrt{K}}{\sqrt{\tilde{K}n}} \right) = O_p \left(\frac{\sqrt{K}}{\eta_n} \right) = O_p \left(\frac{\sqrt{\beta_n}}{\eta_n} \right). \end{split}$$

The last term is discussed under Assumption S3.4.

For Σ_{4n} , we have

$$\Sigma_{4n} = \frac{1}{\tilde{K}} \sum_{\mathcal{S}_m \in \mathcal{S}^c} \frac{1}{2\beta_n(\beta_n - 1)} \sum_{k \in \mathcal{S}_m} \sum_{l \neq k, l \in \mathcal{S}_m} \alpha^\top \delta_{kl} \delta_{kl}^\top \alpha = KO_p\left(\frac{1}{\tilde{K}}\right) = KO_p\left(\frac{\beta_n}{n}\right) = O_p\left(\frac{\beta_n}{n}\right).$$

Therefore, together with the results about Σ_{in} for $i = 1, \dots, 4$, we conclude that

$$\alpha^{\top} (\Sigma_{pooled,n} - \Sigma_{pooled}) \alpha = O_p \left(\frac{1}{\sqrt{n}}\right) + O_p \left(\frac{1}{\sqrt{n\beta_n}}\right) + O_p \left(\frac{\sqrt{\beta_n}}{n}\right) + O_p \left(\frac{\beta_n}{n}\right)$$
$$= O_p \left(\frac{1}{\sqrt{n}}\right) + O_p \left(\frac{\beta_n}{n}\right).$$

To sum up, we conclude that

$$\alpha^{\top}(\Delta_n - \Delta)\alpha = 3\Sigma_{1n} + o_p(1) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{9}{n^2}\sum_{i=1}^n Var(\alpha^{\top}\epsilon_i\epsilon_i^{\top}\alpha)\right)$$

when $\beta_n/n \to 0$. Furthermore, we have

$$\alpha^{\top} (\Delta_n - \Delta) \alpha = \alpha^{\top} (M_n - M) \alpha - 2\alpha^{\top} (\Sigma_{pooled,n} - \Sigma_{pooled}) \alpha$$
$$= O_p \left(\frac{1}{\sqrt{n}}\right) + O_p \left(\frac{\beta_n}{n}\right).$$

Therefore, we have, recalling the definitions at the beginning of the proof

of this theorem,

$$\hat{\lambda}_s(\Delta_n) - \lambda_s(\Delta) = O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{\beta_n}{n}\right), s = 1, 2, \cdots, p_s$$

and

$$\hat{\nu}_s(\Delta_n) - \nu_s(\Delta) = O_p\left(\sqrt{\frac{p}{n}}\right) + O_p\left(\frac{\sqrt{p}\beta_n}{n}\right).$$

Then we get

$$||\Delta_n - \Delta||_F = \sqrt{tr(\Delta_n - \Delta)(\Delta_n - \Delta)^{\top}} = \sqrt{\sum_{k=1}^p \lambda_k^2(\Delta_n - \Delta)}$$
$$= O_p\left(\sqrt{\frac{p}{n}}\right) + O_p\left(\frac{\sqrt{p}\beta_n}{n}\right),$$

where $\lambda_k(\Delta_n - \Delta)$ is the eigenvalue of the matrix $\Delta_n - \Delta$. These results imply that

$$\max_{1 \le s \le p} \left| \hat{\lambda}_s(\Delta_n) - \lambda_s(\Delta) \right| = ||\Delta_n - \Delta||_2$$
$$\leq ||\Delta_n - \Delta||_F$$
$$= O_p \left(\sqrt{\frac{p}{n}} \right) + O_p \left(\frac{\sqrt{p}\beta_n}{n} \right),$$

where $|| \cdot ||_2$ is L_2 matrix norms. Thus, we

$$||B_n - B||_F = O_p\left(\sqrt{\frac{pq}{n}}\right) + O_p\left(\frac{\sqrt{pq}\beta_n}{n}\right).$$

The proof is finished.

Proof of Theorem 2.4. We follow the similar arguments of proving Theorem 2.2 in Zhu et al. [2021] to prove this theorem. Write $\tilde{\eta}_n = \max\left\{\sqrt{\frac{p}{n}}, \frac{\sqrt{p}\beta_n}{n}\right\}$. $\hat{\lambda}_s(\Delta_n)$ and $\lambda_s(\Delta)$ as $\hat{\lambda}_s$ and λ_s in short. From Lemma 1, we can get

$$\max_{1 \le s \le p} \left| \hat{\lambda}_s - \lambda_s \right| = O_p(\tilde{\eta}_n).$$

The following deduction is in a sense with a probability tending to 1. The above implies there exists a constant C such that the following inequality holds:

$$\max_{1 \le s \le p} \left| \hat{\lambda}_s - \lambda_s \right| \le C \tilde{\eta}_n.$$

Then, we have $\lambda_s - C\tilde{\eta}_n \leq \hat{\lambda}_s \leq \lambda_s + C\tilde{\eta}_n, \forall 1 \leq s \leq p$. This implies that

$$-C\tilde{\eta}_n \le \min_{q+1 \le s \le p} \hat{\lambda}_s \le \max_{q+1 \le s \le p} \hat{\lambda}_s \le C\tilde{\eta}_n.$$

Since $\lambda_q > 0$ and $\lambda_{q+1} = 0$, we can obtain

$$\frac{-C\tilde{\eta}_n + c_n}{\lambda_q + C\tilde{\eta}_n + c_n} \le \frac{\hat{\lambda}_{(q+1)} + c_n}{\hat{\lambda}_q + c_n} \le \frac{C\tilde{\eta}_n + c_n}{\lambda_q - C\tilde{\eta}_n + c_n}$$

Due to the conditions $c_n \to 0$ and $c_n/\tilde{\eta}_n \to \infty$, and $c_n/\lambda_q \to 0$, we have

$$\frac{\hat{\lambda}_{(q+1)} + c_n}{\hat{\lambda}_q + c_n} \to 0.$$

Further, since for any l > q, $\lambda_l = 0$ and $c_n / \tilde{\eta}_n \to \infty$, we achieve

$$\min_{l>q} \frac{\hat{\lambda}_{(l+1)} + c_n}{\hat{\lambda}_l + c_n} \ge \frac{\min_{l>q} \hat{\lambda}_q + c_n}{\max_{l>q} \hat{\lambda}_q + c_n} \ge \frac{-C\tilde{\eta}_n + c_n}{C\tilde{\eta}_n + c_n} \to 1 > \tau.$$

Therefore, we conclude that $P(\hat{q} = q) \to 1$.

Proofs of Theorems S1.1, S1.2, and S1.3. The arguments used for proving Theorems 2.1, 2.2, and 2.3 can be used to prove these theorems; we then omit the details here.

Proof of Theorem 4.1. For any basis matrix $B \in \mathcal{R}^{p_Z \times q_\kappa}$ of $S^{\kappa}_{\{X_i\}_{i=1}^n}$, we have $\operatorname{Span}(B) = S^{\kappa}_{\{X_i\}_{i=1}^n}$. Assume $X_i \in \mathbb{R}^p$ for $i = 1, \dots, n$ belongs to a union of d categories $\{\mathcal{C}_k\}_{k=1}^d$ and $B^{\top}Z_i \in \mathbb{R}^{q_\kappa}$ for $i = 1, \dots, n$ belongs to a union of \tilde{K} categories $\{\tilde{\mathcal{C}}_k\}_{k=1}^{\tilde{K}}$.

First, for any pair X_i and X_j with $i \neq j$ belonging to the same category \mathcal{C}_k , we have $\mu_{Z,i} = E(Z_i) = E(Z_j) = \mu_{Z,j}$ and then $E(B^{\top}Z_i) = E(B^{\top}Z_j)$. This implies that $B^{\top}Z_i$ and $B^{\top}Z_j$ simultaneously belong to the some category $\tilde{\mathcal{C}}_l$. Therefore, we can conclude that any category \mathcal{C}_k can belong to some category $\tilde{\mathcal{C}}_l$.

On the other hand, for any $B^{\top}Z_i$ and $B^{\top}Z_j$ with $i \neq j$ in the same category \tilde{C}_k , namely, $E(B^{\top}Z_i) = E(B^{\top}Z_j)$. Then we can get that $B^{\top}\mu_{Z,i} = B^{\top}\mu_{Z,j}$. Therefore, $\mu_{Z,i} - \mu_{Z,j}$ is vertical to the subspace Span(B):

$$\mu_{Z,i} - \mu_{Z,j} \perp \operatorname{Span}(B). \tag{S3.3}$$

By the definition of the central κ -moment deviation subspace $S^{\kappa}_{\{X_i\}_{i=1}^n}$, we have $\mu_{Z,i} - \mu_{Z,j} \in S^{\kappa}_{\{X_i\}_{i=1}^n}$. As $\text{Span}(B) = S^{\kappa}_{\{X_i\}_{i=1}^n}$, we conclude that

$$\mu_{Z,i} - \mu_{Z,j} \in \operatorname{Span}(B). \tag{S3.4}$$

Together the results in (S3.3) and (S3.4), we can get that $\mu_{Z,i} - \mu_{Z,j} = 0$. This produces that X_i and X_j are simultaneously in some category C_l . Hence, any category \tilde{C}_k belongs to some category C_l . Therefore, for any basis matrix $B \in \mathcal{R}^{p_Z \times q_\kappa}$ of $S^{\kappa}_{\{X_i\}_{i=1}^n}$, both the sequences $\{B^{\top}Z_i\}_{i=1}^n$ and $\{X_i\}_{i=1}^n$ have the same clustering results.

The argument for proving Theorem 2.2 can be adopted to derive the rest of this theorem. Hence we omit the details here. The proof is finished.

Case	p_z	a	method	\hat{k}	MSE	RI	p_z	a	method	\hat{k}	MSE	RI
1	65	0.3	E-Divisive _{dr}	4.133	1.196	0.883	20		E-Divisive _{dr}	2.702	4.073	0.906
			E-Divisive	0.148	15.048	0.239			E-Divisive	0.093	15.391	0.224
			ks-cp30 _{dr}	5.460	5.784	0.886		0.9	$ks-cp3o_{dr}$	6.031	8.651	0.838
			ks-cp3o	6.298	10.118	0.769		0.5	ks-cp3o	6.265	10.550	0.773
			KCP_{dr}	3.363	6.695	0.704			KCP_{dr}	2.094	11.890	0.488
			KCP	0.006	15.966	0.200			KCP	0.079	15.555	0.212
1			E-Divisive _{dr}	4.203	0.257	0.983	20		E-Divisive _{dr}	4.116	0.144	0.972
		0.5	E-Divisive	2.491	4.943	0.718			E-Divisive	0.557	12.807	0.333
			ks-cp $3o_{dr}$	4.061	0.146	0.983		0 5	ks-cp $3o_{dr}$	4.487	1.534	0.960
			ks-cp3o	6.248	10.222	0.775		0.5	ks-cp3o	6.181	9.751	0.776
			KCP_{dr}	4.678	1.944	0.979			KCP_{dr}	4.613	2.395	0.949
			KCP	4.013	1.479	0.935			KCP	4.327	11.267	0.785
2	65	0.3	E-Divisive _{dr}	4.074	0.092	0.969	20		E-Divisive _{dr}	3.499	1.822	0.841
			E-Divisive	0.311	14.155	0.275			E-Divisive	0.148	15.072	0.238
			$ks-cp3o_{dr}$	4.500	1.673	0.950		0.2	$ks-cp3o_{dr}$	5.721	7.487	0.867
			ks-cp3o	6.218	9.774	0.763		0.5	ks-cp3o	6.189	9.847	0.766
			KCP_{dr}	4.891	7.417	0.891			KCP_{dr}	2.427	12.665	0.516
			KCP	0.000	16.000	0.198			KCP	0.060	15.688	0.209
			E-Divisive _{dr}	4.050	0.084	0.980			E-Divisive _{dr}	4.056	0.084	0.969
		0.5	E-Divisive	0.600	12.626	0.336			E-Divisive	0.282	14.322	0.268
			ks-cp $3o_{dr}$	4.128	0.319	0.976		05	ks-cp $3o_{dr}$	4.417	1.394	0.955
			ks-cp3o	6.165	9.655	0.762		0.5	ks-cp3o	6.126	9.364	0.769
			KCP_{dr}	5.417	7.455	0.964			KCP_{dr}	4.974	8.482	0.882
			KCP	0.446	14.512	0.267			KCP	0.546	14.766	0.269
3	65		E-Divisive _{dr}	3.981	1.866	0.845	20		E-Divisive _{dr}	2.263	5.563	0.632
			E-Divisive	0.084	15.466	0.223			E-Divisive	0.099	15.367	0.226
		0.3	ks-cp $3o_{dr}$	5.606	7.152	0.863		0.3	ks-cp $3o_{dr}$	6.128	9.256	0.809
			ks-cp3o	6.312	9.944	0.772		0.5	ks-cp30	6.317	10.399	0.768
			KCP_{dr}	3.146	8.238	0.637			KCP_{dr}	2.362	17.662	0.441
			KCP	0.001	15.993	0.199			KCP	0.028	15.818	0.203
		0.5	E-Divisive _{dr}	4.152	0.176	0.977	20		$\operatorname{E-Divisive}_{dr}$	4.023	0.521	0.936
			E-Divisive	0.190	14.886	0.246			E-Divisive	0.178	14.896	0.245
			ks-cp $3o_{dr}$	4.201	0.621	0.967		0.5	ks-cp $3o_{dr}$	4.898	3.228	0.930
			ks-cp3o	6.322	10.206	0.774		0.0	ks-cp30	6.240	9.910	0.765
			KCP_{dr}	5.299	4.731	0.961			KCP_{dr}	3.967	6.703	0.778
			KCP	0.031	15.827	0.205			KCP	0.437	17.323	0.236
4	65	0.3	E-Divisive _{dr}	4.137	2.105	0.850			E-Divisive _{dr}	2.054	6.200	0.626
			E-Divisive	0.133	15.173	0.268			E-Divisive	0.085	15.422	0.259
			ks-cp $3o_{dr}$	5.797	8.496	0.818		0.3	ks-cp $3o_{dr}$	6.327	10.061	0.773
			ks-cp3o	6.291	9.990	0.733	20	0.0	ks-cp30	6.407	10.753	0.731
			KCP_{dr}	4.110	8.951	0.727			KCP_{dr}	1.607	12.371	0.438
			KCP	0.000	16.000	0.236			KCP	0.034	15.787	0.243
		0.5	E-Divisive _{dr}	4.242	0.433	0.970			E-Divisive _{dr}	3.835	0.639	0.936
			E-Divisive	0.270	14.349	0.300			E-Divisive	0.147	15.140	0.268
			ks-cp $3o_{dr}$	4.281	1.533	0.942		0.5	ks-cp $3o_{dr}$	5.053	5.390	0.874
			ks-cp3o	5.977	8.638	0.734		0.0	ks-cp30	6.141	9.828	0.725
			KCP_{dr}	6.010	9.077	0.946			KCP_{dr}	5.154	9.923	0.845
			KCP	0.000	16.000	0.236			KCP	0.180	15.278	0.260

Table 1: Changes in the covariance matrix in ${\it Experiment}~4$



Figure 1: Change point detection for aCGH data, the four figures plot the locations detected by E-Divisive, E-Divisive_{dr}, SBS and SBS_{dr} , respectively.

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